Chapter 2
Linear Spatial Dependence Models for Cross-Section Data

Abstract This chapter gives an overview of all linear spatial econometric models with different combinations of interaction effects that can be considered, as well as the relationships between them. It also provides a detailed overview of the direct and indirect effects estimates that can be derived from these models. In addition, it critically discusses the stationarity conditions that need to be imposed on the spatial interaction parameters and the spatial weights matrix, as well as the row-normalization procedure of the spatial weights matrix. The well-known cross-sectional dataset of Anselin (1988), explaining the crime rate by household income and housing values in 49 Columbus, Ohio neighborhoods, is used for illustration purposes.

Keywords Interaction effects · Model overview · Stationarity conditions · Normalization · Spatial spillover effects · Software · Crime

2.1 Introduction

Starting with a standard linear regression model, three different types of interaction effects in a spatial econometric model can be distinguished: endogenous interaction effects among the dependent variable (Y), exogenous interaction effects among the independent variables (X), and interaction effects among the error terms (ε). Originally, the central focus of spatial econometrics has been the spatial lag model, also known as the spatial autoregressive (SAR) model, and the spatial error model (SEM), both with one type of interaction effect.[1] The first model contains endogenous interaction effects, and the second model interaction effects among the error terms. The seminal book by Anselin (1988) and the testing procedure for a spatial lag or a spatial error model based on the robust Lagrange Multiplier tests

[1] In this book, we use the acronyms most commonly used in the spatial econometrics literature to refer to the model specifications (see e.g., LeSage and Pace 2009).

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developed by Anselin et al. (1996) may be considered as the main pillar behind this way of thinking.

In 2007 the interest for models containing more than one spatial interaction effect increased. In his keynote speech at the first World Conference of the Spatial Econometrics Association in 2007, Harry Kelejian advocated models that include both endogenous interaction effects and interaction effects among the error terms (based on Kelejian and Prucha 1998 and related work). This model is denoted by the term SAC in LeSage and Pace (2009, p. 32), though without pointing out what this acronym is standing for. Elhorst (2010) labels this model the Kelejian–Prucha model after their article in 1998 since they were the first to set out an estimation method for this model, also when the spatial weights matrix used to specify the spatial lag and the spatial error structure is the same. Kelejian and Prucha themselves alternately use the terms SARAR or Cliff-Ord type spatial model.

In his presidential address at the 54th North American Meeting of the Regional Science Association International in 2007, James LeSage advocated models that include both endogenous and exogenous interaction effects. This idea is worked out in the textbook which he published together with Kelley Pace in 2009 (LeSage and Pace 2009). In analogy to Durbin (1960) for the time series case, Anselin (1988) labeled the latter model as the spatial Durbin model (SDM).

Gibbons and Overman (2012) criticize the SAR, SEM and SDM models for reasons of identification, and advocate the SLX (spatial lag of *X*) model. To provide a better understanding, Section 2.2 first gives an overview of all linear spatial econometric models with different combinations of interaction effects that can be considered, as well as the relationships between them. Section 2.3 discusses the stationarity conditions that need to be imposed on the spatial interaction parameters and the spatial weights matrix. Section 2.4 explains and, more importantly, also critically discusses the row-normalization procedure of the spatial weights matrix. Too often this procedure leads to a misspecification problem, which can easily be avoided. Section 2.5 examines the parameter space on which the spatial interaction parameters are defined. Too often this parameter space is simply assumed to be $(-1, 1)$, just as in a time-series model. It is shown that this interval in a second order spatial autoregressive process is too restrictive, because it would lead to the exclusion of feasible and perhaps also relevant parameter combinations. Section 2.6 discusses some strengths and weakness of different estimation methods of spatial econometric models. Section 2.7 gives a detailed overview of direct and indirect effects estimates. The latter are also known as spatial spillover effects. Until recently, empirical studies used the coefficient estimates of a spatial econometric model to test the hypothesis as to whether or not spatial spillovers exist. However, LeSage and Pace (2009) point out that a partial derivative interpretation of the impact from changes to the variables represents a more valid basis for testing this hypothesis. By considering these partial derivatives, it is shown that some models are more flexible in modeling spatial spillovers than others. Section 2.8 lists software to estimate the models discussed in this chapter and presents Matlab routines the author of this book has made available at his Web site. Section 2.9 empirically illustrates the results of different spatial econometric models. Finally, Section 2.10 concludes.

2.2 A Taxonomy of Linear Spatial Dependence Models for Cross-Section Data

The standard approach in most spatial analyses is to start with a non-spatial linear regression model and then to test whether or not this so-called benchmark model needs to be extended with spatial interaction effects. This approach is known as the specific-to-general approach.[2] The non-spatial linear regression model takes the form

$$\boldsymbol{Y} = \alpha \boldsymbol{\iota}_N + \boldsymbol{X\beta} + \boldsymbol{\varepsilon} \tag{2.1}$$

where $\boldsymbol{Y}$ denotes an $N \times 1$ vector consisting of one observation on the dependent variable for every unit in the sample ($i = 1,\ldots,N$), $\boldsymbol{\iota}_N$ is an $N \times 1$ vector of ones associated with the constant term parameter α to be estimated, $\boldsymbol{X}$ denotes an $N \times K$ matrix of exogenous explanatory variables, $\boldsymbol{\beta}$ is an associated $K \times 1$ vector with unknown parameters to be estimated, and $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots, \varepsilon_N)^T$ is a vector of disturbance terms, where ε_i is assumed to be independently and identically distributed for all i with zero mean and variance σ^2.[3] Since the linear regression model is commonly estimated by Ordinary Least Squares (OLS), it is often labeled the OLS model.

The opposite approach is to start with a more general model containing, nested within it as special cases, a series of simpler models that ideally should represent all the alternative economic hypotheses requiring consideration. Generally, three different types of interaction effects may explain why an observation associated with a specific location may be dependent on observations at other locations. The first are endogenous interaction effects, where the dependent variable of a particular unit A depends on the dependent variable of other units, among which, say, unit B, and vice versa,

$$\text{Dependent variable } y \text{ of unit } A \leftrightarrow \text{Dependent variable } y \text{ of unit } B \tag{2.2}$$

Endogenous interaction effects are typically considered as the formal specification for the equilibrium outcome of a spatial or social interaction process, in which the value of the dependent variable for one agent is jointly determined with that of neighboring agents. In the empirical literature on strategic interaction among local governments, for example, endogenous interaction effects are

[2] For an explanation of this terminology see Hendry (1995).

[3] The superscript T indicates the transpose of a vector or matrix.

theoretically consistent with the situation where taxation and expenditures on public services interact with taxation and expenditures on public services in nearby jurisdictions (Brueckner 2003).

The second are exogenous interaction effects, where the dependent variable of a particular unit depends on independent explanatory variables of other units

$$\text{Independent variable } x \text{ of unit } B \leftrightarrow \text{ Dependent variable } y \text{ of unit } A \tag{2.3}$$

Consider, for example, the savings rate. According to standard economic theory, saving and investment are always equal. People cannot save without investing their money somewhere, and they cannot invest without using somebody's savings. This is true for the world as a whole, but it is not true for individual economies. Capital can flow across borders; hence the amount an individual economy saves does not have to be the same as the amount it invests. In other words, per capita income in one economy also depends on the savings rates of neighboring economies. It should be stressed that, if the number of independent explanatory variables in a linear regression model is K, the number of exogenous interaction effects might also be K, provided that the intercept is considered as a separate variable. In other words, not only the savings rate but also other explanatory variables may affect per capita income in neighboring economies. It is for this reason that in both the theoretical and the empirical literature on economic growth and convergence among countries or regions, the economic growth variable is taken to depend not only on the initial income level and the rates of saving, population growth, technological change and depreciation in the own economy, but also on those variables in neighboring economies (Ertur and Koch 2007; Elhorst et al. 2010).

The third type of interaction effects are those among the error terms

$$\text{Error term } u \text{ of unit } A \leftrightarrow \text{ Error term } u \text{ of unit } B \tag{2.4}$$

Interaction effects among the error terms do not require a theoretical model for a spatial or social interaction process, but instead, are consistent with a situation where determinants of the dependent variable omitted from the model are spatially autocorrelated, or with a situation where unobserved shocks follow a spatial pattern. Interaction effects among the error terms may also be interpreted to reflect a mechanism to correct rent-seeking politicians for unanticipated fiscal policy changes (Allers and Elhorst 2005).

A full model with all types of interaction effects takes the form

$$\boldsymbol{Y} = \delta \boldsymbol{WY} + \alpha \boldsymbol{\iota}_N + \boldsymbol{X\beta} + \boldsymbol{WX\theta} + \boldsymbol{u} \tag{2.5a}$$

$$\boldsymbol{u} = \lambda \boldsymbol{Wu} + \boldsymbol{\varepsilon} \tag{2.5b}$$

where $\boldsymbol{WY}$ denotes the endogenous interaction effects among the dependent variable, $\boldsymbol{WX}$ the exogenous interaction effects among the independent variables, and $\boldsymbol{Wu}$ the interaction effects among the disturbance term of the different units. We

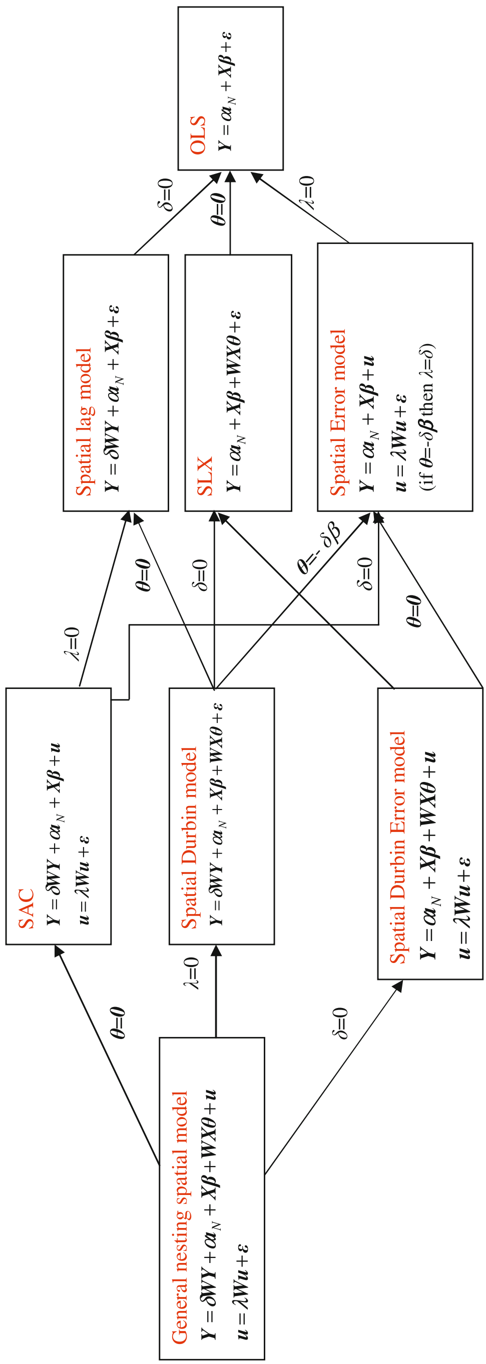

Fig. 2.1 The relationships between different spatial dependence models for cross-section data (*source* Halleck Vega and Elhorst 2012)

will refer to model (2.5a, b) as the general nesting spatial (GNS) model[4] since it includes all types of interaction effects. δ is called the spatial autoregressive coefficient, λ the spatial autocorrelation coefficient, while $\boldsymbol{\theta}$, just as $\boldsymbol{\beta}$, represents a $K \times 1$ vector of fixed but unknown parameters to be estimated. $\boldsymbol{W}$ is a nonnegative $N \times N$ matrix describing the spatial configuration or arrangement of the units in the sample. The next section discusses stationarity conditions that need to be imposed on $\boldsymbol{W}$ to obtain consistent estimators of the parameters in the GNS model.

Figure 2.1 summarizes a family of nine linear spatial econometric models, among which are the OLS model in (2.1) on the right-hand side and the GNS model in (2.5a, b) on the left-hand side. Each model to the right of the GNS model can be obtained from that model by imposing restrictions on one or more of its parameters. The restrictions are shown next to the arrows in Fig. 2.1. This figure shows that there are spatial econometric models that are hardly considered or used in econometric-theoretic and empirical research. The spatial Durbin error model (SDEM), which contains exogenous interaction effects and interaction effects among the error terms, is the best example. In this respect, it should be stressed that there is a large gap in the level of interest in different types of interaction effects between theoreticians and practitioners. Theoreticians are mainly interested in the SAR and SEM models, as well as the SAC model that combines endogenous interaction effects and interaction effects among the error terms, because of the econometric problems accompanying the estimation of these models. Some of these problems will be dealt with in the remainder of this chapter. The reason they generally do not focus on spatial econometric models with exogenous interaction effects is because the estimation of such models does not pose any econometric problems; standard estimation techniques suffice under these circumstances. Consequently, the SLX model is generally not part of the toolbox of researchers interested in the econometric theory of spatial models.

2.3 Stationarity Conditions for δ, λ and W

Spatial weights matrices commonly used in applied research are: (i) p-order binary contiguity matrices (if $p = 1$ only first-order neighbors are included, if $p = 2$ first and second order neighbors are considered, and so on); (ii) inverse distance matrices (with or without a cut-off point); (iii) q-nearest neighbor matrices (where q is a positive integer); (iv) block diagonal matrices where each block represents a group of spatial units that interact with each other but not with observations in other groups. Generally, spatial weights matrices are symmetric, but there are exceptions in which the spatial weights matrix is asymmetric. One example is a commuting flow matrix used to explain regional labor market performance.

[4] LeSage and Pace (2009) neither name nor assign an equation number to model (2.5a, b), which reflects the fact that this model is typically not used in applied research.

A symmetric matrix has the property that all its characteristic roots are real, also when it is row-normalized (see Sect. 2.4) and becomes asymmetric as a result, while an asymmetric matrix will also have complex characteristic roots.

Kelejian and Prucha (1998, 1999) and Lee (2004) make the following assumptions to prove consistency of respectively the GMM estimator of the parameters in the SAR and SAC models and the ML estimator in the SAR model. The spatial weights matrix $\boldsymbol{W}$ is a nonnegative matrix of known constants. The diagonal elements are set to zero by assumption, since no spatial unit can be viewed as its own neighbor. The matrices $\boldsymbol{I}_N$-$\delta\boldsymbol{W}$ and $\boldsymbol{I}_N$-$\lambda\boldsymbol{W}$ are non-singular, where $\boldsymbol{I_N}$ represents the identity matrix of order N. For a symmetric $\boldsymbol{W}$, this condition is satisfied as long as δ and λ are in the interior of $(1/\omega_{min}, 1/\omega_{max})$, where ω_{min} denotes the smallest (i.e. most negative) and ω_{max} the largest real characteristic root of $\boldsymbol{W}$. If $\boldsymbol{W}$ is normalized subsequently, the latter interval takes the form $(1/\omega_{min}, 1)$, since the largest characteristic root of $\boldsymbol{W}$ equals unity in this situation. If $\boldsymbol{W}$ is an asymmetric matrix before it is normalized, it may have complex characteristic roots. LeSage and Pace (2009, pp. 88–89) demonstrate that in that case δ and λ are restricted to the interval $(1/r_{min}, 1)$, where r_{min} equals the most negative purely real characteristic root of $\boldsymbol{W}$ after this matrix is row-normalized. Kelejian and Prucha (1998, 1999) assume that δ and λ are restricted to the interval $(-1, 1)$. We come back to this in Sect. 2.5. Finally, one of the following two conditions should be satisfied: (a) the row and column sums of the matrices $\boldsymbol{W}$, $(\boldsymbol{I}_N\text{-}\delta\boldsymbol{W})^{-1}$ and $(\boldsymbol{I}_N\text{-}\lambda\boldsymbol{W})^{-1}$ before $\boldsymbol{W}$ is row-normalized should be uniformly bounded in absolute value as N goes to infinity, or (b) the row and column sums of $\boldsymbol{W}$ before $\boldsymbol{W}$ is row-normalized should not diverge to infinity at a rate equal to or faster than the rate of the sample size N. Condition (a) is originated by Kelejian and Prucha (1998, 1999), and condition (b) by Lee (2004). Both conditions limit the cross-sectional correlation to a manageable degree, i.e. the correlation between two spatial units should converge to zero as the distance separating them increases to infinity. Below we discuss which of the four matrices introduced above satisfy both conditions (a) and (b), which only satisfy (b), and which satisfy neither (a) and (b).

When the spatial weights matrix is a p-order binary contiguity matrix and p is small, (a) is satisfied. Normally, no spatial unit is assumed to be a neighbor to more than a given number, say q, of other units. Automatically, condition (b) is also satisfied. By contrast, when the spatial weights matrix is an inverse distance matrix, (a) may not be satisfied. To see this, consider an infinite number of spatial units that are arranged linearly. Let the distance of each spatial unit to its first left- and right-hand neighbor be d; to its second left- and right-hand neighbor, the distance $2d$; and so on. When $\boldsymbol{W}$ is an inverse distance matrix and its off-diagonal elements are of the form $1/d_{ij}$, where d_{ij} is the distance between two spatial units i and j, each row sum is $2 \times (1/d + 1/(2d) + 1/(3d) + \ldots)$, representing a series that is not finite. This is perhaps the reason why some empirical applications introduce a cut-off point d^* such that $w_{ij} = 0$ if $d_{ij} > d^*$. However, since the ratio $2 \times (1/d + 1/(2d) + 1/(3d) + \ldots)/N \rightarrow 0$ as N goes to infinity, condition (b) is satisfied, which implies that an inverse distance matrix without a cut-off point does not necessarily have to be excluded in an empirical study for reasons of consistency. Nevertheless, an inverse distance matrix is a border case, which explains why it sometimes leads to numerical problems or unexpected outcomes in empirical applications. This is because the number of units in the sample generally does not go to infinity, but is finite.

Another situation occurs when all cross-sectional units are assumed to be neighbors of each other and are given equal weights. In that case all off-diagonal elements of the spatial weights matrix are $w_{ij} = 1$. Since the row and column sums are $N - 1$, these sums diverge to infinity as N goes to infinity. In contrast to the previous case, however, $(N - 1)/N \rightarrow 1$ instead of 0 as N goes to infinity. This implies that a spatial weights matrix that has equal weights and that is row-normalized subsequently, $w_{ij} = 1/(N - 1)$, must be excluded for reasons of consistency since it satisfies neither condition (a) nor (b). The alternative is a group interaction matrix, introduced by Case (1991). Suppose there are G groups and that there are N_g cross-sectional units in each group. Let $w_{ij} = 1/(N_g - 1)$ if units i and j belong to the same group, and zero otherwise. If both N and N_g tend to infinity, with at least two units in each group, or if the number of units in each group does not tend to infinity faster than or equal to the number of groups, condition (b) is restored (Lee 2007).

2.4 Normalizing W

For ease of interpretation, it is common practice to normalize $\boldsymbol{W}$ such that the elements of each row sum to unity. Since $\boldsymbol{W}$ is nonnegative, this ensures that all weights are between 0 and 1, and has the effect that the weighting operation can be interpreted as an averaging of neighboring values.

As an alternative, $\boldsymbol{W}$ might be normalized such that the elements of each column sum to one. This type of normalization is sometimes used in the new social economics literature (Leenders 2002). Note that the column elements of a spatial weights matrix display the impact *of* a particular unit *on* all other units, while the row elements of a spatial weights matrix display the impact *on* a particular unit *by* all other units. Consequently, row normalization has the effect that the impact on each unit by all other units is equalized, while column normalization has the effect that the impact of each unit on all other units is equalized.

Although common practice, row normalization is not free of criticism. Kelejian and Prucha (2010) demonstrate that normalization of the elements of the spatial weights matrix by a different factor for each row as opposed to a single factor is likely to lead to misspecification problem. This problem occurs especially when an inverse distance matrix is row normalized, because its economic interpretation in terms of distance decay will then no longer be valid (Anselin 1988, pp. 23–24; Elhorst 2001). There are (at least) two reasons for this. First of all, because of row-normalization the spatial weights matrix may become asymmetric, as a result of which the impact of unit i on unit j is not the same as that of unit j on unit *i*. Secondly, as a consequence of row normalization remote and central regions will end up having the same impact, i.e. independent on their relative location. The following example may illustrate this. Consider a centrally located spatial unit and a remote unit that both have two neighbors. The distance of the first unit to its neighbors is *d*, while the distance of the second unit to its neighbors is a multiple of *d*. Despite this difference in location, the entries in the inverse distance matrix describing the spatial arrangement of the units in the sample will be 1/2 in both cases, provided that the spatial weights matrix is row-normalized.

If $\boldsymbol{W_0}$ denotes the spatial weights matrix before normalization, Elhorst (2001) and Kelejian and Prucha (2010) propose a normalization procedure where each element of $\boldsymbol{W_0}$ is divided by its largest characteristic root, $r_{0,max}$, to get $\boldsymbol{W} = (1/r_{0,max})\boldsymbol{W_0}$.[5] Alternatively, one may normalize $\boldsymbol{W_0}$ by $\boldsymbol{W} = \boldsymbol{D}^{-1/2}\boldsymbol{W_0}\boldsymbol{D}^{-1/2}$, where $\boldsymbol{D}$ is a diagonal matrix containing the row sums of the matrix $\boldsymbol{W_0}$. The first operation has the effect that the characteristic roots of $\boldsymbol{W_0}$ are also divided by $r_{0,max}$, as a result of which $r_{max} = 1$, just like the largest characteristic root of a row-normalized matrix. However, the smallest (purely) real characteristic root of a matrix that is normalized by a single factor is generally not the same as that of a matrix that is row-normalized. The second operation has been proposed by Ord (1975) and has the effect that the characteristic roots of $\boldsymbol{W}$ are identical to the characteristic roots of a row-normalized $\boldsymbol{W_0}$. Importantly, the mutual proportions between the elements of $\boldsymbol{W}$ remain unchanged as a result of these two normalizations. This is an important property when $\boldsymbol{W}$ represents an inverse distance matrix, since it avoids that this matrix would lose its economic interpretation of distance decay.

2.5 The Parameter Space of δ and λ

To investigate the asymptotic properties of the GMM estimator, Kelejian and Prucha (1998, 1999, and related work) presume that δ is restricted to the interval $(-1, 1)$. This presumption is based on earlier work of Kelejian and Robinson (1995), who demonstrate that the restriction $1/r_{min} < \delta < 1/r_{max}$, before $\boldsymbol{W}$ is row-normalized, may be unnecessarily restrictive since any first-order spatial autoregressive process is defined for every δ as long as the matrix ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$) is non-singular. The following example taken from Elhorst (2001) illustrates this. Let $N = 2$ and $\boldsymbol{W}$ the corresponding spatial weights matrix of two spatial units whose off-diagonal elements are unity, as a result of which $r_{min} = -1$ and $r_{max} = 1$. If $\boldsymbol{\varepsilon} \sim \mathrm{N}(\boldsymbol{0}, \sigma^2\boldsymbol{I}_N)$, then $\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{0}, (1 + \delta)/(1 - \delta^2)\sigma^2\boldsymbol{I}_N)$, which shows that the variance of $\boldsymbol{Y}$ is finite when the variance of $\boldsymbol{\varepsilon}$ is finite for every δ unless $\delta = 1/r_{min}$ or $\delta = 1/r_{max}$. In other words, the matrix ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$) is non-singular and its inverse is finite only when δ is not

[5] We use the symbol r rather that ω to denote that both symmetric and asymmetric spatial weights matrices are covered here.

equal to the reciprocal of just one of the two characteristic roots of the spatial weights matrix $\boldsymbol{W}$ (see Kelejian and Prucha 2010 for a generalization). According to Bell and Bockstael (2000), this feature is rather curious. The values of δ that make the problem undefined are related directly to the characteristic roots of $\boldsymbol{W}$, which will change if the sample size changes. With no further restrictions, the problem is characterized by a non-continuous parameter space, changing with the addition or the elimination of any observation. To avoid these difficulties and to facilitate the estimation of δ, as well as to ensure the invertibility of the matrix ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$), Ord (1981) suggests to restrict δ to $1/r_{min} < \delta < 1/r_{max}$ before $\boldsymbol{W}$ is row-normalized and to $1/r_{min} < \delta < 1$ after this. Kelejian and Robinson (1995), on their turn, suggest to restrict δ to $-1 < \delta < 1$, to stress the similarity between time-series and spatial econometrics. A first-order serial autoregressive process

$$y_t = \rho y_{t-1} + \varepsilon_t \tag{2.6}$$

with T observations is stationary if ρ lies in the interval $(-1, 1)$. However, the same interval for a first-order spatial autoregressive process would be too restrictive. For normalized spatial weights, the largest characteristic root is indeed +1, but no general result holds for the smallest characteristic root, and the lower bound will be typically less than -1.

Although there might be some similarities for first-order models, substantive differences occur when considering second-order models. The time-series literature (see Beach and MacKinnon 1978, and the references therein) has pointed out that a second-order serial autoregressive process

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t \tag{2.7}$$

with T observations is stationary if $\rho_1 + \rho_2 < 1$, $1 + \rho_2 - \rho_1 > 0$ and $\rho_2 > -1$. These constraints define a triangular region with vertices at $(-2, -1)$, $(0, 1)$ and $(2, -1)$.

A second-order spatial autoregressive process takes the form

$$\boldsymbol{Y} = \delta_1 \boldsymbol{W}_1 \boldsymbol{Y} + \delta_2 \boldsymbol{W}_2 \boldsymbol{Y} + \boldsymbol{\varepsilon} \tag{2.8}$$

where $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are assumed to be normalized. This model as well as some extensions of it have been considered in many studies. Examples are Brandsma and Ketellapper (1979), Sherrell (1990), Hepple (1995), Bell and Bockstael (2000), Bordignon et al. (2003), Lacombe (2004), Allers and Elhorst (2005), McMillen et al. (2007), Ward and Gleditsch (2008); Dall'Erba et al. (2008), Elhorst and Fréret (2009), Lee and Liu (2010), Badinger and Egger (2011), and Elhorst et al. (2012). However, most of these studies do not specify a parameter space for δ_1 and δ_2. Only Lee and Liu (2010) and Badinger and Egger (2011) mention that the sum of the absolute values of the two spatial parameters should be less than one ($|\delta_1| + |\delta_2| < 1$). However, it can be easily seen that this constraint proves to be too restrictive. The fact that δ in a first-order spatial autoregressive process should lie in the interval $(1/r_{min}, 1)$ immediately determines four coordinates of the stationarity region: $(1, 0)$ and $(1/r_{1,min}, 0)$ in case $\delta_2 = 0$, and $(0, 1)$

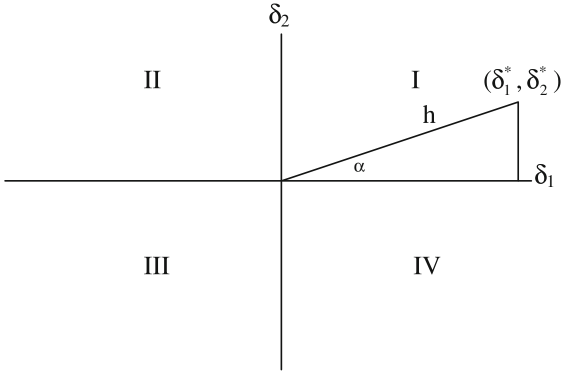

Fig. 2.2 δ_1 and δ_2 and the four quadrants in a two-dimensional space

and (0, $1/r_{2,min}$) in case $\delta_1 = 0$. These four coordinates define a region that is wider than the one assumed in Lee and Liu (2010) and Badinger and Egger (2011). Additionally, these four coordinates demonstrate that the stationarity region does not coincide with the time-series version of the model.

Elhorst et al. (2012) have developed the following procedure to determine the exact boundaries of the curves connecting these four coordinates, depending on the specification of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. Consider the four quadrants defined by the two axes in Fig. 2.2 and the angle α between δ_1 and the hypotenuse (h) that connects the origin with the coordinates of a point located at the border of the feasible region, denoted by (δ_1^*, δ_2^*). Since $\tan(\alpha) = \delta_2^*/\delta_1^*$, we get

$$\delta_2^* = \tan(\alpha)\delta_1^*, \quad \text{for } -270^\circ < \alpha < -90^\circ \quad \text{or } -90^\circ < \alpha < 90^\circ \tag{2.9}$$

Consequently, Eq. (2.8) can be rewritten as

$$\boldsymbol{Y} = \delta_1^*[\boldsymbol{W}_1 + \tan(\alpha)\boldsymbol{W}_2]\boldsymbol{Y} + \varepsilon = \delta_1^*\boldsymbol{W}^*\boldsymbol{Y} + \varepsilon \tag{2.10}$$

This implies that the model is stationary for the following parameter combinations (depending on α)

$$0 < \delta_1 < 1/r_{\max}[\boldsymbol{W}^*], \quad 0 \le \delta_2 \le \tan(\alpha)/r_{\max}[\boldsymbol{W}^*], \quad 0^\circ \le \alpha < 90^\circ \tag{2.11a}$$

$$0 < \delta_1 < 1/r_{\max}[\boldsymbol{W}^*], \quad \tan(\alpha)/r_{\max}[\boldsymbol{W}^*] < \delta_2 < 0, \quad -90^\circ < \alpha < 0^\circ \tag{2.11b}$$

$$\delta_1 = 0, \quad 1/r_{\min}[\boldsymbol{W}_2] < \delta_2 < 0 \quad \alpha = -90^\circ \tag{2.11c}$$

$$1/r_{\min}[\boldsymbol{W}^*] < \delta_1 < 0, \quad \tan(\alpha)/r_{\max}[\boldsymbol{W}^*] \le \delta_2 \le 0, \quad -180^\circ \le \alpha < -90^\circ \tag{2.11d}$$

$$1/r_{\min}[\boldsymbol{W}^*] < \delta_1 < 0, \quad 0 < \delta_2 < \tan(\alpha)/r_{\min}[\boldsymbol{W}^*], \quad -270^\circ < \alpha < -180^\circ \tag{2.11e}$$

$$\delta_1 = 0, \quad 0 < \delta_2 < 1/r_{\max}[\boldsymbol{W}_2], \quad \alpha = -270^\circ \tag{2.11f}$$

where r_{max}[.] and r_{min}[.] are the largest (positive) and smallest (negative) purely real characteristic roots of the matrix in square brackets. When $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are

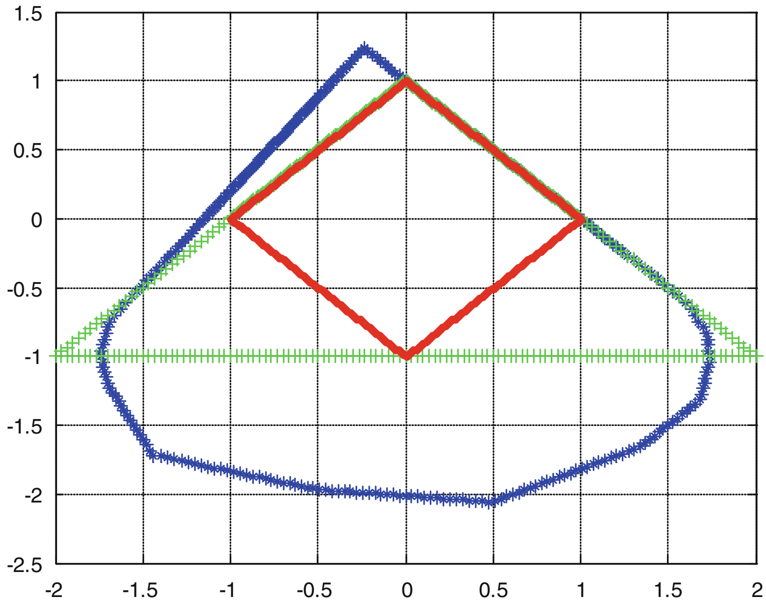

Fig. 2.3 The potential shape of the stationarity region of a second-order spatial autoregressive process (*source* Elhorst et al. 2012)

normalized, the largest characteristic root of the matrix $\delta_1 \boldsymbol{W}_1 + \delta_2 \boldsymbol{W}_2$ in the first quadrant is $\delta_1 + \delta_2$. This implies that the model is stationary for values of δ_1 and δ_2 in the first quadrant if $\delta_1 + \delta_2 < 1$. This expression shows that the parameter space in the first quadrant is independent of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ and thus identical to both the time-series form and the definition given by Lee and Liu (2010) and Badinger and Egger (2011). The curves connecting the coordinates in the other quadrants, depend on $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ and, therefore, might define different parameter spaces.

Elhorst et al. (2012) illustrate the potential shape of the stationarity region for different pairs of spatial weights matrices. Typically, the stationarity region takes the form graphed (in blue) in Fig. 2.3. In addition, Fig. 2.3 graphs the rhombus (in red) implied by the restriction $|\delta_1| + |\delta_2| < 1$ and the triangle (in green) that corresponds to the stationarity region of the second-order serial autoregressive process.

The rhombus in Fig. 2.3 shows that the naïve adoption of the restriction $|\delta_1| + |\delta_2| < 1$ in a second-order spatial autoregressive process is not recommended, because it would lead to the exclusion of feasible and perhaps also relevant parameter combinations. Up to now, positive spatial autocorrelation has been encountered in empirical data more frequently than negative spatial autocorrelation, and researchers tend to consider negative autocorrelation less relevant. Typically, if a particular variable increases (decreases) in one area, it also tends to increase (decrease) in neighboring areas. However, Griffith and Arbia (2010) present three examples of negatively spatially autocorrelated phenomena that are all based on the economic notion of competitive locational processes. If the manifestation of a certain phenomenon in one area is at the expense of its neighboring areas, then negative spatial autocorrelation is likely to occur. Let $(\delta_1,\delta_2) = (0.8, -0.3)$ or $(\delta_1,\delta_2) = (1.1,-0.3)$ be the outcome of a spatial econometric model. Based on the restriction $|\delta_1| + |\delta_2| < 1$, these combinations of values would be rejected but based on the results presented in this section, they should not be excluded. Additionally, as discussed in Griffith and Arbia (2010), they also make sense from a theoretical point of view.

The triangle that corresponds to the stationarity region of the second-order serial autoregressive process shows that the naïve adoption of the time-series region is not recommended either. It would not only lead to the exclusion of feasible parameter combinations, but also to include some infeasible ones.

In other words, the knowledge of the exact boundary is important, both for estimation and inference. More discussion on these issues can also be found in LeSage and Pace (2011). Elhorst et al. (2012) have made a Matlab routine downloadable for free on their web sites to determine the exact boundaries of any second-order spatial autoregressive process.[6]

2.6 Methods of Estimation

Spatial econometric models can be estimated by maximum likelihood (ML) (Ord 1975), quasi-maximum likelihood (QML) (Lee 2004), instrumental variables (IV) (Anselin 1988, pp. 82–86), generalized method of moments (GMM) (Kelejian and Prucha 1998, 1999), or by Bayesian Markov Chain Monte Carlo methods (Bayesian MCMC) (LeSage 1997). In the next two chapters we extensively discuss the ML estimation procedure of (dynamic) spatial panel data models. Due to the overlap with the ML estimation procedure of cross-sectional spatial econometric models, this section only discusses some strengths and weaknesses of the different estimation methods. Furthermore, updated overviews of these estimation methods can be found in the Handbook of Regional Science that appeared in 2013.

One advantage of QML and IV/GMM estimators is that they do not rely on the assumption of normality of the disturbances ε. Nonetheless, both estimators assume that the disturbance terms ε_i are independently and identically distributed for all i with zero mean and variance σ^2. One disadvantage of the IV/GMM estimator is the possibility of ending up with a coefficient estimate for δ in the SAR model or for λ in the SEM model outside its parameter space. Whereas these coefficients are restricted to the interval $(1/r_{min}, 1)$ by the Jacobian term in the log-likelihood function of ML estimators or in the conditional distribution of the spatial parameter of Bayesian estimators, they are unrestricted using IV/GMM since these estimators ignore the Jacobian term.

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[6] http://www.regroningen.nl/elhorst and http://community.wvu.edu/~djl041/.

To avoid computational difficulties was one of the reasons to develop IV/GMM estimators (Kelejian and Prucha, 1998, 1999). Estimation of spatial econometric models involves the manipulation of $N \times N$ matrices, such as matrix multiplication, matrix inversion, the computation of characteristic roots and/or Cholesky decomposition. These manipulations may be computationally intensive and/or may require significant amounts of memory if N is large. Since IV/GMM estimators ignore the Jacobian term, many of these problems could be avoided. In Chap. 4 of their book, however, LeSage and Pace (2009) produce conclusive evidence that many of these computational difficulties have become a thing of the past for ML and Bayesian estimators.

In spite of this, Fingleton and Le Gallo (2007, 2008), Drukker et al. (2013) and Liu and Lee (2013) show that IV/GMM estimators are extremely useful in those cases where linear spatial dependence models contain one or more endogenous explanatory variables (other than the spatially lagged dependent variable) that need to be instrumented, because of measurement errors in explanatory variables, omitted variables correlated with included explanatory variables, or because of the existence of an underlying (perhaps unspecified or unknown) set of simultaneous structural equations. ML or Bayesian estimators of single equation models with a spatial lag (i.e. the spatial lag model and the spatial Durbin model) and additional endogenous variables do not feature in the spatial econometrics literature and would be difficult, if not impossible, to derive. The same applies to single equation models with a spatial error process (i.e. the spatial error model and the spatial Durbin error model). By contrast, models including a spatial lag and additional endogenous variables can be straightforwardly estimated by two-stage least squares (2SLS). To instrument the spatially lagged dependent variable, Kelejian et al. (2004) suggest $[\boldsymbol{X}\ \boldsymbol{WX}\ \ldots\ \boldsymbol{W}^g\boldsymbol{X}]$, where g is a pre-selected constant.[7] Typically, researchers take $g = 1$ or $g = 2$, dependent on the number of regressors and the type of model. One potential problem in case of the spatial Durbin model is that g should be at least two, since this model already contains the variables $\boldsymbol{X}$ and $\boldsymbol{WX}$ on the right-hand side. This means that the number of potential strong instruments diminishes considerably.

If one or more of the explanatory variables are endogenous, the set of instruments must be limited to $[\boldsymbol{X}^{ex}\ \boldsymbol{WX}^{ex}\ \ldots\ \boldsymbol{W}^d\boldsymbol{X}^{ex}]$, where 'ex' denotes the $\boldsymbol{X}$ variables that are exogenous. Furthermore, this set should be used to instrument the additional endogenous explanatory variables. A similar type of extension applies to Kelejian and Prucha's (1999) GMM estimator for models including a spatial error process together with endogenous explanatory variables (Fingleton and Le Gallo 2007). In addition, Fingleton and Le Gallo (2008) consider a mixed 2SLS/GMM estimator of the Kelejian-Prucha model extended to include endogenous explanatory variables.

[7] Lee (2003) introduces the optimal instrument 2SLS estimator, but Kelejian et al. (2004) show that the 2SLS estimator based on this set of instruments has quite similar small sample properties.

Liu and Lee (2013) consider the IV estimation of the spatial lag model with endogenous regressors when the number of instruments grows with the sample size. They suggest a bias-correction procedure based on the leading-order many-instrument bias. To choose among different instruments, they also suggest minimizing an approximation of the mean square error of both the 2SLS and bias-corrected 2SLS estimators.

In an overview paper, Drukker et al. (2013) consider the GMM estimation of the Kelejian-Prucha model with endogenous regressors. Since the model contains more endogenous explanatory variables than $\boldsymbol{WY}$, they suggest (p. 693) "to use a set of instruments as above … augmented by other exogenous variables expected to be part of the reduced form of the system".

One major weakness of spatial econometric models is that the spatial weights matrix $\boldsymbol{W}$ cannot be estimated but needs to be specified in advance and that economic theory underlying spatial econometric applications often has little to say about the specification of $\boldsymbol{W}$ (Leenders 2002). For this reason, it has become common practice to investigate whether the results are robust to the specification of $\boldsymbol{W}$. The same spatial econometric model is estimated, say, S times, every time with a different spatial weights matrix, to investigate whether the estimation results are sensitive to the choice of $\boldsymbol{W}$. One advantage of the Bayesian MCMC estimator is that it offers a criterion, the Bayesian posterior model probability, to select the spatial weights matrix that best describes the data. Whereas tests for significant differences between log-likelihood function values, such as the LR-test, can formally not be used if models are non-nested (i.e. based on different spatial weights matrices), Bayesian posterior model probabilities do not require nested models to carry out these comparisons. The basic idea is to set prior probabilities equal to $1/S$, making each model equally likely a priori, to estimate each model by Bayesian methods, and then to compute posterior probabilities based on the data and the estimation results of this set of S models. Successful applications of this methodology can be found in LeSage and Page (2009, Chap. 6) and Seldadyo et al. (2010).

A Monte-Carlo study of Stakhovych and Bijmolt (2009) demonstrates that a weights matrix selection procedure that is based on 'goodness-of-fit' criteria increases the probability of finding the true specification. If a spatial interaction model is estimated based on S different spatial weights matrices and the log-likelihood function value of every model is estimated, one may select the spatial weights matrix exhibiting the highest log-likelihood function value. However, since LR-tests may formally not be used, one better selects the spatial weights matrix exhibiting the highest Bayesian posterior model probability. Alternatively, one may use J-type statistics to discriminate between different specifications of $\boldsymbol{W}$ (Anselin 1986; Kelejian 2008; Burridge and Fingleton 2010; Burridge 2012).

Harris et al. (2011) criticize these empirical approaches, because they would only find a local maximum among the competing spatial weights matrices and not necessarily a correctly specified $\boldsymbol{W}$ (unless it is unknowingly included in the set of competing matrices considered). However, the Monte Carlo results found by Stakhovych and Bijmolt (2009) partly refute this critique. Although there is a serious probability of selecting the wrong spatial weights matrix if spatial dependence is weak (δ or λ are relatively small in magnitude), the consequences of this poor choice are limited because the coefficient estimates are quite close to the true ones. Conversely, although the wrong choice of a spatial weights matrix can distort the coefficient estimates severely, the probability that this really happens is small if spatial dependence is strong (δ or λ are relatively large in magnitude).

Corrado and Fingleton (2012) strongly argue for the use of more substantive theory in empirical spatial econometric modeling, especially regarding $\boldsymbol{W}$. Despite their criticism, they point out that alternatives to $\boldsymbol{W}$ that have been proposed by e.g., Folmer and Oud (2008) and Harris et al. (2011), such as entering variables in the regression model that proxy spillovers, also require identifying assumptions. In other words, this approach also involves an a priori specification of the spatial relation between units in the sample.

2.7 Direct and Indirect (or Spillover) Effects

Many empirical studies use the point estimates of one or more spatial regression model specifications (δ, $\boldsymbol{\theta}$ and/or λ) to draw conclusions as to whether or not spatial spillovers exist. One of the key contributions of LeSage and Pace's book (2009, p. 74) is the observation that this may lead to erroneous conclusions, and that a partial derivative interpretation of the impact from changes to the variables of different model specifications represents a more valid basis for testing this hypothesis. To illustrate this, they give an example of a spatially lagged independent variable *WX* whose coefficient is negative and insignificant (ibid, Table 3.3), while it's spatial spillover effect is positive and significant (ibid, Table 3.4). The explanation for this can be seen by the derivation below.

By rewriting the general nesting spatial (GNS) model in (2.5a, b) as

$$\boldsymbol{Y} = (\boldsymbol{I} - \delta\boldsymbol{W})^{-1}(\boldsymbol{X\beta} + \boldsymbol{WX\theta}) + \boldsymbol{R} \tag{2.12}$$

where $\boldsymbol{R}$ is a rest term containing the intercept and the error terms, the matrix of partial derivatives of the expected value of $\boldsymbol{Y}$ with respect to the kth explanatory variable of X in unit 1 up to unit N in time can be seen to be

$$\begin{aligned} \begin{bmatrix} \frac{\partial E(\boldsymbol{Y})}{\partial x_{1k}} & \cdot & \frac{\partial E(\boldsymbol{Y})}{\partial x_{Nk}} \end{bmatrix} &= \begin{bmatrix} \frac{\partial E(y_1)}{\partial x_{1k}} & \cdot & \frac{\partial E(y_1)}{\partial x_{Nk}} \\ \cdot & \cdot & \cdot \\ \frac{\partial E(y_N)}{\partial x_{1k}} & \cdot & \frac{\partial E(y_N)}{\partial x_{Nk}} \end{bmatrix} \\ &= (\boldsymbol{I} - \delta\boldsymbol{W})^{-1} \begin{bmatrix} \beta_k & w_{12}\theta_k & \cdot & w_{1N}\theta_k \\ w_{21}\theta_k & \beta_k & \cdot & w_{2N}\theta_k \\ \cdot & \cdot & \cdot & \cdot \\ w_{N1}\theta_k & w_{N2}\theta_k & \cdot & \beta_k \end{bmatrix} \end{aligned} \tag{2.13}$$

where w_{ij} is the (i, j)th element of $\boldsymbol{W}$. This result illustrates that the partial derivatives of E($\boldsymbol{Y}$) with respect to the kth explanatory variable have three important properties. First, if a particular explanatory variable in a particular unit changes, not only will the dependent variable in that unit itself change but also the dependent variables in other units. The first is called a *direct effect* and the second an *indirect effect*. Note that every diagonal element of the matrix of partial derivatives represents a direct effect, and that every off-diagonal element represents an indirect effect. Consequently, indirect effects do not occur if both $\delta = 0$ and $\theta_k = 0$, since all off-diagonal elements will then be zero [see (2.13)].

Second, direct and indirect effects are different for different units in the sample. Direct effects are different because the diagonal elements of the matrix $(\boldsymbol{I}_N\text{-}\delta\boldsymbol{W})^{-1}$ are different for different units, provided that $\delta \neq 0$ [see the diagonal elements of (2.13)]. Indirect effects are different because both the off-diagonal elements of the matrix $(\boldsymbol{I}_N\text{-}\delta\boldsymbol{W})^{-1}$ and of the matrix $\boldsymbol{W}$ are different for different units, provided that $\delta \neq 0$ and/or $\theta_k \neq 0$ [see the off-diagonal elements of (2.13)].

Third, indirect effects that occur if $\theta_k \neq 0$ are known as *local effects*, as opposed to indirect effects that occur if $\delta \neq 0$ and that are known as *global effects*. Local effects got their name because they arise only from a unit's neighborhood set; if the element w_{ij} of the spatial weights matrix is non-zero (zero), then the effect of x_{jk} on y_i is also non-zero (zero). Global effects got their name because they also arise from units that do not belong to a unit's neighborhood set. This follows from the fact that the matrix $(\boldsymbol{I}_N\text{-}\delta\boldsymbol{W})^{-1}$, in contrast to $\boldsymbol{W}$, does not contain zero elements (provided that $\delta \neq 0$) [see $\boldsymbol{W}$ and $(\boldsymbol{I}_N\text{-}\delta\boldsymbol{W})^{-1}$ in (2.13)]. If both $\delta \neq 0$ and $\theta_k \neq 0$, both global and local effects occur which cannot be separated from each other.

Since both the direct and indirect effects are different for different units in the sample, the presentation of these effects is a problem. If we have N spatial units and K explanatory variables, we obtain K different $N \times N$ matrices of direct and indirect effects. Even for small values of N and K, it may already be rather difficult to report these results compactly. To improve the surveyability of the estimation results of spatial regression model specifications, LeSage and Pace (2009) therefore propose to report one summary indicator for the direct effect, measured by the average of the diagonal elements of the matrix on the right-hand side of (2.13), and one summary indicator for the indirect effect, measured by the average of either the row sums or the column sums of the off-diagonal elements of that matrix. The average row effect represents the impact on a particular element of the dependent variable as a result of a unit change in all elements of an exogenous variable, while the average column effect represents the impact of changing a particular element of an exogenous variable on the dependent variable of all other units. However, since the numerical magnitudes of these two calculations of the indirect effect are the same, it does not matter which one is used. Generally, the indirect effect is interpreted as the impact of changing a particular element of an exogenous variable on the dependent variable of all other units, which corresponds to the average column effect.

2.7.1 Direct and Indirect Effects of Different Spatial Econometric Models

The direct and indirect effects corresponding to the different spatial econometric models introduced in Sect. 2.2 and presented in Fig. 2.1 and for an arbitrary spatial weights matrix are reported in Table 2.1 (Halleck Vega and Elhorst 2012).

If the OLS model is adopted, the direct effect of an explanatory variable is equal to the coefficient estimate of that variable (β_k), while its indirect effect is zero by construction. If the OLS model is augmented with a spatially autocorrelated error term to obtain the SEM model, the direct and the indirect effects remain the same. This is because the disturbances do not come into play when considering the partial derivative of the dependent variable with respect to changes in the explanatory variables (see [2.13]). This property also holds for the extension of the SAR, SLX and the SDM model with spatial autocorrelation, i.e. the SAC, SDEM and the GNS model, respectively.

If the SLX or the SDEM model is adopted, the direct effect of an explanatory variable is equal to the coefficient estimate of that variable (β_k), while its indirect effect is equal to the coefficient estimate of its spatial lagged value (θ_k). The advantage of these models is that the direct and indirect effects do not require further calculations and that both these effects might be different from one explanatory variable to another.

Things get complicated when moving to one of the other models due to the multiplication with the spatial multiplier matrix $(\boldsymbol{I}\text{-}\delta\boldsymbol{W})^{-1}$. Whereas the direct effect of the kth explanatory variable in the OLS, SEM, SLX and SDEM models is β_k, the direct effect in the SAR and SAC models is β_k premultiplied with a number that will eventually be greater than or equal to unity. This can be seen by decomposing the spatial multiplier matrix as follows

$$(\boldsymbol{I}-\delta\boldsymbol{W})^{-1} = \boldsymbol{I} + \delta\boldsymbol{W} + \delta^2\boldsymbol{W}^2 + \delta^3\boldsymbol{W}^3\ldots \tag{2.14}$$

Since the non-diagonal elements of the first matrix term on the right-hand side (the identity matrix $\boldsymbol{I}$) are zero, this term represents a direct effect of a change in X only. Conversely, since the diagonal elements of the second matrix term on the right-hand side ($\delta\boldsymbol{W}$) were assumed to be zero (see Sect. 2.2), this term represents

Table 2.1 Direct and spillover effects of different model specifications

	Direct effect	Indirect effect
OLS/SEM	β_k	0
SAR/SAC	Diagonal elements of $(\boldsymbol{I}-\delta\boldsymbol{W})^{-1}\beta_k$	Off-diagonal elements of $(\boldsymbol{I}-\delta\boldsymbol{W})^{-1}\beta_k$
SLX/SDEM	β_k	θ_k
SDM/GNS	Diagonal elements of $(\boldsymbol{I}-\delta\boldsymbol{W})^{-1}(\beta_k + \boldsymbol{W}\theta_k)$	Off-diagonal elements of $(\boldsymbol{I}-\delta\boldsymbol{W})^{-1}(\beta_k + \boldsymbol{W}\theta_k)$

Source Halleck Vega and Elhorst (2012) an indirect effect of a change in X only. Furthermore, since $\boldsymbol{W}$ is taken to the power 1 here, this indirect effect is limited to first-order neighbors only, i.e. the units that belong to the neighborhood set of every spatial unit. All other terms on the right-hand side represent second- and higher-order direct and indirect effects. Higher-order direct effects arise as a result of feedback effects, i.e. impacts passing through neighboring units and back to the unit itself (e.g. $1 \to 2 \to 1$ and $1 \to 2 \to 3 \to 2 \to 1$). It is these feedback effects that are responsible for the fact that the overall direct effect is eventually greater than unity.[8]

One important limitation of the spatial lag model is that the ratio between the indirect and the direct effect of a particular explanatory variable is independent of β_k. This is because β_k in the numerator and β_k in the denominator of this ratio cancel each other out. This property implies that the ratio between the indirect and direct effects in the spatial lag model is the same for every explanatory variable, and that its magnitude depends on the spatial autoregressive parameter δ and the specification of the spatial weights matrix $\boldsymbol{W}$ only. In many empirical applications, this is not very likely.

If the SDM model is adopted, both the direct effect and the indirect effect of a particular explanatory variable will also depend on the coefficient estimate θ_k of the spatially lagged value of that variable (see Table 2.1). The result is that no prior restrictions are imposed on the magnitude of both the direct and indirect effects and thus that the ratio between the indirect and the direct effect may be different for different explanatory variables, just as in the SLX and SDEM model. Due to this flexibility, the SLX, SDM, SDEM models are a more attractive point of departure in an empirical study than other spatial regression specifications.

Figure 2.1 and Table 2.1 seem to indicate that the best strategy to test for spatial interaction effects and to determine indirect effects is to start with the most general model. The direct and indirect effects of the GNS model, which were derived in Eq. (2.13), are similar to those of the SDM model. However, one major problem is that the parameters of this GNS model are only weakly identified. The empirical illustration in Sect. 2.9 will show that the SDM and the SDEM models are already difficult to distinguish from each other. This problem is strengthened when estimating the GNS model; it often leads to a model that is overparameterized. Parameters have the tendency to become insignificant as a result of which this model does not outperform the SDM and SDEM models.

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[8] This also holds if the spatial autoregressive parameter is negative. The first term that produces feedback effects is $\delta^2\boldsymbol{W}^2$. This term will always be positive. The second term is $\delta^3\boldsymbol{W}^3$. Since δ is restricted to the interval $(1/r_{min}, 1)$ and the non-negative elements of $\boldsymbol{W}$ after row-normalisation are smaller than or equal to 1, the diagonal elements of $\delta^3\boldsymbol{W}^3$ are smaller in absolute value than those of $\delta^2\boldsymbol{W}^2$. Since the series $\delta^2\boldsymbol{W}^2 + \delta^3\boldsymbol{W}^3 + \delta^4\boldsymbol{W}^4 + \ldots$ alternates in sign if δ is negative, the sum of the diagonal elements of the matrix represented by this series will always be positive.

2.7.2 Testing for Spatial Spillovers

The estimated indirect effects of the independent explanatory variables should eventually be used to test the hypothesis as to whether or not spatial spillovers exist, rather than the coefficient estimate of endogenous interaction effects (***WY***) and/or the coefficients estimates of the exogenous interaction effects (***WX***). However, one difficulty is that it cannot be seen from the coefficient estimates and the corresponding standard errors or t-values (derived from the variance–covariance matrix) whether the indirect effects in models containing endogenous interaction effects (SAR, SAC, SDM, GNS) are significant. This is because the indirect effects are composed of different coefficient estimates according to complex mathematical formulas and the dispersion of these indirect effects depends on the dispersion of all coefficient estimates involved (see Table 2.1). For example, if the coefficients δ, β_k and θ_k in the spatial Durbin model happen to be significant, this does not automatically mean that the indirect effect of the k^{th} explanatory variable is also significant. Conversely, if one or two of these coefficients are insignificant, the indirect effect may still be significant.

One possible way to calculate the dispersion of the direct and indirect effects is to apply formulas for the sum, the difference, the product and the quotient of random variables (see, among others, Mood et al. 1974, pp. 178–181). However, due to the complexity of the matrix of partial derivatives and because every empirical application will have its own unique number of observations (N) and spatial weights matrix (***W***), it is almost impossible to derive one general approach that can be applied under all circumstances. In order to draw inferences regarding the statistical significance of the direct and indirect effects, LeSage and Pace (2009, p. 39) therefore suggest simulating the distribution of the direct and indirect effects using the variance–covariance matrix implied by the maximum likelihood estimates.

The variance–covariance matrix of the parameter estimates of the GNS model takes the form (rewritten from Anselin 1988, pp. 64–65 without heteroskedasticity)

$$Var(\hat{\alpha}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\theta}}, \hat{\delta}, \hat{\lambda}, \hat{\sigma}^2) = \begin{bmatrix} \frac{1}{\sigma^2}(\boldsymbol{B}\tilde{\boldsymbol{X}})^T\boldsymbol{B}\tilde{\boldsymbol{X}} & \frac{1}{\sigma^2}(\boldsymbol{B}\tilde{\boldsymbol{X}})^T\boldsymbol{B}\tilde{\boldsymbol{W}}_\delta\tilde{\boldsymbol{X}}\hat{\gamma} & \boldsymbol{0} & \boldsymbol{0} \\ . & trace(\tilde{\boldsymbol{W}}_\delta\tilde{\boldsymbol{W}}_\delta + \boldsymbol{B}\tilde{\boldsymbol{W}}_\delta\boldsymbol{B}^{-1}) + \frac{1}{\sigma^2}(\boldsymbol{B}\tilde{\boldsymbol{W}}_\delta\tilde{\boldsymbol{X}}\hat{\gamma})^T(\boldsymbol{B}\tilde{\boldsymbol{W}}_\delta\tilde{\boldsymbol{X}}\hat{\gamma}) & trace(\tilde{\boldsymbol{W}}_\lambda^T\boldsymbol{B}\tilde{\boldsymbol{W}}_\delta\boldsymbol{B}^{-1} + \boldsymbol{W}\tilde{\boldsymbol{W}}_\delta\boldsymbol{B}^{-1}) & \frac{1}{\sigma^2}trace(\boldsymbol{B}\tilde{\boldsymbol{W}}_\delta\boldsymbol{B}^{-1}) \\ . & . & trace(\tilde{\boldsymbol{W}}_\lambda\tilde{\boldsymbol{W}}_\lambda + \tilde{\boldsymbol{W}}_\lambda^T\tilde{\boldsymbol{W}}_\lambda) & 0 \\ . & . & . & \frac{N}{2\sigma^4} \end{bmatrix}^{-1} \tag{2.15}$$

where $\mathbf{B} = \boldsymbol{I} - \hat{\lambda}\boldsymbol{W}$, $\tilde{\boldsymbol{W}}_\delta = \boldsymbol{W}(\boldsymbol{I} - \hat{\delta}\boldsymbol{W})^{-1}$, $\tilde{\boldsymbol{W}}_\lambda = \boldsymbol{W}(\boldsymbol{I} - \hat{\lambda}\boldsymbol{W})^{-1}$, $\tilde{\boldsymbol{X}} = [\,\boldsymbol{\iota}_N \quad \boldsymbol{X} \quad \boldsymbol{W}\boldsymbol{X}\,]$ and $\hat{\gamma} = \begin{bmatrix} \hat{\alpha} & \hat{\boldsymbol{\beta}}^T & \hat{\boldsymbol{\theta}}^T \end{bmatrix}^T$ to simplify notation. Since this matrix is symmetric the lower diagonal elements are not shown.

One particular parameter combination drawn from this variance–covariance matrix (indexed by d) can be obtained by

$$[\,\alpha_d \quad \boldsymbol{\beta}_d^T \quad \boldsymbol{\theta}_d^T \quad \delta_d \quad \lambda_d \quad \sigma_d^2\,]^T = \boldsymbol{P}^T\vartheta + [\,\hat{\alpha} \quad \hat{\boldsymbol{\beta}} \quad \hat{\boldsymbol{\theta}} \quad \hat{\delta} \quad \hat{\lambda} \quad \hat{\sigma}^2\,]^T \tag{2.16}$$

where $\boldsymbol{P}$ denotes the upper-triangular Cholesky decomposition of the variance–covariance matrix and ϑ is a vector of length $4 + 2K$ (the number of parameters that have been estimated) containing random values drawn from a normal distribution with mean zero and standard deviation one. If D parameter combinations are drawn like this[9] and the (in)direct effect of a particular explanatory variable is determined for every parameter combination, the overall (in)direct effect can be approximated by computing the mean value over these D draws and its significance level (t-value) by dividing this mean by the corresponding standard deviation. If μ_{kd} denotes the indirect effect of the kth explanatory variable of draw d, the overall indirect effect over all draws and the corresponding t-value will be

$$\bar{\mu}_k \text{ (ind. eff. kth var.)} = \frac{1}{D}\sum_{d=1}^{D} \mu_{kd} \tag{2.17a}$$

$$\text{t - value (of ind.eff. kth var.)} = \bar{\mu}_k \Bigg/ \left[\frac{1}{D-1}\sum_{d=1}^{D} (\mu_{kd} - \bar{\mu}_k)^2\right] \tag{2.17b}$$

Given the t-value of this indirect effect, one can finally test whether the kth variable causes spatial spillover effects.

There are two possible approaches to program this. One is to determine the matrix on the right-hand side of (2.13) for every draw and then to calculate the direct and indirect effects corresponding to this draw. The disadvantage of using this approach is that ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$) needs to be inverted for every draw, which will be rather time-consuming and even might break down due to memory problems in case N is large. The other approach, proposed by LeSage and Pace (2009, pp. 114–115), is to exploit the decomposition shown in Eq. (2.14) and to store the traces of the matrices $\boldsymbol{I}$ up to and including $\boldsymbol{W}^{100}$ on the right-hand side of (2.14) in advance. The calculation of the direct and indirect effects then no longer requires the inversion of the matrix ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$) for every parameter combination drawn from the variance–covariance matrix in (2.15), but only a matrix operation based on the stored traces which, as a result, does not require much computational effort.

[9] The default value is 1,000, but for models with large N this number might be decreased.

2.8 Software

Software packages and/or routines to estimate spatial econometric models are Stata, Geoda, R and Matlab. The latter three are all freely downloadable. The results to be reported in the next section have been estimated using Matlab routines. At www.spatial-econometrics.com, the routines SAR, SEM and SAC can be downloaded, written by James LeSage, to estimate to SAR, SEM and SAC models, respectively. By changing the argument $\boldsymbol{X}$ of these routines into [$\boldsymbol{X}$ $\boldsymbol{WX}$] it is also possible to estimate the SDM, SDEM and GNS models.

One disadvantage of the routines made available at this Web site is that the Jacobian term, ln| $\boldsymbol{I}_N$-$\delta\boldsymbol{W}$ |, ln| $\boldsymbol{I}_N$-$\lambda\boldsymbol{W}$ | or both, in the log-likelihood functions of these models is approached by a numerical approach. To overcome potential numerical difficulties one might face in evaluating the log determinant, Pace and Barry (1997) and Barry and Pace (1999) propose computing this determinant *once* over a grid of values for the parameter δ (λ) ranging from $1/r_{min}$ to $1/r_{max}$ prior to estimation. This only requires the determination of the smallest and largest characteristic root of $\boldsymbol{W}$. They suggest a grid based on 0.001 increments for δ over the feasible range. Given these predetermined values for the determinant of ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$), one can quickly evaluate the log determinant of ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$) for a particular value of δ. To compute the log determinant over the feasible range for small values of N (< 500), they compute $\Sigma_i \log|\varsigma_{ii}|$, where ς_{ii} (i = 1,…, N) denotes the diagonal elements of the upper triangular LU decomposition matrix of ($\boldsymbol{I}_N$-$\delta\boldsymbol{W}$). When the sparse structure of the spatial weights matrix is exploited, the required computation time of this decomposition can be reduced from order N^3 to order N^2. For larger values of N ($\geq$ 500), they suggest approaching the log determinant for a particular value of δ over the feasible range by

$$\frac{1}{J}\sum_{j=1}^{J}\left[-N\sum_{k=1}^{H}\left(\boldsymbol{z}_j'\boldsymbol{W}^k\boldsymbol{z}_j\Big/\boldsymbol{z}_j'\boldsymbol{z}_j\right)\left(\delta^k/k\right)\right] \tag{2.18}$$

where $\boldsymbol{z}_j$ denotes an $N \times 1$ vector of independent standard normal variates. The precision of this estimate can be manipulated by means of the tuning parameters J (the number of simulations generated over which the estimate is averaged) and H (the number of elements in the sum of ratios of quadratic forms). The required computation time of this simulation approach can be reduced to order $N\log N$ and allows for the estimation of models with very large numbers of observations in the cross-sectional domain.

The disadvantage of this approach is that the parameter estimate of δ (λ) and therefore of $\boldsymbol{\beta}$ changes slightly every time the routine is run again. Generally, researchers do not appreciate this. This problem can be avoided by calculating the log determinant by

$$\ln|\boldsymbol{I} - \delta\boldsymbol{W}| = \sum\nolimits_i \ln(1 - \delta\omega_i) \tag{2.19}$$

where ω_i (i = 1,…,N) denote the characteristic roots of $\boldsymbol{W}$. Generally, this calculation works well for values of N smaller than 1,000. At www.regroningen.nl the routines SARp, SEMp and SACp have been made available, written by Paul Elhorst, to estimate the SAR, SEM and SAC models based on (2.19). Furthermore, by changing the argument $\boldsymbol{X}$ of these routines into [$\boldsymbol{X}$ $\boldsymbol{WX}$] it is also possible to estimate the SDM, SDEM and GNS models. The advantage of these routines is that the estimation results will be exactly the same every time this routine is run.[10] The routine "demo_crime_rates" posted at this Web site can be used to reproduce the estimation results reported in Tables 2.2 and 2.3 in the next section. By changing the specification of $\boldsymbol{Y}$, $\boldsymbol{X}$, $\boldsymbol{W}$ and N in these routines and by reading a different data set, researchers can use this file to estimate these models for their own research problems.

2.9 Empirical Illustration

To demonstrate the performance of the different spatial econometric models in an empirical setting, Anselin's (1988) cross-sectional dataset of 49 Columbus, Ohio neighborhoods is used to explain the crime rate as a function of household income and housing values. The spatial weights matrix $\boldsymbol{W}$ is specified as a row-normalized binary contiguity matrix, with elements $w_{ij} = 1$ if two spatial neighborhoods share a common border, and zero otherwise. It should be stressed that this specification of the spatial weights matrix is also used in Anselin (1988). The estimation results are reported in Table 2.2 and the direct and spatial spillover effects in Table 2.3. Eight different models are considered. The GNS model includes all types of interaction effects, while the other models ignore one or more interaction effects. If a particular entry in Table 2.2 is empty, the interaction effect reported in the left column is not present in the model. The parameter estimates are obtained by applying ML.

One of the main questions is which model best describes the data. One of the criterions that may be used for this purpose is the likelihood ratio (LR) test based on the log-likelihood function values of the different models. The LR test is based on minus two times the difference between the value of the log-likelihood function in the restricted model and the value of the log-likelihood function of the unrestricted model: $-2*(\text{logL}_{\text{restricted}} - \text{logL}_{\text{unrestricted}})$. This test statistic has a Chi squared distribution with degrees of freedom equal to the number of restrictions imposed.

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[10] One of the constants in the log-likelihood function of the routines of James LeSage is $\ln(\pi)$, while this should be $\ln(2\pi)$. This error is probably based on Anselin's (1988) textbook, where the same mistake is made. See, e.g., Eqs. (6.15), (8.4), and p. 181. This innocent error has been removed from the SARp, SEMp and SACp routines.

Table 2.2 Model comparison of the estimation results explaining the crime rate

	OLS	SAR	SEM	SLX	SAC	SDM	SDEM	GNS
Intercept	0.686**	0.451**	0.599**	0.750**	0.478**	0.428**	0.735**	0.509
	(14.49)	(6.28)	(11.32)	(11.32)	(4.83)	(3.38)	(8.37)	(0.75)
Income	−1.597**	−1.031**	−0.942**	−1.109**	−1.026**	−0.914**	−1.052**	−0.951**
	(−4.78)	(−3.38)	(−2.85)	(−2.97)	(−3.14)	(−2.76)	(−3.29)	(−2.16)
House value	−0.274**	−0.266**	−0.302**	−0.290**	−0.282**	−0.294**	−0.276**	−0.286**
	(−2.65)	(−3.01)	(−3.34)	(−2.86)	(−3.13)	(−3.29)	(−3.02)	(−2.87)
W * Crime rate		0.431**			0.368*	0.426**		0.315
		(3.66)			(1.87)	(2.73)		(0.33)
W * Income				−1.371**		−0.520	−1.157**	−0.693
				(−2.44)		(−0.92)	(−2.00)	(−0.41)
W * House value				0.192		0.246	0.112	0.208
				(0.96)		(1.37)	(0.56)	(0.73)
W * Error term			0.562**		0.166		0.425**	0.154
			(4.19)		(0.56)		(2.69)	(0.15)
R^2	0.552	0.652	0.651	0.609	0.651	0.665	0.663	0.651
Log-Likelihood	13.776	43.263	42.273	17.075	43.419	44.260	44.069	44.311

**Significant at 5 %; *Significant at 10 %; T-values in parentheses, W = Binary contiguity matrix

Table 2.3 Model comparison of the marginal effects of the explanatory variables on the crime rate

	OLS	SAR	SEM	SLX	SAC	SDM	SDEM	GNS
Direct effects								
Income	−1.597**	−1.086**	−0.942**	−1.109**	−1.063**	−1.024**	−1.052**	−1.032**
	(−4.78)	(−3.44)	(−2.85)	(−2.97)	(−3.25)	(−3.19)	(−3.29)	(−3.28)
House value	−0.274**	−0.280**	−0.302**	−0.290**	−0.292**	−0.279**	−0.276**	−0.277
	(−2.65)	(−2.96)	(−3.34)	(−2.86)	(−3.10)	(−3.13)	(−3.02)	(0.32)
Indirect or spatial spillover effects								
Income		−0.727*		−1.371**	−0.560	−1.477*	−1.157**	−1.369
		(−1.95)		(−2.44)	(−0.18)	(−1.83)	(−2.00)	(0.02)
House value		−0.188*		0.192	−0.154	0.195	0.112	0.163
		(−1.71)		(0.96)	(−0.39)	(0.66)	(0.56)	(−0.03)

**Significant at 5 %; *Significant at 10 %; T-values in parentheses, W = Binary contiguity matrix

The log-likelihood function value of the OLS model increases from 13.776 to 17.075 when this model is extended to include exogenous interaction effects (***WX***), known as the SLX model. The LR-test of the SLX model versus the OLS model takes the value of 6.598 with 2 degrees of freedom (df), while the 5 % critical value is 6.0. This implies that the OLS model needs to be rejected in favor of the SLX model. However, if the OLS is extended to include endogenous interaction effects (***WY***) or interaction effects among the error terms (***Wu***), the log-likelihood function value increases even more, even though in these two cases only one interaction effect is added to the model. Whether it is this SAR or SEM model that better describes the data is difficult to say, since these two models are not nested. One solution is to test whether the spatial lag model or the spatial error model is more appropriate to describe the data, provided that the OLS model is taken as point of departure. For this purpose, one may use the classic LM-tests proposed by Anselin (1988), or the robust LM-tests proposed by Anselin et al. (1996).[11] Both the classic and the robust tests are based on the residuals of the OLS model and follow a Chi squared distribution with 1 degree of freedom. Using the classic tests, both the hypothesis of no spatially lagged dependent variable and the hypothesis of no spatially autocorrelated error term must be rejected at five per cent significance; the LM test for the spatial lag amounts to 9.36 and for the spatial error to 5.72. When using the robust tests, the hypothesis of no spatially lagged dependent variable must still be rejected, though only at ten per cent significance, whereas the hypothesis of no spatially autocorrelated error term can no longer be rejected; the robust LM test for the spatial lag amounts to 3.72 and for the spatial error to 0.08. This indicates that on the basis of these robust LM tests the spatial lag model is more appropriate.

Another solution is to consider the SAC model, which considers both endogenous interaction effects and interaction effects among the error terms, and therefore nests both the SAR and SEM models. The SAC model produces coefficient estimates of the ***WY*** and the ***Wu*** variables that are not significantly different from their counterparts in the SAR model and the SEM model, respectively.[12] Similarly, the LR-test of the SAC model versus the SAR model takes the value of 0.312 with 1 df, and the LR-test of the SAC model versus the SEM model the value of 2.292 with 1 df, while the 5 % critical value in both cases is 3.84. This implies that it is difficult to choose among these three models. However, since the coefficient of ***WY*** is significant in the SAC model, whereas the coefficient of ***Wu*** is not, and the log-likelihood function value of the SAR model is higher than that of the SEM model, the SAR model seems to be the better choice.

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[11] The latter tests are called robust because the existence of one type of spatial dependence does not bias the test for the other type of spatial dependence.

[12] The coefficient of the spatially autocorrelated error term in the SAC model amounts to 0.166. The corresponding t-value is so low that this coefficient plus two times its standard error also covers the coefficient estimate of the spatially autocorrelated error term in the SEM model of 0.562. The fact that the latter is significant does not change this conclusion.

Another way to look at the SAR, SEM and SLX models, on their turn, is to consider the SDM model, since the SDM model nests these three models. The SDM appears to outperform the SLX model (LR-test 54.370, 2 df, critical value 5.99), but not the SAR model (LR-test 1.994, 1 df, critical value 3.84) and the SEM model (LR-test 3.974, 2 df, critical value 5.99). Alternatively, one might consider the SDEM model which also nests the SLX and the SEM models. The SDEM model also appears to outperform the SLX model (LR-test 53.998, 1 df, critical value 3.84) but not the SEM model (LR-test, 3.592, 1 df, critical value 3.84). Whether it is the SDM model or the SDEM model that better describes the data is difficult to say, since these two models are not nested. Unfortunately, estimation of the GNS model which nests these two models does not provide an answer. The increase of the log-likelihood function value when estimating this model is so small that, on the basis of the results reported in Table 2.2, it is impossible to draw any conclusion as to whether it is SDM, SDEM or GNS that best describes the data. In contrast to the SAC model, the extension to the GNS model also provides no answer whether endogenous interaction or error correlation effects are more important.

We now consider the direct and indirect effects estimates of the different explanatory variables (see Sect. 2.7) to see whether they can be used as an additional mean to select the best model. The general pattern that emerges from Table 2.3 is the following. First, the differences between the direct effects and the coefficient estimates reported in Table 2.2 are relatively small. In the OLS, SEM, SLX and SDEM models they are exactly the same by construction; in the SAR, SDM, SAC and the GNS models they may be different due to the endogenous interaction effects $\boldsymbol{WY}$. These interaction effects cause feedback effects, i.e., impacts affecting crime rates in certain neighborhoods that pass on to surrounding neighborhoods and back to the neighborhood instigating the change. For example, the direct effect of the income variable in the GNS model amounts to −1.032, while the coefficient estimate of this variable is −0.951. This implies that the feedback effect is $-1.032-(-0.951) = -0.081$. This feedback effect corresponds to 8.5 % of the coefficient estimate.

Second, the differences between the direct effects estimates in the different models appear to be relatively small. The direct effects of the income variable range between −0.942 in the SEM model and −1.109 in the SLX model. Only in the OLS model the magnitude of the direct effect of −1.597 is much greater. Just as the LM and LR test results, it indicates that the OLS model needs to be rejected. Since this model accounts neither for spatial interaction effects nor for spatial spillover effects, the direct effect is overestimated (in absolute value). Similarly, the coefficient of the house value variable ranges between −0.274 in the OLS model and −0.302 in the SEM model. Overall, it seems as if it does not matter which model is used to obtain the direct effects estimates. Also the t-values do not differ to any great extent, except for the t-value of the direct effect generated for the house value variable. There are two explanations for this. One is that the significance level of the spatial autoregressive coefficient of the $\boldsymbol{WY}$ variable in the GNS models falls considerably, because this variable competes with the spatial autocorrelation coefficient of the ***Wu*** variable. This result also occurred in the SAC model. If endogenous interaction effects and interaction effects among the error terms are separated from each other, both coefficients turn out to be significant, but if they are combined both become insignificant. Another explanation is that the t-values of the coefficient estimates in the different models are relatively stable, except for the GNS model. The t-values of the variables in this model have the tendency to go down.

In contrast to the direct effects estimates, the differences between the spillover effects are extremely large. Still, one can observe some general patterns. The OLS, SAR, SEM and SAC models produce no or wrong spillover effects compared to the SDM, SDEM and GNS models. For example, whereas the spillover effect of the house value variable is positive in the SLX, SDM, SDEM and GNS models, it is zero by construction in the OLS and SEM models, negative in the SAC model, and negative and weakly significant in the SAR model. The negative and also weakly significant effect in the SAR model can be explained by the fact that this model suffers from the problem that the ratio between the spillover effect and the direct effect is the same for every explanatory variable. Consequently, this model is too rigid to model spillover effects adequately. The negative but insignificant effect in the SAC model can be explained by the fact that this model resembles the SAR model: the spatial autocorrelation coefficient of ***Wu*** appears to be insignificant, whereas the spatial autoregressive coefficient of ***WY*** does not, as a result of which the SAC model is hampered by the same problem as the SAR model. This was pointed out earlier in Table 2.1; mathematically, the SAR and SAC models share the same direct and indirect effects estimates.

The spillover effects produced by the SLX, SDM, SDEM and GNS models are more or less comparable to each other. In these models, the spillover effect of the income variable ranges from −1.157 to −1.477 and of the house value variable from 0.112 to 0.192. By contrast, the t-values do not. The t-values in the SLX model are relatively high. This can be explained by the fact that the SLX has been rejected in favor of the SDM and the SDEM models based on the LR tests. Furthermore, the t-values in the GNS model are relatively low. As recently pointed out by Gibbons and Overman (2012), the explanation for this finding is that interaction effects among the dependent variable on the one hand and interaction effects among the error terms on the other hand are only weakly identified. Considering them both, as in the GNS model, strengthens this problem; it leads to a model that is overparameterized, as a result of which the significance levels of all variables tend to go down. This finding is worrying since the interpretation of both types of interaction effects is completely different. A model with endogenous interaction effects posits that the crime rate in one neighborhood depends on that in other neighborhoods, and on a set of neighborhood characteristics. By contrast, a model with interaction effects among the error terms assumes that the crime rate in one neighborhood depends on a set of observed neighborhood characteristics and unobserved characteristic omitted from the model that neighborhoods have in common. Nevertheless, both models appear to produce spillover effects that are comparable to each other, both in terms of magnitude and significance.

2.9.1 Conclusion

The conclusion from the empirical analysis is twofold. First, for various reasons the OLS, SAR, SEM, SLX, SAC and GNS models need to be rejected. The OLS and SLX models are outperformed by other, more general models. The spillover effects of the SEM model are zero by construction, while the results of more general models show that the spillover effect of the income variable is significant. The SAR and SAC models suffer from the problem that the ratio between the spillover effect and the direct effect is the same for every explanatory variable. Consequently, the spillover effect of the housing value variable gets a wrong sign. Finally, the GNS model is overparameterized, as a result of which the t-values of the coefficient estimates and the effects estimates have the tendency to go down. In sum, only the SDM and SDEM model produce acceptable results. Second, it is not clear which of these two models best describes the data. Even though both models produce spillover effects that are comparable to each other, both in terms of magnitude and significance, this is worrying since these two models have a different interpretation.

2.10 Conclusion

Originally, the central focus of spatial econometrics has been on the spatial lag model (SAR) and the spatial error model (SEM) with one type of interaction effect. The results shown in this chapter make clear that this approach is too limited and that the focus should shift to the spatial Durbin model (SDM) and the spatial Durbin error model (SDEM). At the same time, new test procedures should be developed to choose among these two models, which is difficult because both models tend to produce spillover effects that are comparable to each other in terms of magnitude and significance, and because interaction effects among the dependent variable on the one hand and interaction effects among the error terms on the other hand are only weakly identified. Precisely for this reason, the general nesting spatial (GNS) model is not of much help either. It generally leads to a model that is overparameterzed, as a result of which the significance levels of the variables tend to go down.

Recently, Gibbons and Overman (2012) criticized the SAR, SEM and SDM models. They demonstrate that the reduced form of these models is similar to a model with first, second and higher order exogenous interaction effects, and argue that this reduced form can hardly be distinguished from the SLX model that only contains first order exogenous interaction effects.

Another major weakness of spatial econometric models is that the spatial weights matrix $\boldsymbol{W}$ needs to be specified in advance, although there are exceptions, and that economic theory underlying spatial econometric applications often has little to say about the specification of $\boldsymbol{W}$. For this reason, it has become common practice to investigate whether the results are robust to the specification of $\boldsymbol{W}$, or to test different specifications against each other using the log-likelihood function value, Bayesian posterior model probabilities, or the J-test. In this respect, Corrado and Fingleton (2012) strongly argue for the use of more substantive theory in empirical spatial econometric modeling, especially regarding the modeling of $\boldsymbol{W}$.

In view of these critical notes, Halleck Vega and Elhorst (2012) are currently doing research on finding a better and broader[13] modeling strategy to determine the spatial econometric model, including the spatial weights matrix $\boldsymbol{W}$, that best describes the data.

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[13] Broader in the sense that it is based on theory, statistics, flexibility and possibilities to parameterize $\boldsymbol{W}$.

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