Chapter 2
Composite Asymptotic Expansions: General Study

In this chapter, we present the general theory of CAsEs: their definition and their behavior with respect to the basic operations of addition, multiplication, division, differentiation, integration, composition and analytic continuation. In Sect. 2.4, we also link our CAsEs to the inner and outer expansions of the classical method of matching. Using these inner and outer expansions is also a good method for determining the coefficients of a composite expansion in practice, provided one can show the existence of a composite expansion independently.

Many problems solved using CAsEs have their origin in the real variable, so a purely "real" presentation might seem enough. However, an essential element in solving some problems is the Gevrey character of the CAsEs, which will be developed in Chap. 3. In order to obtain Gevrey properties, the only method known so far is to apply our "key-theorem" 4.1 of Ramis–Sibuya type, for which the complex framework is essential. Therefore the presentation here uses the complex variable; a presentation of the results in the real domain can be found in Fruchard/Schäfke [27, 28].

2.1 Notation

The notation $\mathbb{N}$ refers to the set of all natural numbers, including 0. The open disk of center 0 and radius r is denoted by $D(0,r)$. Given $\alpha < \beta \leq \alpha + 2\pi$ and $0 < r \leq \infty$, $S(\alpha,\beta,r)$ is the sector

$$S(\alpha,\beta,r) = \{x \in \mathbb{C}\,;\ 0 < |x| < r,\ \alpha < \arg x < \beta\}.$$

A sector is usually considered as part of the Riemann surface of the logarithm $\widetilde{\mathbb{C}^*}$. Since our sectors will always have an opening less than 2π, however, we consider them as subsets of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

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We say that a function f holomorphic and bounded on a sector S has an *asymptotic expansion at* $x = 0$ *(in the sense of Poincaré)* if there exists a formal series $\sum_{\nu\geq 0} a_\nu x^\nu$ and for all $N \in \mathbb{N}$ there is some constant C_N such that

$$|x|^{-N} \left| f(x) - \sum_{\nu=0}^{N-1} a_\nu x^\nu \right| \leq C_N$$

for all $x \in S$. In that case, we write

$$f(x) \sim \sum_{\nu\geq 0} a_\nu x^\nu,\ S \ni x \to 0.$$

We say that a function g holomorphic and bounded on an infinite sector S has an *asymptotic expansion at* $X = \infty$ if the function $f : x \mapsto g(1/x)$ has an asymptotic expansion at $x = 0$.

Given a sector $S = S(\alpha, \beta, r)$ and $\mu > 0$, $V(\alpha, \beta, r, \mu)$ denotes the union of the sector S and the disk $D(0, \mu)$:

$$V(\alpha, \beta, r, \mu) = \{x \in \mathbb{C}\ ;\ (|x| < r \text{ and } \alpha < \arg x < \beta) \text{ or } |x| < \mu\}. \tag{2.1}$$

For $\mu < 0$, we define

$$V(\alpha, \beta, r, \mu) = \{x \in \mathbb{C}\ ;\ -\mu < |x| < r \text{ and } \alpha < \arg x < \beta\}. \tag{2.2}$$

In the sequel, we call these sets *quasi-sectors*, for μ positive or negative (Fig. 2.1). For simplicity, we often only consider the case $\mu > 0$. The necessary changes in the case $\mu < 0$ are minor and will be indicated.

Given an infinite quasi-sector $V = V(\alpha, \beta, \infty, \mu)$, $\mathcal{G}(V)$ denotes the vector space of holomorphic functions g bounded in V and having an asymptotic expansion in the Poincaré sense at infinity *without constant term* $g(X) \sim \sum_{\nu\geq 1} g_\nu X^{-\nu}$, $V \ni X \to \infty$, i.e.

$$\forall N \in \mathbb{N} \quad \exists C_N > 0 \quad \forall X \in V, \qquad |X|^N \left| g(X) - \sum_{\nu=1}^{N-1} g_\nu X^{-\nu} \right| \leq C_N.$$

Let $\mathbf{T} : \mathcal{G}(V) \to \mathcal{G}(V)$ denote the operator which, to a function g, associates the function $\mathbf{T}g$ given by

$$\mathbf{T}g(X) = Xg(X) - g_1 \tag{2.3}$$

where $g_1 X^{-1}$ is the first term of the asymptotic expansion of g at infinity.

Sometimes V will be an annulus: $V = A(r, \infty) = \{x \in \mathbb{C}\ ;\ r < |x| < \infty\}$. In that case, the Banach space $\mathcal{G}(V)$ is therefore the space of functions holomorphic and bounded on V, tending to 0 as $X \to \infty$.

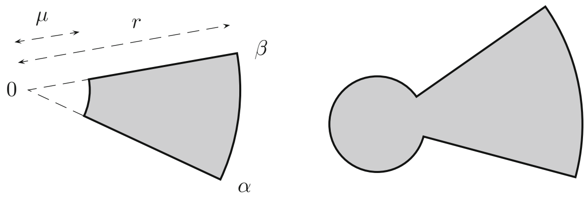

Fig. 2.1 Two examples of quasi-sectors, on the left $\mu < 0$, on the right $\mu > 0$

Given a number $r_0 > 0$, $\mathscr{H}(r_0)$ denotes the vector space of functions a holomorphic and bounded in the disk $D(0, r_0)$ of radius r_0 centered in 0. Similarly to $\mathbf{T}$, let $\mathbf{S} : \mathscr{H}(r_0) \to \mathscr{H}(r_0)$ be the operator which, to a function a, associates the function $\mathbf{S}a$ given by

$$\mathbf{S}a(x) = \frac{a(x) - a(0)}{x}. \tag{2.4}$$

On the expansions, the operators $\mathbf{S}$ and $\mathbf{T}$ act as a shift to the left: if $g(X) \sim \sum_{\nu \geq 1} g_\nu X^{-\nu}$, then $\mathbf{T}g(X) \sim \sum_{\nu \geq 1} g_{\nu+1} X^{-\nu}$ and if $a(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$, then $\mathbf{S}a(x) = \sum_{\nu=0}^{\infty} a_{\nu+1} x^\nu$.

2.2 Composite Formal Series

Definition 2.1. Let $V = V(\alpha, \beta, \infty, \mu)$ be an infinite quasi-sector (with μ positive or negative) and $r_0 > 0$. A *composite formal series* associated to V and $D(0, r_0)$ is an expression of the form

$$\widehat{y}(x, \eta) = \sum_{n \geq 0} \Big(a_n(x) + g_n\big(\tfrac{x}{\eta}\big) \Big) \eta^n, \tag{2.5}$$

where $a_n \in \mathscr{H}(r_0)$ and $g_n \in \mathscr{G}(V)$.

The functions a_n form the *slow part* of the composite formal series, and the g_n the *fast part*.

Remarks. 1. More precisely, a composite formal series is an element of $\big(\mathscr{H}(r_0) \times \mathscr{G}(V)\big)^{\mathbb{N}}$. As for classical formal series, we could therefore represent a composite formal series in the form $\sum_{n \geq 0} \big(a_n(x) + g_n(X)\big)\eta^n$—or in the form $\sum_{n \geq 0} \big(a_n(x), g_n(X)\big)\eta^n$—using three variables. We will however not have to consider functions of three variables asymptotic to a composite formal series: the three variables are always related by $x = \eta X$.

2. In this section, the symbol η introduced in the above definition is arbitrary, because we are just dealing with formal series. Later on, however, η will be a new independent variable and we will consider the behavior of functions as η tends to 0. In the context of singularly perturbed differential equations, as was the case in the introductory examples, η will be connected to ε by $\eta^p = \varepsilon$ with a suitable integer p.
3. The names "slow part" and "fast part" are motivated by their behavior with respect to differentiation d/dx. In general, differentiation does not change the η-order of a slow term. For a fast term, however, differentiation introduces a (large) factor $1/\eta$. For details, see below (Definition 2.3).

The "Differential" Algebra $\widehat{\mathscr{C}}(r_0, V)$. Let $\widehat{\mathscr{C}}(r_0, V)$ denote the vector space of composite formal series associated with V and $D(0, r_0)$, endowed with the canonical addition and multiplication by constants, the ultrametric distance

$$d(\widehat{y}_1, \widehat{y}_2) = 2^{-\mathrm{val}(\widehat{y}_1 - \widehat{y}_2)}, \text{ where } \mathrm{val}(\widehat{y}) = \min\{n \geq 0\ ;\ a_n \text{ or } g_n \not\equiv 0\} \tag{2.6}$$

and the topology induced by this distance.

Let $\mathbf{I}$ denote the canonical inclusion of $\mathscr{H}(r_0)$ in $\widehat{\mathscr{C}}(r_0, V)$ and, to simplify the notation, let the same letter denote the inclusion of $\mathscr{G}(V)$ in $\widehat{\mathscr{C}}(r_0, V)$. The symbol η denotes the real number, the function of (x, η) with value η and the composite formal series with a single term $a_1 = 1$.

Due to the fact that $g_n(X)$ has an asymptotic expansion when $X \to \infty$, the operators $\mathbf{S}$ and $\mathbf{T}$ endow $\widehat{\mathscr{C}}(r_0, V)$ with a structure of algebra as follows.

In order to define the product of two composite formal series $\widehat{y}$ and $\widehat{z}$, we expand the product term by term: if $\widehat{y}(x, \eta) = \sum_{n \geq 0} \left(a_n(x) + g_n\left(\frac{x}{\eta}\right)\right)\eta^n$ and $\widehat{z}(x, \eta) = \sum_{n \geq 0} \left(b_n(x) + h_n\left(\frac{x}{\eta}\right)\right) \eta^n$, then we put

$$\widehat{y} \cdot \widehat{z}(x, \eta) = \sum_{n \geq 0} \left(\sum_{\nu=0}^{n} \Big(\mathbf{I}(a_\nu)(x, \eta) + \mathbf{I}(g_\nu)(x, \eta)\Big) \cdot \cdot \Big(\mathbf{I}(b_{n-\nu})(x, \eta) + \mathbf{I}(h_{n-\nu})(x, \eta)\Big) \right) \eta^n .$$

This is a convergent series with respect to the topology of $\widehat{\mathscr{C}}(r_0, V)$ and it remains to define products of images by $\mathbf{I}$. The sets $\mathbf{I}\big(\mathscr{H}(r_0)\big)$ and $\mathbf{I}\big(\mathscr{G}(V)\big)$ are naturally equipped with a structure of algebra, hence we just have to define the product of an element $\mathbf{I}(a)(x, \eta)$, $a \in \mathscr{H}(r_0)$, and an element $\mathbf{I}(g)(x, \eta)$, $g \in \mathscr{G}(V)$. For this purpose, with the notation $a(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$ and $g(X) \sim \sum_{\nu > 0} g_\nu X^{-\nu}$, we observe that $a(x)g\left(\frac{x}{\eta}\right) = \left(a_0 + x\mathbf{S}a(x)\right) g\left(\frac{x}{\eta}\right)$ and $xg\left(\frac{x}{\eta}\right) = \left(g_1 + \mathbf{T}g\left(\frac{x}{\eta}\right)\right)\eta$. In other words, a composite asymptotic expansion of the product of functions $a(x)\,g\left(\frac{x}{\eta}\right)$ with respect to η can be obtained by

$$a(x)\,g\left(\tfrac{x}{\eta}\right) = a_0\,g\left(\tfrac{x}{\eta}\right) + g_1\,\mathbf{S}a(x)\eta + \mathbf{S}a(x)\,\mathbf{T}g\left(\tfrac{x}{\eta}\right)\eta. \tag{2.7}$$

By iterating this formula, we define with the convention $g_0 = 0$

$$\mathbf{I}(a)(x,\eta)\,\mathbf{I}(g)(x,\eta) = \sum_{\nu\geq 0}\left(g_\nu\,(\mathbf{S}^\nu a)(x) + a_\nu\,(\mathbf{T}^\nu g)\left(\tfrac{x}{\eta}\right)\right)\eta^\nu. \tag{2.8}$$

Remarks. 1. The above formula implies that the product of composite formal series is again a composite formal series. It also shows the need to have an asymptotic expansion of g; otherwise we could not define $\mathbf{T}^\nu g$. This is the main reason why we require the functions g_n in Definition 2.1 to have asymptotic expansions as $X \to \infty$.

2. Classical composite series [4,59] are a special case of our CASEs: the functions g_n decay exponentially. Recall that a function $g : J =]\mu, +\infty[\to \mathbb{R}$ has an *exponential decay* if there exist $C, A > 0$ such that

$$|g(X)| \leq C\exp(-AX) \text{ for all } X \in J.$$

A function g with exponential decay satisfies $g(X) = \mathcal{O}(X^{-N})$, $X \to +\infty$ for all integer N, so is *flat*: it admits the zero series as asymptotic expansion.

3. In the case of classical composite series, the slow part of a product depends only on the slow parts of the factors. This can be seen for example on (2.8): when all the g_ν are zero, the product of a slow term and a fast term generates only fast terms; cf. also Remark 2, p. 11 of [4]. In contrast, for our composite series, formula (2.8) shows that the product of a slow term and a fast term yields many slow terms, so that everything is intertwined.

Composite series are also compatible with the left composition.

Lemma 2.2. *Let $\widehat{y} \in \eta\widehat{\mathscr{C}}(r_0, V)$ be a composite formal series without constant term, i.e. with $a_0(x) \equiv 0$ and $g_0(X) \equiv 0$. Let $\widehat{P} \in \widehat{\mathscr{C}}(r_0, V)[[y,\eta]]$ be a formal series in two variables whose coefficients are composite formal series: $\widehat{P}(x,y,\eta) = \sum_{j,k\geq 0} p_{j,k}(x,\eta)y^j\eta^k$ with $p_{j,k} \in \widehat{\mathscr{C}}(r_0, V)$. Then the expression*

$$\widehat{\mathbf{Q}}(\widehat{y})(x,\eta) = \widehat{P}(x, \widehat{y}(x,\eta), \eta) := \sum_{j,k\geq 0} p_{j,k}(x,\eta)\widehat{y}(x,\eta)^j\eta^k$$

defines a composite formal series. Moreover, the application $\widehat{\mathbf{Q}} : \eta\widehat{\mathscr{C}}(r_0, V) \to \widehat{\mathscr{C}}(r_0, V)$ is well defined and 1-Lipschitz.

The proof is immediate, thanks to the convergence of the series defining $\widehat{\mathbf{Q}}(\widehat{y})(x,\eta)$ for the ultrametric topology induced by the distance (2.6).

Let us now define the derivative of a composite formal series. Since derivatives of functions in $\mathscr{H}(r_0)$ are not necessarily bounded and those of functions in $\mathscr{G}(V)$ have no longer necessarily an asymptotic expansion, this differentiation is somewhat more difficult to treat, although the formula is simple. In particular, it is required to reduce slightly the definition domains of the functions. In the real framework, it is not always possible to define the derivative of a composite formal series.

Definition 2.3. If $\widehat{y} \in \widehat{\mathscr{C}}\big(r_0, V(\alpha,\beta,\infty,\mu)\big)$ is a composite formal series given by

$$\widehat{y}(x,\eta) = \sum_{n\geq 0} \Big(a_n(x) + g_n\big(\tfrac{x}{\eta}\big)\Big)\,\eta^n,$$

such that the first fast term g_0 is identically zero, then its *derivative with respect to* x, $\frac{d\widehat{y}}{dx}$, is given by

$$\frac{d\widehat{y}}{dx}(x,\eta) = \sum_{n\geq 0} \Big(a_n'(x) + g_{n+1}'\big(\tfrac{x}{\eta}\big)\Big)\,\eta^n$$

This formula defines an element of $\widehat{\mathscr{C}}\big(\widetilde{r}_0, V(\widetilde{\alpha},\widetilde{\beta},\infty,\widetilde{\mu})\big)$ for any $\widetilde{r}_0 \in]0, r_0[$, $\alpha < \widetilde{\alpha} < \widetilde{\beta} < \beta$ and any $\widetilde{\mu} < \mu$.

Remark. A priori, the derivative of a composite formal series is only defined if the first fast term g_0 is identically zero. The operator $\eta\,\frac{d}{dx} = \frac{d}{dx}(\eta\ \cdot\ )$, however, is defined without condition on g_0.

Exercise 2.4. Give a detailed proof of Lemma 2.2.

2.3 Composite Expansions: Definition and Basic Properties

Until now, the objects considered were formal expressions. We now want to define the composite expansion of a function of two variables x and η. The simplest and most natural way would be to consider functions defined on a product of sectors in x and η. For some applications, however, it will be convenient that the x-domain contains a neighborhood of 0 of size proportional to $|\eta|$. For other applications, it will be necessary to remove a neighborhood of 0. That is why we introduced the quasi-sectors (2.1) and (2.2).

Definition 2.5. Let $V = V(\alpha,\beta,\infty,\mu)$ denote an infinite quasi-sector, let $S_2 = S(\alpha_2,\beta_2,\eta_0)$ denote a finite sector and let $\alpha_1 < \beta_1$ be such that $\alpha \leq \alpha_1 - \beta_2 < \beta_1 - \alpha_2 \leq \beta$. Let $y(x,\eta)$ be a holomorphic function defined for $\eta \in S_2$ and $x \in$

$V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$. Finally, let $\widehat{y}(x,\eta) = \sum_{n\geq 0} \Big(a_n(x) + g_n\big(\frac{x}{\eta}\big)\Big)\,\eta^n \in \widehat{\mathscr{C}}(r_0, V)$. We say that y *has* $\widehat{y}$ *as* CASE and we write

$$y(x,\eta) \sim \widehat{y}(x,\eta), \text{ as } S_2 \ni \eta \to 0, x \in V(\alpha_1, \beta_1, r_0, \mu\,|\eta|),$$

if, for any integer N, there exists a constant K_N such that for all $\eta \in S_2$ and all $x \in V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$

$$\left| y(x,\eta) - \sum_{n=0}^{N-1} \Big(a_n(x) + g_n\big(\tfrac{x}{\eta}\big)\Big)\,\eta^n \right| \leq K_N\, |\eta|^N . \tag{2.9}$$

Again, the functions a_n are the *slow part* of the CASE and the g_n are its *fast part.* The conditions on the angles α_j, β_j ensure the implication: if $\eta \in S_2$ and $x \in V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$ then $x/\eta \in V$.

Remarks. 1. In the case of an annulus $V = A(r,\infty)$, $r > 0$, there is no condition on the angles. A composite expansion $y(x,\eta) \sim \sum_{n\geq 0} \Big(a_n(x) + g_n\big(\frac{x}{\eta}\big)\Big)\eta^n$ is then another form of a monomial expansion introduced in Canalis-Durand/Mozo/Schäfke [8]. If we put $u = \eta/x$, then $\eta = xu$ and the function $z(x,u) = y(x,xu)$ is defined on a *sector in* xu, defined in [8] after Definition 3.4, i.e. the set of (x,u) such that $|x| < r_0, |u| < \min\big(\frac{1}{r}, \frac{\eta_0}{r_0}\big)$ and $\arg(xu) \in\,]\alpha_1, \beta_1[$, and admits the monomial expansion $z(x,u) \sim \sum_{n\geq 0} \big(a_n(x) + b_n(u)\big)(xu)^n$ defined in [8], Definition 3.6, with $b_n(u) = g_n(1/u)$.

2. For the sake of simplicity, we ask the functions a_n to be holomorphic in the whole disk $D(0,r_0)$, while y itself is only defined for $x \in V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$. In Sect. 2.4 we shall have to generalize the definition of CASE to a situation where the functions a_n are holomorphic on a more general domain containing 0, cf. the remark after Proposition 2.20.
3. A function $y(x,\eta)$ cannot have two different CASEs as $S_2 \ni \eta \to 0$ and $x \in V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$. Indeed, one has $\lim_{\eta\to 0} y(x,\eta) = a_0(x)$ for $x \in S(\alpha_1, \beta_1, r_0)$, hence the holomorphic function $a_0 \in \mathscr{H}(r_0)$ is uniquely determined, therefore also $a_0(0)$. We continue with $\lim_{\eta\to 0} y(\eta X, \eta) = a_0(0) + g_0(X)$, and so on. It should be noted that, to prove this uniqueness, only the property that $g_n(X)$ tends to 0 as $X \to \infty$ was used; this will be useful in Sect. 4.1.

It is immediate that CASEs are compatible with addition and scalar multiplication. For compatibility with the multiplication of expansions, the only less obvious point is to show that a product $a(x)g\big(\frac{x}{\eta}\big)$, $a \in \mathscr{H}(r_0)$, $g \in \mathscr{G}(V)$ has a CASE. This is a consequence of Formula (2.7) and of the fact that **S** and **T** are endomorphisms. Definition (2.8) was made so that we have $a(x)\, g\big(\frac{x}{\eta}\big) \sim I(a)(x,\eta)\, I(g)(x,\eta)$.

Composition. The CASEs are also compatible with the left and right composition by a holomorphic function, as expressed in the following proposition. Statement (a) concerns left composition by a function of three variables, but in the case of a CASE without term in η^0. Statement (b) treats the case of left composition without this restriction, but by a function of one variable only. These two statements are complementary. For the right composition, we have considered only functions of one variable x for the sake of simplicity, but it is possible to generalize the result to the case of a function φ of the two variables x and η, such that $\varphi(0,0) = 0$ and $\frac{\partial\varphi}{\partial x}(0,0) = 1$. The Gevrey version of this generalization is given in Sect. 4.6, Theorem 4.7. We have not formulated any statement concerning a change of the variable η because we do not need it.

Proposition 2.6. (a) *Let $P(x,z,\eta)$ be a holomorphic function defined when $|z| < r$, $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $x \in V(\alpha_1, \beta_1, r_0, \mu|\eta|)$ such that all coefficients P_n of the expansion $P(x,z,\eta) = \sum_{n\geq 0} P_n(x,\eta)z^n$ have a* CASE *$P_n(x,\eta) \sim \widehat{P}_n(x,\eta)$ as $S_2 \ni \eta \to 0$, $x \in V(\alpha_1, \beta_1, r_0, \mu|\eta|)$. Let $y(x,\eta) = \mathscr{O}(\eta)$ be a function having a* CASE *$\widehat{y}(x,\eta)$ as $S_2 \ni \eta \to 0$ and $x \in V(\alpha_1, \beta_1, r_0, \mu|\eta|)$ without terms in η^0. Suppose that $\sup_{x,\eta} |y(x,\eta)| < r$. Then the function $u : (x,\eta) \mapsto P(x, y(x,\eta), \eta)$ has the* CASE

$$\widehat{\mathbf{Q}}(\widehat{y})(x,\eta) = \sum_{n\geq 0} \widehat{P}_n(x,\eta)\widehat{y}(x,\eta)^n.$$

as $S_2 \ni \eta \to 0$, $x \in V$.

(b) *Consider a holomorphic function y defined for $\eta \in S_2$ and $x \in V$ where $V = V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$, with range in a bounded set $W \subset \mathbb{C}$ and having a* CASE *as $\eta \to 0$. Let f be a holomorphic function in a neighborhood of the closure of W. Then the function $z = f \circ y$ has a* CASE *as $S_2 \ni \eta \to 0$, $x \in V$.*

(c) *Let φ be a holomorphic function defined for $|x| < x_1$ such that $\varphi(0) = 0$ and $\varphi'(0) = 1$ and let $z = z(u,\eta)$ be a function with a* CASE *$\sum_{n\geq 0}\Big(a_n(u) + g_n\big(\frac{u}{\eta}\big)\Big)\eta^n$ as $S_2 \ni \eta \to 0$ and $u \in V(\alpha_1, \beta_1, r_0, \mu|\eta|)$, with $a_n \in \mathscr{H}(r_0)$ and $g_n \in \mathscr{G}(V)$. Then for all $\widetilde{\alpha}_1, \widetilde{\beta}_1$ with $\alpha_1 < \widetilde{\alpha}_1 < \widetilde{\beta}_1 < \beta_1$ and all $\tilde{\mu} < \mu$ there are $\widetilde{r}, \widetilde{\eta}_0 > 0$ such that the function $y : (x,\eta) \mapsto z\big(\varphi(x),\eta\big)$ has a* CASE *as $S(\alpha_2, \beta_2, \widetilde{\eta}_0) \ni \eta \to 0$ and $x \in V\big(\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{r}, \widetilde{\mu}|\eta|\big)$.*

Remark. In (a), the assumption "y bounded by r" is not essential: simply reduce the η-domain if it is not satisfied.

Proof. (a) For all $N \in \mathbb{N}^*$, the finite sum $\sum_{0\leq n\leq N-1} P_n(x,\eta)y(x,\eta)^n$ has a CASE (compatibility with product and sum). It remains to verify that a constant $L = L(N)$ exists such that the remainder is bounded by $L\,|\eta|^N$. This is evident from the assumptions.

(b) By modifying f and y if necessary, we may assume that $a_0(0) = 0$. Using a Taylor expansion, it suffices to prove that $f\big(a_0(x) + g_0(\frac{x}{\eta})\big)$ has a CASE.
Set $h(u, v) = f(u + v)$. It suffices to show that $h\big(a_0(x), g_0(\frac{x}{\eta})\big)$ has a CASE as η tends to 0. To show this, we write

$$h(x, y) = h(x, 0) + h(0, y) - h(0, 0) + xy\,k(x, y)$$

with some holomorphic function k of two variables x, y; therefore

$$\begin{aligned} h\big(a_0(x), g_0(\tfrac{x}{\eta})\big) =& h(a_0(x), 0) + h\big(0, g_0(\tfrac{x}{\eta})\big) - h(0, 0) \\ &+ a_0(x) g_0(\tfrac{x}{\eta})\, k\big(a_0(x), g_0(\tfrac{x}{\eta})\big). \end{aligned}$$

Since $a_0(0) = 0$, the product $a_0(x)g_0\big(\frac{x}{\eta}\big)$ is of the form $\mathscr{O}(\eta)$; we obtain a CASE for $h\big(a_0(x), g_0(\frac{x}{\eta})\big)$ by iterating this procedure.
Note that the leading term of the CASE of $f\big(y(x, \eta)\big)$ (without the reduction to $a_0(0) = 0$) has $f\big(a_0(x)\big)$ as slow part and $f\big(a_0(0) + g_0(\frac{x}{\eta})\big) - f\big(a_0(0)\big)$ as fast part.

(c) If $\widetilde{r}, \widetilde{\eta}_0$ are small enough, then $\varphi\big(V(\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{r}, \widetilde{\mu}|\eta|)\big) \subset V(\alpha_1, \beta_1, r_0, \mu|\eta|)$ if $\eta \in S_2, |\eta| < \widetilde{\eta}_0$. It suffices to show that $b\big(\frac{\varphi(x)}{\eta}\big)$ has a CASE, if b is in $\mathscr{G}(V)$. For that purpose, we introduce the functions h and ψ defined by $\frac{1}{\varphi(x)} - \frac{1}{x} = h(x)$ and $\psi(x, t) = x/\big(1 + txh(x)\big)$. The function h can be analytically continued to a function defined for $|x| < x_1$, still denoted h by abuse of notation, and $\psi(x, 0) = x, \psi(x, 1) = \varphi(x)$. The Taylor expansion of $b\big(\frac{\varphi(x)}{\eta}\big) = b\big(\frac{\psi(x,1)}{\eta}\big)$ with respect to t gives for all $N \in \mathbb{N}$

$$b\big(\tfrac{\varphi(x)}{\eta}\big) = \sum_{n=0}^{N-1} \tfrac{1}{n!} \tfrac{\partial^n}{\partial t^n} b\big(\tfrac{\psi(x,t)}{\eta}\big)\ |_{t=0} + \tfrac{1}{(N-1)!} \int_0^1 \tfrac{\partial^N}{\partial t^N} b\big(\tfrac{\psi(x,t)}{\eta}\big)\ |_{t=\tau}\ (1-\tau)^{n-1}\, d\tau.$$

Using the fact that $\frac{\partial}{\partial t}\big[f\big(\frac{\psi(x,t)}{\eta}\big)\big] = \eta h(x)(\Delta f)\big(\frac{\psi(x,t)}{\eta}\big)$ with the operator Δ defined by $(\Delta f)(X) = -X^2 f'(X)$, we obtain

$$\begin{aligned} b\big(\tfrac{\varphi(x)}{\eta}\big) =& \sum_{n=0}^{N-1} \tfrac{\eta^n}{n!}\, h(x)^n (\Delta^n b)\big(\tfrac{x}{\eta}\big) \\ &+ \tfrac{\eta^N}{(N-1)!}\, h(x)^N \int_0^1 (\Delta^N b)\big(\tfrac{\psi(x,\tau)}{\eta}\big)(1-\tau)^{n-1}\, d\tau \end{aligned} \tag{2.10}$$

and one can verify that the last term is $\mathscr{O}(\eta^N)$. The compatibility of CASEs with addition and multiplication then yields the existence of a CASE for $b\big(\frac{\varphi(x)}{\eta}\big)$. □

Differentiation. As was the case for composite formal series, CASEs are compatible with differentiation if the domains are slightly reduced and if the first fast term is identically zero.

Recall and complete the notation of Definition 2.5: let $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in \mathbb{R}$ with $\alpha \le \alpha_1 - \beta_2 < \beta_1 - \alpha_2 \le \beta$ and $\alpha_2 < \beta_2$, let $\eta_0, r_0 > 0$ and let $\mu \in \mathbb{R}$. Let $V = V(\alpha, \beta, \infty, \mu)$, $S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $V_1(\eta) = V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$. Moreover, let $\widetilde{r}_0 \in]0, r_0[$, $\widetilde{\mu} < \mu$, $\widetilde{\alpha}_1, \widetilde{\beta}_1$ be such that $\alpha_1 < \widetilde{\alpha}_1 < \widetilde{\beta}_1 < \beta_1$ and $\beta_2 - \alpha_2 < \widetilde{\beta}_1 - \widetilde{\alpha}_1$ and $\widetilde{\alpha}, \widetilde{\beta}$ such that

$$\alpha < \widetilde{\alpha} \le \widetilde{\alpha}_1 - \beta_2 < \widetilde{\beta}_1 - \alpha_2 \le \widetilde{\beta} < \beta.$$

Let $\widetilde{V} = V(\widetilde{\alpha}, \widetilde{\beta}, \infty, \widetilde{\mu})$ and $\widetilde{V}_1(\eta) = V(\widetilde{\alpha}_1, \widetilde{\beta}_1, \widetilde{r}_0, \widetilde{\mu}\,|\eta|)$.

Lemma 2.7. *Let $y(x,\eta)$ be a function defined for $\eta \in S_2$ and $x \in V_1$ such that $y(x,\eta) \sim \sum_{n\ge0}\left(a_n(x) + g_n\left(\frac{x}{\eta}\right)\right)\eta^n =: \widehat{y}(x,\eta) \in \widehat{\mathscr{C}}(r_0, V)$ as $S_2 \ni \eta \to 0$. Assume that $g_0(X) \equiv 0$. Then one has*

$$\frac{dy}{dx}(x,\eta) \sim \frac{d\widehat{y}}{dx}(x,\eta) = \sum_{n\ge0}\left(a_n'(x) + g_{n+1}'\left(\tfrac{x}{\eta}\right)\right)\eta^n$$

as $S_2 \ni \eta \to 0$ and $x \in \widetilde{V}_1(\eta)$, where $\dfrac{d\widehat{y}}{dx}(x,\eta) \in \widehat{\mathscr{C}}(\widetilde{r}_0, \widetilde{V})$.

Proof. Let $\delta = \min(|\eta|(\mu - \widetilde{\mu}), r_0 - \widetilde{r}_0)$ and, for $N \in \mathbb{N}$ arbitrary,

$$R_N(x,\eta) = y(x,\eta) - \sum_{n<N}\left(a_n(x) + g_n\left(\tfrac{x}{\eta}\right)\right)\eta^n. \tag{2.11}$$

One has $\frac{1}{\delta} = \mathscr{O}\left(\frac{1}{|\eta|}\right)$. The Cauchy formula for the derivative gives

$$\left|\frac{dR_{N+1}}{dx}(x,\eta)\right| = \left|\frac{1}{2\pi i}\int_{|u-x|=\delta}\frac{R_{N+1}(u,\eta)}{(u-x)^2}du\right| \le \frac{1}{\delta}\max_{|u-x|=\delta}|R_{N+1}(u,\eta)|,$$

which yields $\dfrac{dR_{N+1}}{dx}(x,\eta) = \mathscr{O}(|\eta|^N)$. Since by the Cauchy formula the functions a_N' and g_{N+1}' are bounded in $D(0,\widetilde{r}_0)$, resp. in $\widetilde{V}$, we deduce that $\dfrac{dR_N}{dx}(x,\eta) = \mathscr{O}\left(|\eta|^N\right)$. □

Remark. Lemma 2.7 is not valid in the real framework. For example, it is well-known that there are small functions with unbounded derivatives. We have the following result in real framework, however: if the derivative of a function with CASE also has a CASE, then formula (2.12) below implies that the CASE of the derivative can be obtained by differentiating the CASE of the function term by term.

Integration. Integration of a CASE does not always yield a CASE because of possible terms $\frac{1}{X}$ in the expansions of functions in $\mathscr{G}(V)$. If all these terms are absent, integration poses no problem, as the following statement shows.

Proposition 2.8. *Consider a* CASE $y(x,\eta) \sim \sum_{n\ge 0}\Big(a_n(x) + g_n\big(\frac{x}{\eta}\big)\Big)\eta^n$ *defined for* $\eta \in S_2$ *and* $x \in V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$, *such that all functions* g_n *satisfy* $g_n(X) = \mathscr{O}(X^{-2})$ *as* $X \to \infty$. *Let* $r \in S(\alpha_1, \beta_1, r_0)$.

Then the function $(x,\eta) \mapsto \displaystyle\int_r^x y(t,\eta)\,dt$ *has a* CASE. *More precisely, one has*

$$\begin{aligned}\int_r^x y(t,\eta)\,dt &\sim \widehat{Y}(x,\eta) - \widehat{Y}(r,\eta), \text{ where} \\ \widehat{Y}(x,\eta) &= A_0(x) + \sum_{n=1}^{\infty}\Big(A_n(x) + G_{n-1}\big(\tfrac{x}{\eta}\big)\Big)\eta^n\end{aligned} \tag{2.12}$$

with $A_n(x) = \displaystyle\int_r^x a_n(t)\,dt$ *and* $G_n(X) = -\displaystyle\int_X^\infty g_n(T)\,dT$.

Here we have identified $\widehat{Y}(r,\eta)$ with the formal series in which $G_{n-1}(\frac{r}{\eta})$ has been replaced by its asymptotic expansion as $\eta \to 0$. The proof is immediate: one has $A_n \in \mathscr{H}(r_0)$ and, by hypothesis, $G_n \in \mathscr{G}(V)$. Before stating the result in the general case, we introduce some notation. Let ℓ be an analytic function in the quasi-sector $V = V(\alpha, \beta, \infty, \mu)$ such that its derivative ℓ' has an asymptotic expansion at infinity starting with $\frac{1}{X}$:

$$\ell'(X) \sim \sum_{n\ge 1} c_n X^{-n} \text{ with } c_1 = 1.$$

One can choose e.g. $\ell(X) = \log(X - \gamma)$ with $\gamma \notin V$. If one wants a function having real values on the real axis, one may use $\ell(X) = \frac{1}{2}\log(X^2 + L^2)$ with L large enough. Observe that the expression $\ell\left(\frac{x}{\eta}\right)$ will *not* be bounded for $\eta \in S_2$ and x in some quasi-sector $V(\alpha, \beta, r_0, \mu\,|\eta|)$, but we have $\ell\left(\frac{x}{\eta}\right) = \mathscr{O}\left(|\log(|\eta|)|\right)$ there, because $\ell(X) = \log X + C + \mathscr{O}\left(X^{-1}\right)$ as $V \ni X \to \infty$ with some constant C.

The statement in the general case is as follows.

Proposition 2.9. *Given a* CASE $y(x,\eta) \sim \sum_{n\ge 0}\Big(a_n(x) + g_n\big(\frac{x}{\eta}\big)\Big)\eta^n$, *let* $\widehat{R}$ *denote the series of residues of the* $g_n(X)$: $\widehat{R}(\eta) = \sum_{n\ge 0} g_{n1}\eta^n$. *Let* $r \in S(\alpha_1, \beta_1, r_0)$. *Then one has* $\displaystyle\int_r^x y(t,\eta)\,dt \sim \widehat{Y}(x,\eta) - \widehat{Y}(r,\eta)$, *with*

$$\widehat{Y}(x,\eta)=\eta\widehat{R}(\eta)\Big(\ell\big(\tfrac{x}{\eta}\big)-\ell\big(\tfrac{r}{\eta}\big)\Big)+A_0(x)+\sum_{n=1}^{\infty}\Big(A_n(x)+H_{n-1}\big(\tfrac{x}{\eta}\big)\Big)\eta^n, \tag{2.13}$$

where $A_n(x)=\int_r^x a_n(t)\,dt$ *and* $H_n(X)=-\int_X^\infty \big(g_n(T)-g_{n1}\ell'(T)\big)\,dT.$

Again, we have identified $\widehat{Y}(r,\eta)$ with the formal expression obtained by replacing $\ell\big(\frac{r}{\eta}\big)$ and $H_n\big(\frac{r}{\eta}\big)$ by their expansions as η tends to 0.

Proof. The classical Borel–Ritt theorem (see below) provides a function $R(\eta)$ with $\widehat{R}(\eta)$ as asymptotic expansion. The difference $y(x,\eta)-R(\eta)\ell'\big(\frac{x}{\eta}\big)$ satisfies the condition of Proposition 2.8, so its integral has a CASE. The "generalized CASE" for y follows. □

We also have a statement similar to the classical Borel–Ritt theorem. The statement of the classical theorem is as follows: given any sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers and any sector $S(\alpha,\beta,\eta_0)$, there exists a function $a=a(\eta)$ defined and holomorphic on S and having the formal series $\sum_{n=0}^{\infty}a_n\eta^n$ as asymptotic expansion. The result is also true when $(a_n)_{n\in\mathbb{N}}$ is a sequence of bounded analytic functions of a complex variable x. In the case of our CASEs, the statement is as follows.

Lemma 2.10. *(Borel–Ritt) Let* $V=V(\alpha,\beta,\infty,\mu)$ *be an infinite quasi-sector* ($\mu>0$ *or* <0), $S_2=S(\alpha_2,\beta_2,\eta_0)$ *a finite sector,* $r_0>0$ *and let* $\alpha_1<\beta_1$ *be such that* $\alpha\le\alpha_1-\beta_2<\beta_1-\alpha_2\le\beta$. *Given a composite formal series* $\widehat{y}(x,\eta)=\sum_{n\ge0}\Big(a_n(x)+g_n\big(\frac{x}{\eta}\big)\Big)\eta^n\in\widehat{\mathscr{C}}(r_0,V)$, *there exists a holomorphic function* $y(x,\eta)$ *defined for* $\eta\in S_2$ *and* $x\in V(\alpha_1,\beta_1,r_0,\mu\,|\eta|)$ *such that* $y(x,\eta)\sim\widehat{y}(x,\eta)$ *as* $\eta\to0$.

Proof. Simply use the Borel–Ritt theorem for classical uniform asymptotic expansion twice: once for $\sum a_n(x)\eta^n$, once for $\sum g_n(X)\eta^n$. □

Exercise 2.11.

(a) Prove that the equation $y+\frac{\varepsilon}{y}=2x+2x^2$ in the complex domain has a unique solution $y=y(x,\varepsilon)$ holomorphic on the annulus $2\,|\varepsilon|^{1/2}\le|x|\le\frac12$ satisfying $y(x,\varepsilon)=2x+2x^2+o(1)$ as $\varepsilon\to0$ uniformly on this annulus.

(b) Using the properties of CASEs discussed in this section, show that $z(x,\eta)=y(x,\eta^2)$ has a CASE in the annulus $2\,|\eta|<|x|<\frac12$, as $\eta\to0$.

Exercise 2.12. Let $f=f(x,\eta)$ be a holomorphic function defined when $\eta\in S_2=S(\alpha_2,\beta_2,\eta_0)$ and $x\in V(\alpha_1,\beta_1,r_0,\mu|\eta|)$ and having a CASE

$$f(x,\eta)\sim\sum_{n\ge0}\Big(a_n(x)+g_n\big(\tfrac{x}{\eta}\big)\Big)\eta^n\in\widehat{\mathscr{C}}(r_0,V).$$

(a) Suppose that $a_0(0) \neq 0$ and $g_0 = 0$ identically. Prove that the function $1/f$ has a CASE, $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $x \in V(\alpha_1, \beta_1, r_0, \mu|\eta|)$ (with the same μ).
(b) Assume only $a_0(0) \neq 0$ and g_0 arbitrary. Prove that $1/f$ has a CASE, $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $x \in V(\alpha_1, \beta_1, r_0, \mu|\eta|)$ if $a_0(0) + g_0(X)$ does not vanish on the closure of $V(\alpha_1 - \beta_2, \beta_1 - \alpha_2, \mu, \infty)$.

Exercise 2.13. Give a detailed proof of Lemma 2.7.

Exercise 2.14. This example comes from Skinner's book [54]. Prove that the function z given by $z(x,\eta) = \frac{\eta}{x+2x^3+\eta}$ has a CASE for $\eta > 0$ and $x \in]\mu\eta, +\infty[$ for any $\mu > -1$. Compute an asymptotic expansion (containing a term in $\ln \eta$) for

$$F(\eta) = \int_0^1 z(x,\eta)dx.$$

Exercise 2.15. Suppose that $y(x,\eta)$ is a function holomorphic and bounded on the set of all complex (x,η) with $|\eta| < \eta_0$, $K|\eta| < |x| < L$, where η_0, K, L are some positive numbers. Using the Laurent decomposition of y, prove that y has a CASE as $\eta \to 0$, uniformly on the given annulus and that this CASE is actually convergent. Using this result, solve again the Exercises 2.11(b) and 2.14, except for the value of μ.

2.4 Composite Expansions and Matching

Our concept of composite expansion combines the classical asymptotic expansion in the sense of Poincaré of the form $y(x,\eta) \sim \sum_{n\geq 0} c_n(x)\eta^n$ and an expansion of the form $y(\eta X, \eta) \sim \sum_{n\geq 0} h_n(X)\eta^n$. The former expansions are called "outer", the latter are called "inner" expansions. These inner and outer expansions are central in the method of matched asymptotic expansion. Although CASEs are different from both, there are close links with inner and outer expansions.

On the one hand, we show that a function with a CASE also has an inner and an outer expansion, and that these two expansions have a common region of validity. In other words, a proof of existence of a CASE can provide a solid foundation for the method of matching.

On the other hand, the converse is true: if the method of matching is valid, i.e. if a function has inner and outer expansions with a common region of validity, and if moreover such expansions satisfy an additional property, then the function also has a CASE.

We emphasize that the results of this section, especially Proposition 2.17, are not new. They present the classical relations between inner, outer and uniform expansions adapted to our framework (see Chap. 7 for some more details).

The first result is the following.

Proposition 2.16. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a family of functions of* $\mathscr{H}(r_0)$ *and* $(g_n)_{n\in\mathbb{N}}$ *a family of functions of* $\mathscr{G}(V)$ *with* $V = V(\alpha,\beta,\infty,\mu)$. *Let* $a_n(x) = \sum_{m=0}^{\infty} a_{nm}x^m$ *and* $g_n(X) \sim \sum_{m>0} g_{nm}X^{-m}$ *denote their expansions. Suppose that*

$$y(x,\eta) \sim \sum_{n\geq 0} \Big(a_n(x) + g_n\big(\tfrac{x}{\eta}\big)\Big)\,\eta^n$$

as $S_2 \ni \eta \to 0$ *and* $x \in V(\alpha_1,\beta_1,r_0,\mu\,|\eta|)$ *in the sense of Definition 2.5.*

Then, for fixed $x \in S(\alpha_1,\beta_1,r_0)$, *one has*

$$y(x,\eta) \sim \sum_{n\geq 0} c_n(x)\eta^n \quad \text{as } S_2 \ni \eta \to 0, \tag{2.14}$$

where $c_n(x) = a_n(x) + \sum_{0\leq l\leq n-1} g_{l,n-l}\,x^{l-n}$. *Moreover, for all* $r > 0$, *this expansion is uniform with respect to* x *on all* $x \in S(\alpha_1,\beta_1,r_0)$ *such that* $|x| > r$.

Similarly, if $X \in V$ *and* α_3,β_3,η_3 *are such that* $\eta \in S(\alpha_3,\beta_3,\eta_3)$ *implies* $\eta \in S_2$ *and* $\eta X \in V(\alpha_1,\beta_1,r_0,\mu\,|\eta|)$, *then one has*

$$y(\eta X,\eta) \sim \sum_{n\geq 0} h_n(X)\eta^n \text{ as } S(\alpha_3,\beta_3,\eta_3) \ni \eta \to 0, \tag{2.15}$$

where $h_n(X) = g_n(X) + \sum_{0\leq l\leq n} a_{n-l,l}\,X^l$. *The expansion is uniform with respect to* X *on compact subsets of* V *satisfying the above condition.*

Remarks. 1. According to the literature, we will call the first expansion (2.14) *outer expansion* and the second (2.15) *inner expansion*. Each function c_n of the outer expansion may have a singularity at $x = 0$ but only a pole of order at most n; similarly each function h_n of the inner expansion has polynomial growth of order at most n as $X \to \infty$. Thus the *restraint index* in the sense of Wasow [62], Chap. VIII equals 1.

2. One can show that for every $\kappa \in]0,1[$, the outer expansion (2.14) is uniform on $|x| > |\eta|^\kappa$, and that the inner expansion (2.15) is uniform on $|X| < |\eta|^{-\kappa}$, which justifies the method of matched asymptotic expansions when a CASE exists. In both cases, we need to use $\frac{N}{1-\kappa}$ terms in order to obtain an approximation with remainder $\mathscr{O}\big(\eta^N\big)$. It is often preferable, however, to have uniform approximations throughout the domain instead of two different expansions on overlapping regions. Such uniform approximations seem indispensable if we want to obtain estimates of Gevrey kind.

3. In cases where the existence of a composite expansion for a function $y(x,\eta)$ can be shown indirectly, but the functions a_n and g_n are not yet known, one method for determining them is to apply the preceding proposition. For fixed non-zero x, one computes the outer expansion $y(x,\eta) \sim \sum_{n\geq 0} c_n(x)\eta^n$, then one eliminates the terms with negative powers of x to obtain the slow parts $a_n(x)$. Analogously, one computes the inner expansion $y(\eta X,\eta) \sim \sum_{n\geq 0} h_n(X)\eta^n$ and throws away the terms with non-negative powers of X, which gives $g_n(X)$. In practice, the calculation of inner and outer expansions often leads to recurrence equations for their coefficients. This allows to compute a_n, g_n without having to use the cumbersome formulas for multiplication of composite formal series.
In the case of singularly perturbed differential equations, as noted by Gautheron/Isambert [29], the computation of the inner expansion is more involved than the outer one. The latter only needs algebraic operations (if the Taylor expansions of the coefficients of the equation are known). The former, however, requires solving linear differential equations and choosing the constant of integration such that the solution has a certain asymptotic behavior; this introduces transcendence. For this reason, Isambert [32] calls these outer and inner expansions *algebraic* and *transcendental expansions*, respectively.

Proof of Proposition 2.16: Let $N \in \mathbb{N}^*$ be fixed and recall the notation (2.11). Furthermore, set

$$r_{lk}(X) = g_l(X) - \sum_{0<m<k} g_{lm} X^{-m}.$$

By hypothesis, there are positive constants C_N, A_{kn} and C_{lk} such that

$$\forall \eta \in S_2 \ \forall x \in V_{1,\eta} := V(\alpha_1, \beta_1, r_0, \mu|\eta|) \qquad |R_N(x,\eta)| \leq C_N |\eta|^N,$$

$$\forall x \in V_{1,\eta} \qquad \Big| a_k(x) - \sum_{l<n} a_{kl} x^l \Big| \leq A_{kn} |x|^n \tag{2.16}$$

and

$$\forall X \in V \qquad |r_{lk}(X)| \leq C_{lk} |X|^{-k}. \tag{2.17}$$

An elementary calculation gives

$$\begin{aligned} y(x,\eta) - \sum_{n<N} c_n(x)\eta^n &= R_N(x,\eta) + \sum_{l<N} g_l\big(\tfrac{x}{\eta}\big)\eta^l - \sum_{0<n<N} \Big(\sum_{l<n} g_{l\,n-l} x^{l-n} \Big) \eta^l \\ &= R_N(x,\eta) + \sum_{l<N} r_{l\,N-l}\big(\tfrac{x}{\eta}\big)\eta^l, \end{aligned} \tag{2.18}$$

hence, as $|x| > r$,

$$\Big| y(x,\eta) - \sum_{n<N} c_n(x)\eta^n \Big| \leq \Big(C_N + \sum_{l<N} C_{l\,N-l}\, r^{l-N} \Big) |\eta|^N. \tag{2.19}$$

Similarly, one has

$$\begin{aligned} y(\eta X,\eta) - \textstyle\sum_{n<N} h_n(X)\eta^n &= R_N(\eta X,\eta) + \sum_{n<N}\Big(a_n(\eta X) - \sum_{l\le n} a_{n-l\,l} X^l\Big)\eta^n \\ &= R_N(\eta X,\eta) + \sum_{k<N}\Big(a_k(\eta X) - \sum_{l<N-k} a_{kl}\,\eta^l X^l\Big)\eta^k \end{aligned}$$

therefore, for all $R > 0$ and for $|X| \le R$,

$$\Big|y(\eta X,\eta) - \sum_{n<N} h_n(X)\eta^n\Big| \le \Big(C_N + \sum_{k<N} A_{k\,N-k} R^{N-k}\Big)|\eta|^N. \tag{2.20}$$

□

Conversely, one has the following statement.

Proposition 2.17. *Let y be a function defined for $\eta \in S_2 = S(\alpha_2,\beta_2,\eta_0)$ and $x \in V(\eta) = V(\alpha_1,\beta_1,r_0,\mu\,|\eta|)$. Assume that there are real numbers a, b, κ with $0 < a < b$ and $0 < \kappa < 1$, and for each $n \in \mathbb{N}$ a function c_n, $c_n(x) = P_n\big(\frac{1}{x}\big) + a_n(x)$, P_n polynomial without constant term, $a_n \in \mathscr{H}(r_0)$ and a function $h_n = Q_n + g_n$, Q_n polynomial and $g_n \in \mathscr{G}(V)$, $V = V(\alpha,\beta,\infty,\mu)$, $\alpha \le \alpha_1 - \beta_2 < \beta_1 - \alpha_2 \le \beta$, with the following properties.*

Assumption 1. *For all $N \in \mathbb{N}$, there is a constant $C > 0$ such that*

$$\Big|y(x,\eta) - \sum_{n=0}^{N-1} c_n(x)\eta^n\Big| \le C\,|\eta|^{N(1-\kappa)} \tag{2.21}$$

for all $\eta \in S_2$ and all $x \in V(\eta)$ with $|x| > a|\eta|^\kappa$ and

$$\Big|y(\eta X,\eta) - \sum_{n=0}^{N-1} h_n(X)\eta^n\Big| \le C\,|\eta|^{N\kappa} \tag{2.22}$$

for all $\eta \in S_2$ and all $X \in V$ such that $\eta X \in V(\eta)$ with $|X| < b|\eta|^{\kappa-1}$.

Assumption 2. *For any $n \in \mathbb{N}$, the polynomials P_n and Q_n have degree less than $n+1$.*

Then y has a CASE *for $\eta \in S_2$ and $x \in V(\eta)$; precisely*

$$y(x,\eta) \sim \sum_{n=0}^{\infty}\Big(a_n(x) + g_n\big(\tfrac{x}{\eta}\big)\Big)\eta^n.$$

Remarks. 1. As there is a common region for expansions (2.21) and (2.22), these expansions are necessarily consistent, as shown in the proof, cf. (2.25).

2. In general we cannot get better than $|\eta|^{N(1-\kappa)}$ in the remainder of (2.21) and $|\eta|^{N\kappa}$ in that of (2.22), as the first neglected terms have this size when P_N and Q_N are of degree N.
3. This statement is a special case of a general theorem of Eckhaus' book [17]. In the classical method of matched asymptotic expansions, one first establishes inner and outer expansions on growing domains as $\eta \to 0$ having a nonempty intersection. Then one constructs so-called "composite" expansions of which our CASEs are an example, cf. also Chap. 7.

Proof of Proposition 2.17: Let $c_n(x) = \sum_{m=-n}^{+\infty} c_{nm}x^m$ and M_{nN} be the largest integer M such that $M\kappa + n \le N(1-\kappa)$. Then (2.21) implies that for any $N \in \mathbb{N}$ there exists $C_2 > 0$ such that

$$\left| y(x,\eta) - \sum_{n=0}^{N-1} \sum_{m=-n}^{M_{nN}} c_{nm}x^m\eta^n \right| \le C_2\, |\eta|^{N(1-\kappa)}$$

as $\eta \in S_2$ and $x \in V(\eta)$, $a\,|\eta|^\kappa < |x| < b\,|\eta|^\kappa$. For any integer S, we can find a constant C_3 such that

$$\left| y(x,\eta) - \sum_{n\ge 0, m\ge -n, m\kappa+n<S} c_{nm}x^m\eta^n \right| \le C_3\, |\eta|^S \tag{2.23}$$

as $\eta \in S_2$ and $x \in V(\eta)$, $a\,|\eta|^\kappa < |x| < b\,|\eta|^\kappa$.

Similarly, noting $h_n(X) \sim \sum_{m=-n}^{+\infty} z_{nm}X^{-m}$, one can find replacing X by $\frac{x}{\eta}$ that for any integer S there exists a constant C_4 such that

$$\left| y(x,\eta) - \sum_{p\ge 0, q\ge -p, -q(\kappa-1)+p<S} z_{pq}x^{-q}\eta^{p+q} \right| \le C_4\, |\eta|^S \tag{2.24}$$

as $\eta \in S_2$ and $x \in V(\eta)$, $a\,|\eta|^\kappa < |x| < b\,|\eta|^\kappa$.

As (2.23) and (2.24) uniquely determine the coefficients c_{nm} and z_{pq}, they must coincide, i.e. $c_{nm} = z_{n+m,-m}$ for any $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, $m \ge -n$. One thus has the formal equality

$$\sum_{n=0}^{\infty} \widehat{h}_n\big(\tfrac{x}{\eta}\big)\eta^n = \sum_{n=0}^{\infty} \widehat{c}_n(x)\eta^n, \tag{2.25}$$

where $\widehat{h}_n$ and $\widehat{c}_n$ denote the series associated with h_n and c_n.

Now consider the sum $Y_N(x,\eta) = \sum_{n=0}^{N} \left(a_n(x) + g_n\left(\frac{x}{\eta}\right)\right)\eta^n$. When $a\,|\eta|^\kappa < |x|$, we find with $\left| g_n\left(\frac{x}{\eta}\right) - \sum_{q=1}^{N-n-1} z_{nq} x^{-q}\eta^q \right| \le C_5\,|\eta|^{(N-n)(1-\kappa)}$ and hence with $z_{nq} = c_{n+q,-q}$

$$\left|y(x,\eta) - Y_N(x,\eta)\right| \le \left| y(x,\eta) - \sum_{n=0}^{N-1}\left(a_n(x) + \sum_{m=1}^{n} c_{n,-m}x^{-m}\right)\eta^n \right| + C_6\,|\eta|^{N(1-\kappa)}.$$

This implies

$$\begin{aligned}\left|y(x,\eta) - Y_N(x,\eta)\right| &\le \left|y(x,\eta) - \textstyle\sum_{n=0}^{N-1} c_n(x)\eta^n\right| + C_6\,|\eta|^{N(1-\kappa)} \\ &\le C_7\,|\eta|^{N(1-\kappa)}\end{aligned} \tag{2.26}$$

as $\eta \in S_2$, $x \in V(\eta)$, $a\,|\eta|^\kappa < |x|$.

Using the expansions of the a_n, we similarly find that

$$|y(x,\eta) - Y_N(x,\eta)| \le C_8\,|\eta|^{N\kappa}$$

also when $\eta \in S_2$, $x \in V(\eta)$, $|x| < b\,|\eta|^\kappa$. Together with (2.26), this shows that for all N, there is a constant C_9 such that for all $\eta \in S_2$ and $x \in V(\eta)$ one has $|y(x,\eta) - Y_N(x,\eta)| \le C_9\,|\eta|^{N\lambda}$ with $\lambda = \min(\kappa, 1-\kappa)$.

The statement to be proven corresponds to $|\eta|^N$ instead of $|\eta|^{N\lambda}$ in this last inequality. It is obtained in two steps. On the one hand, this last assertion can also be written: there is C_{10} with $|y(x,\eta) - Y_S(x,\eta)| \le C_{10}\,|\eta|^N$, if $S\lambda > N$. On the other hand the fact that all functions $a_n(x)$ and $g_n\left(\frac{x}{\eta}\right)$ are bounded on all x, η in question implies that there is a constant C_{11} such that $|Y_S(x,\eta) - Y_N(x,\eta)| \le C_{11}\,|\eta|^N$. □

Exercise 2.18. Add details for Remark 2 after Proposition 2.16.

2.5 Continuation of Composite Expansions

In connection with the inner and outer expansions of the method of matching, we also have two results of continuation of CASEs, which will be very useful for solutions of differential equations.

The first result says essentially that a function with a CASE for x in a quasi-sector, whose inner expansion exists on a larger quasi-sector, admits the CASE also on the larger quasi-sector. The precise result is as follows.

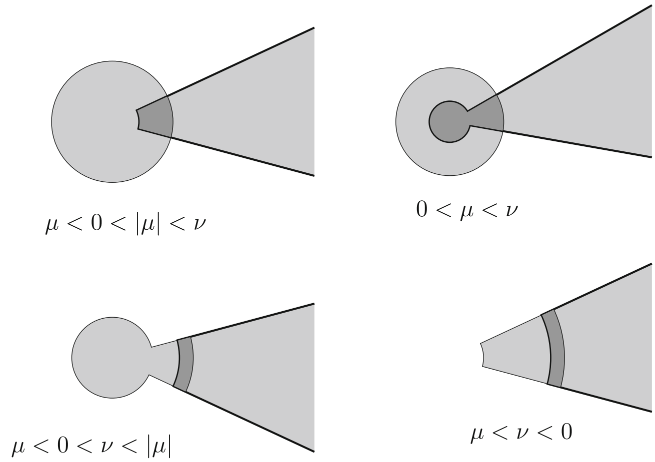

Fig. 2.2 Some domains V and Ω with different signs of μ, $|\mu| - \nu$ and ν. *Bold line*: the boundary of V, *thin line*: that of Ω, their intersection in *dark gray*

Proposition 2.19. *Let y be a function defined for $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $x \in V_1(\eta) = V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$ and having a* CASE $\sum_{n\geq 0}\Big(a_n(x) + g_n\big(\frac{x}{\eta}\big)\Big)\eta^n$, *as $S_2 \ni \eta \to 0$ and $x \in V_1(\eta)$, with $a_n \in \mathscr{H}(r_0)$ and $g_n \in \mathscr{G}(V)$, where $V = V(\alpha, \beta, \infty, \mu)$, $\alpha = \alpha_1 - \beta_2$ and $\beta = \beta_1 - \alpha_2$. Let $\nu > \mu$. In the case where $\nu > |\mu|$, set $\Omega = D(0, \nu)$, otherwise set $\Omega = V(\alpha, \beta, -\mu + \gamma, \nu)$ with $\gamma > 0$ arbitrarily small (Fig. 2.2).*

Assume that the function $Y : (X, \eta) \mapsto y(\eta X, \eta)$ can be analytically continued on $\Omega \times S_2$ and that it has an asymptotic expansion $Y(X, \eta) \sim \sum_{n=0}^{\infty} h_n(X)\eta^n$ as η tends to 0, uniformly on Ω.

Then y can be analytically continued to the set of all (x, η) with $\eta \in S_2$ and with $x \in V(\alpha_1, \beta_1, r_0, \nu\,|\eta|)$ and has a CASE *there as $\eta \to 0$.*

Remarks. 1. The domain Ω has been chosen bounded, with $\Omega \cap V \neq \emptyset$ and $V \cup \Omega = V(\alpha, \beta, \infty, \nu)$. Signs of μ and ν are arbitrary. We will use this result particularly in the case $\mu < 0 < \nu$.

2. The assumption on the domain and the asymptotic expansion of Y may be slightly weakened (the domain with respect to X may depend on the argument of η), but the version presented is sufficient for our applications to differential equations.
3. It is possible to show this result using Propositions 2.16 and 2.17, but we prefer to present an independent proof. One reason for this choice is that this proof will serve for the Gevrey analog Proposition 3.8. In contrast to this, we have no Gevrey analog of Proposition 2.17.

Proof of Proposition 2.19: The expansion of Y in the assumption and the inner expansion corresponding to the CASE of y given by Proposition 2.16 coexist on some open region, hence coincide by the uniqueness of an asymptotic expansion. Thus, the functions $h_n(X)$ of the assumption are necessarily the analytic continuations of the coefficients of this inner expansion.

The hypothesis implies that $y(x,\eta)$ can be analytically continued to the set of all (x,η) such that $\eta \in S_2$ and $x \in \widetilde{V}_1(\eta) = V(\alpha_1, \beta_1, r_0, \nu\,|\eta|)$, precisely by putting $y(x,\eta) = Y\big(\frac{x}{\eta}, \eta\big)$. We now use again the notation introduced in the proof of Proposition 2.16. We have to show that the remainder $R_N(x,\eta)$ given by (2.11) is bounded by a constant times $|\eta|^N$, also for $x \in \widetilde{V}_1(\eta) \setminus V_1(\eta)$. The assumption on Ω ensures that for all $\eta \in S_2$ and all $x \in \widetilde{V}_1(\eta) \setminus V_1(\eta)$, one has $x/\eta \in \Omega$. By hypothesis, there exists D_N such that

$$\Big| y(x,\eta) - \sum_{n<N} h_n\big(\tfrac{x}{\eta}\big)\eta^n \Big| \le D_N |\eta|^N .$$

Now, the equality above (2.20) can be written

$$R_N(x,\eta) = y(x,\eta) - \sum_{n<N} h_n\big(\tfrac{x}{\eta}\big)\eta^n - \sum_{k<N}\Big(a_k(x) - \sum_{l<N-k} a_{kl}x^l\Big)\eta^k .$$

Moreover, modifying the constants A_{kn} if necessary, inequality (2.16) is valid for all $x \in D(0, r_0)$, hence in particular for $x \in \widetilde{V}_1(\eta) \setminus V_1(\eta)$. This shows that for all $\eta \in S_2$ and for all $x \in \widetilde{V}_1(\eta) \setminus V_1(\eta)$

$$|R_N(x,\eta)| \le \Big(D_N + \sum_{k<N} A_{k\,N-k} M^{N-k}\Big)|\eta|^N \tag{2.27}$$

with $M = \sup_{X\in\Omega} |X| = -\mu + \delta$ or ν. □

The second result concerns outward continuation.

Proposition 2.20. *Let $0 < r_0 < \widetilde{r}_0$ and let y be a function defined for $\eta \in S_2 = S(\alpha_2, \beta_2, \eta_0)$ and $x \in \widetilde{V}_1(\eta) = V(\alpha_1, \beta_1, \widetilde{r}_0, \mu\,|\eta|)$. Suppose that y has a* CASE *$\sum_{n\ge 0}\big(a_n(x) + g_n\big(\frac{x}{\eta}\big)\big)\eta^n$, as $S_2 \ni \eta \to 0$ and $x \in V_1(\eta) = V(\alpha_1, \beta_1, r_0, \mu\,|\eta|)$, with $a_n \in \mathscr{H}(r_0)$ and $g_n \in \mathscr{G}(V)$, $V = V(\alpha, \beta, \infty, \mu)$ such that $\alpha \le \alpha_1 - \beta_2 < \beta_1 - \alpha_2 \le \beta$.*

Assume moreover that y has an asymptotic expansion $y(x,\eta) \sim \sum_{n=0}^{\infty} c_n(x)\eta^n$ as η tends to 0, uniformly for $x \in V(\alpha_1, \beta_1, r_0 - \gamma, \widetilde{r}_0)$ with $\gamma > 0$ arbitrarily small.

Then (2.9) is satisfied for all $\eta \in S_2$ and all $x \in \widetilde{V}_1(\eta)$.

Remark. By abuse of notation, we say that y has a CASE for $\eta \in S_2$ and $x \in \widetilde{V}_1(\eta)$, although the functions a_n are not necessarily defined on the whole disk $D(0, \widetilde{r}_0)$.

Proof. First, we can use Proposition 2.16 on the quasi-sector $V(\alpha_1, \beta_1,\ r_0 - \gamma, r_0)$, and by comparing (2.14) with the second hypothesis, we obtain that the functions c_n

of the hypothesis are analytic continuations of those of the proposition. Therefore we can also continue the functions a_n analytically on $D(0, r_0) \cup V(\alpha_1, \beta_1, r_0-\gamma, \widetilde{r}_0)$.

It remains to estimate R_N for $x \in \widetilde{V}_1(\eta) \setminus V_1(\eta)$. By (2.18), one has

$$R_N(x, \eta) = y(x, \eta) - \sum_{n<N} c_n(x)\eta^n - \sum_{l<N} r_{l\,N-l}\left(\tfrac{x}{\eta}\right)\eta^l. \tag{2.28}$$

and by (2.17) $|r_{l\,N-l}(X)| \le C_{l\,N-l}|X|^{l-N}$. By hypothesis, there are $A_N > 0$ such that

$$|y(x, \eta) - \sum_{n<N} c_n(x)\eta^n| \le A_N|\eta|^N$$

for all $\eta \in S_2$ and all $x \in V_1(\eta) \setminus \widetilde{V}_1(\eta)$. We then obtain $|R_N(x, \eta)| \le C|\eta|^N$ with $C = A_N + \sum_{l<N} C_{l\,N-l} r_0^{l-N}$. □

Exercise 2.21. Use Propositions 2.16 and 2.17 to prove Proposition 2.19.

2.6 Quotients of CASEs

Here we investigate under which conditions the multiplicative inverse of a function with a CASE has a CASE.

If the first slow term a_0 is non zero at $x = 0$, then this inverse has a CASE thanks to composition with the function $f \mapsto 1/f$, see Exercise 2.12. Here we investigate a more general situation.

Let $y = y(x, \eta)$ be a function defined and analytic for $\eta \in S = S(-\delta, \delta, \eta_0)$ and $x \in V(\alpha, \beta, r_0, \mu|\eta|)$, having a CASE $y(x, \eta) \sim \sum_{n\ge 0}\left(a_n(x) + g_n\left(\tfrac{x}{\eta}\right)\right)\eta^n$ as $\eta \to 0$. We propose a slightly more general statement, more useful in practice: it establishes conditions under which there exists $k \in \mathbb{N}$ such that the function $(x, \eta) \mapsto \eta^k/y(x, \eta)$ has a CASE.

By Proposition 2.16, y has an inner expansion

$$y(\eta X, \eta) \sim \sum_{n\ge 0} h_n(X)\eta^n \quad \text{as } S_2 \ni \eta \to 0,$$

uniformly with respect to X on compact subsets of $S(\alpha_1, \beta_1, \infty)$ where $\alpha_1 = \alpha - \delta$, $\beta_1 = \beta + \delta$ and

$$h_n(X) = g_n(X) + \sum_{0\le l\le n} a_{l,n-l} X^{n-l}. \tag{2.29}$$

For all n, let $\displaystyle\sum_{m=-n}^{+\infty} h_{nm}X^{-m}$ denote the asymptotic expansion of h_n at infinity and let $\mathbf{v}_n = \operatorname{val}(h_n)$ denote the least integer $m \ge -n$ such that $h_{nm} \ne 0$. If h_n is flat, we put $\mathbf{v}_n = \mathrm{val}(h_n) = +\infty$. We say that y is *degenerate*, if it is flat or if there exists $N \in \mathbb{N}$ such that $h_0 = \ldots = h_{N-1} = 0$ and $h_N \neq 0$ is flat. If y is nondegenerate, let $C(y)$ denote the pair (N, M) with $N \in \mathbb{N}$ such that $h_0 = \ldots = h_{N-1} = 0$, $h_N \neq 0$ and $M = \mathrm{val}(h_N) \geq -N$.

Proposition 2.22. *With the previous notation, the following three conditions are equivalent.*

(a) *There exist $k \in \mathbb{N}$, $\widetilde{\eta}_0 < \eta_0$, $\widetilde{r}_0 < r_0$ and $\widetilde{\mu} \leq \mu$ such that the function $(x, \eta) \mapsto \eta^k / y(x, \eta)$ has a* CASE *as $\eta \to 0$ in $\widetilde{S} = S(-\delta, \delta, \widetilde{\eta}_0)$ and $x \in V(\alpha, \beta, \widetilde{r}_0, \widetilde{\mu})$.*
(b) *y is non-degenerate and, if $C(y) = (N, M)$, one has val$(h_n) \geq M - n + N$ for all $n \geq N$.*
(c) *There exist $k \in \mathbb{N}$, $\ell \in \mathbb{Z}$ such that the function $(x, \eta) \mapsto \eta^{-k} x^\ell y(x, \eta)$ has a* CASE *whose first slow coefficient $\widetilde{a}_0$ satisfies $\widetilde{a}_0(0) \neq 0$.*

Remarks. 1. The second condition in (b) can also be written in terms of the outer expansion and the expansions of its coefficients. This is a consequence of the relation (2.25).

2. Graphically, the second condition in (b) means that the points with coordinates (n, m) such that $h_{nm} \neq 0$ (the "support" of the inner expansion) are all in the quadrant on the right of the vertical line and above the line of slope -1 passing through $C(y)$, see Fig. 2.3. Since h_n has a polynomial part of degree at most n, we already know that this support is in the quadrant on the right of the axis and above the second bisector. The change of variable $y \to z : (x, \eta) \mapsto \eta^{-k} x^\ell y(x, \eta)$ induces a shift of just $-C(y) = (-N, -M)$ on the supports, with $N = k - \ell$ and $M = \ell$.
3. The proof also provides a procedure to calculate the CASE for $\eta^k / y(x, \eta)$. Using the above shift, the situation is reduced to the case where y has a first slow term a_0 non-zero at $x = 0$. Thus we obtain the CASE by left composition with the function $u \mapsto 1/u$.

Proof of Proposition 2.22: We show the implications (b)⇒(c)⇒(a)⇒(b). Suppose that condition (b) is satisfied.

If $M < 0$, first consider $z(x, \eta) = \left(\frac{\eta}{x}\right)^{-M} y(x, \eta)$. As a product of two functions having CASEs, z has a CASE on the same domain as y. The corresponding inner expansion is that of $X^M y(\eta X, \eta)$ and satisfies therefore a condition similar to (b) with $(N, 0)$ instead of (N, M).

If $M > 0$, consider $z(x, \eta) = x^M y(x, \eta)$. As before, z has a CASE and the corresponding inner expansion is that of $\eta^M X^M y(\eta X, \eta)$, thus satisfies a condition similar to (b) with $(N + M, 0)$ instead of (N, M). Therefore both cases $M > 0$ and $M < 0$ can be reduced to the case $M = 0$.

If $M = 0$, then formula (2.29) and the condition on the h_n show that $a_{sm} = 0$ for $0 \leq s < N$ and $m \geq 0$, and that $a_{N0} \neq 0$. Since the functions a_s are analytic, this implies $a_s = 0$ for $s = 0, \ldots, N - 1$. Then the function $\widetilde{y} : (x, \eta) \mapsto \eta^{-N} y(x, \eta)$ has also a CASE on the same domain as y and satisfies the condition (c). To sum up, in the three cases y satisfies condition (c) with $k = N + M$ and $\ell = M$.

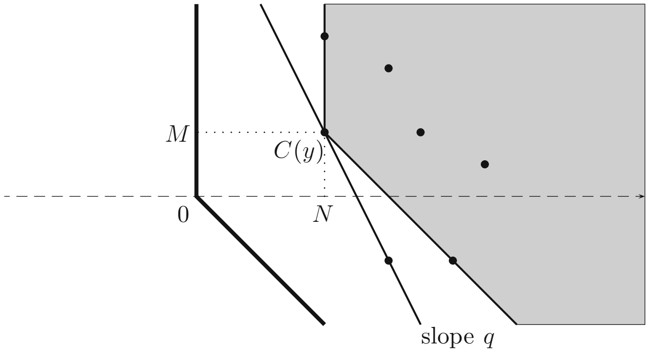

Fig. 2.3 In *gray*, the part of the plane containing the support of the inner expansion of y. In *bold*, the boundary of the analogous part for z

Now suppose the condition (c) satisfied and set $z(x,\eta) = \eta^{-k}x^{\ell}y(x,\eta)$. For $\widetilde{r}_0$ and $\widetilde{\eta}_0$ small enough and $\widetilde{\mu} \le \mu$ suitable, the function z does not vanish when $\eta \in \widetilde{S} = S(-\delta,\delta,\widetilde{\eta}_0)$ and $x \in V(\alpha,\beta,\widetilde{r}_0,\widetilde{\mu})$. Proposition 2.6(b) applies with the function $f : u \mapsto 1/u$ and we deduce that $\frac{1}{z}$ has a CASE. In the case where $\ell \ge 0$, the function $(x,\eta) \mapsto x^{\ell}/z(x,\eta)$ has therefore a CASE, which proves (a). In the case $\ell < 0$, the function $(x,\eta) \mapsto \left(\frac{x}{\eta}\right)^{\ell}/z(x,\eta)$ has a CASE, which gives (a) with $k-\ell$ instead of k.

Finally, suppose that condition (a) is satisfied, let $\widetilde{y}(x,\eta) = \eta^k/y(x,\eta)$ and let $\widetilde{h}_n$ denote the coefficients of the inner expansion of $\widetilde{y}$. Since $(y\widetilde{y})(x,\eta) = \eta^k$, there is a first term h_r which is not identically zero in the inner expansion of y and a first term $\widetilde{h}_s$ for $\widetilde{y}$, with $r+s=k$ and $h_r\widetilde{h}_s = 1$. Each of these functions is of polynomial growth as $X \to \infty$, so none can be flat. Thus the two functions y and $\widetilde{y}$ are nondegenerate. Set $(N,M) = C(y)$ and $(\widetilde{N},\widetilde{M}) = C(\widetilde{y})$.

For a proof by contradiction, suppose there exists $n > N$ such that $\mathrm{val}(h_n) < M - n + N$. Let $q = \min\left\{\frac{\mathrm{val}(h_s)-M}{s-N}\ ;\ s > N\right\} < -1$ and $\mathscr{M} = \{s \ge N\ ;\ \mathrm{val}(h_s) - sq = M - Nq\}$; it is a set of cardinal at least 2 (containing at least N and some s for which the minimum q is attained) and finite (since $\mathrm{val}(h_s) \ge -s$).

If $\widetilde{y}$ satisfies condition (b), we set $\widetilde{q} = -1$, otherwise $\widetilde{q}$ is the analog of q for y. Switching $\widetilde{y}$ and y if necessary, we can assume without loss of generality that $q \le \widetilde{q}$. Let then $K = \min\{\mathrm{val}(\widetilde{h}_s) - sq\ ;\ s > \widetilde{N}\}$ and $\mathscr{N}$ denote the finite and nonempty set of all $s \in \mathbb{N}$ such that $\mathrm{val}(\widetilde{h}_s) - sq = K$. Note that $\mathscr{N} = \{\widetilde{N}\}$ if $\widetilde{q} > q$ and that the cardinal of $\mathscr{N}$ is at least 2 if $\widetilde{q} = q$.

Recall that the minimum of $\mathscr{M}$ is $n_1 = N$, and let $n_2 = \max\mathscr{M}$; let $\widetilde{n}_1$ and $\widetilde{n}_2$ denote the minimum and the maximum of $\mathscr{N}$. Consider the inner expansion of $p = y\widetilde{y}$: $p(\eta X,\eta) \sim \sum_{n\ge 0} p_n(X)\eta^n$. Thus one has $p_n = \sum_{r+s=n} h_r\widetilde{h}_s$ for all $n \ge 0$. If $n = n_1 + \widetilde{n}_1$ then $h_{n_1}\widetilde{h}_{\widetilde{n}_1}$ has valuation $M + \widetilde{n}_1 + K = M - Nq + nq + K$; if $r+s = n = n_1 + \widetilde{n}_1$ with $r \ne n_1$, then the valuation of $h_r\widetilde{h}_s$ is greater than that number because of the choice of n_1 and $\widetilde{n}_1$. So we obtain $p_{n_1+\widetilde{n}_1} \ne 0$. Similarly, we also get $p_{n_2+\widetilde{n}_2} \neq 0$. Since $n_1 < n_2$ and $\widetilde{n}_1 \leq \widetilde{n}_2$, this contradicts the assumption that the product $y\widetilde{y}$ is reduced to the monomial η^k. □

2.7 Multiple CASEs

In this section we call *vertex* of a function y a point near which y has a CASE which is not a classical asymptotic expansion, i.e. at least one of the fast terms g_n is non zero. Here we would like to discuss the case of a domain with *two* vertices on its boundary. The following statement shows that it not necessary to generalize the concept of CASE for uniform expansions on domains that have several vertices on their boundary, because we can reduce this situation to the case of a single vertex. For simplicity, we only study the case of a real interval.

Proposition 2.23. *Let $a < b < c < d$ be four real numbers and let $y :]a,d[\times]0,\eta_1] \to \mathbb{R}$ be a function having a* CASE

$$y(x,\eta) \sim \sum_{n=0}^{\infty} \Big(a_n(x) + g_n\big(\tfrac{x-a}{\eta}\big)\Big)\eta^n$$

as $\eta \to 0$, uniformly on $]a,c[$, with a_n holomorphic in a neighborhood of $[a,c]$ and $g_n \in \mathcal{G}(S)$, $S = S(-\delta,\delta,\infty)$ with some $\delta > 0$.

Assume that y also has a CASE

$$y(x,\eta) \sim \sum_{n=0}^{\infty} \Big(b_n(x) + h_n\big(\tfrac{d-x}{\eta}\big)\Big)\eta^n$$

as $\eta \to 0$, uniformly on $]b,d[$, with b_n holomorphic in a neighborhood of $[b,d]$ and $h_n \in \mathcal{G}(S)$.

Then y has an asymptotic expansion

$$y(x,\eta) \sim \sum_{n=0}^{\infty} \Big(c_n(x) + g_n\big(\tfrac{x-a}{\eta}\big) + h_n\big(\tfrac{d-x}{\eta}\big)\Big)\eta^n \tag{2.30}$$

as $\eta \to 0$, uniform on $]a,d[$ with the g_n, h_n of the previous formulas and with functions c_n holomorphic in a neighborhood of $[a,d]$.

More precisely, if $g_n(X) \sim \sum_{m=1}^{\infty} g_{nm}X^{-m}$ and $h_n(X) \sim \sum_{m=1}^{\infty} h_{nm}X^{-m}$ as $X \to \infty$, then $c_n(x) = a_n(x) - \sum_{\ell=0}^{n-1} h_{\ell\, n-\ell}(d-x)^{\ell-n}$ when x is in a neighborhood of $[a,c]$ and $c_n(x) = b_n(x) - \sum_{\ell=0}^{n-1} g_{\ell\, n-\ell}(x-a)^{\ell-n}$ when x is in a neighborhood of $[b,d]$.

Remark. The functions c_n are the non-polar parts of the functions b_n at the point $x = a$ and the non-polar parts of the functions a_n at the point $x = d$.

Proof. By the classical Borel–Ritt theorem, we construct two functions g, h with $g(X, \eta) \sim \sum_{n\geq 0} g_n(X)\eta^n$ and $h(X, \eta) \sim \sum_{n\geq 0} h_n(X)\eta^n$ uniformly on S. Then we consider the difference $z(x, \eta) = y(x, \eta) - g\big(\frac{x-a}{\eta}, \eta\big) - h\big(\frac{d-x}{\eta}, \eta\big)$. Proposition 2.16, applied to h respectively g, shows that z has two slow expansions uniform on $[a, c]$ and on $[b, d]$. By the uniqueness of asymptotic expansions, they must coincide on $[b, c]$. This implies that their coefficients must be continuations of each other and we obtain the statement. □

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