Chapter 2
Feynman Integrals: Basic Definitions and Tools

In this chapter, basic definitions for Feynman integrals are given, ultraviolet (UV), infrared (IR) and collinear divergences are characterized, and basic tools such as alpha parameters are presented. Various kinds of regularizations, in particular dimensional one, are presented and properties of dimensionally regularized Feynman integrals are formulated and discussed.

2.1 Feynman Rules and Feynman Integrals

In perturbation theory, any quantum field model is characterized by a Lagrangian, which is represented as a sum of a free-field part and an interaction part, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}}$. Amplitudes of the model, e.g. S-matrix elements and matrix elements of composite operators, are represented as power series in coupling constants. Starting from the S-matrix represented in terms of the time-ordered exponent of the interaction Lagrangian which is expanded with the application of the Wick theorem, or from Green functions written in terms of a functional integral treated in the perturbative way, one obtains that, in a fixed perturbation order, the amplitudes are written as finite sums of Feynman diagrams which are constructed according to Feynman rules: lines correspond to $\mathcal{L}_0$ and vertices are determined by $\mathcal{L}_{\mathrm{I}}$. The basic building block of the Feynman diagrams is the propagator that enters the relation

$$T\phi_i(x_1)\phi_i(x_2) =: \phi_i(x_1)\phi_i(x_2) : + D_{F,i}(x_1 - x_2). \tag{2.1}$$

Here $D_{F,i}$ is the Feynman propagator of the field of type i and the colons denote a normal product of the free fields. The Fourier transforms of the propagators have the form

$$\tilde{D}_{F,i}(p) \equiv \int d^4x\, \mathrm{e}^{\mathrm{i}p\cdot x} D_{F,i}(x) = \frac{\mathrm{i}Z_i(p)}{(p^2 - m_i^2 + \mathrm{i}0)^{a_i}}, \tag{2.2}$$

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where m_i is the corresponding mass, Z_i is a polynomial and $a_i = 1$ or 2 (for the gluon propagator in the general covariant gauge). The powers of the propagators a_l will be also called *indices*. For the propagator of the scalar field, we have $Z = 1$, $a = 1$. This is not the most general form of the propagator. For example, in the axial or Coulomb gauge, the gluon propagator has another form. We usually omit the causal i0 for brevity. Polynomials associated with vertices of graphs can be taken into account by means of the polynomials Z_l. We also omit the factors of i and $(2\pi)^4$ that enter in the standard Feynman rules (in particular, in (2.2)); these can be included at the end of a calculation.

Eventually, we obtain, for any fixed perturbation order, a sum of Feynman amplitudes labelled by Feynman graphs[1] constructed from the given type of vertices and lines. In the commonly accepted physical slang, the graph, the corresponding Feynman amplitude and the integral are all often called the 'diagram'. A Feynman graph differs from a graph by distinguishing a subset of vertices which are called *external*. The external momenta or coordinates on which a Feynman integral depends are associated with the external vertices.

Thus quantities that can be computed perturbatively are written, in any given order of perturbation theory, through a sum over Feynman graphs. For a given graph Γ, the corresponding Feynman amplitude

$$G_\Gamma(q_1, \ldots, q_{n+1}) = (2\pi)^4 \, \mathrm{i} \, \delta\left(\sum_i q_i\right) F_\Gamma(q_1, \ldots, q_n) \tag{2.3}$$

can be written in terms of an integral over loop momenta

$$F_\Gamma(q_1, \ldots, q_n) = \int \mathrm{d}^4 k_1 \ldots \int \mathrm{d}^4 k_h \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l), \tag{2.4}$$

where $\mathrm{d}^4 k_i = \mathrm{d}k_i^0 \, \mathrm{d}\boldsymbol{k}_i$, and a factor with a power of 2π is omitted, as we have agreed. The Feynman integral F_Γ depends on n linearly independent external momenta $q_i = (q_i^0, \boldsymbol{q}_i)$; the corresponding integrand is a function of L internal momenta p_l, which are certain linear combinations of the external momenta and $h = L - V + 1$ chosen loop momenta k_i, where L, V and h are numbers of lines, vertices and (independent) loops, respectively, of the given graph.

One can choose the loop momenta by fixing a *tree* T of the given graph, i.e. a maximal connected subgraph without loops, and correspond a loop momentum to each line not belonging to this tree. Then we have the following explicit formula for the momenta of the lines:

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[1] When dealing with graphs and Feynman integrals one usually does not bother about the mathematical definition of the graph and thinks about something that is built of lines and vertices. So, a graph is an ordered family $\{\mathcal{V}, \mathcal{L}, \pi_\pm\}$, where $\mathcal{V}$ is the set of vertices, $\mathcal{L}$ is the set of lines, and $\pi_\pm : \mathcal{L} \to \mathcal{V}$ are two mappings that correspond the initial and the final vertex of a line. By the way, mathematicians use the word 'edge', rather than 'line'.

$$p_l = \sum_{i=1}^{h} e_{il} k_i + \sum_{i=1}^{n} d_{il} q_i , \tag{2.5}$$

where $e_{il} = \pm 1$ if l belongs to the jth loop and $e_{il} = 0$ otherwise, $d_{il} = \pm 1$ if l lies in the tree T on the path with the momentum q_i and $d_{il} = 0$ otherwise. The signs in both sums are defined by orientations.

After a tensor reduction is performed one can deal only with scalar Feynman integrals. For the tensor reduction, various projectors can be applied. For example, in the case of Feynman integrals contributing to the electromagnetic formfactor (see Fig. 2.1) $\Gamma^{\mu}(p_1, p_2) = \gamma^{\mu} F_1(q^2) + \sigma^{\mu\nu} q_{\nu} F_2(q^2)$, where $q = p_1 - p_2$, γ^{μ} and $\sigma^{\mu\nu}$ are γ- and σ-matrices, respectively, the following projector can be applied to extract scalar integrals which contribute to the formfactor F_1 in the massless case (with $F_2 = 0$):

$$F_1(q^2) = \frac{\mathrm{Tr}\left[\gamma_{\mu} \not{p}_2 \Gamma^{\mu}(p_1, p_2) \not{p}_1\right]}{2(d-2)\, q^2} , \tag{2.6}$$

where $\not{p} = \gamma^{\mu} p_{\mu}$ and d is the parameter of dimensional regularization (to be discussed shortly in Sect. 2.4).

Anyway, after applying some projectors, one obtains, for a given graph, a family of Feynman integrals which have various powers of the scalar parts of the propagators, $1/(p_l^2 - m_l^2)^{a_l}$, and various monomials in the numerator. The denominators p_l^2 can be expressed linearly in terms of scalar products of the loop and external momenta. The factors in the numerator can also be chosen as quadratic polynomials of the loop and external momenta raised to some powers. It is convenient to consider both types of the quadratic polynomials on the same footing and treat the factors in the numerators as extra factors in the denominator raised to negative powers. The set of the denominators for a given graph is linearly independent. It is natural to complete this set by similar factors coming from the numerator in such a way that the whole set will be linearly independent.

Therefore we come to the following family of scalar integrals generated by the given graph:

$$F(a_1, \ldots, a_N) = \int \cdots \int \frac{\mathrm{d}^4 k_1 \ldots \mathrm{d}^4 k_h}{E_1^{a_1} \ldots E_N^{a_N}} , \tag{2.7}$$

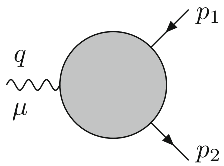

Fig. 2.1 Electromagnetic formfactor

where k_i, $i = 1, \ldots, h$, are loop momenta, a_i are integer indices, and the denominators are given by

$$E_l = \sum_{i \geq j \geq 1} A_l^{ij} \, r_i \cdot r_j - m_l^2 \,, \tag{2.8}$$

with $l = 1, \ldots, N$. The momenta r_i are either the loop momenta $r_i = k_i$, $i = 1, \ldots, h$, or independent external momenta $r_{h+1} = q_1, \ldots, r_{h+n} = q_l$ of the graph.

For a usual Feynman graph, the denominators E_r determined by some matrix A are indeed quadratic. However, a more general class of Feynman integrals where the denominators are linear with respect to the loop and/or external momenta also often appears in practical calculations. Linear denominators usually appear in asymptotic expansions of Feynman integrals within the strategy of expansion by regions [1, 31]. Such expansions provide a useful link of an initial theory described by some Lagrangian with various effective theories where, indeed, the denominators of propagators can be linear with respect to the external and loop momenta. For example, one encounters the following denominators: $p \cdot k$, with an external momentum p on the light cone, $p^2 = 0$, for the Sudakov limit and with $p^2 \neq 0$ for the quark propagator of Heavy Quark Effective Theory (HQET) [17, 22, 25]. Some non-relativistic propagators appear within threshold expansion and in the effective theory called Non-Relativistic QCD (NRQCD) [4, 5, 21, 37], for example, the denominator $k_0 - \boldsymbol{k}^2/(2m)$.

2.2 Divergences

As has been known from early days of quantum field theory, Feynman integrals suffer from divergences. This word means that, taken naively, these integrals are ill-defined because the integrals over the loop momenta generally diverge. The *ultraviolet* (UV) *divergences* manifest themselves through a divergence of the Feynman integrals at large loop momenta. Consider, for example, the Feynman integral corresponding to the one-loop graph Γ of Fig. 2.2 with scalar propagators. This integral can be written as

$$F_\Gamma(q) = \int \frac{\mathrm{d}^4 k}{(k^2 - m_1^2)\left[(q-k)^2 - m_2^2\right]}, \tag{2.9}$$

where the loop momentum k is chosen as the momentum of the first line. Introducing four-dimensional (generalized) spherical coordinates $k = r\hat{k}$ in (2.9), where $\hat{k}$ is on the unit (generalized) sphere and is expressed by means of three angles, and counting powers of propagators, we obtain, in the limit of large r, the following divergent behaviour: $\int_\Lambda^\infty \mathrm{d}r\, r^{-1}$. For a general diagram, a similar power counting at large values of the loop momenta gives $4h(\Gamma) - 1$ from the Jacobian that arises when one introduces generalized spherical coordinates in the $(4 \times h)$-dimensional space of h

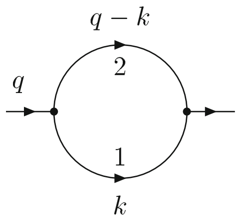

Fig. 2.2 One-loop self-energy diagram

loop four-momenta, plus a contribution from the powers of the propagators and the degrees of its polynomials, and leads to an integral $\int_\Lambda^\infty \mathrm{d}r\, r^{\omega-1}$, where

$$\omega = 4h - 2L + \sum_l n_l \tag{2.10}$$

is the (UV) *degree of divergence* of the graph. (Here n_l are the degrees of the polynomials Z_l.)

This estimate shows that the Feynman integral is UV convergent overall (no divergences arise from the region where all the loop momenta are large) if the degree of divergence is negative. We say that the Feynman integral has a logarithmic, linear, quadratic, etc. overall divergence when $\omega = 0, 1, 2, \ldots$, respectively. To ensure a complete absence of UV divergences it is necessary to check convergence in various regions where some of the loop momenta become large, i.e. to satisfy the relation $\omega(\gamma) < 0$ for all the subgraphs γ of the graph. We call a subgraph UV divergent if $\omega(\gamma) \geq 0$. In fact, it is sufficient to check these inequalities only for *one-particle-irreducible* (1PI) subgraphs (which cannot be made disconnected by cutting a line). It turns out that these rough estimates are indeed true—see some details in Sect. 4.4.

If we turn from momentum space integrals to some other representation of Feynman diagrams, the UV divergences will manifest themselves in other ways. For example, in coordinate space, the Feynman amplitude (i.e. the inverse Fourier transform of (2.3)) is expressed in terms of a product of the Fourier transforms of propagators

$$\prod_{l=1}^{L} D_{F,l}(x_{l_\mathrm{i}} - x_{l_\mathrm{f}}) \tag{2.11}$$

integrated over four-coordinates x_i corresponding to the internal vertices. Here l_i and l_f are the beginning and the end, respectively, of a line l.

The propagators in coordinate space,

$$D_{F,l}(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p\, \tilde{D}_{F,l}(p) \mathrm{e}^{-\mathrm{i}x\cdot p}, \tag{2.12}$$

are singular at small values of coordinates $x = (x_0, \boldsymbol{x})$. To reveal this singularity explicitly let us write down the propagator (2.2) in terms of an integral over a so-called alpha-parameter

$$\tilde{D}_{F,l}(p) = \mathrm{i}\, Z_l \left(\frac{1}{2\mathrm{i}} \frac{\partial}{\partial u_l} \right) \mathrm{e}^{2\mathrm{i}u_l \cdot p} \bigg|_{u_l=0} \frac{(-\mathrm{i})^{a_l}}{\Gamma(a_l)} \int_0^\infty \mathrm{d}\alpha_l\, \alpha_l^{a_l-1} \mathrm{e}^{\mathrm{i}(p^2-m^2)\alpha_l}. \quad (2.13)$$

which turns out to be a very useful tool both in theoretical analyses and practical calculations.

To present an explicit formula for the scalar (i.e. for $a = 1$ and $Z = 1$) propagator

$$\tilde{D}_F(p) = \int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}(p^2-m^2)\alpha} \quad (2.14)$$

in coordinate space we insert (2.14) into (2.12), change the order of integration over p and α and take the Gaussian integrations explicitly using the formula

$$\int \mathrm{d}^4k\, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} = -\mathrm{i}\pi^2 \alpha^{-2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}, \quad (2.15)$$

which is nothing but the product of four one-dimensional Gaussian integrals:

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}k_0\, \mathrm{e}^{\mathrm{i}(\alpha k_0^2 - 2q_0 k_0)} &= \sqrt{\frac{\pi}{\alpha}}\, \mathrm{e}^{-\mathrm{i}q_0^2/\alpha + \mathrm{i}\pi/4}, \\ \int_{-\infty}^{\infty} \mathrm{d}k_j\, \mathrm{e}^{-\mathrm{i}(\alpha k_j^2 - 2q_j k_j)} &= \sqrt{\frac{\pi}{\alpha}}\, \mathrm{e}^{\mathrm{i}q_j^2/\alpha - \mathrm{i}\pi/4}, \quad j = 1, 2, 3 \end{aligned} \quad (2.16)$$

(without summation over j in the last formula).

The final integration is then performed using [28] or in `MATHEMATICA` [39] with the following result:

$$\begin{aligned} D_F(x) &= -\frac{\mathrm{i}m}{4\pi^2\sqrt{-x^2+\mathrm{i}0}} K_1\left(\mathrm{i}m\sqrt{-x^2+\mathrm{i}0}\right) \\ &= -\frac{1}{4\pi^2} \frac{1}{x^2 - \mathrm{i}0} + O\left(m^2 \ln m^2\right) \end{aligned} \quad (2.17)$$

where K_1 is a Bessel special function [15]. The leading singularity at $x = 0$ is given by the value of the coordinate space massless propagator.

Thus, the inverse Fourier transform of the convolution integral (2.9) equals the square of the coordinate-space scalar propagator, with the singularity $(x^2 - \mathrm{i}0)^{-2}$. Power-counting shows that this singularity produces integrals that are divergent in the vicinity of the point $x = 0$, and this is the coordinate space manifestation of the UV divergence.

The divergences caused by singularities at small loop momenta are called *infrared* (IR) *divergences*. First we distinguish IR divergences that arise at general values of the external momenta. A typical example of such a divergence is given by the graph of Fig. 2.2 when one of the lines contains the second power of the corresponding propagator, so that $a_1 = 2$. If the mass of this line is zero we obtain a factor $1/(k^2)^2$ in the integrand, where k is chosen as the momentum of this line. Then, keeping in mind the introduction of generalized spherical coordinates and performing power-counting at small k (i.e. when all the components of the four-vector k are small), we again encounter a divergent behaviour $\int_0^{\Lambda} \mathrm{d}r\, r^{-1}$ but now at small values of r. There is a similarity between the properties of IR divergences of this kind and those of UV divergences. One can define, for such off-shell IR divergences, an IR degree of divergence, in a similar way to the UV case. A reasonable choice is provided by the value

$$\tilde{\omega}(\gamma) = -\omega(\Gamma/\overline{\gamma}) \equiv \omega(\overline{\gamma}) - \omega(\Gamma), \tag{2.18}$$

where $\overline{\gamma} \equiv \Gamma \backslash \gamma$ is the completion of the subgraph γ in a given graph Γ and Γ/γ denotes the reduced graph which is obtained from Γ by reducing every connectivity component of γ to a point. The absence of off-shell IR divergences is guaranteed if the IR degrees of divergence are negative for all massless subgraphs γ whose completions $\overline{\gamma}$ include all the external vertices in the same connectivity component [11, 30]. (See details in Sect. 4.4.) The off-shell IR divergences are the worst but they are in fact absent in physically meaningful theories. However, they play an important role in asymptotic expansions of Feynman diagrams—see [31] and Chap. 9.

The other kinds of IR divergences arise when the external momenta considered are on a surface where the Feynman diagram is singular: either on a mass shell or at a threshold. Consider, for example, the graph Fig. 2.2, with the indices $a_1 = 1$ and $a_2 = 2$ and the masses $m_1 = 0$ and $m_2 = m \neq 0$ on the mass shell, $q^2 = m^2$. With k as the momentum of the second line, the corresponding Feynman integral is of the form

$$F_\Gamma(q; d) = \int \frac{\mathrm{d}^4 k}{k^2 (k^2 - 2q \cdot k)^2}. \tag{2.19}$$

At small values of k, the integrand behaves like $1/[4k^2(q \cdot k)^2]$, and, with the help of power counting, we see that there is an *on-shell IR divergence* which would not be present for $q^2 \neq m^2$.

If we consider Fig. 2.2 with equal masses and indices $a_1 = a_2 = 2$ at the threshold, i.e. at $q^2 = 4m^2$, it might seem that there is a *threshold IR divergence* because, choosing the momenta of the lines as $q/2 + k$ and $q/2 - k$, we obtain the integral

$$\int \frac{\mathrm{d}^4 k}{(k^2 + q \cdot k)^2 (k^2 - q \cdot k)^2}, \tag{2.20}$$

with an integrand that behaves at small k as $1/(q \cdot k)^4$ and is formally divergent. However, the divergence is in fact absent. (The threshold singularity at $q^2 = 4m^2$ is, of course, present.) Nevertheless, threshold IR divergences do exist. For example, the sunset[2] diagram of Fig. 2.3 with general masses at threshold, $q^2 = (m_1 + m_2 + m_3)^2$,

[2] called also the sunrise diagram, or the London transport diagram.

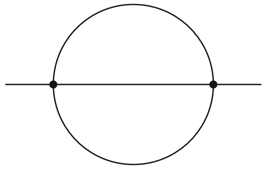

Fig. 2.3 Sunset diagram

is divergent in this sense when the sum of the integer powers of the propagators is greater than or equal to five (see, e.g., [14]).

The IR divergences characterized above are local in momentum space, i.e. they are connected with special points of the loop integration momenta. *Collinear* divergences arise at lines parallel to certain light-like four-vectors. A typical example of a collinear divergence is provided by the massless triangle graph of Fig. 2.4. Let us take $p_1^2 = p_2^2 = 0$ and all the masses equal to zero. The corresponding Feynman integral is

$$\int \frac{\mathrm{d}^4 k}{(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)k^2}. \tag{2.21}$$

At least an on-shell IR divergence is present, because the integral is divergent when $k \to 0$ (componentwise). However, there are also divergences at non-zero values of k that are collinear with p_1 or p_2 and where $k^2 \sim 0$. This follows from the fact that the product $1/[(k^2 - 2p{\cdot}k)k^2]$, where $p^2 = 0$ and $p \neq 0$, generates collinear divergences. To see this let us take residues in the upper complex half plane when integrating this product over k_0. For example, taking the residue at $k_0 = -|\boldsymbol{k}| + \mathrm{i}0$ leads to an integral containing $1/(p{\cdot}k) = 1/[p^0|\boldsymbol{k}|(1 - \cos\theta)]$, where θ is the angle between the spatial components $\boldsymbol{k}$ and $\boldsymbol{p}$. Thus, for small θ, we have a divergent integration over angles because of the factor $\mathrm{d}\cos\theta/(1 - \cos\theta) \sim \mathrm{d}\theta/\theta$. The second residue generates a similar divergent behaviour—this can be seen by making the change $k \to p - k$.

Another way to reveal the collinear divergences is to introduce the light-cone coordinates $k_\pm = k_0 \pm k_3$, $\underline{k} = (k_1, k_2)$. If we choose p with the only non-zero component p_+, we will see a logarithmic divergence coming from the region $k_- \sim \underline{k}^2 \sim 0$ just by power counting.

These are the main types of divergences of usual Feynman integrals. Various special divergences arise in more general Feynman integrals (2.7) that can contain linear propagators and appear on the right-hand side of asymptotic expansions in momenta

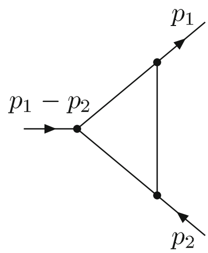

Fig. 2.4 One-loop triangle diagram

and masses and in associated effective theories. For example, in the Sudakov limit, one encounters divergences that can be classified as UV collinear divergences. Another situation with various non-standard divergences is provided by threshold expansion and the corresponding effective theories, NRQCD and pNRQCD, where special power counting is needed to characterize the divergences.

2.3 Alpha Representation

A useful tool to analyze the divergences of Feynman integrals is the so-called alpha representation based on (2.13). It can be written down for any Feynman integral. For example, for (2.9), one inserts (2.13) for each of the two propagators, takes the four-dimensional Gaussian integral by means of (2.15) to obtain

$$F_\Gamma(q) = \mathrm{i}\pi^2 \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, (\alpha_1 + \alpha_2)^{-2} \\ \times \exp\left(\mathrm{i}q^2 \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - \mathrm{i}(m_1^2\alpha_1 + m_2^2\alpha_2) \right). \qquad (2.22)$$

For a usual general Feynman integral, this procedure can also explicitly be implemented. Using (2.13) for each propagator of a general usual Feynman integral (i.e., with usual propagators (2.2)) one takes (see, e.g., [23]) $4h$-dimensional Gauss integrals by means of a generalization of (2.15) to the case of an arbitrary number of loop integration momenta:

$$\int \mathrm{d}^4k_1 \ldots \mathrm{d}^4k_h \exp\left[\mathrm{i}\left(\sum_{i,j} A_{ij} k_i \cdot k_j + 2\sum_i q_i \cdot k_i \right) \right] \\ = \mathrm{i}^{-h}\pi^{2h} (\det A)^{-2} \exp\left[-\mathrm{i} \sum_{i,j} A^{-1}_{ij} q_i \cdot q_j \right]. \qquad (2.23)$$

Here A is an $h \times h$ matrix and A^{-1} its inverse.[3]

The elements of the inverse matrix involved here are rewritten in graph-theoretical language (see details in [7, 23]), and the resulting alpha representation takes the form [8]

[3] In fact, the matrix A involved here equals $e\beta e^+$ with the elements of an arbitrarily chosen column and row with the same number deleted. Here e is the incidence matrix of the graph, i.e. $e_{il} = \pm 1$ if the vertex i is the beginning/end of the line l, e^+ is its transpose and β consists of the numbers $1/\alpha_l$ on the diagonal—see, e.g., [23].

$$F_\Gamma(q_1,\ldots,q_n;d) = \frac{\mathrm{i}^{-a-h}\pi^{2h}}{\prod_l \Gamma(a_l)} \times \int_0^\infty \mathrm{d}\alpha_1 \ldots \int_0^\infty \mathrm{d}\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-2} Z \mathrm{e}^{\mathrm{i}\mathcal{V}/\mathcal{U}-\mathrm{i}\sum m_l^2\alpha_l}, \tag{2.24}$$

where $a = \sum a_l$, and $\mathcal{U}$ and $\mathcal{V}$ are the well-known functions

$$\mathcal{U} = \sum_{T\in T^1} \prod_{l\notin T} \alpha_l, \tag{2.25}$$

$$\mathcal{V} = \sum_{T\in T^2} \prod_{l\notin T} \alpha_l \left(q^T\right)^2. \tag{2.26}$$

In (2.25), the sum runs over trees of the given graph, and, in (2.26), over *2-trees*, i.e. subgraphs that do not involve loops and consist of two connectivity components; $\pm q^T$ is the sum of the external momenta that flow into one of the connectivity components of the 2-tree T. (It does not matter which component is taken because of the conservation law for the external momenta.) The products of the alpha parameters involved are taken over the lines that do not belong to the given tree T. The functions $\mathcal{U}$ and $\mathcal{V}$ are homogeneous functions of the alpha parameters with the homogeneity degrees h and $h+1$, respectively. See also [6] for various properties of these two basic functions.

The factor Z is responsible for the non-scalar structure of the diagram:

$$Z = \prod_l Z_l \left(\frac{1}{2\mathrm{i}} \frac{\partial}{\partial u_l}\right) \mathrm{e}^{\mathrm{i}(2B-K)/\mathcal{U}} \bigg|_{u_1=\ldots u_L=0}, \tag{2.27}$$

where (see, e.g., [30, 40])

$$B = \sum_l u_l \sum_{T\in T_l^1} q_T \prod_{l'\notin T} \alpha_{l'}, \tag{2.28}$$

$$K = \sum_{T\in T^0} \prod_{l\notin T} \alpha_l \left(\sum_l \pm u_l\right)^2. \tag{2.29}$$

In (2.28), the sum is taken over trees T_l^1 that include a given line l, and q_T is the total external momentum that flows through the line l (in the direction of its orientation). In (2.29), the sum is taken over pseudotrees T^0 (a *pseudotree* is obtained from a tree by adding a line), and the sum in l is performed over the loop (circuit) of the pseudotree T, with a sign dependent on the coincidence of the orientations of the line l and the pseudotree T.

Let me emphasize that this terrible-looking machinery for evaluating the determinant of the matrix A that arises from Feynman integrals, as well as for evaluating the elements of the inverse matrix, together with interpreting these results from the graph-theoretical point of view, is exactly the same as that used in the problem of the solution of Kirchhoff's laws for electrical circuits, a problem typical of the nineteenth century. Recall, for example, that the parameters α_l play the role of ohmic resistances and that the expression (2.25) for the function $\mathcal{U}$ as a sum over trees is a Kirchhoff result.

In practical calculations, one often derives the alpha representation for concrete diagrams by hand, rather than deduces it from the general formulae presented above. For the derivation, one can also use the public code `UF.m` [29] which is applicable also for general quadratic and linear propagators.

2.4 Regularization

The standard way of dealing with divergent Feynman integrals is to introduce a *regularization*. This means that, instead of the original ill-defined Feynman integral, we consider a quantity which depends on a regularization parameter, λ, and formally tends to the initial, meaningless expression when this parameter takes some limiting value, $\lambda = \lambda_0$. This new, regularized, quantity turns out to be well-defined, and the divergence manifests itself as a singularity with respect to the regularization parameter. Experience tells us that this singularity can be of a power or logarithmic type, i.e. $\ln^n(\lambda - \lambda_0)/(\lambda - \lambda_0)^i$.

Although a regularization makes it possible to deal with divergent Feynman integrals, it does not actually remove UV divergences, because this operation is of an auxiliary character so that sooner or later it will be necessary to switch off the regularization. To provide finiteness of physical observables evaluated through Feynman diagrams, another operation, called *renormalization*, is used. This operation is described, at the Lagrangian level, as a redefinition of the bare parameters of a given Lagrangian by inserting counterterms. The renormalization at the diagrammatic level is called *R-operation* and removes the UV divergence from individual Feynman integrals. It is, however, beyond the scope of the present book. (See, however, some details in Sect. 14.6, where the method of IR rearrangement is briefly described.)

An obvious way of regularizing Feynman integrals is to introduce a cut-off at large values of the loop momenta. Another well-known regularization procedure is the Pauli–Villars regularization [26], which is described by the replacement

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2}$$

and its generalizations. For finite values of the regularization parameter M, this procedure clearly improves the UV asymptotics of the integrand. Here the limiting value of the regularization parameter is $M = \infty$.

If we replace the integer powers a_l in the propagators by general complex numbers λ_l we obtain an *analytically regularized* [32] Feynman integral where the divergences of the diagram are encoded in the poles of this regularized quantity with respect to the analytic regularization parameters λ_l. For example, power counting at large values of the loop momentum in the analytically regularized version of (2.9) leads to the divergent behaviour $\int_\Lambda^\infty \mathrm{d}r\, r^{\lambda_1+\lambda_2-3}$, which results in a pole $1/(\lambda_1+\lambda_2-2)$ at the limiting values of the regularization parameters $\lambda_l = 1$.

For example, in the case of the analytically regularized integral of Fig. 2.2, we obtain

$$F_\Gamma(q;\lambda_1,\lambda_2) = \frac{\mathrm{e}^{-\mathrm{i}\pi(\lambda_1+\lambda_2+1)/2}\pi^2}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \int_0^\infty\int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\, \frac{\alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}}{(\alpha_1+\alpha_2)^2} \times \exp\left(\mathrm{i}q^2\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)} - \mathrm{i}(m_1^2\alpha_1+m_2^2\alpha_2)\right). \quad (2.30)$$

After the change of variables $\eta = \alpha_1+\alpha_2,\ \xi = \alpha_1/(\alpha_1+\alpha_2)$ and explicit integration over η, we arrive at

$$F_\Gamma(q;\lambda_1,\lambda_2) = \mathrm{e}^{\mathrm{i}\pi(\lambda_1+\lambda_2)}\frac{\mathrm{i}\pi^2\Gamma(\lambda_1+\lambda_2-2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \times \int_0^1 \mathrm{d}\xi\, \frac{\xi^{\lambda_1-1}(1-\xi)^{\lambda_2-1}}{\left[m_1^2\xi+m_2^2(1-\xi)-q^2\xi(1-\xi)-\mathrm{i}0\right]^{\lambda_1+\lambda_2-2}}. \quad (2.31)$$

Thus the UV divergence manifests itself through the first pole of the gamma function $\Gamma(\lambda_1+\lambda_2-2)$ in (2.31), which results from the integration over small values of η due to the power $\eta^{\lambda_1+\lambda_2-3}$.

The alpha representation turns out to be very useful for the introduction of *dimensional* regularization, which is a commonly accepted computational technique successfully applied in practice and which will serve as the main kind of regularization in this book. Let us imagine that the number of space–time dimensions differs from four. To be more precise, the number of space dimensions is considered to be $d-1$, rather than three. (But we still think of an integer number of dimensions.) The derivation of the alpha representation does not change much in this case. The only essential change is that, instead of (2.15), we need to apply its generalization to an arbitrary number of dimensions, d:

$$\int \mathrm{d}^d k\, \mathrm{e}^{\mathrm{i}(\alpha k^2-2q\cdot k)} = \mathrm{e}^{\mathrm{i}\pi(1-d/2)/2}\pi^{d/2}\alpha^{-d/2}\mathrm{e}^{-\mathrm{i}q^2/\alpha}. \quad (2.32)$$

So, instead of (2.22), we have the following formula in d dimensions:

$$F_\Gamma(q;d) = \mathrm{e}^{-\mathrm{i}\pi(1+d/2)/2}\pi^{d/2}\int_0^\infty\int_0^\infty \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,(\alpha_1+\alpha_2)^{-d/2}$$
$$\times\exp\left(\mathrm{i}q^2\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - \mathrm{i}(m_1^2\alpha_1+m_2^2\alpha_2)\right). \tag{2.33}$$

The only two places where something has been changed are the exponent of the combination $(\alpha_1+\alpha_2)$ in the integrand and the exponents of the overall factors.

Now, in order to introduce dimensional regularization, we want to consider the dimension d as a complex number. So, by definition, the dimensionally regularized Feynman integral for Fig. 2.2 is given by (2.33) and is a function of q^2 as given by this integral representation. We choose $d = 4-2\varepsilon$, where the value $\varepsilon = 0$ corresponds to the physical number of the space–time dimensions. By the same change of variables as used after (2.30), we obtain

$$F_\Gamma(q;d) = \mathrm{e}^{-\mathrm{i}\pi(1+d/2)/2}\pi^{d/2}\int_0^\infty \mathrm{d}\eta\,\eta^{\varepsilon-1}$$
$$\times\int_0^1 \mathrm{d}\xi\,\exp\{\mathrm{i}q^2\xi(1-\xi)\eta - \mathrm{i}[m_1^2\xi+m_2^2(1-\xi)]\eta\}. \tag{2.34}$$

This integral is absolutely convergent for $0 < \mathrm{Re}\,\varepsilon < \Lambda$ (where $\Lambda = \infty$ if both masses are non-zero and $\Lambda = 1$ otherwise; this follows from an IR analysis of convergence, which we omit here) and defines an analytic function of ε, which is extended from this domain to the whole complex plane as a meromorphic function.

After evaluating the integral over η, we arrive at the following result:

$$F_\Gamma(q;d) = \mathrm{i}\pi^{d/2}\Gamma(\varepsilon)\int_0^1 \frac{\mathrm{d}\xi}{\left[m_1^2\xi+m_2^2(1-\xi)-q^2\xi(1-\xi)-\mathrm{i}0\right]^\varepsilon}. \tag{2.35}$$

The UV divergence manifests itself through the first pole of the gamma function $\Gamma(\varepsilon)$ in (2.35), which results from the integration over small values of η in (2.34).

This procedure of introducing dimensional regularization is easily generalized [8, 9, 11] to an arbitrary usual Feynman integral. Instead of (2.23), we use

$$\int \mathrm{d}^dk_1\ldots\mathrm{d}^dk_h\exp\left[\mathrm{i}\left(\sum_{i,j}A_{ij}k_i\cdot k_j + 2\sum_i q_i\cdot k_i\right)\right]$$
$$= \mathrm{e}^{\mathrm{i}\pi h(1-d/2)/2}\pi^{hd/2}(\det A)^{-d/2}\exp\left[-\mathrm{i}\sum_{i,j}A^{-1}_{ij}q_i\cdot q_j\right], \tag{2.36}$$

and the resulting d-dimensional alpha representation takes the form [8, 9]

$$F_\Gamma(q_1,\ldots,q_n;d) = (-1)^a \frac{e^{i\pi[a+h(1-d/2)]/2}\pi^{hd/2}}{\prod_l \Gamma(a_l)} \times \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} Z e^{i\mathcal{V}/\mathcal{U} - i\sum m_l^2\alpha_l}. \tag{2.37}$$

Let us now define[4] the *dimensionally regularized* Feynman integral by means of (2.37), treating the quantity d as a complex number. This is a function of kinematical invariants constructed from the external momenta and contained in the function $\mathcal{V}$. In addition to this, we have to take care of polynomials in the external momenta and the auxiliary variables u_l hidden in the factor Z. We treat these objects q_i and u_l, as well as the metric tensor $g_{\mu\nu}$, as elements of an algebra of covariants, where we have, in particular,

$$\left(\frac{\partial}{\partial u_l^\mu}\right) u_{l'}^\nu = g_\mu^\nu \delta_{l,l'}, \qquad g_\mu^\mu = d.$$

This algebra also includes the γ-matrices with anticommutation relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ so that $\gamma^\mu\gamma_\mu = d$, the tensor $\varepsilon_{\kappa\mu\nu\lambda}$, etc.

Thus the dimensionally regularized Feynman integrals are defined as linear combinations of tensor monomials in the external momenta and other algebraic objects with coefficients that are functions of the scalar products $q_i \cdot q_j$. However, this is not all, because we have to see that the α-integral is well-defined. Remember that it can be divergent, for various reasons.

The alpha representation is not only an important technique to evaluate Feynman integrals either analytically (as explained in the next chapter) or numerically (as explained in Sects. 4.2 and 4.3) but also a convenient tool to analyze their convergence. The adequate technique for the numerical evaluation and the analysis of convergence is the same: these are sector decompositions (appeared, for the first time, in [18]) of a given alpha-parametric integral where new variables are introduced in such a way that the integrand factorizes, i.e. takes the form of a product of some powers of the sector variables with a non-zero function. Eventually, in the new

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4 An alternative definition of algebraic character [19, 35, 38] (see also [13]) exists and is based on certain axioms for integration in a space with non-integer dimension. It is unclear how to perform the analysis within such a definition, for example, how to apply the operations of taking a limit, differentiation, etc. to algebraically defined Feynman integrals in d dimensions, in order to say something about the analytic properties with respect to momenta and masses and the parameter of dimensional regularization. After evaluating a Feynman integral according to the algebraic rules, one arrives at some concrete function of these parameters but, *before integration*, one is dealing with an abstract algebraic object. Let us remember, however, that, in practical calculations, one usually does not bother about precise definitions. From the purely pragmatic point of view, it is useless to think of a diagram when it is not calculated. On the other hand, from the pure theoretical and mathematical point of view, such a position is beneath criticism.

variables, the analysis of convergence reduces to power counting in one-dimensional integrals.

For Feynman integrals considered at *Euclidean* external momenta q_i, i.e. when any sum of incoming momenta is spacelike,

$$\left(\sum_{i\in I} q_i\right)^2 < 0 \qquad (2.38)$$

this analysis is described in Sect. 4.4.

As a result of this analysis, any Feynman integral at *Euclidean* external momenta is defined as meromorphic function of d with series of UV and IR poles [9, 27, 30, 33, 34, 36]. Here it is also assumed that there are no massless detachable subgraphs, i.e. massless subdiagrams with zero external momenta. For example, a *tadpole*, i.e. a line with coincident end points, is a detachable subgraph. However, such diagrams are naturally put to zero in case they are massless—see a discussion below.

Observe that increasing $\mathrm{Re}\,\varepsilon$ improves UV convergence and decreasing $\mathrm{Re}\,\varepsilon$ improves IR convergence. If a given Feynman integral is only UV or IR divergent one can apply sector decompositions and choose an appropriate domain of ε to provide the convergence and then analytically continue the integral to the whole complex plane of ε. If there are both UV and IR divergences in a given Feynman integral so that changing ε improves one kind of convergence and spoils the other kind. As explained in Sect. 4.4 one can, however, exploit an auxiliary analytic regularization and provide an ambiguous definition[5] [11] of dimensionally regularized Feynman integrals in this situation.

There are no similar mathematical results for general Feynman integrals in cases where at least some of external momenta squared are non-negative. However, one can follow a simple recipe which is implicitly adopted at least by the authors of so-called modern sector decompositions initiated in [2, 3]. According to this recipe, various subintegrals appearing in parametric representations are considered in their own domains of ε where they are convergent. We will continue this discussion in the end of Sect. 4.4 after presenting various sector decompositions in Sects. 4.1–4.3.

Let me now emphasize that one is forced to evaluate Feynman integrals on a mass shell or a threshold because they are really needed in practice. In fact, such integrals will be mainly considered in this book as examples illustrating the methods described. However, in every concrete example considered below, we will see that every Feynman diagram is indeed an analytical function of d, both in intermediate steps of a calculation and, of course, in our results.

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[5] Besides [11], the problem of defining UV and IR divergent Feynman integrals within dimensional regularization was studied in [10] where Mellin–Barnes integrals were applied for this purpose.

2.5 Properties of Dimensionally Regularized Feynman Integrals

We can formally write down dimensionally regularized Feynman integrals as integrals over d-dimensional vectors k_i:

$$F_\Gamma(q_1, \ldots, q_n; d) = \int \mathrm{d}^d k_1 \ldots \int \mathrm{d}^d k_h \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l). \tag{2.39}$$

If a tensor reduction was already performed we deal with the corresponding scalar integrals represented by the d-dimensional version of (2.7)

$$F(q_1, \ldots, q_n; a_1, \ldots, a_N; d) = \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_N^{a_N}}, \tag{2.40}$$

where the denominators E_i have the form (2.8). The indices a_i can be either positive or non-positive so that numerators in Feynman integrals correspond to negative indices.

In order to obtain dimensionally regularized integrals with their dimension independent of ε, a factor of $\mu^{-2\varepsilon}$ per loop, where μ is a massive parameter, is introduced. This parameter serves as a renormalization parameter for schemes based on dimensional regularization. Therefore, we obtain logarithms and other functions depending not only on ratios of given parameters, e.g. q^2/m^2, but also on q^2/μ^2 etc. However, we will usually omit this μ-dependence for brevity (i.e. set $\mu = 1$) so that you will meet sometimes quantities like $\ln q^2$ which should be understood in the sense of $\ln(q^2/\mu^2)$.

We have reasons for using the notation (2.39), because dimensionally regularized Feynman integrals as defined above possess the standard properties of integrals of the usual type in integer dimensions. In particular,

- the integral of a linear combination of integrands equals the same linear combination of the corresponding integrals;
- one may cancel the same factors in the numerator and denominator of integrands.

These properties follow directly from the above definition. A less trivial property is that

- a derivative of an integral with respect to a mass or momentum equals the corresponding integral of the derivative.

This is also a consequence (see [11, 30]) of the definition of dimensionally regularized Feynman integrals based on the alpha representation and the corresponding analysis of convergence presented in Sect. 4.4. To prove this statement, one uses standard algebraic relations between the functions entering the alpha representation [9, 23]. (We note again that these are relations quite similar to those encoded in the solutions of Kirchhoff's laws for a circuit defined by the given graph.) A corollary of the last property is the possibility of integrating by parts and always neglecting surface terms:

$$\int \mathrm{d}^d k_1 \ldots \int \mathrm{d}^d k_h \left(\frac{\partial}{\partial k_i} \cdot r_j\right) \prod_{l=1}^{L} \tilde{D}_{F,l}(p_l) = 0\ , \quad i = 1, \ldots, h, \qquad (2.41)$$

where r_j is a loop or external momentum.

This property is the basis for solving the reduction problem for Feynman integrals using IBP relations [12]—see Chap. 6.

The next property says that

- any diagram with a detachable massless subgraph is zero.

Let us consider, for example, the massless tadpole diagram, which can be reduced by means of alpha parameters to a scaleless one-dimensional integral:

$$\int \frac{\mathrm{d}^d k}{k^2} = -\mathrm{i}^\varepsilon \pi^{d/2} \int_0^\infty \mathrm{d}\alpha\, \alpha^{\varepsilon-2}. \qquad (2.42)$$

We divide this integral into two pieces, from 0 to 1 and from 1 to ∞, evaluate these two integrals and find results that are equal except for opposite signs, which lead to the zero value.[6] It should be emphasized here that the two pieces that contribute to the right-hand side of (2.42) are convergent in *different* domains of the regularization parameter ε, namely, $\mathrm{Re}\,\varepsilon > -1$ and $\mathrm{Re}\,\varepsilon < -1$, with no intersection.

A massless Feynman integral with a zero external momentum can appear either in the beginning when using Feynman rules, or after some manipulations: after using partial fractions, integration by parts, etc. We can also include in this second class all such integrals that appear on the right-hand side of asymptotic expansions in momenta and masses [1, 31]—see Chap. 9. In any case, one sets such integrals to zero. In fact, in any massless Feynman integral at zero external momenta, one can reveal an internal one-dimensional integral with a pure power, similar to the above integral for the tadpole (2.42). We will come back to this point in Chap. 4.

On-shell and threshold Feynman integrals have been already mentioned many times, so that let us consider several typical one-loop examples. We must realize that, generally, an on-shell or threshold Feynman integral is *not* the value of the given Feynman integral $F_\Gamma(q^2, \ldots)$, defined as a function of q^2 and other kinematical variables, at a value of q^2 on a mass shell or at a threshold. Consider, for example, the Feynman integral corresponding to Fig. 2.2, with $m_1 = 0,\ m_2 = m,\ a_1 = 1,\ a_2 = 2$. We know an explicit result for the diagram given by (1.5). There is a logarithmic singularity at threshold, $q^2 = m^2$, so that we cannot strictly speak about the value of the integral there. Still we can certainly define the threshold Feynman integral by putting $q^2 = m^2$ *in the integrand* of the integral over the loop momentum or over the alpha parameters. And this is what was really meant and will be meant by 'on-shell' and 'threshold' integrals. In this example, we obtain an integral which can be evaluated by means of (10.13) (to be derived in Chap. 3):

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[6] These arguments can be found, for example, in [20], and, ironically, even in a pure mathematical book [16].

$$\int \frac{\mathrm{d}^d k}{k^2(k^2 - 2q \cdot k)^2} = \mathrm{i}\pi^{d/2} \frac{\Gamma(\varepsilon)}{2(m^2)^{1+\varepsilon}}. \tag{2.43}$$

This integral is divergent, in contrast to the original Feynman integral defined for general q^2.

Thus on-shell or threshold dimensionally regularized Feynman integrals are defined by the alpha representation or by integrals over the loop momenta with restriction of some kinematical invariants to appropriate values in the corresponding integrands. In this sense, these regularized integrals are 'formal' values of general Feynman integrals at the chosen variables.

Note that the products of the free fields in the Lagrangian are not required to be normal-ordered, so that products of fields of the same sort at the same point are allowed. The formal application of the Wick theorem therefore generates values of the propagators at zero. For example, in the case of the scalar free field, with the propagator

$$D_F(x) = \frac{\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4 k \frac{\mathrm{e}^{-\mathrm{i}x \cdot k}}{k^2 - m^2}, \tag{2.44}$$

which satisfies $(\Box + m^2) D_F(x) = -\mathrm{i}\delta(x)$, we have

$$T\phi(x)\phi(x) =: \phi^2(x) : + D_F(0). \tag{2.45}$$

The value of $D_F(x)$ at $x = 0$ does not exist, because the propagator is singular at the origin according to (2.17). However, we imply the *formal* value at the origin rather than the 'honestly' taken value. This means that we set x to zero in some integral representation of this quantity. For example, using the inverse Fourier transformation, we can define $D_F(0)$ as the integral (2.44) with x set to zero *in the integrand*. Thus, by definition,

$$D_F(0) = \frac{\mathrm{i}}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{k^2 - m^2}. \tag{2.46}$$

This integral is, however, quadratically divergent, as Feynman integrals typically are. So, we understand $D_F(0)$ as a dimensionally regularized formal value when we put $x = 0$ in the Fourier integral and obtain, using (10.1) (which we will derive shortly),

$$\int \frac{\mathrm{d}^d k}{k^2 - m^2} = -\mathrm{i}\pi^{d/2} \Gamma(\varepsilon - 1)(m^2)^{1-\varepsilon}. \tag{2.47}$$

This Feynman integral in fact corresponds to the tadpole ϕ^4 theory graph shown in Fig. 2.5. The corresponding quadratic divergence manifests itself through an UV pole in ε—see (2.47).

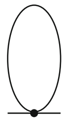

Fig. 2.5 Tadpole

Observe that one can trace the derivation of the integrals tabulated in Sect. 10.1 and see that the integrals are convergent in some non-empty domains of the complex parameters λ_l and ε and that the results are analytic functions of these parameters with UV, IR and collinear poles.

Before continuing our discussion of setting scaleless integrals to zero, let us present an analytic result for the one-loop massless triangle integral with two on-shell external momenta, $p_1^2 = p_2^2 = 0$. Using (10.28) (which we will derive in Chap. 3), we obtain

$$\int \frac{\mathrm{d}^d k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2} = -\mathrm{i}\pi^{d/2}\frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(1-2\varepsilon)(-q^2)^{1+\varepsilon}}. \tag{2.48}$$

A double pole at $\varepsilon = 0$ arises from the IR and collinear divergences.

A similar formula with a monomial in the numerator can be obtained also straightforwardly:

$$\int \frac{\mathrm{d}^d k\, k^\mu}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2} = \mathrm{i}\pi^{d/2}\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}\frac{p_1^\mu + p_2^\mu}{(-q^2)^{1+\varepsilon}}. \tag{2.49}$$

Now only a simple pole is present, because the factor k^μ kills the IR divergence.

Consider now a massless one-loop integral with the external momentum on the massless mass shell, $p^2 = 0$:

$$\int \frac{\mathrm{d}^d k}{(p-k)^2 k^2}. \tag{2.50}$$

If we write down the alpha representation for this integral we obtain the same expression (2.42) as for $p = 0$ because only p^2, equal to zero in both cases, is involved there. In spite of this obvious fact, there is still a qualitative difference: for $p = 0$, there are UV and IR poles which enter with opposite signs and, for $p^2 = 0$ (but with $p \neq 0$ as a d-dimensional vector), there is a similar interplay of UV and collinear poles.

Now we follow the arguments presented in [24] and write down the following identity for (2.50), with $p = p_1$:

$$\begin{aligned}\int &\frac{\mathrm{d}^d k}{(k^2 - 2p_1\cdot k)k^2} \\ &= \int \frac{\mathrm{d}^d k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)} - \int \frac{\mathrm{d}^d k\, 2p_2\cdot k}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)k^2},\end{aligned} \tag{2.51}$$

where $p_2^2 = 0$ and $p_1 \cdot p_2 \neq 0$. We then evaluate the integrals on the right-hand side by means of (10.7) and (2.49), respectively, and obtain a zero value. This fact again exemplifies the consistency of our rules.

Thus we are going to systematically apply the properties of dimensionally regularized Feynman integrals in any situation, no matter where the external momenta are considered to be. Moreover, we will believe that these properties are also valid for more general Feynman integrals given by the dimensionally regularized version of (2.7) which can contain linear propagators.

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