Chapter 1
Definitions and Basic Results

1.1 Regular Functions

Let $\Omega$ be a domain in the space of quaternions $\mathbb{H}$, namely, an open connected subset of $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ and let

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$$

denote the 2-sphere of quaternionic imaginary units. We define the notion of regular function as follows.

**Definition 1.1.** Let $f$ be a quaternion-valued function defined on a domain $\Omega$. For each $I \in \mathbb{S}$, let $\Omega_I = \Omega \cap L_I$ and let $f_I = f|_{\Omega_I}$ be the restriction of $f$ to $\Omega_I$. The restriction $f_I$ is called **holomorphic** if it has continuous partial derivatives and

$$\bar{\partial}_I f(x + yI) = \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) \equiv 0. \quad (1.1)$$

The function $f$ is called **regular** if, for all $I \in \mathbb{S}$, $f_I$ is holomorphic.

**Remark 1.2.** It is useful to note that if $f$ is regular in a domain $\Omega$ and if $r$ is a real number, then $g(q) := f(q + r)$ is obviously regular on $\Omega - r$. Note, however, that, in general, the composition of two regular functions is not regular.

The following lemma clarifies the relation between quaternionic regularity and complex holomorphy. For each $I \in \mathbb{S}$, let us identify $L_I$ with $\mathbb{C}$. Notice, moreover, that for all $J \in \mathbb{S}$ with $J \perp I$, the following equality holds:

$$\mathbb{H} = L_I + L_I J.$$
Lemma 1.3 (Splitting). Let $f$ be a regular function defined on a domain $\Omega$. Then for any $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ with $J \perp I$, there exist two holomorphic functions $F, G : \Omega_I \to L_I$ such that for every $z = x + yI$, it is

$$f_I(z) = F(z) + G(z)J.$$  

The previous Lemma can be reformulated in a way that will be useful in the sequel.

Lemma 1.4. Let $I \in \mathbb{S}$, let $\Omega_I$ be open in $L_I$, and let $f_I : \Omega_I \to \mathbb{H}$. The function $f_I$ is holomorphic if and only if, for all $J \in \mathbb{S}$ with $J \perp I$ and every $z = x + yI$, it is

$$f_I(z) = F(z) + G(z)J$$  \hspace{1cm} (1.2)

where $F, G : \Omega_I \to L_I$ are complex-valued holomorphic functions of one complex variable.

Let us now review the first examples and the basic properties of regular functions.

Example 1.5. The identity function $q \mapsto q$ is regular in $\mathbb{H}$. The same holds for any polynomial function of the type $q \mapsto a_0 + qa_1 + \ldots + q^na_n$, $a_i \in \mathbb{H}$ for all $l$.

This class of examples extends as follows. For all $R \in (0, +\infty]$, let us denote by

$$B(0, R) = \{q \in \mathbb{H} : |q| < R\}$$

the Euclidean ball of radius $R$ centered at $0$ in $\mathbb{H}$.

Theorem 1.6 (Abel’s Theorem). Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{H}$ and let

$$R = \frac{1}{\lim \sup_{n \in \mathbb{N}} |a_n|^{1/n}}.$$  \hspace{1cm} (1.3)

If $R > 0$, then the power series

$$\sum_{n \in \mathbb{N}} q^n a_n$$  \hspace{1cm} (1.4)

converges absolutely and uniformly on compact sets in $B(0, R)$. Moreover, its sum defines a regular function on $B(0, R)$.

The proof is completely analogous to that of the complex Abel theorem. The converse result holds; in other words all regular functions on $B(0, R)$ can be expressed as power series. In order to prove this, we first introduce an appropriate notion of derivative.

Definition 1.7. Let $f : \Omega \to \mathbb{H}$ be a regular function. For each $I \in \mathbb{S}$, the $I$-derivative of $f$ is defined as

$$\partial_I f(x + yI) = \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI)$$  \hspace{1cm} (1.5)
on \( \Omega_I \). The slice derivative of \( f \) is the function \( f' = \partial_I f : \Omega \to \mathbb{H} \) defined by 
\[ \partial_I f \] 
on \Omega_I, for all \( I \in \mathbb{S} \).

The definition is well posed because, by direct computation, \( \partial_I f = \partial_J f \) in \( \Omega_I \cap \Omega_J \) for any choice of \( I, J \in \mathbb{S} \). Furthermore, the following can be proven making use of the fact that \( \partial_I \) and \( \tilde{\partial}_I \) commute.

**Remark 1.8.** For any regular function \( f : \Omega \to \mathbb{H} \), the slice derivative \( f' \) is regular in \( \Omega \).

It is thus possible to iterate the derivation process. Let us denote the \( n \)th slice derivative as \( f^{(n)} \) for each \( n \in \mathbb{N} \). We now come to the announced result.

**Theorem 1.9 (Series expansion).** Let \( R > 0 \) and let \( f : B = B(0, R) \to \mathbb{H} \) be a regular function. Then

\[ f(q) = \sum_{n \in \mathbb{N}} q^n \frac{1}{n!} f^{(n)}(0) \tag{1.6} \]

for all \( q \in B \). In particular, \( f \in C^\infty(B) \).

**Proof.** Fix \( I \in \mathbb{S} \) and identify \( L_I \) with \( \mathbb{C} \). Choose \( J \in \mathbb{S} \) such that \( J \perp I \); by the Splitting Lemma 1.3, there exist holomorphic functions \( F, G : B_I \to L_I \) such that \( f_I = F + GJ \). Notice that, for all \( z \in B_I \),

\[ f'(z) = \partial_I f(z) = \frac{\partial F}{\partial z}(z) + \frac{\partial G}{\partial z}(z)J \]

and, similarly,

\[ f^{(n)}(z) = \frac{\partial^n F}{\partial z^n}(z) + \frac{\partial^n G}{\partial z^n}(z)J. \]

Now observe that the complex series

\[ \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0) \]

converges to \( F(z) \) for \( z \in B_I \) (absolutely and uniformly on its compact subsets). The same can be proved for \( G \), so that for all \( z \in B_I \)

\[ f(z) = F(z) + G(z)J = \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0) + \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} \frac{\partial^n G}{\partial z^n}(0)J = \]

\[ = \sum_{n \in \mathbb{N}} z^n \frac{1}{n!} f^{(n)}(0) \]

as desired. The thesis follows from the arbitrariness of \( I \in \mathbb{S} \). Finally, \( f \in C^\infty(B) \) because each addend \( q^n \frac{1}{n!} f^{(n)}(0) \) is clearly in \( C^\infty(B) \) and because the convergence is uniform on compact sets. \( \Box \)
Propositions 1.6 and 1.9 are fundamental in the study of regular quaternionic functions on balls $B = B(0, R)$ centered at the origin of $\mathbb{H}$. For instance, they allowed the proof of an identity principle in [62], stating that if, for some $I \in \mathbb{S}$, two regular functions $f, g : B \to \mathbb{H}$ coincide on a subset of $B_I$ having an accumulation point in $B_I$, then $f = g$ in $B$. This principle does not hold for an arbitrarily chosen domain in $\mathbb{H}$, as shown by the next example.

**Example 1.10.** Let $I \in \mathbb{S}$ and let $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be defined as follows:

$$f(q) = \begin{cases} 0 & \text{if } q \in \mathbb{H} \setminus L_I \\ 1 & \text{if } q \in L_I \setminus \mathbb{R} \end{cases}$$

This function is clearly regular.

The previous example proves that if the domain $\Omega$ is not carefully chosen, then a regular function $f : \Omega \to \mathbb{H}$ does not even need to be continuous. It is possible to prevent such pathologies by imposing further conditions on the domain $\Omega$.

**Definition 1.11.** Let $\Omega$ be a domain in $\mathbb{H}$ that intersects the real axis. $\Omega$ is called a *slice domain* if, for all $I \in \mathbb{S}$, the intersection $\Omega_I$ with the complex plane $L_I$ is a domain of $L_I$.

The identity principle holds true on all slice domains.

**Theorem 1.12 (Identity Principle).** Let $f, g$ be regular functions on a slice domain $\Omega$. If, for some $I \in \mathbb{S}$, $f$ and $g$ coincide on a subset of $\Omega_I$ having an accumulation point in $\Omega_I$, then $f = g$ in $\Omega$.

**Proof.** The restrictions $f_I, g_I$ are holomorphic functions. Under the hypotheses, $f_I$ and $g_I$ must coincide in $\Omega_I$. In particular, $f$ must coincide with $g$ in $\Omega \cap \mathbb{R}$. For all $K \in \mathbb{S}$, the intersection $\Omega \cap \mathbb{R}$ is a subset of $\Omega_K$ that has an accumulation point in $\Omega_K$. Thus, $f_K = g_K$ in $\Omega_K$ for all $K \in \mathbb{S}$, and we conclude that $f = g$ in $\Omega$.

Notice that, in the proof of Theorem 1.12, both properties that define slice domains are essential: the fact that $\Omega \cap \mathbb{R} \neq \emptyset$ and the connectedness of $\Omega_I$ for all $I \in \mathbb{S}$. In the next section we will present a symmetry condition for the domains of definition which guarantees continuity and differentiability for regular functions.

### 1.2 Affine Representation

We now present a very peculiar property of regular functions, which allows to identify the quaternionic analogs of the domains of holomorphy. Consider the following property of quaternionic powers, which is a direct consequence of the (complex) binomial theorem.
Remark 1.13. For each $x, y \in \mathbb{R}$, there exist sequences $\{\alpha_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $(x + yI)^n = \alpha_n + \beta_n I$ for all $I \in \mathbb{S}$.

As a consequence, the following formula holds for a regular function $f(q) = \sum_{n \in \mathbb{N}} q^n a_n$:

$$f(x + yI) = \sum_{n \in \mathbb{N}} \alpha_n a_n + I \sum_{n \in \mathbb{N}} \beta_n a_n.$$  

This formula has a nice geometric interpretation: the restriction of $f$ to the sphere

$$x + yS = \{x + yI : I \in \mathbb{S}\}$$

is affine in the imaginary unit $I$, that is, there exist $b, c \in \mathbb{H}$ such that

$$f(x + yI) = b + Ic$$  

for all $I \in \mathbb{S}$. This is not only true for power series, but for all regular functions on the slice domains that have the following property.

**Definition 1.14.** A set $T \subset \mathbb{H}$ is called *axially symmetric* if, for all points $x + yI \in T$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, the set $T$ contains the whole sphere $x + yS$.

Since no confusion can arise, we will refer to such a set as *symmetric*, tout court.

The most general statement is the following.

**Theorem 1.15 (Representation Formula).** Let $f$ be a regular function on a symmetric slice domain $\Omega$ and let $x + yS \subset \Omega$. For all $I, J, K \in \mathbb{S}$ with $J \neq K$

$$f(x + yI) = (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)] + I(J - K)^{-1} [f(x + yJ) - f(x + yK)]$$  

The quaternions $b = (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)]$ and $c = (J - K)^{-1} [f(x + yJ) - f(x + yK)]$ do not depend on $J, K$ but only on $x, y$.

**Proof.** Fix $J, K \in \mathbb{S}$ with $J \neq K$, and set

$$\phi(x + yI) = (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)] +$$

$$+ I(J - K)^{-1} [f(x + yJ) - f(x + yK)] =$$

$$= [(J - K)^{-1} J + I(J - K)^{-1}] f(x + yJ) +$$

$$- [(J - K)^{-1} K + I(J - K)^{-1}] f(x + yK)$$

for all $I \in \mathbb{S}$ and for all $x, y \in \mathbb{R}$ such that $x + yS \subset \Omega$. Computing the above formulae for $y = 0$ shows that $\phi(x) = f(x)$ for all $x \in \Omega \cap \mathbb{R}$. If we prove that $\phi$ is regular in $\Omega$, then we will conclude that $\phi \equiv f$ thanks to the Identity Principle 1.12.

The regularity of $\phi$ is proved computing
\[ \frac{\partial \phi(x + yI)}{\partial x} = \left[ (J - K)^{-1}J + I(J - K)^{-1} \right] \frac{\partial f(x + yJ)}{\partial x} + \]

\[ - \left[ (J - K)^{-1}K + I(J - K)^{-1} \right] \frac{\partial f(x + yK)}{\partial x} \]

and

\[ I \frac{\partial \phi(x + yI)}{\partial y} = \left[ I(J - K)^{-1}J - (J - K)^{-1}J \right] \frac{\partial f(x + yJ)}{\partial y} + \]

\[ - \left[ I(J - K)^{-1}K - (J - K)^{-1}K \right] \frac{\partial f(x + yK)}{\partial y} = \]

\[ = - \left[ I(J - K)^{-1} + (J - K)^{-1}J \right] \frac{\partial f(x + yJ)}{\partial x} + \]

\[ + \left[ I(J - K)^{-1} + (J - K)^{-1}K \right] \frac{\partial f(x + yK)}{\partial x}, \]

so that

\[ \frac{\partial \phi(x + yI)}{\partial x} + I \frac{\partial \phi(x + yI)}{\partial y} \equiv 0 \]

as desired. The formula is thus proven. The last statement follows because the choice of \( J, K \in \mathbb{S} \) is arbitrary.

The following special case will prove particularly useful in the next chapters.

**Corollary 1.16.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \) and let \( x + y\mathbb{S} \subset \Omega \). For all \( I, J \in \mathbb{S} \)

\[ f(x + yJ) = \frac{1 - JJ}{2} f(x + yI) + \frac{1 + JJ}{2} f(x - yI) \]

(1.9)

We presently deduce an alternative formula, proving that the restriction of \( f \) to a sphere \( x + y\mathbb{S} \) is actually affine in the variable \( q \).

**Corollary 1.17.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \) and let \( S = x + y\mathbb{S} \subset \Omega \). For all \( q, q_1, q_2 \in S \) with \( q_1 \neq q_2 \)

\[ f(q) = (q_1 - q_2)^{-1} [\bar{q}_2 f(q_2) - \bar{q}_1 f(q_1)] + q(q_1 - q_2)^{-1} [f(q_1) - f(q_2)] \]

(1.10)

where \( A = (q_1 - q_2)^{-1} [f(q_1) - f(q_2)] \) and \( B = (q_1 - q_2)^{-1} [\bar{q}_2 f(q_2) - \bar{q}_1 f(q_1)] \) do not depend on \( q_1, q_2 \) but only on \( S \).

**Proof.** If \( q = x + yI, q_1 = x + yJ, q_2 = x + yK \), then we deduce the thesis from (1.8) by direct computation:
The fact that \( f \) is affine in each sphere \( x + yI \subseteq \mathbb{S} \) justifies the following definition, given in [72] in a more general setting.

**Definition 1.18.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \). The **spherical derivative** of \( f \) is defined by the formula

\[
\partial_s f(q) = (q - \bar{q})^{-1} [f(q) - f(\bar{q})] = [2\text{Im}(q)]^{-1} [f(q) - f(\bar{q})]
\]

while the **spherical value** is the function

\[
v_s f(q) = \frac{1}{2} [f(q) + f(\bar{q})].
\]

We conclude by proving a consequence of (1.7), which will be useful in the study of uniform convergence of infinite \(*\)-products of regular functions, Sect. 4.5.

**Proposition 1.19.** Let \( f \) be a regular function on a symmetric slice domain \( \Omega \subseteq \mathbb{H} \). Let \( T \subseteq \Omega \) be a symmetric compact set. For every \( I \in \mathbb{S} \), \( p \in \mathbb{H} \), and \( R > 0 \) such that

\[
f_I(T_I) \subseteq B(p, R),
\]

we have

\[
f(T) \subseteq B(p, 2R).
\]

**Proof.** Let \( I \), \( p \), \( R \) be as in the hypothesis. If \( T \) is symmetric, then

\[
T = \bigcup_{x + yI \in T} x + y\mathbb{S};
\]

hence,

\[
f(T) = \bigcup_{x + yI \in T} f(x + y\mathbb{S}).
\]
It is enough to prove that $f(x + yS) \subseteq B(p, 2R)$ for all $x, y \in \mathbb{R}$ such that $x + yI \in T$. Let $x, y \in \mathbb{R}$ be such that $x + yS \subseteq T$. By (1.7) there exist $b, c \in \mathbb{H}$ such that

$$f(x + yJ) = b + Jc$$

for all $J \in S$. Since $f_I(T_I) \subseteq B(p, R)$, we have that

$$f(x + yI) = b + Ic \in B(p, R)$$

and that

$$f(x - yI) = b - Ic \in B(p, R).$$

Since two antipodal points (i.e., $b + Ic$ and $b - Ic$) of the 2-sphere $f(x + yS) = b + Sc$ belong to $B(p, R)$, the center $b$ of $b + Sc$ also belongs to $B(p, R)$ and the radius of $b + Sc$ is less than or equal to $R$. Hence,

$$f(x + yS) \subseteq B(p, 2R).$$

\[ \Box \]

### 1.3 Extension Results

The results presented in the previous section show that a regular function on a symmetric slice domain $\Omega$ is uniquely determined by its restriction to a slice $\Omega_I$ (or to two “half slices” $\Omega^+_J = \{x + yJ : y \geq 0\}$ and $\Omega^+_K = \{x + yK : y > 0\}$). This suggests the following definition and proposition:

**Definition 1.20.** The (axially) symmetric completion of a set $T \subseteq \mathbb{H}$ is the smallest symmetric set $\widetilde{T}$ that contains $T$. In other words,

$$\widetilde{T} = \bigcup_{x + yI \in T} (x + yS). \quad (1.13)$$

**Proposition 1.21 (Extension Formula).** Let $J, K$ be distinct imaginary units, let $T$ be a domain in $L_J$, intersecting the real axis, let $U = \{x + yK : x + yJ \in T\}$, and let $\Omega$ be the symmetric completion $\widetilde{T} = \widetilde{U}$. For any choice of holomorphic functions $r : T \to \mathbb{H}, s : U \to \mathbb{H}$ such that $r|_{T \cap \mathbb{R}} = s|_{U \cap \mathbb{R}}$, the function $f : \Omega \to \mathbb{H}$ defined, for all $x + yI \in \Omega$, by

$$f(x + yI) = (J - K)^{-1} [Jr(x + yJ) - Ks(x + yK)] + (1.14)$$

$$+ I(J - K)^{-1} [r(x + yJ) - s(x + yK)]$$

is the (unique) regular function on $\Omega$ such that $f|_T = r$ and $f|_U = s$. 
1.3 Extension Results

Proof. The function $f$ is proved to be regular in $\Omega$ by the same reasoning used for Formula (1.8). Furthermore, $f|_r = r$ by direct computation, since

$$(J - K)^{-1} J + J(J - K)^{-1} = |J - K|^{-2}[(K - J)J + J(K - J)] =$$

$$= [(J - K)(K - J)]^{-1}(2 + JK + KJ) = 1$$

and

$$(J - K)^{-1} K + J(J - K)^{-1} = |J - K|^{-2}[(K - J)K + J(K - J)] =$$

$$= |J - K|^{-2}(-1 - JK + JK + 1) = 0.$$ 

Similarly, $f|_u = s$. The uniqueness is a consequence of the Identity Principle 1.12.

The following special case (where $J = I, K = -I$) will be particularly useful in the sequel.

Lemma 1.22. Let $\Omega$ be a symmetric slice domain and let $I \in \mathbb{S}$. If $f_I : \Omega_I \to \mathbb{H}$ is holomorphic, then there exists a unique regular function $g : \Omega \to \mathbb{H}$ such that $g|_I = f_I$ in $\Omega_I$.

The function $g$ will be denoted by $\text{ext}(f_I)$ and called the regular extension of $f_I$.

In analogy with what is done in the complex plane, we give the following definition:

Definition 1.23. A slice domain $\Omega \subset \mathbb{H}$ is a domain of regularity if there exists a regular function on $\Omega$ that cannot be extended as a regular function to a larger domain.

It is well known that every domain in $\mathbb{C}$ is a domain of holomorphy. The next theorem shows that this is not the case for $\mathbb{H}$. 

Theorem 1.24 (Extension). Let $f$ be a regular function on a slice domain $\Omega$. There exists a unique regular function $\tilde{f} : \tilde{\Omega} \to \mathbb{H}$ that extends $f$ to the symmetric completion of $\Omega$.

Proof. By hypothesis $\Omega \cap \mathbb{R} \neq \emptyset$. Since $\Omega$ is open, it is possible to choose a neighborhood $D$ of $\Omega \cap \mathbb{R}$ in $\mathbb{H}$ that is a symmetric slice domain contained in $\Omega$. Now let $M$ be the largest symmetric slice domain with $D \subseteq M \subseteq \tilde{\Omega}$ to which $f$ extends as a regular function. The domain $M$ cannot be strictly contained in $\tilde{\Omega}$ because of the following reasoning.

If $\tilde{\Omega} \setminus M \neq \emptyset$, then there exists a $p \in \partial M \cap \tilde{\Omega}$. From $p = u + Lv \in \tilde{\Omega}$, we deduce the existence of $J \in \mathbb{S}$ such that $u + vJ \in \Omega$. Moreover, choosing $K \in \mathbb{S}$ sufficiently near to $J$, but distinct, there exists an $\varepsilon > 0$ such that $\Omega$ contains the discs $\Delta_J = \{z \in L_J : |z - (u + vJ)| < \varepsilon\}$ and $\Delta_K = \{z \in L_K : |z - (u + vK)| < \varepsilon\}$.

Now, if $\tilde{\Delta}_J$ is the symmetric completion of $\Delta_J$, setting

$$g(x + yJ) = (J - K)^{-1} [Jf(x + yJ) - Kf(x + yK)] +$$

$$+ I(J - K)^{-1} [f(x + yJ) - f(x + yK)]$$
for all $x + yI \in \tilde{\Delta}_J$ will define a regular function $g : \tilde{\Delta}_J \to \mathbb{H}$. The latter coincides with $f$ in $\tilde{\Delta}_J \cap M$ because formula (1.8) holds in $M$. Hence, setting
\[
\tilde{f} = \begin{cases} 
  f & \text{in } M \\
  g & \text{in } \tilde{\Delta}_J
\end{cases}
\]
extends $f$ to a regular function $\tilde{f}$ on the symmetric slice domain $M \cup \tilde{\Delta}_J$, a contradiction with the hypotheses on $M$. \hfill \square

Since every domain in $\mathbb{C}$ is a domain of holomorphy, and by Lemma 1.22, it is immediate to see that on every symmetric slice domain, there exists a regular function that cannot be extended. Thus, we proved the following corollary.

**Corollary 1.25.** A slice domain $\Omega \subseteq \mathbb{H}$ is a domain of regularity if and only if it is a symmetric slice domain.

We point out that, as a consequence of Theorem 1.24, considering regular functions on symmetric slice domains is not more restrictive than considering regular functions on slice domains. For this reason, we will often impose the symmetry condition in our presentation.

### 1.4 Algebraic Structure

In this section we present the algebraic structure of the set of regular functions. It is not hard to see that the class of regular functions is endowed with an addition operation: if $f, g$ are regular functions on $\Omega$, then $f + g$ is regular in $\Omega$, too. The same does not hold for pointwise multiplication: $f \cdot g$ is not regular, except for some special cases. This is easily seen even in the simplest case when $f(q) = qa$ and $g(q) = q$, with $a \in \mathbb{H} \setminus \mathbb{R}$. Then
\[
f(q)g(q) = qaq
\]
which is clearly not regular. We instead use the multiplicative operation described below, following the classical approach used for polynomials in noncommutative algebra (see, e.g., [84]).

**Definition 1.26.** Let $f, g : B(0, R) \to \mathbb{H}$ be regular functions and let $f(q) = \sum_{n \in \mathbb{N}} q^n a_n, g(q) = \sum_{n \in \mathbb{N}} q^n b_n$ be their power series expansions. The **regular product** of $f$ and $g$ (sometimes referred to as their $\ast$-product) is the regular function defined by
\[
f \ast g(q) = \sum_{n \in \mathbb{N}} q^n \sum_{k=0}^{n} a_k b_{n-k}
\]
on the same ball $B(0, R)$. 
Notice that if \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \), then \( f \ast g(q) = f(q)g(q) \). It turns out that the set of regular functions on a ball \( B(0, R) \) is a ring with +, ⋆, and that this multiplication can be extended to all regular functions on symmetric slice domains. The definition of regular product in this new setting relies upon the Splitting Lemma 1.3 and upon Lemma 1.22.

**Definition 1.27.** Let \( f, g \) be regular functions on a symmetric slice domain \( \Omega \). Choose \( I, J \in \mathcal{S} \) with \( I \perp J \) and let \( F, G, H, K \) be holomorphic functions from \( \Omega_I \) to \( L_I \) such that \( f_I = F + GJ, g_I = H + KJ \). Consider the holomorphic function defined on \( \Omega_I \) by

\[
f_I \ast g_I(z) = \left[ F(z)H(z) - G(z)\overline{K(\overline{z})} \right] + \left[ F(z)K(z) + G(z)\overline{H(\overline{z})} \right] J.
\] (1.16)

Its regular extension \( \text{ext}(f_I \ast g_I) \) is called the regular product (or ⋆-product) of \( f \) and \( g \), and it is denoted by \( f \ast g \).

It is possible to check directly that this definition is coherent with the previous one in the special case \( \Omega = B(0, R) \). Note that Definition 1.27 apparently depends upon the choices of \( I, J \). The next result shows that this is not the case.

**Proposition 1.28.** Let \( \Omega \) be a symmetric slice domain. The definition of regular product is well posed, and the set of regular functions on \( \Omega \) is a (noncommutative) ring with respect to + and ⋆.

**Proof.** Let \( f, g \) be regular functions on \( \Omega \). By hypothesis, \( \Omega \) intersects the real axis at some \( r \in \mathbb{R} \). By possibly substituting \( f(q + r) \) for \( f(q) \) and \( g(q + r) \) for \( g(q) \), we may suppose \( r = 0 \). Then there exists a ball \( B = B(0, R) \subseteq \Omega \), with \( R > 0 \) on which the restrictions \( f|_B \) and \( g|_B \) are power series. We already observed that, for all \( I \in \mathcal{S} \), \( \text{ext}(f_I \ast g_I) \) coincides with \( f|_B \ast g|_B \) in \( B \). In particular, for all \( I, J \in \mathcal{S} \), \( \text{ext}(f_I \ast g_I) \) equals \( \text{ext}(f_J \ast g_J) \) in \( B \). By the Identity Principle 1.12,

\[
\text{ext}(f_I \ast g_I) = \text{ext}(f_J \ast g_J)
\]
in \( \Omega \). This proves that \( f \ast g \) is well defined on \( \Omega \). The operation ⋆ is associative: \( f \ast (g \ast h) = (f \ast g) \ast h \) because

\[
f|_B \ast (g|_B \ast h|_B) = (f|_B \ast g|_B) \ast h|_B
\]
The distributive law can be proven using the same technique. Finally, ⋆ is clearly noncommutative.

As in the case of power series, the regular product coincides with the pointwise product for a special class of regular functions.

**Definition 1.29.** A regular function \( f : \Omega \to \mathbb{H} \) such that \( f(\Omega_I) \subseteq L_I \) for all \( I \in \mathcal{S} \) is called a slice preserving regular function.
Lemma 1.30. Let \( f, g \) be regular functions on a symmetric slice domain \( \Omega \). If \( f \) is slice preserving, then \( fg \) is a regular function on \( \Omega \) and \( f \ast g = fg \).

**Proof.** For any \( I, J \in \mathbb{S} \) with \( I \perp J \), let \( F, G, H, K \) be holomorphic functions \( \Omega_I \to L_I \) such that \( f_I = F + GJ, g_I = H + KJ \). If \( f(\Omega_I) \subseteq L_I \), then \( G \) must vanish identically, so that

\[
f_I(z)g_I(z) = F(z)H(z) + F(z)K(z)J.
\]

Since \( FH \) and \( FK \) are holomorphic functions from \( \Omega_I \) to \( L_I, f_Ig_I = (fg)_I \) is holomorphic. By the arbitrariness of \( I \in \mathbb{S} \), the Splitting Lemma 1.3 implies that \( fg : \Omega \to \mathbb{H} \) is regular.

Now fix \( I \in \mathbb{S} \). By the equation above, \( (fg)_I = FH + FKJ = f_I \ast g_I \). Hence, \( fg \) and \( f \ast g = \text{ext}(f_I \ast g_I) \) are two regular functions on \( \Omega \) coinciding in \( \Omega_I \). By the Identity Principle 1.12, they must coincide in \( \Omega \). \( \square \)

In the special case when \( \Omega \) is a ball centered in a real point (which we may assume to be the origin by Remark 1.2), Lemma 1.30 captures what we already observed for power series thanks to the following remark.

**Remark 1.31.** Let \( f \) be a regular function on \( \Omega = B(0, R) \) (for some \( R > 0 \)). Then \( f \) is slice preserving if, and only if, the power series expansion \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \) has real coefficients \( a_n \in \mathbb{R} \).

As we pointed out in Remark 1.2, the composition of two regular functions is not, in general, a regular function. However, the following lemma can be proven by direct computation.

**Lemma 1.32.** Let \( f : \Omega \to \Omega' \subseteq \mathbb{H} \) and \( g : \Omega' \to \mathbb{H} \) be regular functions. If \( f \) is a slice preserving function, then the composition \( g \circ f \) is regular.

We can define two additional operations on regular functions. We begin with the case of power series.

**Definition 1.33.** Let \( f : B(0, R) \to \mathbb{H} \) be a regular function and let \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \) be its power series expansion. The **regular conjugate** of \( f \) is the regular function defined by

\[
f^c(q) = \sum_{n \in \mathbb{N}} q^n \tilde{a}_n \tag{1.17}
\]

on the same ball \( B(0, R) \). The **symmetrization** of \( f \) is the function

\[
f^s = f \ast f^c = f^c \ast f. \tag{1.18}
\]

These operations are defined in order to study the zero set, as we will explain in Chap. 3, but they also allow us to construct the ring of quotients of regular functions (see Chap. 5). In the general case of symmetric slice domains, they are defined in the following way.
Definition 1.34. Let $f$ be a regular function on a symmetric slice domain $\Omega$. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let $F, G$ be holomorphic functions from $\Omega_I$ to $L_I$ such that $f_I = F + GJ$. If $f^c_I$ is the holomorphic function defined on $\Omega_I$ by
\[
f_I^c(z) = \overline{F(z)} - G(z)J.
\]
then the regular conjugate of $f$ is the regular function defined on $\Omega$ as $f^c = \text{ext}(f^c_I)$.

Definition 1.35. Let $f$ be a regular function on a symmetric slice domain $\Omega$. The symmetrization of $f$ is the regular function defined on $\Omega$ as $f^s = f \ast f^c = f^c \ast f$.

Remark 1.36. Using the Splitting Lemma, we can write
\[
f_I(z) = F(z) + G(z)J,
\]
with $F, G : \Omega_I \to L_I$ holomorphic functions. Therefore,
\[
f_I^s = f_I \ast f_I^c = f_I^c \ast f_I = (F(z) + G(z)J) \ast (\overline{F(z)} - G(z)J)
\]
\[
= [F(z)\overline{F(z)} + G(z)\overline{G(z)}] + [-F(z)G(z) + G(z)F(z)]J
\]
\[
= F(z)\overline{F(z)} + G(z)\overline{G(z)}.
\]
This shows that $f^s(\Omega_I) \subseteq L_I$ for every $I \in \mathbb{S}$, that is, that $f^s$ is slice preserving.

The previous definitions are well posed and coherent with those given in the special case $\Omega = B(0, R)$ (by direct computation).

We conclude this section by proving a few simple but important consequences of Definition 1.27.

Proposition 1.37. Let $f$ and $g$ be regular functions on a symmetric slice domain $\Omega$. Then $(f \ast g)^c = g^c \ast f^c$.

Proof. Follows immediately from Definitions 1.27 and 1.34.

Proposition 1.38. Let $f$ and $g$ be regular functions on a symmetric slice domain $\Omega$. Then $(f \ast g)^s = f^s \ast g^s = g^s \ast f^s$.

Proof. Follows immediately from Definitions 1.27, 1.35, and from the fact that $f^s$ is slice preserving.

Proposition 1.39. Let $f$ be a regular function on a symmetric slice domain $\Omega$. If $f$ is slice preserving then $f^s(q) = f(q)$ and $f^s(q) = f(q)^2$ for all $q \in \Omega$.

Proof. Follows from Definition 1.34 and Lemma 1.30.

We finally prove the Leibniz formula for slice derivatives.

Proposition 1.40. Let $f$ and $g$ be regular functions on a symmetric slice domain $\Omega$. Then $(f \ast g)' = f' \ast g + f \ast g'$.
Proof. Let $I$ be any element of $\mathbb{S}$ and let $f_I, g_I$ be the restrictions of $f, g$ to $L_I$. Since for every regular function $h$ defined on $\Omega$ the equality $(h')_I = (h_I)'$ holds, by the definition of $*$-product and the identity principle, we have

$$(f \ast g)' = ext \left\{ [(f \ast g)']_I \right\} = ext \left\{ [(f' \ast g)_I] \right\} = ext \left\{ [(f_I \ast g_I)'] \right\}.$$ 

Therefore, it is enough to prove that

$$(f_I \ast g_I)' = f'_I \ast g_I + f_I \ast g'_I,$$

which can be easily done by applying the Splitting Lemma 1.3 to both $f$ and $g$, so reducing the problem to the case of the Leibniz rule for holomorphic functions. □

For the sake of completeness, let us also mention the Leibniz formula for the spherical derivative (from [72]).

**Proposition 1.41.** Let $f$ and $g$ be regular functions on a symmetric slice domain $\Omega$. Then $\partial_s(f \ast g) = (\partial_s f)(v_s g) + (v_s f)(\partial_s g)$.

**Bibliographic Notes**

Regular functions were introduced (under the name of Cullen regular functions) in [61, 62]. The same articles proved the basic properties presented in Sect. 1.1 (such as the Identity Principle 1.12) for Euclidean balls centered at 0. The definition of slice regular quaternionic function (which requires “slicewise” differentiability instead of global differentiability) was given in [19]. All the aforementioned properties are extended to slice domains in the same paper [19] (except for the Identity Principle 1.12, whose extension was proven in [119]). The notation $\partial_c f$ for the slice derivative of a regular function $f$ derives from the original papers [61, 62] where the subscript $c$ stood for Cullen.

The Representation Formula (1.8) was proven in [19], while (1.9) was proven in [18] (the special case of power series had been considered in [56]). The new formula (1.10) is presented here for the first time, while Proposition 1.19 derives from [69].

The extension results in Sect. 1.3 are all original contributions of [19]. Finally, the algebraic structure presented in Sect. 1.4 was constructed in [55] for power series, and in [19] for regular functions on symmetric slice domains.
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