Chapter 2
Worldlines and Proper Time

2.1 Introduction

Having introduced the mathematical framework of special relativity, we may move to the basics of (non-quantum) physics, namely, the description of the motion of a particle or a physical system idealized to a pointlike particle. We shall notably see the interpretation of the metric tensor $\boldsymbol{g}$ as the operator giving the elapsed time along the trajectory of a particle.

2.2 Worldline of a Particle

Special relativity being a non-quantum theory,[1] particles are described as points, as in classical mechanics. Actually, we shall use the word ***particle*** or ***point particle*** to cover either an elementary particle or a physical system whose spatial extension can be neglected at the scale of the phenomenon under study. A "particle at a given instant" will be represented by a point in the spacetime $\mathscr{E}$ (an *event*), and the "successive positions" of the particle will draw a one-dimensional curve in the affine space $\mathscr{E}$. Let us note that, at this stage, we cannot give some meaning to the phrase "at a given instant" if we wish to preserve the mixed space/time character of $\mathscr{E}$ and not to split it into some space part and some time part. Therefore, we shall define a particle by its entirety in spacetime, namely, a curve of $\mathscr{E}$, which we shall call the ***worldline*** of the particle.

The link between physics and the mathematics introduced in Chap. 1 consists in stating that the so-called *massive* particles do not follow any kind of worldline in Minkowski spacetime, but only those that are timelike:

[1] Minkowski spacetime is however the arena for relativistic quantum field theory.

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Any ***massive particle*** is represented by a piecewise twice continuously differentiable curve $\mathscr{L}$ of Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$ such that any vector tangent to $\mathscr{L}$ is timelike.

Let us recall that a vector $\vec{\boldsymbol{v}} \in E$ is timelike iff $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = \boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) < 0$ (cf. Sect. 1.3.4) and that a *piecewise twice continuously differentiable* curve means that there exists some function

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathscr{E} \\ \lambda &\longmapsto A = \varphi(\lambda) \end{aligned} \tag{2.1}$$

that is (i) twice differentiable with a continuous second derivative (i.e. of class C^2) on each interval of a finite subdivision of $\mathbb{R}$ and (ii) such that $\mathscr{L}$ is the image set of φ: $\mathscr{L} = \varphi(\mathbb{R})$. If φ is injective, it is called a ***parametrization*** of $\mathscr{L}$.

Remark 2.1. Of course, for a given worldline $\mathscr{L}$, there exists an infinite number of parametrizations: if φ is one of them, any bijective function $f : \mathbb{R} \to \mathbb{R}$ of class C^2 induces a new parametrization $\tilde{\varphi} := \varphi \circ f$. A priori, a parametrization of $\mathscr{L}$ is a purely mathematical operation. We shall introduce in Sect. 2.3 a parametrization with physical grounds: that provided by the "elapsed time" (the so-called *proper time*) along $\mathscr{L}$.

Remark 2.2. We may consider the above statement as the formal definition of a *massive particle*. The notion of *mass* will be introduced in Chap. 9, and we shall see that indeed massive particles, as defined above, have a nonvanishing mass.

Remark 2.3. By demanding that the worldline be timelike, we exclude hypothetical particles called ***tachyons*** (Bilaniuk et al. 1962; Feinberg 1967; Recami 1987; Boratav and Kerner 1991; Fayngold 2002). These particles would on the contrary move on spacelike worldlines. Note that there is no consistent relativistic theory that allows a given worldline to change its type on some part of it: a worldline is either always timelike (ordinary massive particles), null (photons, Sect. 2.5) or spacelike (tachyons). We shall elaborate more on tachyons in Sect. 4.3.3.

A parametrization φ of $\mathscr{L}$ induces a one-parameter family of vectors of E—at each point of $\mathscr{L}$, we may consider the ***derivative vector*** of φ at this point:

$$\forall \lambda \in \mathbb{R}, \qquad \vec{\boldsymbol{v}}(\lambda) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overrightarrow{A(\lambda) A(\lambda + \varepsilon)}, \tag{2.2}$$

where we have used the notation $A(\lambda)$ for $\varphi(\lambda)$ [generic point of $\mathscr{L}$, cf. (2.1)]. $\vec{\boldsymbol{v}}$ is called the ***field of tangent vectors*** associated with the parametrization φ. One may give a more "physical" expression to $\vec{\boldsymbol{v}}$: denoting by $\mathrm{d}\lambda$ the increase ε of the parameter λ and by $\mathrm{d}\vec{\boldsymbol{x}}$ the infinitesimal vector joining the point $A(\lambda)$ to the point $A(\lambda + \mathrm{d}\lambda)$ (cf. Fig. 2.1), we get

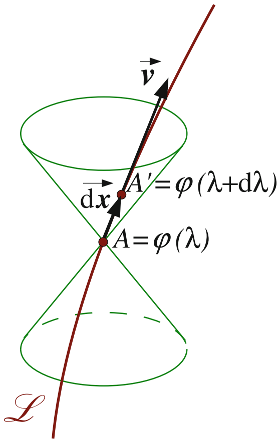

Fig. 2.1 Worldline of a massive particle, with the tangent vector $\vec{v}$ associated with the parametrization $\varphi(\lambda)$. The null cone at point A is shown. Since $\vec{v}$ is timelike, it is located inside the null cone

$$\forall \lambda \in \mathbb{R}, \qquad \boxed{\vec{v}(\lambda) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}}. \tag{2.3}$$

From the definition of a worldline, the vector $\vec{v}(\lambda)$ must be timelike for all values of the parameter λ: $\vec{v}(\lambda) \cdot \vec{v}(\lambda) < 0$.

If $(O; \vec{e}_\alpha)$ is an affine frame of $\mathscr{E}$ (cf. Sect. 1.2.3) and $(x^\alpha(\lambda))$ the affine coordinates of $A = \varphi(\lambda)$ in this frame, the components of the tangent vector $\vec{v}(\lambda)$ with respect to the basis $(\vec{e}_\alpha)$ are the derivatives of the functions $x^\alpha(\lambda)$: $v^\alpha(\lambda) = \mathrm{d}x^\alpha/\mathrm{d}\lambda$; hence,

$$\vec{v}(\lambda) = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\, \vec{e}_\alpha. \tag{2.4}$$

2.3 Proper Time

2.3.1 Definition

We have already noticed in Sect. 1.3.1 that the metric tensor $\boldsymbol{g}$ does not define a metric on $\mathscr{E}$ in the strict mathematical sense and that it should be called instead *pseudo-metric tensor* (cf. Remark 1.6 p. 8). As a consequence, the norm with respect to $\boldsymbol{g}$ introduced in Sect. 1.3.5, $\|\ \|_g$, is not a norm in the mathematical sense. In particular, $\|\vec{v}\|_g = 0$ is not equivalent to $\vec{v} = 0$. However, if the "norm" $\|\ \|_g$ is taken only on timelike vectors (such as the tangent vectors to massive particle worldlines), i.e. if one considers the mapping

$$\begin{array}{ccl} E_{\text{timelike}} & \longrightarrow & \mathbb{R}^+ \\ \vec{v} & \longmapsto & \|\vec{v}\|_g = \sqrt{-\boldsymbol{g}(\vec{v}, \vec{v})}, \end{array} \tag{2.5}$$

then one gets a function that vanishes only for $\vec{v} = 0$, as for any norm.[2] Accordingly, one may use $\boldsymbol{g}$ to measure "lengths" along a given worldline. The fundamental physical interpretation of the metric tensor $\boldsymbol{g}$ consists in stating that these "lengths" correspond to the elapsed time along the worldline:

Let A and A' be two infinitely close events on the worldline $\mathscr{L}$ of a given massive particle (cf. Fig. 2.1). Let $\mathrm{d}\vec{x}$ be the infinitesimal vector connecting A and A'. The vector $\mathrm{d}\vec{x}$ is tangent to $\mathscr{L}$, and from the definition of a worldline, it is timelike. We may then set

$$\boxed{\begin{cases} c\,\mathrm{d}\tau := \|\mathrm{d}\vec{x}\|_{\boldsymbol{g}} = \sqrt{-\boldsymbol{g}(\mathrm{d}\vec{x},\mathrm{d}\vec{x})} & \text{if } \mathrm{d}\vec{x} \text{ is future-directed} \\ c\,\mathrm{d}\tau := -\|\mathrm{d}\vec{x}\|_{\boldsymbol{g}} = -\sqrt{-\boldsymbol{g}(\mathrm{d}\vec{x},\mathrm{d}\vec{x})} & \text{if } \mathrm{d}\vec{x} \text{ is past-directed} \end{cases}}\,. \tag{2.6}$$

Let us recall that the future/past-directed properties have been defined in Sect. 1.4. Thanks to the c factor (cf. Sect. 1.2.4), the dimension of $\mathrm{d}\tau$ is time, $\boldsymbol{g}$ having no dimension and $\mathrm{d}\vec{x}$ having the dimension of length (cf. the convention adopted in Sect. 1.2.4). $\mathrm{d}\tau$ is called the ***proper time*** elapsed between the events A and A' on $\mathscr{L}$.

If the displacement $\mathrm{d}\vec{x}$ is represented by its components $(\mathrm{d}x^\alpha)$ in some orthonormal basis of $(E,\boldsymbol{g})$, the scalar product $\boldsymbol{g}(\mathrm{d}\vec{x},\mathrm{d}\vec{x})$ can be expressed according to (1.18), so that (2.6) becomes

$$\boxed{c\,\mathrm{d}\tau = \pm\sqrt{(\mathrm{d}x^0)^2-(\mathrm{d}x^1)^2-(\mathrm{d}x^2)^2-(\mathrm{d}x^3)^2},}_{\text{orthonormal basis}} \tag{2.7}$$

where the sign $\pm$ corresponds to the two cases considered in (2.6).

Given a parametrization $\varphi(\lambda)$ of $\mathscr{L}$, one may express the proper time in terms of the associated tangent vector field $\vec{v}$. Let us suppose that $\vec{v}$ is future-directed. Would this not be the case, the change of parameter $\lambda \mapsto -\lambda$ would provide a future-directed tangent vector. Then, we have

$$\mathrm{d}\vec{x} = \vec{v}\,\mathrm{d}\lambda, \tag{2.8}$$

where $\mathrm{d}\lambda$ is the difference of parameter between A' and A: $A = \varphi(\lambda)$, $A' = \varphi(\lambda + \mathrm{d}\lambda)$ (cf. Fig. 2.1). Thanks to $\boldsymbol{g}$'s bilinearity, (2.6) can be written

$$c\,\mathrm{d}\tau = \sqrt{-\boldsymbol{g}(\vec{v},\vec{v})}\,\mathrm{d}\lambda. \tag{2.9}$$

[2] It is however still not a norm in the mathematical meaning, for it does not satisfy the triangle inequality $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

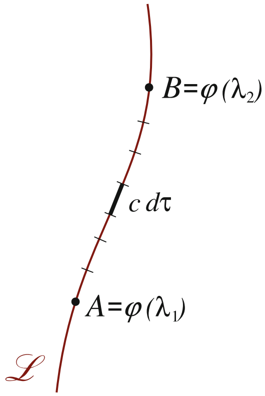

Fig. 2.2 Proper time between events A and B along a worldline $\mathscr{L}$

Remark 2.4. Choosing a parametrization such that $\vec{v}$ is future-directed ensures that $\mathrm{d}\tau$ has the correct sign, namely, is positive (resp. negative) if $\mathrm{d}\vec{x}$ is future-directed (resp. past-directed). Let us stress that, although Eq. (2.9) lets appear the parametrization φ of $\mathscr{L}$, the value of $\mathrm{d}\tau$ is independent of that parametrization, as it is clear on (2.6).

The definition of proper time can be extended to events with a finite separation along a worldline, by integrating (2.6) between these two events. Hence if A and B are two events of some worldline $\mathscr{L}$ (cf. Fig. 2.2) and if φ is a parametrization of $\mathscr{L}$ such that $A = \varphi(\lambda_1)$ and $B = \varphi(\lambda_2)$, we set

$$\boxed{\tau(A, B) := \int_A^B \mathrm{d}\tau = \frac{1}{c}\int_{\lambda_1}^{\lambda_2} \sqrt{-\boldsymbol{g}\,(\vec{v}(\lambda), \vec{v}(\lambda))}\, \mathrm{d}\lambda}, \qquad (2.10)$$

where $\vec{v}(\lambda)$ is the tangent vector field associated with the parametrization φ. As for $\mathrm{d}\tau$, $\tau(A, B)$ does not depend on the choice of the parametrization φ. On the other hand, it depends on the worldline connecting A to B.

2.3.2 *Ideal Clock*

Let us call ***clock*** any physical device that (i) can be reduced to a point particle (at the scale of the phenomenon under study), (ii) follows a timelike worldline $\mathscr{L}$ and (iii) provides a sequence of "signals", i.e. a sequence of events $\ldots, E_{-1}, E_0, E_1, E_2, \ldots$ sampling $\mathscr{L}$ (Fig. 2.3). Each E_k is called a ***tick***.

An ***ideal clock*** is then defined as a clock for which the proper time $\tau(E_k, E_{k+N})$ between two ticks E_k and E_{k+N} is equal to a constant K times the number N of elapsed ticks:

$$\tau(E_k, E_{k+N}) = K\,N. \qquad (2.11)$$

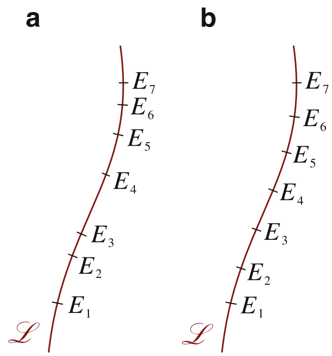

Fig. 2.3 (**a**) Generic clock; (**b**) ideal clock

Among all the clocks, ideal clocks are characterized by the fact that the proportionality; factor K is the same at each point of their worldline (Fig. 2.3). In other words, the time indicated by an ideal clock is the proper time along the clock's worldline.

Remark 2.5. Relativity has banished the concept of *absolute time* (cf. Sect. 1.2.5). It however introduces along each worldline a privileged time: that given by the metric tensor according to (2.6). An ideal clock is a clock that displays this time. Obviously it varies from one worldline to the other, i.e. the quantity $\tau(A, B)$ defined by (2.10) depends upon the worldline connecting A and B. It is in that sense that relativity has suppressed absolute time.

An ideal clock is a "theoretical" device that can be more or less well approximated by an actual device. To know whether a given experimental clock constitutes a good approximation of an ideal clock, one may check if the laws of kinematics and dynamics (which will be developed in the coming chapters and are expressed in terms of the proper time) are satisfied when experiments are described with the time given by this clock. For instance, a pendulum held fixed with respect to the Earth constitutes a relatively good approximation (at the human scale!) of an ideal clock. But this is no longer true in a strongly accelerated frame with respect to the Earth: the pendulum motion looses any periodicity if the acceleration is not constant. An atomic clock constitutes a much better approximation of an ideal clock, because it provides a time that depends very weakly on its state of acceleration, at least for accelerations smaller than the centripetal acceleration of an electron around the atomic nucleus, which is about $10^{23}\ \mathrm{m\,s^{-2}}$.

Remark 2.6. Since it is related to the fundamental object of relativity, namely, the metric tensor $\boldsymbol{g}$, the proper time is the only truly *physical time*, in the following meaning. The definition of time along a given worldline is a priori arbitrary: one can choose the time provided by any clock. The distinctive feature of proper time is that the physical laws expressed in terms of it are simpler than if expressed in terms of an arbitrary time, because the basic physical laws involve the metric tensor, to which proper time is directly related. Considering the example mentioned above, the pendulum beats are periodic functions of the proper time in an inertial frame. To paraphrase Poincaré (1898), we may say that it is for a matter of *commodity* that one uses proper time and not an arbitrary time.

Remark 2.7. When considering a human being, the proper time is also the most convenient one to describe her/his physiological evolution, given the physical nature of physiological processes. Admitting that the physiological time is indeed the one perceived by consciousness, one may think about the proper time along a worldline as the time "felt" by a human observer moving along this worldline.

Remark 2.8. The fundamental concept that appears once the metric tensor and the worldlines have been introduced is that of *time* and not of *length*. We shall discuss this further below.

2.4 Four-Velocity and Four-Acceleration

2.4.1 Four-Velocity

We have seen in Sect. 2.2 that one may associate many tangent vector fields to a given worldline $\mathscr{L}$, namely, the tangent fields linked to all possible parametrizations of $\mathscr{L}$. The introduction of proper time in Sect. 2.3 allows us to select a tangent vector field independent of any parametrization and thereby intrinsic to the worldline: the ***four-velocity***, or ***4-velocity*** for short, of a massive particle evolving along a worldline $\mathscr{L}$ is the vector of E defined at any point $A \in \mathscr{L}$ by

$$\boxed{\vec{\boldsymbol{u}} := \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{x}}}{\mathrm{d}\tau}}, \tag{2.12}$$

where $\mathrm{d}\vec{\boldsymbol{x}}$ is an infinitesimal vector tangent to $\mathscr{L}$ and future-directed (cf. Sect. 1.4) and $\mathrm{d}\tau$ is the proper time interval corresponding to $\mathrm{d}\vec{\boldsymbol{x}}$ via (2.6). If one wishes to give a rigorous mathematical meaning to (2.12), it suffices to parametrize the worldline $\mathscr{L}$ by c times its proper time: $\lambda = c\tau$. Such a parametrization is unique, up to the choice of some origin. The vector $\vec{\boldsymbol{u}}$ is then nothing but the derivative of that parametrization, as defined in Sect. 2.2. As a derivative, it is of course independent of the origin of proper time.

If $\vec{\boldsymbol{v}}$ is a future-directed tangent vector field associated with some parametrization $\varphi(\lambda)$ of $\mathscr{L}$, we may insert (2.8) and (2.9) into (2.12) and get

$$\vec{\boldsymbol{u}} = \frac{\vec{\boldsymbol{v}}}{\sqrt{-\boldsymbol{g}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}})}} = \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|_{\boldsymbol{g}}}. \tag{2.13}$$

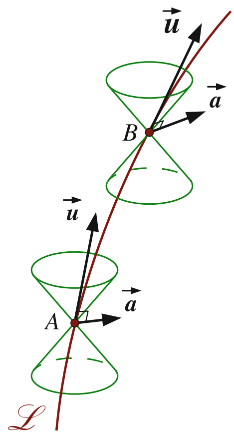

Fig. 2.4 4-velocity $\vec{u}$ and 4-acceleration $\vec{a}$ at two points A and B of a timelike worldline $\mathscr{L}$

This identity can be viewed as the definition of a *unit* tangent vector ($\vec{u}$) from an arbitrary tangent vector ($\vec{v}$). It is actually trivial to check on (2.13) that[3]

$$\boxed{\vec{u} \cdot \vec{u} = -1}. \tag{2.14}$$

We could even have introduced the 4-velocity $\vec{u}$ as the unique future-directed unit vector tangent to $\mathscr{L}$. The definition (2.12) has more the aspect of a "velocity". Note however that $\vec{u}$ is dimensionless, thanks to the factor $1/c$ in (2.12).

Remark 2.9. Many authors define the 4-velocity with the dimension of a velocity, by setting $\vec{u} := \mathrm{d}\vec{x}/\mathrm{d}\tau$ instead of (2.12). Equation (2.14) becomes then $\vec{u} \cdot \vec{u} = -c^2$. We follow here the convention of Landau and Lifshitz (1975), preferring a dimensionless 4-velocity, because many expressions are simplified when $\vec{u}$ is a unit vector. Moreover, from a pedagogical point of view, the dimensionless character of the 4-velocity is valuable in avoiding the confusion with an "ordinary" velocity, which is a different concept (in particular, it is relative to some observer, contrary to the 4-velocity, cf. Remark 2.10 below).

The property (2.14) implies that the 4-velocity belongs to the set $\mathscr{U}^+$ introduced in Sect. 1.4.3:

$$\boxed{\vec{u} \in \mathscr{U}^+}. \tag{2.15}$$

Conversely, any element of $\mathscr{U}^+$ can be considered as a 4-velocity. We conclude that $\mathscr{U}^+$ is nothing but the set of all possible 4-velocities.

The 4-velocity at two points A and B of a worldline is depicted in Fig. 2.4. The null cone of $\boldsymbol{g}$ is also drawn at these two points (cf. Sect. 1.4): as a timelike future-directed vector, $\vec{u}$ is located inside the future null cone $\mathscr{I}^+$.

[3]Let us recall that the notation $\vec{u} \cdot \vec{u}$ stands for $\boldsymbol{g}(\vec{u}, \vec{u})$.

Remark 2.10. The reader might have been surprised by the fact that, in the theory of *relativity*, the 4-velocity has not been defined *relatively* to a frame or an observer. On the contrary, it has been introduced as an *absolute* quantity, which depends only on the considered worldline, the latter being obviously independent of any observer. Actually, the 4-velocity is different from a velocity and is not a directly measurable quantity. After having introduced the concept of observer in Chap. 3, we shall define in Chap. 4 the "ordinary" velocity of a point particle with respect to an observer. It will be a function of the 4-velocities of the particle and the observer, and will be a measurable quantity, as the ratio of a length by a time.

2.4.2 Four-Acceleration

It is natural to define the ***four-acceleration***, or ***4-acceleration*** for short, as the vector of E that measures the variation of the 4-velocity field $\vec{\boldsymbol{u}}$ along the worldline $\mathscr{L}$:

$$\boxed{\vec{\boldsymbol{a}} := \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}}{\mathrm{d}\tau}}, \tag{2.16}$$

where τ stands for the proper time along $\mathscr{L}$. The above expression takes a rigorous mathematical meaning if $\mathscr{L}$ is parametrized by $\lambda = c\tau$: the vector $\vec{\boldsymbol{a}}$ is then nothing but the second derivative of this parametrization.

$\vec{\boldsymbol{u}}$ being dimensionless and $c\tau$ having the dimension of a length [cf. Eq. (2.6)], the dimension of the 4-acceleration is that of the inverse of a length and not that of an acceleration.[4]

Two basic properties of the 4-acceleration follow easily from its definition:

- $\vec{\boldsymbol{a}}$ is orthogonal to $\vec{\boldsymbol{u}}$ (with respect to the metric $\boldsymbol{g}$):

$$\boxed{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{u}} = 0}. \tag{2.17}$$

Proof. One has

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{u}} = \frac{1}{c}\frac{\mathrm{d}\vec{\boldsymbol{u}}}{\mathrm{d}\tau} \cdot \vec{\boldsymbol{u}} = \frac{1}{2c}\frac{d}{\mathrm{d}\tau}\left(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}\right) = \frac{1}{2c}\frac{d}{\mathrm{d}\tau}(-1) = 0. \qquad \square$$

- $\vec{\boldsymbol{a}}$ is either the zero vector or a spacelike vector.

Proof. If $\vec{\boldsymbol{a}} \neq 0$, thanks to (2.17) and by means of the Gram–Schmidt process (Deheuvels 1981), we may find an orthogonal basis of E of the type $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{a}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2)$. In this basis, taking into account (2.14), the matrix of $\boldsymbol{g}$ is

[4] As for the 4-velocity, which has not the dimension of a velocity, cf. Remark 2.9.

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \vec{a}\cdot\vec{a} & 0 & 0 \\ 0 & 0 & \vec{e}_1\cdot\vec{e}_1 & 0 \\ 0 & 0 & 0 & \vec{e}_2\cdot\vec{e}_2 \end{pmatrix}.$$

Since the signature of $\boldsymbol{g}$ is $(-,+,+,+)$, the diagonal terms but -1 are necessarily strictly positive (Sylvester's law of inertia, Sect. 1.3.1); hence, in particular, $\vec{a}\cdot\vec{a} > 0$, which proves that $\vec{a}$ is spacelike. □

We conclude that

$$\boxed{\vec{a}\cdot\vec{a} \geq 0}, \tag{2.18}$$

with $\vec{a}\cdot\vec{a} = 0$ iff $\vec{a} = 0$.

It is worth to note that the above demonstration uses only the fact that $\vec{a}$ is orthogonal to $\vec{u}$; we have therefore established a very useful property:

Any vector orthogonal to a timelike vector is necessarily spacelike or zero.

The 4-acceleration at two points of a worldline is depicted in Fig. 2.4. Being spacelike, this vector is located outside the null cone, contrary to $\vec{u}$. Note that the orthogonality between $\vec{a}$ and $\vec{u}$ does not imply the orthogonality in the usual (Euclidean) sense of the arrows representing $\vec{a}$ and $\vec{u}$ in Fig. 2.4 (cf. the discussion about the graphical representation of vectors in Sect. 1.3.6). We shall see in Sect. 2.7.3 a geometrical interpretation of the 4-acceleration involving the *curvature* of the worldline.

Remark 2.11. As for the 4-velocity (cf. Remark 2.10), the 4-acceleration is an absolute quantity, independent of any frame or observer.

Historical note: *The concepts of* worldline*,* 4-velocity *and* 4-acceleration *have been introduced by Hermann Minkowski (cf. p. 26). They appear in a publication of 1908 (Minkowski 1908) and play a central role in the famous article on spacetime published the year after (Minkowski 1909) and discussed at the end of Chap. 1. Note however that, as soon as 1905, in the "Palermo memoir" (Poincaré 1906), Henri Poincaré (cf. p. 26) let appear a four-dimensional vector that was nothing but the 4-velocity, although without any explicit mention of a worldline. The concept of* proper time*, as exposed above, namely, the length given by the metric tensor along a worldline, is also due to Minkowski: in the publications (Minkowski 1908) and (Minkowski 1909), he wrote the relations (2.6) and (2.10) (making use of the components (1.17) of* $\boldsymbol{g}$ *in an orthonormal basis). Besides, the relation (2.17) expressing the orthogonality of the 4-acceleration and the 4-velocity appears clearly in the 1909 text (Minkowski 1909).*

2.5 Photons

2.5.1 *Null Geodesics*

In Sect. 2.2, we have postulated that massive particles follow worldlines that are timelike. We shall now define the worldlines of massless particles, the first of them being photons. As for any point particle, a photon is represented by a one-dimensional curve in Minkowski spacetime (its worldline). Whereas the worldlines of massive particles show a great variety (all the curves with timelike tangent vectors), photons are compelled to follow quite specific curves, *straight lines*, the direction vector of which is null:

> In vacuum, a ***massless particle***, and in particular a photon, is represented by a straight line of $\mathscr{E}$ whose direction vector is a null vector of the metric $\boldsymbol{g}$, i.e. a vector $\vec{\boldsymbol{v}}$ obeying $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} = 0$. Such a line is called a ***null geodesic*** of spacetime. If the particle is a photon, it is also called a ***light ray***.

This principle justifies the choice of qualifier *lightlike* given to null vectors of $\boldsymbol{g}$ (cf. Sect. 1.3.4). When we shall treat electromagnetism (Chaps. 17–20), we shall verify that the wave solutions to Maxwell equations in vacuum propagate along null directions of the metric tensor.

Remark 2.12. A null geodesic is a special case of a ***null curve***, i.e. a curve of $\mathscr{E}$ whose tangent vectors are null vectors. There exists null curves that are not straight lines and thus not null geodesics. An example is the helix defined by the parametric equation $x^0(\lambda) = r\lambda$, $x^1(\lambda) = r\cos\lambda$, $x^2(\lambda) = r\sin\lambda$, $x^3(\lambda) = 0$, with $r > 0$, in some affine coordinate system (x^α) associated with an orthonormal basis.

Remark 2.13. The notion of *proper time* introduced for massive particles cannot be extended to photons, because (2.6) would result in $d\tau = 0$ (for $d\vec{\boldsymbol{x}}$ is null along a null geodesic). This would mean that an ideal clock carried by a photon is frozen. Consequently, the 4-velocity of a photon cannot be defined. In other words, there does not exist any null vector that is a unit one (since by definition, the scalar square of a null vector is zero).

2.5.2 *Light Cone*

Let us consider an event A in the spacetime $\mathscr{E}$. The worldlines of all the photons that encounter A (photon passing through A, or emitted at A or received at A) form a subset of $\mathscr{E}$ that is the image of the null cone of $\boldsymbol{g}$ in E (Sect. 1.4) under the identification of the pair $(\mathscr{E}, A)$ (affine space $\mathscr{E}$ with A as an origin) with the vector space E (cf. Fig. 2.4). More precisely, let $\vec{\boldsymbol{u}}$ be the 4-velocity of a massive particle passing through A and $(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ three vectors such that $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is an orthonormal basis of $(E, \boldsymbol{g})$. $(A; \vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$ is then an (orthogonal) affine coordinate system of $\mathscr{E}$ (cf. Sect. 1.2.3). A point $M \in \mathscr{E}$ of affine coordinates (x^0, x^1, x^2, x^3) belongs to the worldline of a photon that encounters A iff $\overrightarrow{AM}$ is a null vector: $\boldsymbol{g}(\overrightarrow{AM}, \overrightarrow{AM}) = 0$. From (1.6) and (1.18), this is equivalent to

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0. \tag{2.19}$$

Such an equation defines a three-dimensional cone of apex A in the affine space $\mathscr{E}$, which is called the ***light cone*** of event A. We shall denote it by $\mathscr{I}(A)$, the sheet corresponding to the future (resp. past) null cone being denoted $\mathscr{I}^+(A)$ (resp. $\mathscr{I}^-(A)$). $\mathscr{I}^+(A)$ is called the ***future light cone*** of event A and $\mathscr{I}^-(A)$ the ***past light cone*** of A.

The null cone of apex A separates the events that are related to A by a timelike vector to those that are related to A by a spacelike vector. Figure 2.4 shows the light cones of two points A and B on the worldline of a massive particle.

Remark 2.14. The light cone is entirely determined by the considered event and does not depend upon the worldline passing through this event. Note also that the light cones of different events can be deduced from each other by a mere translation (see Fig. 2.4).

2.6 Langevin's Traveller and Twin Paradox

Having introduced formally the proper time at Sect. 2.3, let us now study it in a specific case, which puts forward its dependency with respect to the considered worldline. The "experiment" to be described has been designed by Paul Langevin[5] in 1911 (Langevin 1911). It is known as *Langevin's traveller* and it illustrates the so-called *twin paradox*. Beside proper time, it provides also a nice illustration of the concepts of 4-velocity and 4-acceleration introduced in Sect. 2.4.

[5]**Paul Langevin** (1872–1946): French physicist, known for his work on the magnetic properties of materials and Brownian motion. As a friend of Einstein since 1911, he contributed a lot in the diffusion of relativity in France (Paty 1999a). He was also president of the French League of Human Rights from 1944 to 1946.

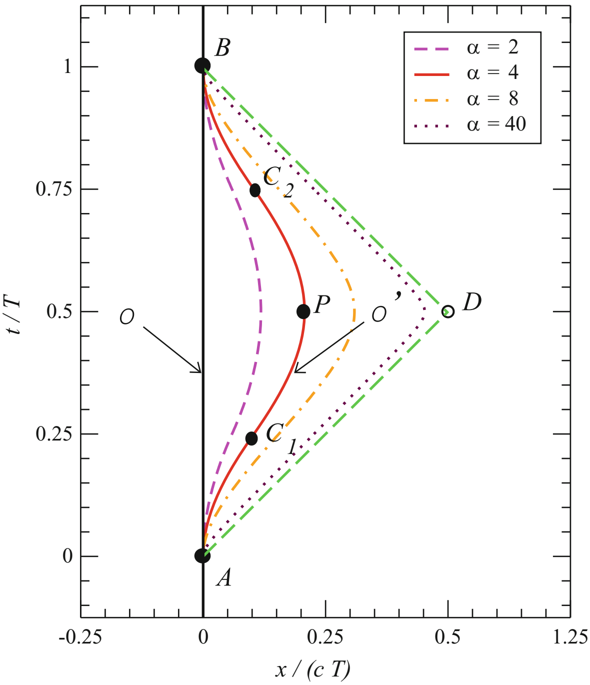

Fig. 2.5 Worldlines of the twins $\mathscr{O}$ and $\mathscr{O}'$: that of $\mathscr{O}$ is the *vertical line* $x = 0$ and that of $\mathscr{O}'$ is represented for different values of the parameter α. Between the events A ($\mathscr{O}'$ departure) and B ($\mathscr{O}'$ return), the worldline of $\mathscr{O}'$ is made of three arcs of hyperbola: AC_1, C_1C_2 and C_2B, defined by (2.20) (the points C_1, C_2 and P have been drawn only for $\alpha = 4$). *Long-dashed lines* indicate a null geodesic issued from A (segment $[AD]$) and a null geodesic arriving in B (segment $[DB]$); $[AD] \cup [DB]$ is thus the worldline of a photon emitted at A and reflected at D in order to meet observer $\mathscr{O}$ in B

2.6.1 Twins' Worldlines

Let us consider two observers $\mathscr{O}$ and $\mathscr{O}'$ that we shall model as two particles on timelike worldlines equipped with ideal clocks.[6] We take for the worldline $\mathscr{L}$ of $\mathscr{O}$ the simplest that one may think of: a straight line of $\mathscr{E}$. The worldline $\mathscr{L}'$ of $\mathscr{O}'$ is chosen to coincide with $\mathscr{L}$ until some event A; this is the very reason why $\mathscr{O}$ and $\mathscr{O}'$ may be called *twins*. At A, $\mathscr{O}'$ separates from $\mathscr{O}$ and travels until the event P. He then moves back and meets up with $\mathscr{O}$ at the event B, after which the worldlines $\mathscr{L}$ and $\mathscr{L}'$ coincide again (cf. Fig. 2.5).

Since $\mathscr{L}$ is a straight line, the 4-velocity $\vec{\boldsymbol{u}}$ of $\mathscr{O}$ is constant. This implies that the 4-acceleration of $\mathscr{O}$ vanishes. Let then $(\vec{\boldsymbol{e}}_\alpha)$ be an orthonormal basis of $(E, \boldsymbol{g})$ such

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[6] We shall define more precisely the concept of *observer* in Chap. 3, the present version being sufficient for the purpose of this section.

that $\vec{u}$ is equal to the (constant) vector $\vec{e}_0$. We consider the affine coordinate system ($x^0 = ct, x^1 = x, x^2 = y, x^3 = z$) defined by this basis and having A as origin (cf. Sect. 1.2.3). The points M of $\mathscr{L}$ obey the relation $\overrightarrow{AM} = ct\,\vec{e}_0$. Differentiating, we get $\mathrm{d}\overrightarrow{AM} = c\,\mathrm{d}t\,\vec{e}_0$, so that formula (2.6) along with the property $\boldsymbol{g}(\vec{e}_0, \vec{e}_0) = -1$ shows that the coordinate t coincides with $\mathscr{O}$'s proper time.

Let us now define precisely the worldline of $\mathscr{O}'$. For simplicity, we suppose that $\mathscr{O}'$ travels always in the same direction, which we shall select to be that of $\vec{e}_1$. The spatial trajectory of $\mathscr{O}'$ as perceived by $\mathscr{O}$ is then a line segment, travelled in one way and then in the reverse way. The corresponding worldline in Minkowski spacetime is contained in the plane through A and generated by the vectors $(\vec{e}_0, \vec{e}_1)$, i.e. the plane (t, x). The precise shape of $\mathscr{L}'$ depends on the velocity of $\mathscr{O}'$ with respect to $\mathscr{O}$. We shall choose $\mathscr{L}'$ between A and B to be made of three arcs of hyperbola, AC_1, C_1C_2 et C_2B (cf. Fig. 2.5), defined in terms of the affine coordinates (ct, x, y, z) by the following equations:

$$\text{for } t \in \left[0, \frac{T}{4}\right]: \quad x(t) = \frac{cT}{\alpha}\left[\sqrt{1+\alpha^2\,(t/T)^2} - 1\right] \tag{2.20a}$$

$$\text{for } t \in \left[\frac{T}{4}, \frac{3T}{4}\right]: \quad x(t) = \frac{cT}{\alpha}\left[-\sqrt{1+\alpha^2\,(t/T-1/2)^2} + 2\sqrt{1+\frac{\alpha^2}{16}} - 1\right] \tag{2.20b}$$

$$\text{for } t \in \left[\frac{3T}{4}, T\right]: \quad x(t) = \frac{cT}{\alpha}\left[\sqrt{1+\alpha^2\,(t/T-1)^2} - 1\right], \tag{2.20c}$$

where T is $\mathscr{O}$'s proper time elapsed between the events A and B, so that $t(A) = 0$, $t(C_1) = T/4$, $t(C_2) = 3T/4$ and $t(B) = T$. The parameter $\alpha \in \mathbb{R}$ is dimensionless and allows us to consider a whole family of worldlines for $\mathscr{O}'$, as shown in Fig. 2.5. If $\alpha = 0$, $\mathscr{L}'$ coincides with $\mathscr{L}$ and for $\alpha \neq 0$, Eq. (2.20a) leads to

$$\left(\alpha\frac{x}{cT} + 1\right)^2 - \left(\alpha\frac{t}{T}\right)^2 = 1, \tag{2.21}$$

which is the equation of a hyperbola in the plane (t, x), having the "horizontal" line $t = 0$ as the foci axis. Similarly, (2.20b) defines a hyperbola of foci axis the line $t = T/2$ and (2.20c) a hyperbola of foci axis the line $t = T$ (cf. Fig. 2.5). The choice of arcs of hyperbola will be justified in Sect. 2.6.4, where we shall see that it implies a constant norm of the 4-acceleration. This constitutes a relativistic generalization of the uniformly accelerated motion, as we shall discuss in Sect. 12.2. We shall call the observer $\mathscr{O}'$ following the worldline defined above ***Langevin's traveller***.

Let P be the mid-journey event (maximal distance from $\mathscr{O}$, cf. Fig. 2.5). Its position depends upon α and is obtained by setting $t = T/2$ in (2.20b):

$$x(P) = \frac{2cT}{\alpha}\left(\sqrt{1+\frac{\alpha^2}{16}} - 1\right) = \frac{\alpha}{8}\,\frac{cT}{\sqrt{1+\alpha^2/16}+1}. \tag{2.22}$$

2.6.2 Proper Time of Each Twin

We have seen above that the proper time of $\mathscr{O}$ coincides with the coordinate t of the affine system (ct, x, y, z). To determine the proper time t' of $\mathscr{O}'$, let us parametrize the worldline $\mathscr{L}'$ by $\lambda = t$. An infinitesimal displacement $\mathrm{d}\vec{\boldsymbol{x}}'$ along $\mathscr{L}'$ has the components $\mathrm{d}x'^{\alpha} = (c\,\mathrm{d}t, \mathrm{d}x, 0, 0)$ in the orthonormal basis $(\vec{\boldsymbol{e}}_{\alpha})$, where $\mathrm{d}x$ is related to $\mathrm{d}t$ by differentiating (2.20):

$$\mathrm{d}x = (-1)^k \frac{\alpha(t/T - k/2)}{\sqrt{1+\alpha^2\,(t/T-k/2)^2}}\,c\,\mathrm{d}t, \tag{2.23}$$

where the integer k takes the following values : $k = 0$ for $0 \le t \le T/4$, $k = 1$ for $T/4 \le t \le 3T/4$ and $k = 2$ for $3T/4 \le t \le T$. The basis $(\vec{\boldsymbol{e}}_{\alpha})$ being orthonormal, the proper time t' along $\mathscr{L}'$ is given by formula (2.7):

$$\mathrm{d}t' = \frac{1}{c}\sqrt{(\mathrm{d}x'^0)^2 - (\mathrm{d}x'^1)^2 - (\mathrm{d}x'^2)^2 - (\mathrm{d}x'^3)^2} = \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - \mathrm{d}x^2}. \tag{2.24}$$

Substituting (2.23) for $\mathrm{d}x$ yields

$$\mathrm{d}t' = \frac{\mathrm{d}t}{\sqrt{1+\alpha^2\,(t/T-k/2)^2}}. \tag{2.25}$$

Thanks to the change of variable $\alpha(t/T - k/2) = \sinh u$, this equation is easily integrated into[7]

$$t' = \frac{T}{\alpha}\operatorname{arsinh}\left[\alpha\left(\frac{t}{T} - \frac{k}{2}\right)\right] + \frac{k}{2}\,T', \tag{2.26}$$

where arsinh stands for the inverse hyperbolic sine ($\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$) and

$$T' := \frac{4T}{\alpha}\operatorname{arsinh}\left(\frac{\alpha}{4}\right). \tag{2.27}$$

[7]Let us recall that $\mathrm{d}(\sinh u) = \cosh u\,\mathrm{d}u$ and $\sqrt{1+\sinh^2 u} = \cosh u$.

The integration constant $kT'/2$, which appears in (2.26), has been chosen in each of the domains $k = 0$ ($t \in [0, T/4]$), $k = 1$ ($t \in [T/4, 3T/4]$) and $k = 2$ ($t \in [3T/4, T]$) in order to enforce the continuity of t', starting from $t' = 0$ at $t = 0$. The relation (2.26) between the proper times t and t' is plotted in Fig. 2.6. At the particular points A ($k = 0, t = 0$), C_1 ($k = 0, t = T/4$), P ($k = 1, t = T/2$), C_2 ($k = 1, t = 3T/4$) and B ($k = 2, t = T$), it results in

$$t'(A) = 0, \quad t'(C_1) = \frac{T'}{4}, \quad t'(P) = \frac{T'}{2}, \quad t'(C_2) = \frac{3T'}{4}, \quad t'(B) = T'. \tag{2.28}$$

We notice that the inequality $t' \leq t$ always holds (cf. Fig. 2.6). In particular, when $\mathscr{O}$ and $\mathscr{O}'$ meet again in B, the elapsed proper time from $\mathscr{O}'$ departure is $t(B) = T$ for $\mathscr{O}$, whereas the elapsed proper time for $\mathscr{O}'$, $t'(B) = T'$, is given by (2.27). Whenever $\alpha \neq 0$, we have $T' \neq T$, and the ratio of the two elapsed proper times is

$$\boxed{\frac{T'}{T} = \frac{t'(B) - t'(A)}{t(B) - t(A)} = \frac{4}{\alpha} \operatorname{arsinh}\left(\frac{\alpha}{4}\right) \leq 1}. \tag{2.29}$$

The ratio T'/T is plotted as a function of α in Fig. 2.7. For the worldlines drawn in Fig. 2.5, its value is 0.96 ($\alpha = 2$), 0.88 ($\alpha = 4$), 0.72 ($\alpha = 8$) and 0.30 ($\alpha = 40$).

2.6.3 The "Paradox"

The result (2.29) constitutes the so-called ***twin paradox***. Actually, this is not a paradox for this does not generate any contradiction in the theory of relativity, as discussed below; this is simply a surprising result for a "nonrelativistic" physicist: in Newtonian theory, the times given by the clocks of each twin would be the same when they meet in B, provided that they have been synchronized in A.

The paradoxical aspect of Langevin's traveller arises from a naive interpretation of the principle of relativity: from the point of view of twin $\mathscr{O}$, the twin $\mathscr{O}'$ is the traveller and the above computation shows that when $\mathscr{O}'$ is back, he is younger than $\mathscr{O}$. But from the point of view of $\mathscr{O}'$, it is $\mathscr{O}$ who is travelling. When the twins meet again, $\mathscr{O}$ should then be younger. Since both points of view should be equally valid according to the principle of relativity, a paradox appears: at the event B, $\mathscr{O}'$ cannot be both younger and older than $\mathscr{O}$. Actually, this argument is false because the two twins $\mathscr{O}$ and $\mathscr{O}'$ do not follow equivalent worldlines in Minkowski spacetime. The worldline of $\mathscr{O}$ is a very peculiar curve: a straight line, which implies that $\mathscr{O}$'s 4-acceleration is vanishing. On the contrary, the 4-acceleration of $\mathscr{O}'$ is nonzero, as we shall see below. Since the two twins are not equivalent, the relativity principle cannot be invoked and the paradox disappears. For a more detailed discussion, we refer the reader to Grandou and Rubin (2009).

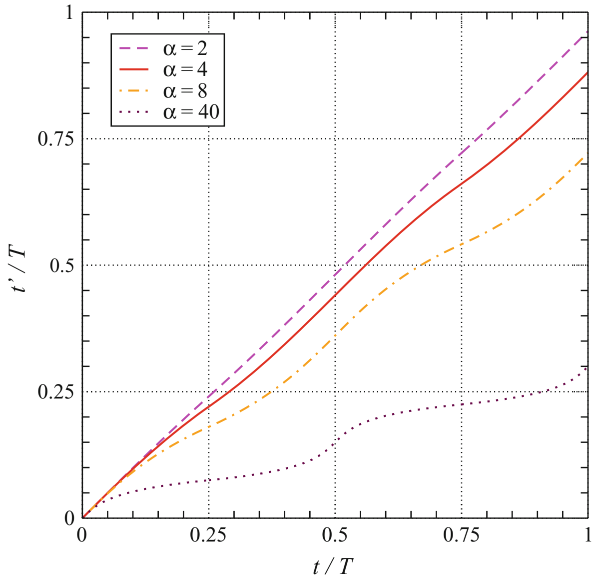

Fig. 2.6 Proper time t' of the twin $\mathscr{O}'$ (Langevin's traveller) as a function of the proper time t of the twin $\mathscr{O}$, for various values of the parameter α. Note that at the instants $t = 0$, $t = T/2$ and $t = T$, where the two worldlines are parallel (cf. Fig. 2.5), the slope of the curve is 45°, which means that t' flows at the same rate as t. On the other side, at the instants $t = T/4$ and $t = 3T/4$, where the inclination of $\mathscr{L}'$ differs the most from that of $\mathscr{L}$, the slope of the curve is the smallest

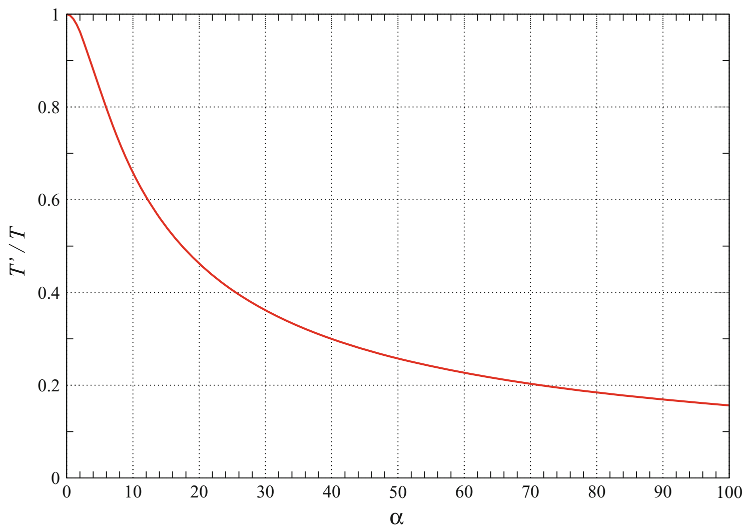

Fig. 2.7 Ratio between the proper time elapsed between A and B for $\mathscr{O}'$ and that elapsed for $\mathscr{O}$, as a function of the parameter α [cf. formula (2.29)]

Remark 2.15. From the four-dimensional point of view adopted in this book, the solution of the twin "paradox" appears rather trivial: the proper time has been defined as the length given by the metric tensor $\boldsymbol{g}$ along a worldline, and it seems obvious that the length between two points A and B depends upon the path chosen between these two points. A sceptical mind could reply: "there is nothing revolutionary in this with respect to the Newtonian time, because everything relies on the definition of proper time as the length of worldlines with respect to $\boldsymbol{g}$; this is an arbitrary definition of "time". It is therefore not surprising that it results in a strange behaviour". However, we have already mentioned in Sect. 2.3 that the time defined from $\boldsymbol{g}$ is the actual physical time, in the sense that the equations of dynamics take a simple form when expressed with it (we shall see it explicitly in Chap. 9). We shall actually see in Sect. 2.6.6 some experimental realizations of the twin paradox, showing that the time provided by atomic clocks between two events A and B do depend on the worldline between these two events. It is therefore not a mere semantic effect!

Remark 2.16. Since $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$, we deduce from formula (2.29) that when $\alpha \to +\infty$, the ratio T'/T goes to 0, behaving as $4 \ln\alpha/\alpha$. This is not surprising if one contemplates Fig. 2.5: when $\alpha \to +\infty$, the worldline $\mathscr{L}'$ approaches the worldline $[AD] \cup [DB]$ of a photon emitted in A and reflected back to B in D. Each segment $[AD]$ and $[DB]$ is a piece of null geodesic and hence as a vanishing metric length (cf. Remark 2.13 in p. 39). Therefore we understand why T', which is nothing but the metric length of $\mathscr{L}'$ between A and B, converges to zero when $\alpha \to +\infty$.

Historical note: *This is in fact Albert Einstein who, in the seminal 1905 article (Einstein 1905b), already pointed out that two clocks initially synchronized and, at the same position, would not show the same time if they are read at the same place after having travelled on different paths. Einstein gave an approximate formula (valid for small velocities) of the delay between t' and t. During a conference in Zurich in 1911, he illustrated the effect by describing a round-trip journey of a living organism locked in a moving box (cf. Damour (2006), p. 34). In order to make the effect more spectacular, Paul Langevin (cf. p. 40) imagined in 1911 a human being leaving the Earth aboard a "projectile", travelling towards some star at a velocity close to that of light and coming back on Earth after 2 years, whereas 200 years have been elapsed on our planet (Langevin 1911; Paty 1999a). Let us note that in the 1911 text (Langevin 1911), Langevin did not speak explicitly of "twins" but of a "traveller" and "the Earth". Besides, he gave clearly the explanation of the dissymmetry between the two by mentioning the acceleration felt by the traveller.*

2.6.4 4-Velocity and 4-Acceleration

Let us compute the 4-velocity $\vec{u}'$ of Langevin's traveller $\mathscr{O}'$ at each point of his worldline. From the definition (2.12), we have

$$\vec{u}' = \frac{1}{c}\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t'}.$$

The components of $\vec{u}'$ in the orthonormal basis $(\vec{e}_\alpha)$ are thus $u'^\alpha = c^{-1}\mathrm{d}x'^\alpha/\mathrm{d}t'$. Since $\mathrm{d}x'^\alpha = (c\,\mathrm{d}t, \mathrm{d}x, 0, 0)$, we get $u'^2 = 0$, $u'^3 = 0$,

$$u'^0 = \frac{1}{c}\frac{dx'^0}{\mathrm{d}t'} = \frac{\mathrm{d}t}{\mathrm{d}t'} \qquad \text{and} \qquad u'^1 = \frac{1}{c}\frac{dx'^1}{\mathrm{d}t'} = \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t'} = \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'}.$$

By means of (2.25) and (2.23), we obtain

$$u'^0 = \sqrt{1+\alpha^2\,(t/T-k/2)^2} \tag{2.30a}$$

$$u'^1 = (-1)^k\alpha\,(t/T-k/2)\,. \tag{2.30b}$$

Given the definition of the integer k, we note that if $\alpha > 0$, then for $0 \le t \le T/2$, $u'^1 \ge 0$ ($\mathscr{O}'$ is moving away from $\mathscr{O}$ in the direction of increasing x), whereas for $T/2 \le t \le T$, $u'^1 \le 0$ ($\mathscr{O}'$ is moving towards $\mathscr{O}$). The vector $\vec{u}'$, as given by (2.30), is drawn at some selected points of $\mathscr{L}'$ in Fig. 2.8.

Let us notice that, from (2.26),

$$\alpha\left(\frac{t}{T}-\frac{k}{2}\right) = \sinh\left[\frac{\alpha}{T}\left(t'-\frac{k}{2}T'\right)\right], \tag{2.31}$$

so that the components of $\vec{u}'$ can be expressed in terms of the proper time t' according to

$$u'^0 = \cosh\left[\frac{\alpha}{T}\left(t'-\frac{k}{2}T'\right)\right] \tag{2.32a}$$

$$u'^1 = (-1)^k\sinh\left[\frac{\alpha}{T}\left(t'-\frac{k}{2}T'\right)\right]. \tag{2.32b}$$

Remark 2.17. Thanks to the identity $\cosh^2 x - \sinh^2 x = 1$, it is easily checked on these formulas that $\vec{u}'\cdot\vec{u}' = -(u'^0)^2 + (u'^1)^2 = -1$, as it should be for any 4-velocity [Eq. (2.14)].

Let us now compute the 4-acceleration $\vec{a}'$ of $\mathscr{O}'$. By definition [cf. Eq. (2.16)],

$$\vec{a}' = \frac{1}{c}\frac{\mathrm{d}\vec{u}'}{\mathrm{d}t'}.$$

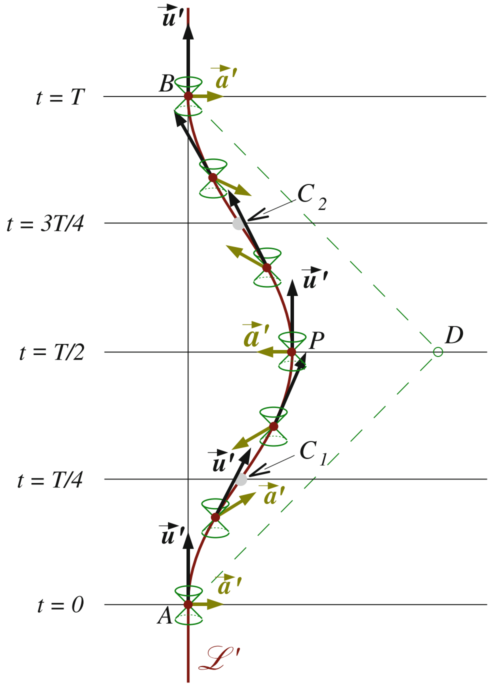

Fig. 2.8 Worldline $\mathscr{L}'$ of Langevin's traveller $\mathscr{O}'$ with the 4-velocity $\vec{u}'$ and 4-acceleration $\vec{a}'$ at some selected points. This figure corresponds to the case $\alpha = 4$ (solid line in Fig. 2.5), i.e. to the acceleration $\gamma = 4c/T$. At the event A, the 4-acceleration changes sharply from 0 to $\vec{a}' = \gamma c^{-2}\,\vec{e}_1$ (ignition of the rocket engine). Its norm stays then constant (equal to γc^{-2}), until the return event B, where $\vec{a}'$ vanishes again (rocket engine stopped). The event C_1 is the sudden change of direction of acceleration by 180° (thrust reversing). $\mathscr{O}'$ is subsequently slowed down until P and then sped up towards $\mathscr{O}$, until C_2. At this point, a new thrust reversing occurs, so that $\mathscr{O}'$ is slowed down until it reaches B

Accordingly, the components of $\vec{a}'$ in the orthonormal basis $(\vec{e}_\alpha)$ are

$$a'^0 = \frac{1}{c}\frac{du'^0}{\mathrm{d}t'}, \quad a'^1 = \frac{1}{c}\frac{du'^1}{\mathrm{d}t'}, \quad a'^2 = 0 \quad \text{and} \quad a'^3 = 0.$$

Taking the derivative of (2.32) with respect to t', we get

$$a'^0 = \frac{\alpha}{cT}\sinh\left[\frac{\alpha}{T}\left(t' - \frac{k}{2}T'\right)\right] \tag{2.33a}$$

$$a'^1 = (-1)^k\frac{\alpha}{cT}\cosh\left[\frac{\alpha}{T}\left(t' - \frac{k}{2}T'\right)\right]. \tag{2.33b}$$

As a check, the orthogonality between the 4-acceleration and the 4-velocity [Eq. (2.17)] is recovered from (2.32) and (2.33): $\vec{a}' \cdot \vec{u}' = -a'^0 u'^0 + a'^1 u'^1 = 0$. Thanks to (2.31), we can express the components of $\vec{a}'$ in terms of t instead of t':

$$a'^0 = \frac{\alpha^2}{cT}\left(\frac{t}{T} - \frac{k}{2}\right) \tag{2.34a}$$

$$a'^1 = (-1)^k \frac{\alpha}{cT}\sqrt{1 + \alpha^2\left(\frac{t}{T} - \frac{k}{2}\right)^2}. \tag{2.34b}$$

We notice that a'^1 has a sudden change of sign when k goes from 0 to 1, i.e. when $t = T/4$, as well as when k goes from 1 to 2, i.e. when $t = 3T/4$. More precisely, if $\alpha > 0$, formula (2.34b) yields

$$t \in \left[0, \frac{T}{4}\right] \Longrightarrow a'^1 > 0, \quad t \in \left[\frac{T}{4}, \frac{3T}{4}\right] \Longrightarrow a'^1 < 0,$$
$$t \in \left[\frac{3T}{4}, T\right] \Longrightarrow a'^1 > 0. \tag{2.35}$$

Physically, if we consider that $\mathscr{O}'$ is travelling in some spaceship, the sudden change of sign of a'^1 corresponds to a thrust reversing operated on the rocket engine (events C_1 and C_2 in Fig. 2.8).

Let us evaluate the scalar square of $\vec{a}'$. The basis $(\vec{e}_\alpha)$ being orthonormal, we have $\vec{a}' \cdot \vec{a}' = -(a'^0)^2 + (a'^1)^2$. From (2.33) or (2.34), we get easily

$$\vec{a}' \cdot \vec{a}' = \frac{\alpha^2}{c^2 T^2}. \tag{2.36}$$

The right-hand side being clearly positive, we recover the property (2.18), namely, that $\vec{a}'$ is a spacelike vector. More remarkably, (2.36) shows that the norm of the 4-acceleration,

$$a' := \left\| \vec{a}' \right\|_g = \sqrt{\vec{a}' \cdot \vec{a}'} = \frac{|\alpha|}{cT}, \tag{2.37}$$

does not depend upon t': it is therefore constant along the worldline $\mathscr{L}'$ between A and B. This property is specific to the spacetime motion along an arc of hyperbola, which we have chosen for $\mathscr{O}'$.

We have seen in Sect. 2.4.2 that the dimension of a' is the inverse of a length, in agreement with (2.37), α being dimensionless. To let appear a quantity with the dimension of an acceleration, it suffices to multiply a' by c^2. We thus introduce the parameter

$$\gamma := \alpha \frac{c}{T}, \tag{2.38}$$

instead of α. γ has the dimension of an acceleration and is related to the norm of the 4-acceleration of $\mathscr{O}'$ by

$$\boxed{a' = \frac{|\gamma|}{c^2}}. \tag{2.39}$$

We shall see in Chap. 12 that γ is actually the acceleration felt by the observer $\mathscr{O}'$ in his local frame.

Remark 2.18. Note that $\gamma \neq \mathrm{d}^2x/\mathrm{d}t^2$, i.e. γ is not the second derivative of the function $x(t)$ defining the worldline of $\mathscr{O}'$. The latter is obtained by taking the derivative of $\mathrm{d}x/\mathrm{d}t$ as given by (2.23). One gets, after substituting γ/c for α/T,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = (-1)^k\gamma\left[1 + \frac{\gamma^2}{c^2}\left(t - \frac{k}{2}T\right)^2\right]^{-3/2}. \tag{2.40}$$

We conclude that one has $|\gamma| \simeq |\mathrm{d}^2x/\mathrm{d}t^2|$ only in the nonrelativistic limit $|\gamma|T \ll c$.

Remark 2.19. In many textbooks,[8] the twin paradox is exposed from a worldline $\mathscr{L}'$ simpler than the three arcs of hyperbola considered here, namely, a straight line segment from A to P as well as from P to B (see Fig. 2.9). The computations are then simpler than those presented above, the equation of $\mathscr{L}'$ being $x(t) = Vt$ for $t \in [0, T/2]$ and $x(t) = V(T - t)$ for $t \in [T/2, T]$, with $V := 2x(P)/T$. We have then $\mathrm{d}x = \pm V\mathrm{d}t$, so that evaluating $\mathrm{d}t'$ according to formula (2.24), we get $\mathrm{d}t' = \sqrt{1 - (V/c)^2}\,\mathrm{d}t$, which is easily integrated and leads to the proper time ratio

$$\frac{T'}{T} = \sqrt{1 - \frac{V^2}{c^2}} \leq 1. \tag{2.41}$$

However, this configuration is not physical for it corresponds to an infinite acceleration of $\mathscr{O}'$ at A (the 4-velocity jumping suddenly from $\vec{\boldsymbol{u}}'(A_-)$ to $\vec{\boldsymbol{u}}'(A_+)$, cf. Fig. 2.9) as well as in P and B. On the contrary, the "tri-hyperbolic" worldline considered here involves always a finite acceleration. It admits thus a clear physical interpretation in terms of a (rocket) engine of constant thrust, switched on at A, inverted in C_1 and C_2 and switched off at B. This demonstrates that the twin paradox is not an artefact resulting from an infinite acceleration.

[8]Two exceptions are the books by Møller (1952) and by Marder (1971).

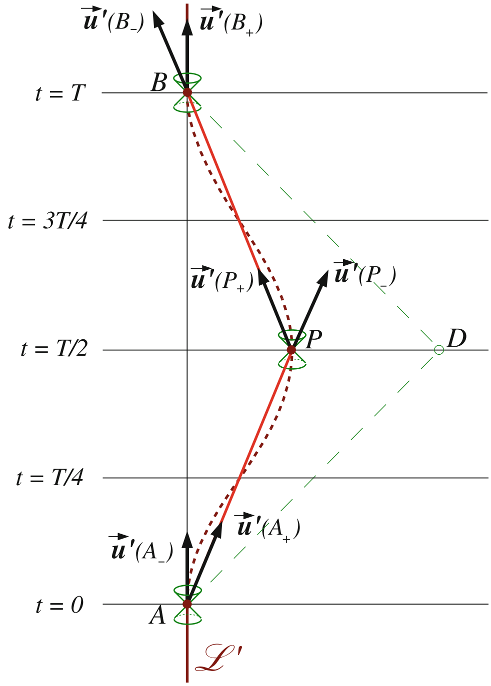

Fig. 2.9 Simplified worldline for the Langevin's traveller $\mathscr{O}'$: $\mathscr{L}'$ is reduced to a line segment between A and P, as well as between P and B. The 4-acceleration of $\mathscr{O}'$ is then infinite at A, P and B, as indicated by the jumps of the 4-velocity $\vec{u}'$ at these points. On the opposite, the "tri-hyperbolic" worldline (*dotted curve*, the same as in Figs. 2.5 and 2.8) yields always a finite 4-acceleration

2.6.5 A Round Trip to the Galactic Centre

Let $d := x(P)$ be the maximal distance of $\mathscr{O}'$ with respect to $\mathscr{O}$. We may reexpress formulas (2.22) and (2.29) in terms of the acceleration γ via (2.38):

$$d = \frac{2c^2}{\gamma}\left[\sqrt{1+\left(\frac{\gamma T}{4c}\right)^2}-1\right] \qquad \text{and} \qquad \frac{T'}{T} = \frac{4c}{\gamma T}\,\mathrm{arsinh}\left(\frac{\gamma T}{4c}\right). \tag{2.42}$$

The second relation allows one to express T in terms of T' as

$$\boxed{T = T_* \sinh\left(\frac{T'}{T_*}\right)}, \qquad \text{with} \quad T_* := \frac{4c}{\gamma} = \frac{4}{\alpha}T. \tag{2.43}$$

T_* is the timescale that can be built from c and the acceleration γ; in Newtonian physics, this would be 4 times the time required to reach the light velocity, starting from a zero velocity with the acceleration γ. Substituting T by the above value in the expression of d and noticing that $\sqrt{1+\sinh^2 x} = \cosh x$, we get

$$\boxed{d = \frac{cT_*}{2}\left[\cosh\left(\frac{T'}{T_*}\right) - 1\right]}. \tag{2.44}$$

When $T' \ll T_*$, the Taylor expansions of (2.43) and (2.44) lead to

$$T' \ll T_* \Longrightarrow \begin{cases} T \simeq T' \\ d \simeq \dfrac{c}{4T_*}(T')^2 = 2 \times \dfrac{1}{2}\gamma\left(\dfrac{T'}{4}\right)^2. \end{cases} \tag{2.45}$$

If $T' \ll T_*$, $\mathscr{O}'$ has not the time to reach a relativistic velocity with respect to $\mathscr{O}$ and (2.45) gives the Newtonian results, as it should be: no differential ageing and the travelled distance $d/2$ at a constant acceleration γ (phase $[AC_1]$ lasting $T'/4$) being equal to $\gamma/2$ times the square of travel time. This is indeed the expected result for a vanishing initial velocity.

Conversely, if $T' \gg T_*$, the ultra-relativistic regime is reached and relations (2.43) and (2.44) lead to

$$T' \gg T_* \Longrightarrow \begin{cases} T \simeq \dfrac{T_*}{2}\exp(T'/T_*) \\ d \simeq \dfrac{cT_*}{4}\exp(T'/T_*) = \dfrac{1}{2}cT. \end{cases} \tag{2.46}$$

Remark 2.20. Formula (2.43), which relates T to T', depends on a single parameter: the acceleration γ, via the time $T_* = 4c/\gamma$. One should however not conclude that the twin paradox is a phenomenon intrinsically linked to acceleration. It should rather be perceived as the reflect of the dissymmetry of the worldlines between A and B. It turns out that in Minkowski spacetime $(\mathscr{E}, \boldsymbol{g})$, the only way for $\mathscr{L}'$ to depart from a straight line (worldline $\mathscr{L}$) is to have some episode of nonvanishing 4-acceleration. If $\mathscr{E}$ is given a topology different from that of an affine space, then it is possible to have $T \neq T'$ with $\mathscr{L}$ and $\mathscr{L}'$ both having a vanishing 4-acceleration. It suffices that $\mathscr{E}$ has a non-simply connected topology,[9] as shown is the study (Uzan et al. 2002).

Let us apply the above formulas to the "concrete" case where $\mathscr{O}'$ is an astronaut in some spaceship. To consider an acceleration bearable for a human being, let us take the value of Earth's gravity: $\gamma = 1\ g = 9.81\ \mathrm{m\,s^{-2}}$. This has even the advantage to create an artificial gravity aboard the spaceship that simulates the terrestrial environment and makes the journey comfortable. The corresponding time parameter defined by (2.43) is $T_* = 4c/\gamma = 1.22 \times 10^8\ \mathrm{s} = 3.87\ \mathrm{yr}$, and formulas (2.43) and (2.44) lead to values of T and d as functions of T' listed in Table 2.1. We observe that if $\mathscr{O}'$ is travelling for more than a year, then the difference between

[9] An example of such spaces is a torus or, more generally, any compact domain with periodic boundary conditions.

Table 2.1 Properties of various trips of Langevin's traveller, when the acceleration is fixed to $\gamma = 9.81\,\mathrm{m\,s^{-2}}$: T' is the round-trip duration measured by him, T is the duration of the same trip but measured by the "sedentary" observer $\mathscr{O}$ and d is the maximal achieved distance between $\mathscr{O}'$ and $\mathscr{O}$ (1 light-year = 9.46×10^{15} m). One checks that if $T \gg T_* = 3.87$ yr, then $d \simeq cT/2$, in agreement with (2.46)

T' [yr]	T [yr]	d [light-year]
1	1.01	0.065
2	2.09	0.26
4	4.75	1.13
8	15.0	5.82
16	120	58
32	7.50×10^3	3.74×10^3
39.5	5.20×10^4	2.60×10^4
56	3.68×10^6	1.84×10^6
64	2.90×10^7	1.45×10^7
80	1.81×10^9	9.03×10^8
90	2.39×10^{10}	1.19×10^{10}
100	3.15×10^{11}	1.58×10^{11}

the "onboard" proper time T' and the "harbour" proper time T is noticeable. With a journey lasting for 8 years, $\mathscr{O}'$ can reach the closest stars from the Solar System. If he is travelling for $T' = 16$ years, when he is back on Earth, $T = 120$ years will have elapsed, implying that he will not be able to report his journey to his acquaintances but to their children. Table 2.1 shows that the centre of the Galaxy, located a roughly 26,000 light-years, can be reached within a journey of round-trip duration of only 39.5 years. In this case, it is not guaranteed that there will be anybody interested by the traveller's account at the return, for 52,000 years will have elapsed on Earth! Let us not speak about a round trip to Andromeda Galaxy, located at 2 million light-years, because while it takes only 56 years for the astronaut, his return will take place on an Earth aged by 3 million years and, at the very least, he will face some language issue…

Of course, in the above description, we have limited ourselves to pure kinematic considerations and have not taken into account the energetic cost of such travels: maintaining an acceleration of 1 g during several years requires an enormous amount of energy and forbids such travels with today technology. Nevertheless, we shall keep in mind that relativity allows one, at least theoretically, to visit the Galactic centre and even to reach the border of the observable universe within a human lifetime ($d \sim 12$ billion light-years for a round trip of 90 year, cf. Table 2.1), travelling at less than the speed of light! ($\mathscr{O}'$'s worldline is always located inside the light cone, cf. Fig. 2.8). Hence it is not correct to say that it is not possible for a person to travel further than a hundred light-year or so away from Earth because relativity forbids to travel faster than light. On the contrary, the solution is offered by relativity itself: it remains true that for any observer that he may encounter on his way, $\mathscr{O}'$ is travelling slower than light[10]; this implies that for people observing him from the Earth, $\mathscr{O}'$ will take at least 26,000 years to reach the Galactic centre. On the contrary, for $\mathscr{O}'$, only 20 years will have elapsed when he will arrive at the Galactic centre.

One lesson from the above example is that relativity allows for ***time travel to the future***: one may say that $\mathscr{O}'$ is travelling to $\mathscr{O}$'s future, since when $\mathscr{O}'$ meets again $\mathscr{O}$ at B, $\mathscr{O}$ is older than him. The numbers listed in Table 2.1 show even that this time travel to the future can be of millions of years. On the other side, special relativity does not allow for *time travel to the past*: even if we take the point of view of $\mathscr{O}$, when $\mathscr{O}$ meets again $\mathscr{O}'$ at B, the latter is younger than him but still older than when he left him at A.

Remark 2.21. It is Minkowski spacetime structure that forbids the time travel to the past: all the light cones being parallel (cf. Figs. 1.8 and 2.8), one can show that it is not possible for the worldline $\mathscr{L}'$ to meet $\mathscr{L}$ at a point B located in the past of A while staying inside the light cone of any of its points. However, if a gravitational field is present, the spacetime structure is no longer that of an affine space, as we shall see in Chap. 22, but that of a "curved" space ruled by general relativity. The light cones are then no longer parallel with respect to each other and it is possible, under certain conditions (quite extreme though…), to have $\mathscr{L}'$ such that B is anterior to A. This is the time machine of science-fiction novels! We shall not discuss this subject further and refer the interested reader (who would not be?) to Lehoucq (2004), Davies (2002), Thorne (1994).

2.6.6 Experimental Verifications

Undoubtedly, the twin paradox puts forward an effect that is not part of everyday life, namely, the dependency of time upon the motion of bodies. Actually the velocities of people and objects around us are very small with respect to the velocity of light, and we have seen that the time shift is sizeable only if T' is of the order T_*, which implies a velocity close to c [cf. (2.43) under the form $V_* := \gamma T_* \sim c$]. Nevertheless, even if the effect is too small for our senses, it can be exhibited by a sufficiently sensitive experiment. This turned out to be possible in the 1970s, thanks to atomic clocks.

2.6.6.1 Hafele–Keating Experiment (1971)

The first experimental reproduction of the twin paradox has been performed in 1971 by J.C. Hafele of Washington University at Saint Louis (Missouri) and Richard E. Keating of the US Naval Observatory (Hafele 1972b; Hafele and Keating 1972a,b)

[10] We shall make precise the notion of velocity in Chap. 4.

(cf. also Hafele (1972a)). Four caesium atomic clocks have been loaded on airline jets for a journey round the Earth; when back to their starting point, they have been compared with atomic clocks stayed on the ground. Two journeys have taken place. The first one has been performed eastward from 4 to 7 October 1971, with 12 stopovers, 3 changes of plane (Boeing 747 and 707) and a total of 41 h of flight. The second one has been performed westward from 13 to 17 October 1971, with 13 stopovers, 2 changes of plane (Boeing 707), totalizing 49 h of flight. The corresponding worldlines are rather different from that of Langevin's traveller defined in Sect. 2.6.1: the motion around the Earth being circular and not linear, the worldlines rather looks like helices. The precise trajectory of the clocks was pretty complicated because of the different stopovers, the experiment being performed on commercial airlines. Thanks to the flight data provided by the pilots, it has been possible to reconstruct the worldline $\mathscr{L}'$ followed by the onboard clocks. Another difference with Sect. 2.6.1 is that the worldline $\mathscr{L}$ of the reference clock staying on the ground is not a straight line but also a helix, due to the Earth rotation. However, even if the worldlines $\mathscr{L}$ and $\mathscr{L}'$ are more complicated than in Sect. 2.6.1, one could similarly compute, by means of (2.10), the proper times T along $\mathscr{L}$ and T' along $\mathscr{L}'$ between the plane departure (event A) and its return (event B). These theoretical values have been then compared with the actual measurements given by the clocks.

In addition to the shape of the worldlines, another complication arises from the fact that the clocks aboard the planes were travelling higher in the gravitational potential of the Earth than the clocks stayed on the ground. A general relativistic effect then takes place: the *gravitational redshift*, that we shall discuss in Chap. 22. It results in a difference between the proper times T' and T, in the direction of increasing T'. This effect has a magnitude comparable to that of the special relativistic effect that we are interested in here. The verification of the twin paradox must thus take this into account.

In Chap. 13, we shall compute the value $T' - T$ in the framework of special relativity, by means of simplified airplane trajectories. The precise computation, relying on the actual trajectories, results in $T' = T - 184 \pm 18\,\mathrm{ns}$ (nanoseconds) for the eastward journey. The 18 ns error bar is related to the uncertainties in the reconstruction of the airplane worldline (uncertainties in position and velocity). Hence, the clocks that have travelled eastward must be younger by 184 ns than those stayed on the ground. This value must be corrected from the general relativistic effect mentioned above; the latter goes in the reverse direction: it increases T' by $144 \pm 14\,\mathrm{ns}$. Accordingly, the theoretical prediction is $T' = T - 40 \pm 32\,\mathrm{ns}$. The observed value, obtained by taking the average over the four clocks, in order to reduce the experimental error, is $T' = T - 59 \pm 10\,\mathrm{ns}$.

Regarding the westward journey (counterrotating with respect to the Earth), the worldline $\mathscr{L}'$ (a helix at first approximation) deviates less from a straight line than the worldline $\mathscr{L}$. We are thus in the case where special relativity predicts $T' > T$, as we shall see explicitly in Chap. 13. The computation leads to $T' = T + 96 \pm 10\,\mathrm{ns}$, to which the gravitational redshift effect must be added (always with the result of increasing T'), to arrive at $T' = T + 275 \pm 21\,\mathrm{ns}$. The measured value is $T' = T + 273 \pm 7\,\mathrm{ns}$.

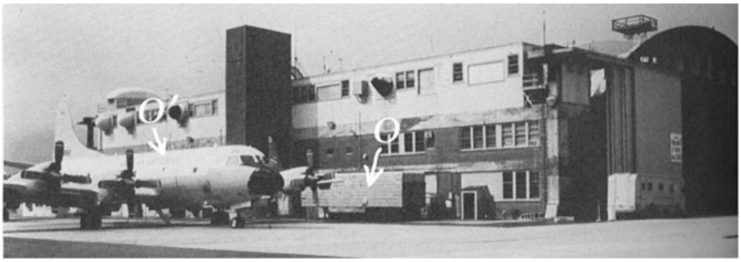

Fig. 2.10 Airplane carrying the atomic clocks (observer $\mathscr{O}'$) in Alley experiment (1975), parked near the truck containing the reference atomic clocks (observer $\mathscr{O}$), at the Naval Air Station Patuxent River (Eastern coast of United States). This picture may be considered as a view of the event A, where $\mathscr{O}$ and $\mathscr{O}'$, who were following the same worldline, are on the verge to separate [Credit: C.O. Alley (1983)]

Given the error bars, we conclude that Hafele–Keating experiment has confirmed that the proper time elapsed between two events does depend on the worldline related them. This may be seen as the experimental demonstration that the actual time is not Newton's absolute time, but relativity's time.

2.6.6.2 Alley Experiment (1975)

A more precise experiment with atomic clocks in airplane has been performed in 1975 by Carroll O. Alley of the University of Maryland (USA) (Alley 1983). This time, an aircraft entirely devoted to the experiment has been used instead of regular airline planes. It was an antisubmarine aircraft Lockheed P-3C Orion, which has the capability to fly non-stop for 16 h. On 22 November 1975 six atomic clocks (three caesium ones and three rubidium ones) have been loaded for a 15-h flight turning around Chesapeake Bay, on the Northeast coast of the United States (worldline $\mathscr{L}'$). A set of identical atomic clocks was installed in a trailer on the base from which the aircraft departed (worldline $\mathscr{L}$) (cf. Fig. 2.10). The average speed of the plane was $540\,\mathrm{km\,h^{-1}} = 150\,\mathrm{m\,s^{-1}} = 5\times10^{-7}\,c$ and the altitude was 7,600 m during the 5 first hours, 9,100 m during the 5 next hours and 10,700 m during the 5 last hours. The computation of the proper times along $\mathscr{L}'$ and $\mathscr{L}$ leads to the following theoretical prediction:

$$T' = T \underbrace{-\,5.7\mathrm{ns}}_{\mathrm{SR}} \underbrace{+\,52.8\mathrm{ns}}_{\mathrm{GR}} = T + 47.1\mathrm{ns}, \tag{2.47}$$

where "SR" labels the contribution of special relativity (kinematic effect considered in this chapter) and "GR" the contribution of general relativity (gravitational redshift). The value measured at the return is in agreement with (2.47) within a relative accuracy of 1.5%. Since the kinematic effect is a tenth of the total effect, we conclude that Alley experiment has confirmed the twin paradox with an accuracy of the order of 15%.

Remark 2.22. Other tests about the dependency of proper time with respect to the motion will be presented in Chaps. 4 and 5. They are much more precise than the experiments described above. We have limited ourselves to the last ones because they are directly interpretable in terms of the twin paradox.

2.7 Geometrical Properties of a Worldline

2.7.1 Timelike Geodesics

In the study of Langevin's traveller, we have observed that $T > T'$ as long as the worldline $\mathscr{L}'$ departs from $\mathscr{L}$ (i.e. as long as $\alpha \neq 0$). Given the definition of proper time, we may state in an equivalent manner that between events A and B, the straight worldline $\mathscr{L}$ has a length (given by the metric tensor $\boldsymbol{g}$) *larger* than that of the curved worldline $\mathscr{L}'$. We shall show now that Langevin's traveller reflects the most general case: if two points of $\mathscr{E}$ can be joined by a timelike straight line, all the other timelike curves joining them have a smaller metric length. This result is of course the exact opposite of what holds in a Euclidean space, where the straight line is always the shortest path between two points.

Let A and B be two points of $\mathscr{E}$ such that B is located inside the future light cone of A. These two points can then be joined by timelike curves (i.e. worldlines of massive particles). A particular worldline is the straight line $\mathscr{L}_0$ through A and B. Let $(\vec{\boldsymbol{e}}_\alpha)$ be an orthonormal basis of $(E, \boldsymbol{g})$ such that $\vec{\boldsymbol{e}}_0$ coincides with the 4-velocity of $\mathscr{L}_0$. We introduce the affine coordinate system $(x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$ associated with $(\vec{\boldsymbol{e}}_\alpha)$ and centred on A (cf. Fig. 2.11). Let $\mathscr{L}$ be a timelike worldline connecting A and B. As $\mathscr{L}$ must stay inside the light cone of each of its points, we can use the affine coordinate t as a regular parameter[11] along $\mathscr{L}$. Let then X, Y and Z be three functions $\mathbb{R} \to \mathbb{R}$ giving the position of $\mathscr{L}$ in terms of the affine coordinates (x^α), according to

$$x = X(t), \quad y = Y(t), \quad z = Z(t). \tag{2.48}$$

The components in the basis $(\vec{\boldsymbol{e}}_\alpha)$ of the elementary displacement vector $\mathrm{d}\vec{\boldsymbol{x}}$ along $\mathscr{L}$ are then

$$\mathrm{d}x^\alpha = (c\,\mathrm{d}t, \dot{X}\mathrm{d}t, \dot{Y}\mathrm{d}t, \dot{Z}\mathrm{d}t), \tag{2.49}$$

[11] If the timelike constraint was relaxed, then $\mathscr{L}$ could move "backward in time" and t would not be a good parameter along it.

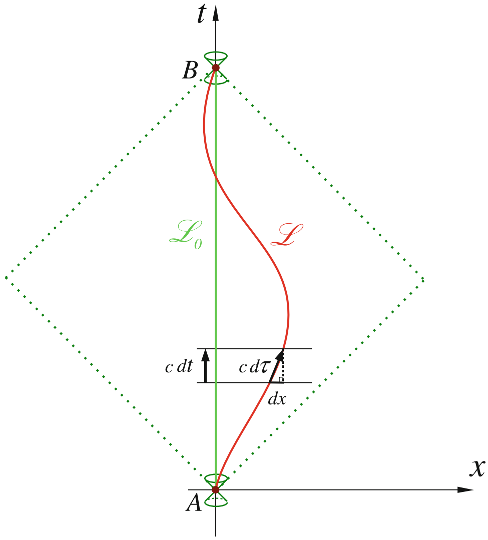

Fig. 2.11 Comparing the metric length (proper time) of two worldlines joining two events A and B: a *straight line* and a *curved line*. Since $c^2 \mathrm{d}\tau^2 = c^2 \mathrm{d}t^2 - \mathrm{d}x^2$, the *curved line* is, with respect to the metric $\boldsymbol{g}$, shorter than the *straight line*

where the derivatives of the functions X, Y and Z are indicated with a dot. The length of $\mathscr{L}$ (with respect to $\boldsymbol{g}$) between the points A and B is given by formula (2.10):

$$c\,\tau(A, B) = c \int_A^B \mathrm{d}\tau = \int_A^B \sqrt{-\boldsymbol{g}(\mathrm{d}\vec{\boldsymbol{x}}, \mathrm{d}\vec{\boldsymbol{x}})}. \tag{2.50}$$

Now, since $(\vec{\boldsymbol{e}}_\alpha)$ is an orthonormal basis, $-\boldsymbol{g}(\mathrm{d}\vec{\boldsymbol{x}}, \mathrm{d}\vec{\boldsymbol{x}}) = -\eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = c^2\mathrm{d}t^2 - (\dot{X}\,\mathrm{d}t)^2 - (\dot{Y}\,\mathrm{d}t)^2 - (\dot{Z}\,\mathrm{d}t)^2$. Hence

$$\begin{aligned} c\,\tau(A, B) &= c \int_A^B \sqrt{1 - \frac{1}{c^2}\left[(\dot{X})^2 + (\dot{Y})^2 + (\dot{Z})^2\right]}\,\mathrm{d}t \\ &\leq c \int_A^B \mathrm{d}t = c[t(B) - t(A)]. \end{aligned} \tag{2.51}$$

Since $c[t(B) - t(A)] = c\,\tau_0(A, B)$ is the length of the straight line $\mathscr{L}_0$ between A and B, we conclude that $\mathscr{L}_0$ *maximizes* the metric length (proper time) between A and B, among all the possible worldlines.

For this reason, one calls ***timelike geodesic*** any timelike straight line of $\mathscr{E}$. Note that the term *geodesic* must be understood as a curve of extremal length, not necessarily minimal. To summarize, the null geodesics introduced in Sect. 2.5.1 correspond to minima of the metric length, whereas the timelike geodesics to maxima.

Remark 2.23. The timelike geodesic between A and B providing the upper bound on the metric length between these two points, one may ask about the lower bound on this length, taking into account that it must be positive or zero [cf. the integral (2.50)]. The answer is given by the example of Langevin's traveller: the lower bound is zero. Indeed, when the parameter α tends to infinity, the length of the worldline $\mathscr{L}'$ between A and B shrinks to zero, as shown by formula (2.29) (see also Remark 2.16 p. 46).

2.7.2 *Vector Field Along a Worldline*

Given a timelike worldline, $\mathscr{L}$ let us say, we have already encountered two kinds of vector fields defined along it: (i) the tangent vector fields associated with the various parametrizations of $\mathscr{L}$, among which the 4-velocity and (ii) the 4-acceleration field introduced in Sect. 2.4.2 (which is nowhere tangent to $\mathscr{L}$). More generally, let us define a ***vector field along the worldline*** $\mathscr{L}$ as a mapping

$$\begin{aligned} \vec{\boldsymbol{v}} : \mathscr{L} &\longrightarrow \quad E \\ A &\longmapsto \vec{\boldsymbol{v}}(A). \end{aligned} \tag{2.52}$$

Since the points A of $\mathscr{L}$ are often labelled with their proper time τ, we shall also write $\vec{\boldsymbol{v}}(\tau)$ for $\vec{\boldsymbol{v}}(A(\tau))$.

One says that the vector field $\vec{\boldsymbol{v}}$ is ***differentiable*** at a point $A(\tau) \in \mathscr{L}$ iff the limit

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}\tau} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\vec{\boldsymbol{v}}(\tau + \varepsilon) - \vec{\boldsymbol{v}}(\tau)\right]} \tag{2.53}$$

exists. The vector $\mathrm{d}\vec{\boldsymbol{v}}/\mathrm{d}\tau$ is then called the ***derivative of*** $\vec{\boldsymbol{v}}$ ***along*** $\mathscr{L}$ at point $A(\tau)$. Given a basis $(\vec{\boldsymbol{e}}_\alpha)$ of E, we may write $\vec{\boldsymbol{v}}(\tau) = v^\alpha(\tau)\, \vec{\boldsymbol{e}}_\alpha$. It is then easy to see that $\vec{\boldsymbol{v}}$ is differentiable iff the components $v^\alpha(\tau)$ are differentiable functions $\mathbb{R} \to \mathbb{R}$. Moreover, the components of the derivative are nothing but the derivatives of the components:

$$\boxed{\frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}\tau} = \frac{\mathrm{d}v^\alpha}{\mathrm{d}\tau}\, \vec{\boldsymbol{e}}_\alpha}\,. \tag{2.54}$$

2.7.3 *Curvature and Torsions*

This section can be skipped during a first reading.

Along any timelike worldline $\mathscr{L}$, one may define, from a pure geometrical viewpoint, an orthonormal basis, the ***Serret–Frenet tetrad*** $(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3)$, which characterizes the *curvature* and *torsion* of the worldline. Usually, the Serret–Frenet tetrad is constructed in a Euclidean space, from the *arc-length parameter* s along the curve. In Minkowski spacetime, which is not Euclidean (cf. Sect. 1.3.1), the Serret–Frenet tetrad is constructed instead from the metric length $c\tau$, τ being the proper time along the curve.

The first vector of the Serret–Frenet tetrad is nothing but the 4-velocity $\vec{u}$ of $\mathscr{L}$: $\vec{e}_0 := \vec{u}$. The vector $\vec{e}_0$ is thus timelike, unit and tangent to $\mathscr{L}$. Let us assume that the 4-acceleration $\vec{a}$ of $\mathscr{L}$ is nonvanishing. If this is not the case, $\mathscr{L}$ is reduced to a straight line and the Serret–Frenet approach is useless. The second vector of the Serret–Frenet tetrad is defined by

$$\vec{e}_1 := \frac{1}{a}\vec{a} = \frac{1}{ac}\frac{\mathrm{d}\vec{e}_0}{\mathrm{d}\tau}, \qquad \text{where} \qquad a := \|\vec{a}\|_g = \sqrt{\vec{a}\cdot\vec{a}}. \tag{2.55}$$

The second equality in a's definition is meaningful for $\vec{a}$ is a spacelike vector [cf. Eq. (2.18)]. The positive number a is called the ***curvature*** of the worldline $\mathscr{L}$ at the considered point. From our conventions (cf. Sect. 2.4.2), the dimension of a is the inverse of a length. The quantity a^{-1} is called the ***curvature radius*** of $\mathscr{L}$ at the considered point. In a Euclidean space, a^{-1} would be the radius of the circle that approximates the best the curve $\mathscr{L}$ at the considered point. However, Minkowski spacetime being not a metric space, the notion of circle is not defined in the present context. A second interpretation of the curvature radius is this time transposable to Minkowski spacetime: a^{-1} is the distance to $\mathscr{L}$ at which two hyperplanes orthogonal to $\vec{u}$ at two neighbouring points of $\mathscr{L}$ intersect. We shall show it at Sect. 3.7.

Let us consider now the derivative of the vector $\vec{e}_1$ along $\mathscr{L}$, following the definition (2.53). Since $\vec{e}_1$ is a unit vector, $\mathrm{d}\vec{e}_1/\mathrm{d}\tau$ is orthogonal to $\vec{e}_1$; it is thus expressible as a linear combination of $\vec{e}_0$ and a unit vector $\vec{e}_2$ orthogonal to both $\vec{e}_0$ and $\vec{e}_1$:

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_1}{\mathrm{d}\tau} = a\,\vec{e}_0 + T_1\,\vec{e}_2. \tag{2.56}$$

The fact that the coefficient of $\vec{e}_0$ in the above formula is a can be checked by expanding the identity $d/\mathrm{d}\tau(\vec{e}_0\cdot\vec{e}_1) = 0$. If $\mathrm{d}\vec{e}_1/\mathrm{d}\tau$ is not collinear to $\vec{e}_0$, relation (2.56) constitutes the definition of both the scalar $T_1 \geq 0$ and the unit vector $\vec{e}_2$. T_1 is called the ***first torsion*** of the worldline $\mathscr{L}$. If $T_1 = 0$, $\mathscr{L}$ is contained in the plane generated by $(\vec{e}_0, \vec{e}_1)$. In the general case, let O be a point of $\mathscr{L}$ and let us set $\tau(O) = 0$. Given a point $A(\tau) \in \mathscr{L}$ close to O and of proper time τ, we may perform a Taylor expansion of the vector $\overrightarrow{OA}$ in terms of the dimensionless parameter

$$\varepsilon := a_0 c\tau, \tag{2.57}$$

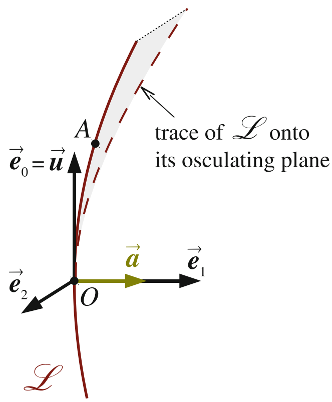

Fig. 2.12 Serret–Frenet tetrad at some point O of the worldline $\mathscr{L}$ (the vector $\vec{e}_3$ is not drawn)

where a_0 is $\mathscr{L}$'s curvature at O. Expanding up to power 3 in ε, we get

$$\overrightarrow{OA}(\tau) = \varepsilon \frac{d\,\overrightarrow{OA}}{d\varepsilon} + \frac{\varepsilon^2}{2}\frac{\mathrm{d}^2\overrightarrow{OA}}{d\varepsilon^2} + \frac{\varepsilon^3}{6}\frac{\mathrm{d}^3\overrightarrow{OA}}{d\varepsilon^3} + O(\varepsilon^4), \tag{2.58}$$

with $d^k\overrightarrow{OA}/d\varepsilon^k = (ca_0)^{-k}\, d^k\overrightarrow{OA}/\mathrm{d}\tau^k$, and from Eqs. (2.12), (2.55) and (2.56),

$$\frac{1}{c}\frac{d\,\overrightarrow{OA}}{\mathrm{d}\tau} = \vec{e}_0 \tag{2.59a}$$

$$\frac{1}{c^2}\frac{\mathrm{d}^2\overrightarrow{OA}}{\mathrm{d}\tau^2} = a\,\vec{e}_1 \tag{2.59b}$$

$$\frac{1}{c^3}\frac{\mathrm{d}^3\overrightarrow{OA}}{\mathrm{d}\tau^3} = a^2\vec{e}_0 + \frac{1}{c}\frac{da}{\mathrm{d}\tau}\vec{e}_1 + aT_1\vec{e}_2. \tag{2.59c}$$

Hence

$$\boxed{\begin{aligned}\overrightarrow{OA}(\tau) = {} & \left(1 + \frac{(ac\tau)^2}{6}\right) c\tau\ \vec{e}_0 + \left(a + \frac{da}{\mathrm{d}\tau}\frac{\tau}{3}\right)\frac{(c\tau)^2}{2}\vec{e}_1 \\ & + \frac{aT_1}{6}(c\tau)^3\ \vec{e}_2 + O((ac\tau)^4).\end{aligned}} \tag{2.60}$$

In this equality, the quantities a, $da/\mathrm{d}\tau$ and T_1, as well as the vectors $\vec{e}_0$, $\vec{e}_1$ and $\vec{e}_2$, have to be taken at the point O.

The expansion (2.60) shows that, up to the order $(ac\tau)^2$, the worldline stays in the plane $(O; \vec{e}_0, \vec{e}_1)$. This plane is called the ***osculating plane*** of $\mathscr{L}$ at O. The first torsion T_1, which appears at the order $(ac\tau)^3$ in the expansion (2.60), measures thus the departure of the worldline from its osculating plane (cf. Fig. 2.12).

Let us assume that $T_1 \neq 0$, i.e. that $\vec{e}_2$ is well defined. Since the latter is a unit vector ($\vec{e}_2 \cdot \vec{e}_2 = 1$), $\mathrm{d}\vec{e}_2/\mathrm{d}\tau$ is orthogonal to $\vec{e}_2$ and thus can be written as a linear combination of $\vec{e}_0$, $\vec{e}_1$ and a unit vector $\vec{e}_3$ orthogonal to $\vec{e}_0$, $\vec{e}_1$ and $\vec{e}_2$:

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}\tau} = \alpha\vec{e}_0 + \beta\vec{e}_1 + T_2\vec{e}_3.$$

The coefficients α and β are determined from the scalar products $\vec{e}_0 \cdot \vec{e}_2 = 0$ and $\vec{e}_1 \cdot \vec{e}_2 = 0$; taking the derivative of the first one with respect to τ, we get $\alpha = 0$, whereas taking the derivative of the second one yields $\beta = -T_1$. Hence

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_2}{\mathrm{d}\tau} = -T_1\vec{e}_1 + T_2\vec{e}_3. \tag{2.61}$$

If $\mathrm{d}\vec{e}_2/\mathrm{d}\tau$ is not collinear to $\vec{e}_1$, this relation constitutes the definition of both the scalar $T_2 \geq 0$ and the unit vector $\vec{e}_3$. T_2 is called the ***second torsion*** of the worldline $\mathscr{L}$. If $T_2 = 0$, $\mathscr{L}$ is contained in the affine subspace of $\mathscr{E}$ of dimension 3 (hyperplane) and generated by $(\vec{e}_0, \vec{e}_1, \vec{e}_2)$. In the general case, (2.60) shows that at the order $(ac\tau)^3$, $\mathscr{L}$ is contained in the hyperplane $(O; \vec{e}_0, \vec{e}_1, \vec{e}_2)$, that we shall call the ***osculating hyperplane*** of the worldline at the point O. It is easy to see that $T_2\vec{e}_3$ is involved at the order $(ac\tau)^4$ in the expansion of $\overrightarrow{OA}(\tau)$. The second torsion measures thus the departure of $\mathscr{L}$ from its osculating hyperplane.

Let us suppose that $T_2 \neq 0$ and evaluate $\mathrm{d}\vec{e}_3/\mathrm{d}\tau$. Since $\vec{e}_3$ is a unit vector, $\mathrm{d}\vec{e}_3/\mathrm{d}\tau$ is orthogonal to $\vec{e}_3$. It is then necessarily a linear combination of the vectors $\vec{e}_0$, $\vec{e}_1$ and $\vec{e}_2$:

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}\tau} = \alpha\vec{e}_0 + \beta\vec{e}_1 + \gamma\vec{e}_2.$$

Taking the derivative with respect to τ of the identities $\vec{e}_0 \cdot \vec{e}_3 = 0$, $\vec{e}_1 \cdot \vec{e}_3 = 0$ and $\vec{e}_2 \cdot \vec{e}_3 = 0$, we get $\alpha = 0$, $\beta = 0$ and $\gamma = -T_2$, so that we may write

$$\frac{1}{c}\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}\tau} = -T_2\vec{e}_2. \tag{2.62}$$

Altogether, (2.55), (2.56), (2.61) and (2.62) can be written as

$$\boxed{c^{-1}\begin{pmatrix} \mathrm{d}\vec{e}_0/\mathrm{d}\tau \\ \mathrm{d}\vec{e}_1/\mathrm{d}\tau \\ \mathrm{d}\vec{e}_2/\mathrm{d}\tau \\ \mathrm{d}\vec{e}_3/\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & T_1 & 0 \\ 0 & -T_1 & 0 & T_2 \\ 0 & 0 & -T_2 & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_0 \\ \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}}. \tag{2.63}$$

We shall see at Sect. 3.5.3 that the matrix appearing in the above formula can be interpreted in terms of the *4-rotation* of the Serret–Frenet tetrad.

Historical note: *The interpretation of the norm of the 4-acceleration as the curvature of the worldline appeared as early as 1908 in an article by Hermann Minkowski (cf. p. 26) (1908) and subsequently in his famous text on spacetime (Minkowski 1909).*

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