Chapter 2
Some Solution Schemes and Game Properties

Even though we are interested in a new characterization of the pre-kernel based on the Fenchel-Moreau conjugation, we will discuss in this chapter some solution concepts and game properties in order to allow the reader to assess the upcoming results in a broader context of game theory. Instead of a comprehensive treatment of this material it is a short reference of the used concepts. For a complete and systematic coverage of solution concepts and game properties, we refer the reader to Driessen (1985, 1988).

A n-person cooperative game with side-payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1, 2, \ldots, n\}$ represents the player set and v is the characteristic function with $v : 2^N \to \mathbb{R}$, and the convention that $v(\emptyset) := 0$. Elements of N are denoted as players. A subset S of the player set N is called a coalition. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. However, the cardinality of the player set N is given by $n := |N|$, and that for a coalition S by $s := |S|$. We assume throughout that $v(N) > 0$ and $n \geq 2$ is valid. Formally, we identify a cooperative game by the vector $v := (v(S))_{S \subseteq N} \in \mathcal{G}^n = \mathbb{R}^{2^{|N|}}$, if no confusion can arise, whereas in case of ambiguity, we identify a game by $\langle N, v\rangle$. Notice that we denote by $\mathcal{G}^n$ the space of all n-person games with player set N. A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a vector $\mathbf{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\mathbf{x}$ with the corresponding measure on S, such that $x(S) := \sum_{k \in S} x_k$. For all $\epsilon \in \mathbb{R}$, the set of vectors $\mathbf{x} \in \mathbb{R}^n$ which satisfies the ϵ-efficiency principle $v(N) - x(N) = \epsilon$ is called the ϵ- **pre-imputation set** and it is defined by

$$\mathcal{I}^{\epsilon}(v) := \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N) - \epsilon\} \qquad \forall\, \epsilon \in \mathbb{R}, \tag{2.1}$$

where an element $\mathbf{x} \in \mathcal{I}^{\epsilon}(v)$ is called an ϵ-pre-imputation. Hence, a vector $\mathbf{x} \in \mathbb{R}^n$ is a pre-imputation if $\mathbf{x} \in \mathcal{I}^0(v)$. Moreover, it should be obvious that each set $\mathcal{I}^{\epsilon}(v)$

H.I. Meinhardt, *The Pre-Kernel as a Tractable Solution for Cooperative Games*,
Theory and Decision Library C 45, DOI 10.1007/978-3-642-39549-9_2,

is a hyper-surface of dimension $n-1$ that slides through $\mathbb{R}^n$ with the property that $\mathcal{I}^{\epsilon}(v) \cap \mathcal{I}^{\epsilon_0}(v) = \emptyset$ for $\epsilon \neq \epsilon_0$. Therefore all sets $\mathcal{I}^{\epsilon}(v)$ describe a partition of $\mathbb{R}^n$.

The set of pre-imputations which satisfies in addition the **individual rationality property** $x_k \geq v(\{k\})$ for all $k \in N$ is called the **imputation set** $\mathcal{I}(v)$.

A vector that results from a vector $\mathbf{x}$ by a **transfer** of size $\delta \geq 0$ between a pair of players $i, j \in N, i \neq j$, is referred to as $\mathbf{x}^{i,j,\delta} = (x_k^{i,j,\delta})_{k \in N}$, which is given by

$$\mathbf{x}^{i,j,\delta}_{N\setminus\{i,j\}} = \mathbf{x}_{N\setminus\{i,j\}}, \; x_i^{i,j,\delta} = x_i - \delta \quad \text{and} \quad x_j^{i,j,\delta} = x_j + \delta. \tag{2.2}$$

A **side-payment** for the players in N is a vector $\mathbf{z} \in \mathbb{R}^n$ such that $z(N) = 0$.

A **solution concept**, denoted as σ, on a non-empty set $\mathcal{G}$ of games is a correspondence on $\mathcal{G}$ that assigns to any game $v \in \mathcal{G}$ a subset $\sigma(N, v)$ of $\mathcal{I}^0(N, v)$. This set can be empty or just be single-valued, in the latter case, the solution σ is a function and is simply called a value.

Given a vector $\mathbf{x} \in \mathcal{I}^{\epsilon}(v)$, we define the **excess** of coalition S with respect to the ϵ-pre-imputation $\mathbf{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S, \mathbf{x}) := v(S) - x(S). \tag{2.3}$$

A non-negative (non-positive) excess of S at $\mathbf{x}$ in the game $\langle N, v\rangle$ represents a gain (loss) to the members of the coalition S unless the members of S do not accept the payoff distribution $\mathbf{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game $v \in \mathcal{G}^n$. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any ϵ-pre-imputation $\mathbf{x} \in \mathcal{I}^{\epsilon}(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S \in \mathcal{G}_{ij}} e^v(S, \mathbf{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{2.4}$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the ϵ-pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\mathbf{x} \in \mathcal{I}^0(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathcal{P}r\mathcal{K}(v) := \left\{\mathbf{x} \in \mathcal{I}^0(v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\right\}. \tag{2.5}$$

The pre-kernel has the advantage of addressing a stylized bargaining process, in which the figure of argumentation is a **pairwise equilibrium procedure** of claims while relying on his best arguments, that is, the coalitions that will best support his claim. The pre-kernel solution characterizes all those imputations in which all pairs of players $i, j \in N, i \neq j$ are in equilibrium with respect to their claims.

Observe that in case that the admissible bargaining range is the imputation set $\mathcal{I}(v)$ rather than $\mathcal{I}^0(v)$, player j cannot get less than $v(\{j\})$, the amount he can assure by himself without relying on the cooperation of the other players. A player i outweighs player j w.r.t. the proposal $\mathbf{x} \in \mathcal{I}(v)$ presented in a bilateral bargaining situation if $x_j > v(\{j\})$ and $s_{ij}(\mathbf{x}, v) > s_{ji}(\mathbf{x}, v)$. The set of imputations $\mathcal{I}(v)$ for which no player outweighs another player is called the **kernel** of a game $v \in \mathcal{G}^n$ referred to as $\mathcal{K}(v)$. More formally, the kernel of a n-person game is the set of imputations $\mathbf{x} \in \mathcal{I}(v)$ satisfying for all $i, j \in N, i \neq j$

$$\left[s_{ij}(\mathbf{x}, v) - s_{ji}(\mathbf{x}, v)\right] \cdot \left[x_j - v(\{j\})\right] \leq 0 \quad \text{and} \tag{2.6}$$

$$\left[s_{ji}(\mathbf{x}, v) - s_{ij}(\mathbf{x}, v)\right] \cdot [x_i - v(\{i\})] \leq 0. \tag{2.7}$$

This solution scheme is related to the pre-kernel $\mathcal{P}r\mathcal{K}(v)$ of a TU game. In addition, the following inclusion $\mathcal{K}(v) \cap \mathcal{I}(v) \subset \mathcal{P}r\mathcal{K}(v)$ is satisfied. The kernel is non-empty, and it is a finite union of closed convex polyhedra (cf. Davis and Maschler (1965)). Therefore, we can infer that the pre-kernel is non-empty and it coincides with the kernel for the class of zero-monotonic TU games (cf. Maschler et al. (1972)).

The kernel as well as the pre-kernel solution are a set-valued solution scheme with the consequence that it is difficult to justify why a selected element from one of these sets should be preferred over the other. To overcome this selection problem, the **nucleolus** of a n-person game, denoted as $\nu(v)$, might be the solution concept of choice, since it is contained in the kernel, $\nu(v) \in \mathcal{K}(v)$, it is non-empty and single-valued. This solution concept is due to Schmeidler (1969).

In order to define the nucleolus $\nu(v)$ of a game $v \in \mathcal{G}^n$, take any $\mathbf{x} \in \mathbb{R}^n$ to define a 2^n-tuple vector $\theta(\mathbf{x})$ whose components are the excesses $e^v(S, \mathbf{x})$ of the 2^n coalitions $S \subseteq N$, arranged in decreasing order, that is,

$$\theta_i(\mathbf{x}) := e^v(S_i, \mathbf{x}) \geq e^v(S_j, \mathbf{x}) =: \theta_j(\mathbf{x}) \qquad \text{if} \qquad 1 \leq i \leq j \leq 2^n. \tag{2.8}$$

Ordering the so-called complaint or dissatisfaction vectors $\theta(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ by the lexicographic order $\leq_L$ on $\mathbb{R}^n$, we shall write

$$\theta(\mathbf{x}) <_L \theta(\mathbf{y}) \qquad \text{if } \exists \text{ an integer } 1 \leq k \leq 2^n, \tag{2.9}$$

such that $\theta_i(\mathbf{x}) = \theta_i(\mathbf{y})$ for $1 \leq i < k$ and $\theta_k(\mathbf{x}) < \theta_k(\mathbf{y})$. Furthermore, we write $\theta(\mathbf{x}) \leq_L \theta(\mathbf{y})$ if either $\theta(\mathbf{x}) <_L \theta(\mathbf{y})$ or $\theta(\mathbf{x}) = \theta(\mathbf{y})$. Now the nucleolus $\mathcal{N}(v)$ of a game $v \in \mathcal{G}^n$ over the set $\mathcal{I}(v)$ is defined as

$$\mathcal{N}(v) = \{\mathbf{x} \in \mathcal{I}(v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{y}) \; \forall \; \mathbf{y} \in \mathcal{I}(v)\}. \tag{2.10}$$

At this set the total complaint $\theta(\mathbf{x})$ is lexicographically minimized over the non-empty compact convex imputation set $\mathcal{I}(v)$. Schmeidler (1969) proved that the nucleolus $\mathcal{N}(v)$ w.r.t. $\mathcal{I}(v)$ is non-empty and it consists of a unique point, which is referred to as $\nu(v)$.

Similar to the pre-kernel, the pre-nucleolus $\mathcal{P}r\mathcal{N}(v)$ over the pre-imputations set $\mathcal{I}^0(v)$ is defined by

$$\mathcal{P}r\mathcal{N}(v) = \left\{\mathbf{x} \in \mathcal{I}^0(v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{y}) \;\forall\; \mathbf{y} \in \mathcal{I}^0(v)\right\}. \tag{2.11}$$

The **pre-nucleolus** of any game $v \in \mathcal{G}^n$ is non-empty as well as unique, and it is referred to as $\nu(v)^*$.

The **reasonable set** of a game $\langle N, v\rangle$, denoted by $\mathcal{R}(v)$, is the collection of all pre-imputation that distribute to a player at most the largest amount he can contribute to a specific coalition. More formally we define

$$\mathcal{R}(v) := \left\{\mathbf{x} \in \mathcal{I}^0(v) \mid x_k \leq r_k \quad \forall k \in N\right\} \tag{2.12}$$

with

$$r_k := \max_{S:k\in S} \left(v(S) - v(S\backslash\{k\})\right) \tag{2.13}$$

Since Maschler et al. (1979) it is known that $\mathcal{K}(v) \subset \mathcal{R}(v)$ and $\mathcal{P}r\mathcal{K}(v) \subset \mathcal{R}(v)$ is satisfied. Hence, at a (pre-)kernel element no player can get more than the largest amount he can contribute to a coalition, this constitutes an upper bound of the (pre-)kernel.

Another single-valued solution scheme is the **Shapley value** $\phi(v) \in \mathbb{R}^n$ introduced by Shapley (1953). The Shapley value is characterized by the following formula

$$\phi_k(v) = \sum_{S\subseteq N\backslash\{k\}} \chi(S) \cdot \left(v(S \cup \{k\}) - v(S)\right) \qquad \forall k \in N, \tag{2.14}$$

with $\chi(S) := (n-1-s)!\; s!/n!$ for all $S \subset N, S \neq N$. The weight $\chi(S)$ can be interpreted as the probability that S is already assembled. In this sense, the Shapley value $\phi(v)$ is a vector of the average marginal contributions of game v. Rather than balancing the maximum surpluses between the pair of players like the pre-kernel, the Shapley value, however, satisfies the balanced contributions property Myerson (1980, Proposition 2), that is, for all $i, j \in N, i \neq j$, the Shapley value satisfies:

$$\phi_i(N, v) - \phi_i(N\backslash\{j\}, v) = \phi_j(N, v) - \phi_j(N\backslash\{i\}, v), \tag{2.15}$$

where $\phi_i(N\backslash\{j\}, v)$ and $\phi_j(N\backslash\{i\}, v)$ are the payoffs distributed by the Shapley value to player i and player j under the subgames $\langle N\backslash\{j\}, v\rangle$ and $\langle N\backslash\{i\}, v\rangle$ respectively.

Note that $\langle\, N\backslash\{k\}, v_{N\backslash\{k\}}\,\rangle$ is specified by $v_{N\backslash\{k\}}(T) = v(T)$ for all $T \subseteq N\backslash\{k\}$ and $k \in N$. We also have $\phi_k(N, v) = \phi_k(v)$ for all $k \in N$.

The **core** of a game $\mathcal{C}(v)$ is the set of imputations satisfying besides the individual rationality property as well as the coalitional rationality property, i.e. the core of a game $v \in \mathcal{G}^n$ is given by

$$\mathcal{C}(v) := \{\mathbf{x} \in \mathcal{I}(v) \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) \;\forall\; S \subset N\}\,. \tag{2.16}$$

The core of a n-person game may be empty. Whenever it is non-empty we have some incentive for mutual cooperation in the grand coalition. A core agreement is preferable over imputations outside the core, since the grand coalition can distribute to its members a value that exceeds the value that the intermediate coalitions can produce to their members. Hence, the formation of a smaller coalition is unattractive. In this sense, a payoff distribution located in the core cannot be blocked by any coalition. Moreover, the nucleolus is contained in the core of the game whenever the core is non-empty, i,e., $\nu(v) \in \mathcal{C}(v)$ if $\mathcal{C}(v) \neq \emptyset$. This does not hold for the Shapley value. As we shall learn below, the Shapley value belongs to the core only for certain subclasses of games.

Imposing on the worth of any proper coalition – namely the set of coalitions excluding the grand coalition N and the empty set – the same cost $\epsilon \in \mathbb{R}$, then we can define the **strong ϵ-core** $\mathcal{C}_\epsilon(v)$ through

$$\mathcal{C}_\epsilon(v) := \{\mathbf{x} \in \mathcal{I}(v) \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) - \epsilon \;\forall\; \emptyset \neq S \subset N\}\,. \tag{2.17}$$

with $\mathcal{C}_0(v) = \mathcal{C}(v)$. For $n \geq 2$ we note that $\mathcal{C}_\epsilon(v) \neq \emptyset$ if ϵ is large enough and $\mathcal{C}_\epsilon(v) = \emptyset$ for small enough ϵ. Furthermore, if $\epsilon_0 < \epsilon_1$ then $\mathcal{C}_{\epsilon_0}(v) \subseteq \mathcal{C}_{\epsilon_1}(v)$ and $\mathcal{C}_{\epsilon_0}(v) \subset \mathcal{C}_{\epsilon_1}(v)$ whenever $\mathcal{C}_{\epsilon_1}(v) \neq \emptyset$. Similar to the core of the game, we have $\nu(v) \in \mathcal{C}_\epsilon(v)$ whenever $\mathcal{C}_\epsilon(v) \neq \emptyset$ and $\epsilon \leq 0$.

To specify a necessary and sufficient condition under which the core of game $v \in \mathcal{G}^n$ is non-empty, we have to treat the term of **balanced collection** and **(totally) balanced games**. For doing so, let $\mathcal{B} = \{S_1, \ldots, S_m\}$ be a collection of non-empty sets of N. We denote the collection $\mathcal{B}$ as balanced whenever there exist positive numbers w_S for all $S \in \mathcal{B}$ such that we have $\sum_{S \in \mathcal{B}} w_S \mathbf{1}_S = 1_N$. The numbers w_S are called weights for the balanced collection $\mathcal{B}$ and $\mathbf{1}_S$ is the **indicator function** or **characteristic vector** $\mathbf{1}_S : N \mapsto \{0, 1\}$ given by $\mathbf{1}_S(k) := 1$ if $k \in S$, otherwise $\mathbf{1}_S(k) := 0$.

We say that a game $v \in \mathcal{G}^n$ is balanced if for every balanced collection $\mathcal{B}$ with weights $\{w_S\}_{S \in \mathcal{B}}$, we obtain

$$\sum_{S \in \mathcal{B}} w_S\, v(S) \leq v(N). \tag{2.18}$$

A game $v \in \mathcal{G}^n$ is called to be totally balanced if all subgames $\langle\, S, v_S \,\rangle, \emptyset \neq S \subseteq N$ are balanced. Notice that a subgame $\langle\, S, v_S \,\rangle$ is specified by $v_S(T) = v(T)$ for all $T \subseteq S$. The subclass of balanced TU games is indicated by $\boldsymbol{\mathcal{B}}^n$. Bondareva (1963) and Shapley (1967) proved independently from one another that a game $v \in \mathcal{G}^n$ is balanced iff the core of the game is non-empty, that is, $\mathcal{C}(v) \neq \emptyset$. Implying that a game is totally balanced iff any subgame has a non-empty core.

An **objection** of player i against a player j w.r.t. a payoff vector $\mathbf{x} \in \mathbb{R}^n$ in game $v \in \mathcal{G}^n$ is a pair $(\mathbf{y}_S, S)$ with $S \in \mathcal{G}_{ij}$ and $\mathbf{y}_S := \{y_k\}_{k\in S}$ satisfying the following properties:

$$v(S) = \sum_{k\in S} y_k \qquad \text{and} \qquad y_k > x_k \quad \text{for } k \in S. \tag{2.19}$$

A **counter-objection** to the objection $(\mathbf{y}_S, S)$ is a pair $(\mathbf{z}_T, T)$ with $T \in \mathcal{G}_{ji}$ and $\mathbf{z}_T := \{z_k\}_{k\in T}$ satisfying

$$v(T) = \sum_{k\in T} z_k \qquad \text{and} \qquad \begin{aligned} z_k &\geq x_k \quad \text{for } k \in T \backslash S \\ z_k &\geq y_k \quad \text{for } k \in T \cap S. \end{aligned} \tag{2.20}$$

Thus, if the pair $(\mathbf{y}_S, S)$ is an objection against vector $\mathbf{x}$, then any member of coalition $S \in \mathcal{G}_{ij}$ can improve upon rather than accepting proposal $\mathbf{x}$. Acceptance would mean that players in $S \in \mathcal{G}_{ij}$ would accept a loss due to $e^v(S, \mathbf{x}) > 0$. Hence, a player i can formulate an objection against player j using coalition $S \in \mathcal{G}_{ij}$ w.r.t. the proposal $\mathbf{x}$ iff the excess $e^v(S, \mathbf{x})$ is positive.

In contrast, a counter-objection $(\mathbf{z}_T, T)$ of player j against player i w.r.t. objection $(\mathbf{y}_S, S)$ uses a coalition T without player i, i.e. $T \in \mathcal{G}_{ji}$, to formulate a proposal that cannot strictly be improved upon to the precedent proposal for players belonging to the set $S \cap T$ and which can also not strictly be improved upon w.r.t. $\mathbf{x}$ for all $k \in T \backslash S$. This means, that player j can only use a coalition $T \in \mathcal{G}_{ji}$ with non-negative excess $e^v(T, \mathbf{x})$ to formulate a counter-objection against player i.

An imputation $\mathbf{x} \in \mathcal{I}(v)$ is an element of the **bargaining set** $\mathcal{M}(v)$ of game $v \in \mathcal{G}^n$ whenever for any objection of a player against another player w.r.t. $\mathbf{x}$ in $v \in \mathcal{G}^n$ exists a counter-objection.

Be reminded that the following property $\mathcal{C}(v) \subseteq \mathcal{M}(v)$ holds for all $v \in \mathcal{G}^n$. This means, that for core allocations the excesses described by formula (2.3) are non-positive, implying that for core allocations there are no objections w.r.t. other core allocations, and for allocations outside the core it is always possible to formulate against an objection a counter-objection. Hence, core allocations can be stabilized by an abstract bargaining procedure while formulating objections and counter-objections. Moreover, the bargaining set $\mathcal{M}(v)$ is non-empty, since $\emptyset \neq \mathcal{K}(v) \subseteq \mathcal{M}(v)$ for all $v \in \mathcal{G}^n$ (Davis and Maschler 1965). This implies that whenever $\mathcal{C}(v) = \emptyset$ is valid, we may fail to achieve cooperation into the grand coalition, however, the bargaining set $\mathcal{M}(v)$ is non-empty there exist allocations which can be stabilized on the basis of the bargaining set. As a consequence, cooperation in a subgroup of players in N is always possible.

In addition, we want to discuss some important game properties. A game $v \in \mathcal{G}^n$ is said to be **monotonic** if

$$v(S) \leq v(T) \qquad \forall \emptyset \neq S \subseteq T. \tag{2.21}$$

Thus, whenever a game is monotonic, a coalition T can guarantee to its member a value at least as high as any sub-coalition S can do. This subclass of games is referred to as $\mathcal{MN}^n$. A game $v \in \mathcal{G}^n$ satisfying the condition

$$v(S) + v(T) \leq v(S \cup T) \qquad \forall S, T \subseteq N, \text{ with } S \cap T = \emptyset, \tag{2.22}$$

is called **superadditive**. This means, that two disjoint coalitions have some incentive to join into a mutual coalition. This can be regarded as an incentive of merging economic activities into larger units. We denote this subclass of games by $\mathcal{SA}^n$. However, if a game $v \in \mathcal{G}^n$ satisfies

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \qquad \forall S, T \subseteq N, \tag{2.23}$$

then it is called **convex**. In this case, we will observe a strong incentive for a mutual cooperation in the grand coalition, due to its achievable over proportionate surpluses while increasing the scale of cooperation. This subclass of games has been introduced by Shapley (1971), and we denote it by $\mathcal{CV}^n$. Convex games having a non-empty core and the Shapley value is the center of gravity of the extreme point of the core (cf. Shapley (1971)), that is, a convex combination of the vectors of marginal contributions, which are core imputations for convex games. It should be evident that $\mathcal{CV}^n \subset \mathcal{SA}^n$ is satisfied. Finally, note that whenever $v \in \mathcal{CV}^n$, then $\mathcal{C}(v) = \mathcal{M}(v)$.

In the next step, we want to discuss a special subclass of convex games, the so-called modest-bankruptcy games. For doing so, we introduce first a **bankruptcy situation** as an ordered pair $(B_{es}, \mathbf{d})$, where $B_{es} \in \mathbb{R}$ is the bankrupt estate and $\mathbf{d} = \{d_1, \ldots, d_n\} \in \mathbb{R}^n$ is a claims or debts vector such that $d_k \geq 0$ for all $k \in N$ and $0 \leq B_{es} \leq \sum_{k=1}^{n} d_k$ is given. This problem is called a bankruptcy situation, since the bankrupt estate is insufficient to meet all claims simultaneously. From this situation a corresponding transferable utility game, a **(modest-)bankruptcy game** $\langle N, v_{B_{es},d} \rangle$, can be derived by

$$v_{B_{es},d}(S) := \max\left(0, B_{es} - \sum_{k \in N \setminus S} d_k\right) \qquad \text{for all} \quad \emptyset \neq S \subseteq N, \tag{2.24}$$

with the convention that $v_{B_{es},d}(\emptyset) = 0$. This game class has been introduced by O'Neill (1982). A coalition of s-creditors in S gets either zero or what remains from the estate B_{es} after the opponents in coalition $N \setminus S$ are payed in accordance with their claims in $\mathbf{d}$.

However, this is not the unique way to derive from a bankruptcy situation a TU game. In an alternative representation, a **greedy bankruptcy game** $\langle N, \tilde{v}_{B_{es},d} \rangle$ is defined as

$$\tilde{v}_{B_{es},d}(S) := \min\left(B_{es}, \sum_{k \in S} d_k\right) \qquad \text{for all} \quad \emptyset \neq S \subseteq N. \tag{2.25}$$

with the convention that $\tilde{v}_{B_{es},d}(\emptyset) = 0$. This game is called to be greedy on the understanding that the s-creditors of S can go to the court in order to attempt to obtain the complete estate B_{es} while the amount that goes beyond this estate is considered as irrelevant to satisfy the claims.

The **dual of game** $v \in \mathcal{G}^n$ is the game $v^* \in \mathcal{G}^n$ defined by

$$v^*(S) := v(N) - v(N \backslash S) \qquad \text{for all } S \subseteq N. \tag{2.26}$$

The worth of $v^*(S)$ can be interpreted as indicating the marginal contribution of coalition S to the grand coalition N. In other words, the value $v^*(S)$ is the amount from which coalition S can not be prevented from when the complement $N \backslash S$ receives $v(N \backslash S)$. It can be easily seen that the **dual game** of the (modest-)bankruptcy game $\langle N, v_{B_{es},d} \rangle$ is the greedy-bankruptcy game $\langle N, \tilde{v}_{B_{es},d} \rangle$, hence we have $\tilde{v}_{B_{es},d}(S) = v_{B_{es},d}(N) - v_{B_{es},d}(N \backslash S)$ for all $S \subseteq N$. For a more detailed discussion of bankruptcy games, see Driessen (1998).

A game that refers to the power of a voter in a voting scheme where exists only two states of the world, that is, either winning or losing, is called a **simple game** (cf. von Neumann and Morgenstern (1944)). A winning coalition gets a worth of one whereas a losing coalition gets zero, or more formally

$$v(S) \in \{0, 1\} \qquad \text{for all} \quad S \subset N \quad \text{and} \quad v(N) = 1. \tag{2.27}$$

Notice that a simple game satisfies the monotonicity property. This class of games is denoted in the sequel as $\mathcal{GS}^n$.

A simple game is referred to as **weighted majority game**, if there exists a quota/threshold $th > 0$ and weights $w_k \geq 0$ for all $k \in N$ such that for all $S \subseteq N$ it holds

$$v(S) := \begin{cases} 1 & \text{if } w(S) \geq th \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

Such a game is generically represented as $[th; w_1, \ldots, w_n]$.

Denote $\mathcal{U}$ as a set of players and let $\mathcal{G}_{\mathcal{U}}$ be the set of all games with players in $\mathcal{U}$. A **potential** is a function $\rho : \mathcal{G}_{\mathcal{U}} \to \mathbb{R}$ satisfying for every game $\langle N, v \rangle \in \mathcal{G}_{\mathcal{U}}$ the following two properties

$$\rho(\emptyset, v) = 0 \qquad \text{and} \qquad \sum_{k \in N} \mathcal{D}^k \rho(N, v) = v(N), \tag{2.29}$$

whereas the marginal contribution $\mathcal{D}^k \rho(N, v)$ of a player k in game $\langle N, v \rangle$ is defined to be

$$\mathcal{D}^k \rho(N, v) := \begin{cases} \rho(N, v) & \text{if } |N| = 1 \\ \rho(N, v) - \rho(N \backslash \{k\}, v) & \text{if } |N| \geq 2 \end{cases} \tag{2.30}$$

with subgame $\langle N \backslash \{k\}, v \rangle$ of game $\langle N, v \rangle$ for all $k \in N$. Whenever function ρ is a potential the allocation of **marginal contributions of players** is efficient (Pareto optimal). The potential has been invented by Hart and Mas-Colell (1989). From formula (2.29), we derive

$$\rho(N, v) = \frac{1}{|N|} \left[v(N) + \sum_{k \in N} \rho(N \backslash \{k\}, v) \right]. \tag{2.31}$$

In Chap. 4 we will see how we finally get from $\rho(\emptyset, v) = 0$ while applying a recursive procedure the expression $\rho(N, v)$. The **Shapley value** $\phi_k(v)$ for each player $k \in N$ in game $\langle N, v \rangle$ is equal to

$$\phi_k(v) = \rho(N, v) - \rho(N \backslash \{k\}, v). \tag{2.32}$$

A proof is given in Hart and Mas-Colell (1989, pp. 591–592).

http://www.springer.com/978-3-642-39548-2

The Pre-Kernel as a Tractable Solution for Cooperative Games
An Exercise in Algorithmic Game Theory
Meinhardt, H.I.
2014, XXXIII, 242 p. 8 illus., 3 illus. in color., Hardcover
ISBN: 978-3-642-39548-2