## Contents

### Part I  Principles of Monte Carlo Methods

1  **Introduction** ......................................................... 3  
   1.1  Why Use Probabilistic Models and Simulations? ................. 3  
      1.1.1  What Are the Reasons for Probabilistic Models? ....... 4  
      1.1.2  What Are the Objectives of Random Simulations? ....... 6  
   1.2  Organization of the Monograph .............................. 9  

2  **Strong Law of Large Numbers and Monte Carlo Methods** .. 13  
   2.1  Strong Law of Large Numbers, Examples of Monte Carlo Methods 13  
      2.1.1  Strong Law of Large Numbers, Almost Sure Convergence . 13  
      2.1.2  Buffon’s Needle ........................................... 15  
      2.1.3  Neutron Transport Simulations ............................ 15  
      2.1.4  Stochastic Numerical Methods for Partial Differential Equations ............................................. 17  
   2.2  Simulation Algorithms for Simple Probability Distributions . 18  
      2.2.1  Uniform Distributions ....................................... 19  
      2.2.2  Discrete Distributions ....................................... 20  
      2.2.3  Gaussian Distributions ...................................... 21  
      2.2.4  Cumulative Distribution Function Inversion, Exponential Distributions ............................................. 22  
      2.2.5  Rejection Method ............................................ 23  
   2.3  Discrete-Time Martingales, Proof of the SLLN ................. 25  
      2.3.1  Reminders on Conditional Expectation .................. 25  
      2.3.2  Martingales and Sub-martingales, Backward Martingales . 27  
      2.3.3  Proof of the Strong Law of Large Numbers ............. 30  
   2.4  Problems ......................................................... 33  

3  **Non-asymptotic Error Estimates for Monte Carlo Methods** .. 37  
   3.1  Convergence in Law and Characteristic Functions .......... 37  
   3.2  Central Limit Theorem ......................................... 40  
      3.2.1  Asymptotic Confidence Intervals ....................... 41  
   3.3  Berry–Esseen’s Theorem ....................................... 42  

| 3.4 | Bikelis’ Theorem | 45 |
| 3.4.1 | Absolute Confidence Intervals | 45 |
| 3.5 | Concentration Inequalities | 47 |
| 3.5.1 | Logarithmic Sobolev Inequalities | 48 |
| 3.5.2 | Concentration Inequalities, Absolute Confidence Intervals | 50 |
| 3.6 | Elementary Variance Reduction Techniques | 54 |
| 3.6.1 | Control Variate | 54 |
| 3.6.2 | Importance Sampling | 55 |
| 3.7 | Problems | 60 |

Part II  
Exact and Approximate Simulation of Markov Processes

4  
Poisson Processes as Particular Markov Processes  
4.1  
Quick Introduction to Markov Processes  
4.1.1  
Some Issues in Markovian Modeling  
4.1.2  
Rudiments on Processes, Sample Paths, and Laws  
4.2  
Poisson Processes: Characterization, Properties  
4.2.1  
Point Processes and Poisson Processes  
4.2.2  
Simple and Strong Markov Property  
4.2.3  
Superposition and Decomposition  
4.3  
Simulation and Approximation  
4.3.1  
Simulation of Inter-arrivals  
4.3.2  
Simulation of Independent Poisson Processes  
4.3.3  
Long Time or Large Intensity Limit, Applications  
4.4  
Problems  
85

5  
Discrete-Space Markov Processes  
5.1  
Characterization, Specification, Properties  
5.1.1  
Measures, Functions, and Transition Matrices  
5.1.2  
Simple and Strong Markov Property  
5.1.3  
Semigroup, Infinitesimal Generator, and Evolution Law  
5.2  
Constructions, Existence, Simulation, Equations  
5.2.1  
Fundamental Constructions  
5.2.2  
Explosion or Existence for a Markov Process  
5.2.3  
Fundamental Simulation, Fictitious Jump Method  
5.2.4  
Kolmogorov Equations, Feynman–Kac Formula  
5.2.5  
Generators and Semigroups in Bounded Operator Algebras  
5.2.6  
A Few Case Studies  
5.3  
Problems  
115

6  
Continuous-Space Markov Processes with Jumps  
6.1  
Preliminaries  
6.1.1  
Measures, Functions, and Transition Kernels  
6.1.2  
Markov Property, Finite-Dimensional Marginals  
6.1.3  
Semigroup, Infinitesimal Generator  
6.2  
Markov Processes Evolving Only by Isolated Jumps  
6.2.1  
Semigroup, Infinitesimal Generator, and Evolution Law  
126
6.2.2 Construction, Simulation, Existence .......................... 130
6.2.3 Kolmogorov Equations, Feynman–Kac Formula, Bounded Generator Case ......................................................... 133
6.3 Markov Processes Following an Ordinary Differential Equation Between Jumps: PDMP .................................................. 136
6.3.1 Sample Paths, Evolution, Integro-Differential Generator .... 136
6.3.2 Construction, Simulation, Existence ............................ 141
6.3.3 Kolmogorov Equations, Feynman–Kac Formula .............. 144
6.3.4 Application to Kinetic Equations ............................... 146
6.3.5 Further Extensions ................................................. 149
6.4 Problems ................................................................. 151

7 Discretization of Stochastic Differential Equations .............. 155
7.1 Reminders on Itô’s Stochastic Calculus .......................... 155
7.1.1 Stochastic Integrals and Itô Processes ......................... 155
7.1.2 Itô’s Formula, Existence and Uniqueness of Solutions of Stochastic Differential Equations .............................. 160
7.1.3 Markov Properties, Martingale Problems and Fokker–Planck Equations .......................................................... 162
7.2 Euler and Milstein Schemes ........................................... 165
7.3 Convergence Rates in $L^p(\Omega)$ Norm and Almost Surely ... 168
7.4 Monte Carlo Methods for Parabolic Partial Differential Equations ................................................................. 176
7.5 The Principle of the Method ........................................... 176
7.5.2 Introduction of the Error Analysis ............................. 177
7.6 Optimal Convergence Rate: The Talay–Tubaro Expansion .... 180
7.7 Romberg–Richardson Extrapolation Methods .................. 185
7.8 Probabilistic Interpretation and Estimates for Parabolic Partial Differential Equations ........................................ 186
7.9 Problems ................................................................. 191

Part III  Variance Reduction, Girsanov’s Theorem, and Stochastic Algorithms

8 Variance Reduction and Stochastic Differential Equations .... 199
8.1 Preliminary Reminders on the Girsanov Theorem .............. 199
8.2 Control Variates Method ............................................. 200
8.3 Variance Reduction for Sensitivity Analysis ..................... 202
8.3.1 Differentiable Terminal Conditions ........................... 202
8.3.2 Non-differentiable Terminal Conditions ................... 204
8.4 Importance Sampling Method ...................................... 206
8.5 Statistical Romberg Method ......................................... 209
8.6 Problems ................................................................. 210

9 Stochastic Algorithms .................................................. 213
9.1 Introduction .......................................................... 213
9.2 Study in an Idealized Framework ................................. 214
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2.1 Definitions</td>
<td>214</td>
</tr>
<tr>
<td>9.2.2 The Ordinary Differential Equation Method, Martingale Increments</td>
<td>216</td>
</tr>
<tr>
<td>9.2.3 Long-Time Behavior of the Algorithm</td>
<td>217</td>
</tr>
<tr>
<td>9.3 Variance Reduction for Monte Carlo Methods</td>
<td>221</td>
</tr>
<tr>
<td>9.3.1 Searching for an Importance Sampling</td>
<td>221</td>
</tr>
<tr>
<td>9.3.2 Variance Reduction and Stochastic Algorithms</td>
<td>223</td>
</tr>
<tr>
<td>9.4 Problems</td>
<td>225</td>
</tr>
</tbody>
</table>

**Appendix**  Solutions to Selected Problems ...................................... 231

**References** ......................................................................................... 253

**Index** ................................................................................................. 257