Introduction

The study of asymptotic properties of solutions of differential equations in the neighborhood of a critical point has almost always accompanied the development of the theory of stability of motion. It was not just as an empty exercise that A.M. Lyapunov, the founder of the classical theory of stability, developed two methods for investigating the behavior of a solution of a system of differential equations in the neighborhood of a critical point. And, if the so-called *second* or *direct Lyapunov method* has mainly a qualitative character and is intended for answering the question "do solutions leave some small neighborhood of a critical point, having begun close to that point?" then *Lyapunov's first method* is dedicated to the analytic representation of solutions in the neighborhood of an equilibrium position. Lyapunov's basic result in this direction, obtained for autonomous systems, consists of the following: if the characteristic equation for the first-order approximation system has s roots with negative real part, then the full system of differential equations has an s-parameter family of solutions beginning in a small neighborhood of the equilibrium solution and converging exponentially toward this solution [133]. In the literature, this result bears the name "Lyapunov's theorem on conditional asymptotic stability". But, as Lyapunov himself remarked [133], this assertion was known yet earlier to Poincaré and was actually contained in Poincaré's doctoral dissertation [151]. There likewise exists a conceptually closely related result, known as the Hadamard-Perron theorem in the literature (although the assertion as it was formulated in the original papers [72, 150] only rather distantly recalls the theorem in its contemporary form): if the characteristic equation of the system of first approximation has s roots with negative real part and p roots with positive real part, then, in the neighborhood of the critical point, there exist two invariant manifolds with respective dimensions s and p, the first consisting of solutions converging exponentially to the critical point as $t \to +\infty$, and the second of solutions converging exponentially to the critical point as $t \to -\infty$. This theorem

The Introduction is substantially a translation of the Preface to the original Russian edition.

and its modern proof can be found in most monographs on differential equations and bifurcation theory [41, 42, 76, 81, 137].

Lyapunov's method is based on asymptotic integration of the system of differential equations being investigated, in the form of certain series containing multiple complex exponentials, the coefficients of which are polynomials in the independent variable. This method was later further developed in the paper [77] and the book [76]. It should be mentioned that, if complex numbers appear among the eigenvalues of the first approximation system, then the construction of real solutions in the form of the aforementioned series is not at all a simple problem, due to the exceedingly awkward appearance of the real solutions, and results in an exorbitant number of computations. Among the classical works devoted to the asymptotics of solutions approaching a critical point as $t \to +\infty$ or $t \to -\infty$, we should also mention the papers of P.G. Bol' [23].

In many concrete problems, the fundamental method of proof of stability is the method for constructing Chetaev functions [39, 40], which make it possible to give a qualitative picture of the behavior of trajectories in a whole domain of phase space bordering the critical point. This method is in a certain sense "excessive", inasmuch as—as was noted by Chetayev himself—"in order to reveal the instability of unperturbed motion, it suffices to observe, among all trajectories, just one that exits from a given region under perturbations of arbitrarily small numerical value" [39, 40]. It was observed long ago that in the majority of cases the condition of instability of a critical point of a system of ordinary differential equations is accompanied by the existence of a particular solution of the system, converging to this point as $t \to -\infty$. But the most general conditions for Chetaev functions, under which asymptotic solutions actually exist, were found considerably later by N.N. Krasovskiy [124]. In problems related to the generalized Lagrange stability theorem, this phenomenon was noted rather long ago. It is worth stressing that the reversibility of the equations of motion of a conservative mechanical system guarantees simultaneously the existence and "entering" of solutions, i.e., the convergence to the equilibrium state as $t \to +\infty$. One of the first papers where this phenomenon is described is due to A. Kneser [101]. It was later the subject of generalization by Bohl [22]. In a paper by our first author [103], requirements on Chetayev functions of a special type were presented and used in the proof of instability of equilibrium states of reversible conservative systems, ensuring the existence of "exiting" solutions.

Practically all the results cited above concern cases where the presence of an asymptotic solution, tending to a critical point during unbounded increase or decrease in the independent variable, is successfully detected based only on an analysis of the linearized equation in cases where the convergence of solutions to critical points displays an exponential character. But the detection of solutions whose convergence to a critical point is nonexponential, and likewise the construction of asymptotics for such solutions, represents a more difficult problem. It is all the more paradoxical that one of the first papers dedicated to a "nonexponential" problem was published long before the appearance of Lyapunov's "General Problem of the Stability of Motion." We are referring to the work of Briot and Bouquet [28], which served as a basis of the development by G.V. Kamenkov [93] of a method for constructing invariant curves, along which a solution of nonexponential type departs from a critical point. For many years, this method remained a reliable technique in mechanics for proving instability in so-called critical cases, based only on linearized equations (see, e.g., the monograph of V.G. Veretennikov [188]). The work of Kamenkov just mentioned is almost entirely unknown to nonmathematicians, for the unfortunate reason that the only available publication containing the formulation and proof of Kamenkov's theorem is the posthumous collection of his papers already cited above [93]. Although the formulation of the theorem was absolutely correct, the proof as presented contained some technical assertions that were regarded as obvious when they really requiredadditional analysis. Among the more contemporary authors who established the beginnings of research into the asymptotics of solutions of systems of differential equations in the neighborhood of nonelementary critical points, we should mention A.A. Shestakov in connection with his paper [164]. One of the first papers which discussed the possibility of constructing asymptotics of solutions to differential equations in power form was the paper of N.V. Bugaev [33]. A.D. Bryuno [29] proposed a general algorithm, based on the technique of the Newton polytope, for computing the leading terms of expansions of solutions, possessing a generalized power asymptotic, of an analytic system of differential equations in the neighborhood of a nonelementary critical point. Newton diagrams and polytopes play an important role in various mathematical areas. At the "top of the list" of our national literature in applying these fascinating techniques to contemporary research, it is a pleasure to cite the paper on Newton's polygon by N.G. Chebotarev [38].

Newly increasing interest toward the end of the 1970s in the inverse Lyapunov problem on stability gave new impetus to the investigation of asymptotics for solutions of differential equations in the neighborhood of an elementary critical point. In a paper by our first author [104], and likewise in the paper [117] written by him incollaboration with V.P. Palamodov, asymptotic solutions of the corresponding equations of motion were found in the form of various series in the quantity $t^{-\alpha j} \ln^k t$, which in form coincide with expansions in just one of the variables x_j that had been proposed by G.V. Kamenkov [93]. It should be remarked that the appearance of logarithmic terms in asymptotic expansions of solutions to nonlinear equations is a very general phenomenon (see, e.g., [15]). In this connection, it was discovered that in a whole multitude of cases the constructed series can diverge, even if the right sides of the investigated equations are holomorphic in the neighborhood of the critical point [106]. The way out of this dilemma is to apply the theory of A.N. Kuznetsov [125, 126], which establishes a correspondence between the formal solutions of the nonlinear system of equations being investigated and certain smooth particular solutions having the required asymptotic. Application of this technique allowed our second author to give an elementary and rigorous proof of Kamenkov's theorem [60].

The question arises as to whether the leading terms of the expansions of nonexponential asymptotic solutions always have the form of a power. As follows from the paper of A.P. Markeev [136], which deals with the existence of asymptotic trajectories of Hamiltonian systems in critical cases, the answer to this question is negative: for example, in the presence of fourth-order resonances between frequencies of the linearized system, asymptotic solutions entering the critical point may have a more complicated form than powers. Other examples of this type can be found in one of the articles by Kuznetsov cited above [126].

In the last 10 or 15 years, the problem of existence of particular solutions of systems of differential equations with nonexponential asymptotic has attracted the attention of theoretical physicists. The situation is that the structure of these solutions is closely connected with the Painlevé property [68, 87]. By the "Painlevé property" in the extensive literature by physicist-geometers (see, e.g., the survey [26]) is meant the following: (a) movable singularities of solutions in the complex domain can only be poles and (b) the formal expansions of these solutions into Laurent series contain $(n-1)$ arbitrary constants as free parameters, where n is the dimension of the phase space. The test of these properties bears the name "Painlevé test" or ARS-test after the names of the authors who were among the first to apply the stated approach to nonlinear problems of mathematical physics [1]. Practically all the systems that pass the ARS-test can be integrated in explicit form [1, 26, 49]. The idea that the solutions of the integrated system must be single-valued meromorphic functions of time goes back to Kovalevsky [102]. It seems a plausible hypothesis that nonintegrable systems cannot satisfy the Painlevé property. However, there are but few rigorous results on nonintegrability that use nonexponential asymptotics. In this connection, it is worthwhile recalling the work of H. Yoshida [197] (where there are some inaccuracies that have been pointed out in [69]) and also a series of rather recent articles, also by Yoshida [198–200], based on a method of S.L. Ziglin [202], which include much stronger conditions on the system considered. It has been observed that the presence of logarithmic terms in the asymptotic expansions of solutions of many concrete systems, ordinarily considered chaotic, actually corresponds to very intricate behavior of trajectories and comes down to the fact that the singularities of these solutions form capricious star-shaped structures in the complex plane that resemble fractals [52, 181]. Very similar effects are engendered by the presence of irrational and complex powers in the asymptotic expansions of solutions [36, 37].

The purpose of our monograph is to systematically set forth the present state of affairs in the problem of investigating the asymptotics of solutions of differential equations in the neighborhood of a nonelementary singular point, to indicate ways of extending this theory to other objects of a dynamical nature and likewise to demonstrate the wide spectrum of applications to mechanics and other fields. In all this the authors make no pretense toward a complete bibliographical survey of work related to our problem. Many of the results indicated in the book have been obtained by the authors themselves, so that the material presented is fundamentally determined by their viewpoints and biases.

Before beginning a brief account of results for our readers' attention, it is necessary to make a remark of a bibliographical nature. At first glance, it may seem that problems of constructing particular solutions of differential equations with exponential or generalized power asymptotics differ so much from one another that, in their construction, we could scarcely succeed in using some idea encompassed in the first Lyapunov method. This misunderstanding becomes attenuated by a cursory acquaintance with these objects. Wherein lies the essence of Lyapunov's first method? In his famous "General Problem of the Stability of Motion", [133] Lyapunov, while considering systems of ordinary differential equations of the form

$$\dot{x} = \mathbf{f}(\mathbf{x}),\ \mathbf{f}(\mathbf{0}) = \mathbf{0},\ \mathbf{x} \in \mathbb{R}^{\mathbf{n}},\ \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \dots,$$

the dots representing the totality of the nonlinear terms, proposed looking for particular solutions in the form of a series

$$\mathbf{x}(t) = \sum_{j_1+\ldots+j_p\geq 1}^{\infty} \mathbf{x}_{j_1,\ldots,j_p}(t)\exp((j_1\lambda_1 + \ldots + j_p\lambda_p)t),\ \ p \leq n,$$

where the functions $x_{j_1,\ldots,j_p}(t)$ depend polynomially on t and, if we wish to limit ourselves to consideration of real solutions, on some trigonometric functions of time.

In all this, it is supposed that the totality of the first terms of theseseries (i.e., where $j_1 + \ldots + j_p = 1$) is a solution of the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

In this way, the application of Lyapunov's first method demands three steps:

1. Selection from the system of some truncated (in the present instance linear) subsystem
2. Construction of a particular solution (or a family of particular solutions) of the given truncated system
3. Adding to the found supporting solution (or family of supportingsolutions) to get the solution of the full system with the aid of some series

This same scheme is also used for constructing particular solutions with generalized power asymptotic. Throughout the reading of the book, other deeper analogies between these two problems will come to light. The more essential results presented herein were published in the article [116].

We likewise turn the attention of readers to the term *strongly nonlinear system*, used in the title, which could raise a number of questions. If none of the eigenvalues of the system, linearized in the neighborhood of a critical point, lies on the imaginary axis, then this system is topologically conjugate to its linear part (Grobman-Hartman theorem) [7, 76]. From this point of view, it is natural to call a system strongly nonlinear if the topological type of its phase portrait in the neighborhood of the critical point is not completely determined by its linear terms. We therefore include in this class any system for which the characteristic equation of its linear part has a pure imaginary or zero root.

This book consists of four chapters and two appendices. The first chapter is dedicated to the theory of so-called *semi-quasihomogeneous systems*, i.e., those that are "almost" invariant with respect to the action of the phase flow of a certain special linear system of equations of Fuchsian type, and the construction of their solutions. The action of this flow introduces a certain small parameter that allows us to select a truncated quasihomogeneous subsystem. In the first section of this chapter, the fundamental theorem is proved that particular solutions of the truncated system lying on the orbits of the indicated flow can be fully constructed with the help of certain series for the solutions of the full system. The second section is dedicated to an analysis of the convergence of those series. As has already been noted, the problem of finding solutions of a system of differential equations with generalized power asymptotic has its roots in Lyapunov's first method. The third section is dedicated to applying the idea of this method to the problem considered. In particular, with the aid of the "quasilinear" technique, a theorem is proved on the existence of multiparameter families of solutions with nonexponential asymptotic for systems of equations of a much broader class than had been considered in the preceding section. The subsequent fourth section contains a large collection of concrete examples from mathematics, mechanics, physics, and other branches of natural science. On the one hand, these examples illustrate the proved theoretical results, and on the other, they have an independent significance. For example, we investigate a new critical case of high codimension n of zero roots of the characteristic equation with one group of solutions. Another interesting application consists of methods for constructing collision trajectories in real time for the Hill problem. In the fifth and final section, we discuss a group theoretical approaches to the problem of constructing particular solutions of systems of differential equations. The proposed method is based on using arbitrary one-parameter groups of transformations of phase space, being in some sense "almost" a symmetry group for the system of equations considered.

The second chapter is dedicated to finding sufficient conditions for the existence of solutions of systems to differential equations that converge to a critical point as $t \to +\infty$ or $t \to -\infty$ when the first approximation system is neutral. In Sect. 2.1 we introduce sufficient conditions for the existence of such solutions for truncations of the Poincaré normal form. The critical case of two pure imaginary roots for a general four-dimensional system of differential equations is considered at length. In Sect. 2.2, the results obtained in Sect. 2.1 are generalized to systems for which the right sides depend periodically on time. An analogous theory is constructed for problems of finding solutions that converge asymptotically to invariant tori, provided these tori are neutral in the first approximation. In Sect. 2.3, we discuss characteristics of solutions of problems considered in the first section induced by the Hamiltonian property of the system considered. Here it is shown by way of illustration precisely how, with the help of the stated results, we obtain known theorems on the instability of the equilibrium position of Hamiltonian systems with two degrees of freedom in the presence of resonances between frequencies of small vibrations.

In the third chapter, problems are considered that the authors call singular. The peculiarity of these problems lies in the fact that series that represent solutions, and have the required asymptotic, diverge even in the case where the system under investigation is analytic. In the first section, dedicated to a method of obtaining enough instability conditions in the critical case where there is at least one zero root of the first approximation system, we prove a theorem on the existence of a formal invariant manifold, to which the linear subspace corresponding tosuch zero roots is tangent. It is shown that nonanalyticity for this manifold is inevitable and is the reason for the divergence of the aforementioned series, whose asymptotic expansions contain both exponentials and negative powers of the independent variable. It was noted long ago that an asymptotic solution of a system of differential equations can contain powers of iterated logarithms of rather high orders. The mechanism of this phenomenon is revealed in Sect. 3.2. It is also connected with certain "critical" cases, where there is a zero present in the spectrum of the so-called Kovalevsky matrix. In Sect. 3.2, another reason for the divergence of the asymptotic series is set forth, which amounts to the fact that, in selecting a quasihomogeneous truncation, even with the aid of the standard methods of Newton polytopes, certain derivatives may vanish. We introduce several theorems on the existence of asymptotic solutions of systems which are implicit (for systems in which only first derivatives occur, explicit means that only individual first-order derivatives appear on the left-hand side) in the derivatives. We briefly set forth the above-mentioned theory of Kuznetsov [125, 126], allowing us to set up a correspondence between formal and actual solutions of such systems. By way of an example with critical positions, we discuss a series of papers dedicated to new integrable cases in the problem of the motion of a massive solid body about a fixed point with the aid of Kovalevsky's method. It is shown that the solutions obtained in these papers are above all *not analytic*, since, for their construction, the authors had to use the truncation procedure for an Euler-Poisson system, leading to loss in differentiability.

The material presented in Chaps. 2 and 3, and in a part of Chap. 1, is first of all a powerful means for proving the instability of a critical point for a system of differential equations. Therefore the majority of theorems concerning the existence of asymptotic solutions are accompanied by dual theorems that give sufficient conditions for instability.

The fourth chapter has an illustrative character. In it we consider a range of problems that in one way or another are connected with the converse to Lagrange's theorem on the stability of equilibrium. In Sect. 4.1, we present (basically without proof) sufficient conditions for stability (including asymptotic stability) of an equilibrium position of generalized gradient systems, for reversible conservative mechanical systems, for mechanical systems on which a dissipative gyroscopic force is acting, and, likewise, for systems whose parameters change with time. These conditions are expressed in terms of the presence or absence of a minimum in potential energy at the equilibrium position considered. We likewise consider the problem of imposing some added constraints on the stability of equilibrium. In particular, we give an instructive example that shows that the stabilization in the first approximation of a reversible conservative system by means of imposition of nonholonomic constraints may be either stable or unstable depending on the arithmetic properties of its frequencies. The following two sections are dedicated to the converse of the theorem stated in the first section, with the aid of construction of asymptotic solutions. Their division bears a purely conditional character. If in Sect. 4.2 we use assertions of a "regular" character, i.e., proofs from Chap. 1 that guarantee convergence of the constructed series in the most important applications, then Sect. 4.3 is based on "singular" methods, leading to the construction of divergent series.

Appendix A is dedicated to the extension of well-developed methods for certain other objects of a dynamical character. In this appendix, for systems of differential equations with a deflecting argument and likewise for systems of integro-differential equations of a specific form, we introduce the concepts of quasihomogeneity and semi-quasihomogeneity. Conditions are indicated that are sufficient for the existence of solutions that tend to equilibrium with an unbounded decrease in time and, based on these, instability theorems are formulated. By way of an example, we discuss the interesting effect of explosive instability in ecological systems of the Volterra-Lotka type at zero values of the Malthusian birth rates.

In Appendix B, we consider the problem of how the presence and structure of particular solutions with generalized power asymptotic influences the integrability of systems of ordinary differential equations. We mentioned above the paper of H. Yoshida [197] where, by way of criteria for integrability, the arithmetic properties of the eigenvalues of the Kovalevsky matrix were used, whose calculation is impossible without finding the principal terms of the asymptotic of the particular solutions of nonexponential type. In Appendix B, Yoshida's criteria are sharpened and the result is applied to some systems of equations that are well-known in mathematical physics.

The book is aimed in the first instance at a wide circle of professional scholars and at those who are preparing to be such: pure and applied mathematicians, as well as students, who are interested in problems connected with ordinary differential equations, and specialists in theoretical mechanics who are occupied with questions about the behavior of trajectories of mechanical systems. The authors are also hopeful that any physicist who is attracted by theoretical research will find much that is useful in the book. In order to extend the circle of readers, the authors have attempted to plan the exposition as much as they could in such a way to make the book accessible to readers whose background includes just the standard program of higher mathematics and theoretical mechanics in the applied mathematics department of a technical university. The proofs of theorems on the convergence of formal series, or of the asymptotic behavior of solutions, containing (not at all complicated) elements of functional analysis, are written in such a way that they may be omitted without detriment to the understanding of the basic circle of ideas. We have included in the text information (without proofs) from the theory of normal forms so as not to interrupt the flow of the exposition. In a few places (and in particular in Sect. 1.5 of Chap. 1, Sect. 2.3 of Chap. 2, and Appendix B), the reader will need some elementary material from algebra (e.g., the concepts of group and

Lie algebra). Nonetheless, a series of indispensable assertions concerning groups of symmetries of systems of differential equations are formulated and proved in the text. Some information from differential topology is used, but only in the proof of a single technical lemma and thus it too will not interrupt the ideal simplicity of the exposition.

The present book represents the fruit of more than two decades of work by the authors: their first publications on this subject go back to 1982. This book contains not only work that was published previously, but also results that appear here for the first time. A portion of the material is based on the specialized courses "asymptotic methods in mechanics", given by the first author in the mechanical-mathematical faculty of Moscow State University, and "analytical methods and celestial mechanics in the dynamics of complex objects", given by the second author in collaboration with P.S. Krasilnikov in the faculty of applied mathematics and physics at the Moscow Aviation Institute. A significant portion of the book is also based on mini-courses given by the second author at the Catholic University of Louvain (Belgium) in 1994 and at the University of Trento (Italy) in 1995.

In the process of working on the book, the authors had frequent opportunity for discussing Lyapunov's first method and its application to strongly nonlinear systems with many scholars. We wish to take this opportunity to thank those who participated in these discussions and listened to the nontraditional and perhaps controversial views of the authors on this subject: P. Hagedorn (Technical University Darmstadt), J. Mawhin and K. Peiffer (Catholic University of Louvain), L. Salvadori (University of Trento), and V.V. Rumyantsev and S.V. Bolotin (Moscow State University). We especially wish to thank Professor L. Salvadori for bringing to the authors' attention a range of problems that were previously unknown to them and whose hospitality helped bring the work on this book nearer to completion. We cannot neglect to mention the dedicated work of M.V. Matveev in carefully reading the manuscript and in making comments that helped the authors eliminate a number of errors. Thanks are also due to the book's translator (into English), L.J. Senechal, who has assumed full responsibility for the integrity of the translated version and who performed the translation task in the shortest possible time.

Chapter 1
Semi-quasihomogeneous Systems of Differential Equations

1.1 Formal Asymptotic Particular Solutions of Semi-quasihomogeneous Systems of Differential Equations

We consider an infinitely smooth system of differential equations for which the origin $\mathbf{x} = 0$ is a critical point:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(\mathbf{0}) = \mathbf{0}, \ \mathbf{x} \in \mathbb{R}^n, \tag{1.1}$$

Let $\mathbf{A} = d\mathbf{f}(\mathbf{0})$ be the Jacobian matrix of the vector field $\mathbf{f}(\mathbf{x})$, computed at the critical point $\mathbf{x} = \mathbf{0}$. We will look for conditions on the right side of (1.1) that are necessary in order that there be a particular solution $\mathbf{x}$ such that $\mathbf{x}(t) \to 0$ either as $t \to +\infty$ or as $t \to -\infty$. In the sequel such solutions will be called *asymptotic*.

We first formulate a well known result. Consider the operator $\mathbf{A}: \mathbb{R}^n \to \mathbb{R}^n$. We decompose the space $\mathbb{R}^n$ into the direct sum of three spaces

$$\mathbb{R}^n = E^{(s)} \oplus E^{(u)} \oplus E^{(c)}$$

(where s connotes *stable*, u—*unstable*, c—*center*). This decomposition is dictated by the following requirements: all three subspaces on the right hand side are invariant under the operator $\mathbf{A}$; the spectrum of the restricted operator $\mathbf{A}|_{E^{(s)}}$ lies in the left half plane, that of $\mathbf{A}|_{E^{(u)}}$—in the right half plane, and that of $\mathbf{A}|_{E^{(c)}}$—on the imaginary axis. The possibility of such a decomposition follows from standard theorems of linear algebra (see e.g. [74]).

Theorem 1.1.1. *To system (1.1) there correspond three smooth invariant manifolds $W^{(s)}$, $W^{(u)}$, $W^{(c)}$ passing through $\mathbf{x} = \mathbf{0}$ and tangent there to $E^{(s)}$, $E^{(u)}$, $E^{(c)}$ respectively and having the same respective dimensions. The solutions with initial conditions on $W^{(s)}$ ($W^{(u)}$) converge exponentially to $\mathbf{x} = \mathbf{0}$ as $t \to +\infty$ ($t \to -\infty$), but the behavior of solutions on $W^{(c)}$ is determined by nonlinear elements.*

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DOI 10.1007/978-3-642-33817-5_1,

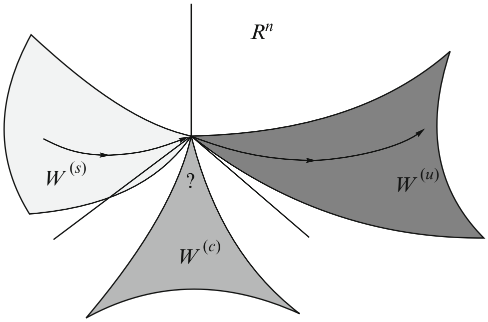

Fig. 1.1 Stable, center and unstable manifolds

The manifold $W^{(s)}$ is called stable, $W^{(u)}$—unstable, and $W^{(c)}$—center (see Fig. 1.1).

The stated theorem is a combination of the Hadamard-Perron theorem and the center manifold theorem [41,76,81,137]. It should be noted that the center manifold $W^{(c)}$ generally has but a finite order of smoothness [41].

Thus the question of the existence of asymptotic particular solutions of (1.1) with exponential asymptotic is solved by studying the first approximation system. We note yet another fact, arising from the general philosophy of Lyapunov's first method: if the matrix $\mathbf{A}$ has at least one nonzero real eigenvalue β, then there exists a real particular solution (1.1), belonging to $W^{(s)}$ or $W^{(u)}$, according to the sign of β, represented in the form

$$\mathbf{x}(t) = e^{-\beta t} \sum_{k=0}^{\infty} \mathbf{x}_k(t) e^{-k\beta t}, \tag{1.2}$$

where the $\mathbf{x}_k(t)$ are certain polynomial functions of the time t and where $\mathbf{x}_0 \equiv$ **const** is an eigenvector of the matrix $\mathbf{A}$ with eigenvalue β. For complex eigenvalues the corresponding decomposition of the real parts of solutions has a much more complicated appearance.

In order to find nonexponential asymptotic solutions, it is necessary to reduce the system on the center manifold. But in this chapter we will assume that the critical case of stability holds both in the future and in the past, i.e. that all eigenvalues of the Jacobian matrix $d\mathbf{f}(0)$ have purely imaginary values; and in this section we will implicitly assume that the operator $d\mathbf{f}(0)$ is nilpotent. Our main task will be to find sufficient conditions for the system of equations (1.1) to have a nonexponential particular solution $\mathbf{x}(t) \to \mathbf{0}$, either as $t \to +\infty$ or as $t \to -\infty$.

In this section we will settle an even more general question. Consider the nonautonomous smooth system of differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t). \tag{1.3}$$

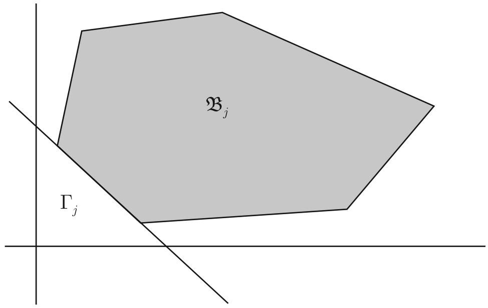

Fig. 1.2 The j-th Newton polytope

and let the vector field $\mathbf{f}(\mathbf{x}, t)$ be such that its components $f^1, \ldots, f^n$ can be represented as formal power series

$$f^i = \sum_{i_1,\ldots,i_n,i_{n+1}} f^j_{i_1,\ldots,i_n,i_{n+1}} (x^1)^i_1 \ldots (x^n)^i_n t^{i_{n+1}}, \tag{1.4}$$

where the indices $i_1, \ldots, i_n$ are nonnegative integers and i_{n+1} is an integer. Thus we waive the requirement $\mathbf{f}(\mathbf{0}, t) \equiv \mathbf{0}$. For systems of this sort we will find conditions sufficient to guarantee the existence of particular solutions whose components have a generalized power asymptotic either as $t \to \pm 0$ or as $t \to \pm\infty$. Here the smoothness requirements on the right hand terms of (1.3)—even over the Cartesian product of some small neighborhood of $\mathbf{x} = \mathbf{0}$ with a set of the form $0 < \underline{t} < |t| < \overline{t} < +\infty$—may turn out to be inadequate. For this reason we will always assume that the right hand members are infinitely differentiable vector functions over that region of space where the desired solution should be found.

A critical point of a system of differential equations is called *elementary* [32] if the Jacobian matrix of the right hand elements of this system, computed at the critical point, has at least one nonzero eigenvalue. We mention some definitions that are used in the theory of stability of systems with nonelementary critical points [32, 100].

Definition 1.1.1. Let $f^j_{i_1,\ldots,i_n,i_{n+1}} (x^1)^i_1 \ldots (x^n)^i_n t^{i_{n+1}}$ be some nontrivial monomial in the expansion (1.4), the j-th component of the nonautonomous vector field $\mathbf{f}(\mathbf{x},t)$. We consider in $\mathbb{R}^{n+1}$ a geometric point with coordinates $(i_1, \ldots, i_n, i_{n+1})$. The collection of all such points is called the j-th *Newton diagram* $\mathfrak{D}_j$ of the vector field $\mathbf{f}(\mathbf{x},t)$, and its convex hull the j-th *Newton polytope* $\mathfrak{B}_j$ (see Fig. 1.2).

Definition 1.1.2. The vector field $\mathbf{f} = \mathbf{f}_{\mathrm{q}}(\mathbf{x},t)$ is called *quasihomogeneous* of degree $\mathrm{q} \in \mathbb{N}$, $\mathrm{q} \neq 1$, with exponents $s_1, \ldots, s_n \in Z$, where the numbers $\mathrm{q} - 1, s_1, \ldots, s_n$ don't have any nontrivial common divisor, if for arbitrary $(\mathbf{x}, t) \in \mathbb{R}^{n+1}$, $t \neq 0$, $\lambda \in \mathbb{R}^+$ the following condition is satisfied:

$$f^j_{\mathrm{q}}(\lambda^{s_1} x^1, \ldots, \lambda^{s_n} x^n, \lambda^{1-\mathrm{q}} t) = \lambda^{\mathrm{q}+s_j-1} f^j_{\mathrm{q}}(x^1, \ldots, x^n, t).$$

It is useful to note that the Newton diagram $\mathfrak{D}_j$ and the Newton polytopes $\mathfrak{B}_j$ of a quasihomogeneous vector field lie on hyperplanes determined by the equations

$$s_1 i_1 + \ldots + s_j(i_j - 1) + \ldots + s_n i_n + (1 - \mathrm{q})(i_{n+1} + 1) = 0. \tag{1.5}$$

The requirement $\mathrm{q} > 1$ is generally redundant: if we examine an individual vector field, then from the sign changes of the quantities $s_1, \ldots, s_n$ we can ascertain that the inequality $\mathrm{q} > 1$ is satisfied.

Definition 1.1.3. Let Γ be some r-dimensional face of the j-th Newton polytope $\mathfrak{B}_j$ $(0 \le r < n + 1)$, lying in the hyperplane given by Eq. (1.5). The face Γ_j will be called *positive* if an arbitrary point $(i_1, \ldots, i_n, i_{n+1}) \in \mathfrak{B}_j \setminus \Gamma_j$ lies in the positive half-space determined by this hyperplane (see Fig. 1.2), i.e. if it satisfies the inequality

$$s_1 i_1 + \cdots + s_j(i_j - 1) + \cdots + s_n i_n + (1 - \mathrm{q})(i_{n+1} + 1) > 0.$$

Conversely, if for each point in $\mathfrak{B}_j \setminus \Gamma_j$ the opposite inequality

$$s_1 i_1 + \cdots + s_j(i_j - 1) + \cdots + s_n i_n + (1 - \mathrm{q})(i_{n+1} + 1) < 0,$$

holds, then the face Γ_j is said to be *negative*.

Definition 1.1.4. We say that the vector field $\mathbf{f}(\mathbf{x}, t)$ is *semi-quasihomogeneous* if it can be represented in the form

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t) + \mathbf{f}^*(\mathbf{x}, t),$$

where $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is some quasihomogeneous vector field determined by all positive, or all negative, faces in the Newton polytope of the total system, and where the exponents of the "perturbing" field $\mathbf{f}^*(\mathbf{x}, t)$ lie strictly in the interior of these polytopes (see Fig. 1.2). We also say that the vector field under consideration is *positive semi-quasihomogeneous* if its quasihomogeneous truncation $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is chosen with positive faces; in the opposite case we call the considered field *negative semi-quasihomogeneous*.

In the "positive" case we look for an asymptotic solution of system (1.1) as $t \to \pm\infty$, and in the "negative" case as $t \to \pm 0$.

We introduce the following notation. Let $\mathbf{S}$ be some diagonal matrix

$$\mathrm{diag}(s_i, \ldots, s_n)$$

with integer entries and λ some real number. The symbol $\lambda^{\mathbf{S}}$ denotes the diagonal matrix $\mathrm{diag}(\lambda^{s_1}, \ldots \lambda^{s_n})$.

It is clear that Definition 1.1.2 of the quasihomogeneous vector field $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ can be written in the following equivalent form: for arbitrary $(\mathbf{x}, t) \in \mathbb{R}^{n+1}$, $t \neq 0$ and arbitrary real λ the following equation must be satisfied:

$$\mathbf{f}_{\mathrm{q}}(\lambda^{\mathbf{S}}\mathbf{x}, \lambda^{1-\mathrm{q}}t) = \lambda^{\mathbf{S}+(\mathrm{q}-1)\mathbf{E}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t), \tag{1.6}$$

where $\mathbf{E}$ is the identity matrix.

The simplest example of a quasihomogeneous vector field is perhaps the homogeneous vector field where $\mathbf{S} = \mathbf{E}$. We will consider other examples of quasihomogeneous vector fields a bit later.

From (1.6) it follows that a quasihomogeneous system of differential equations, i.e. one for which the right side is a quasihomogeneous vector field that is invariant under the quasihomogeneous group of dilations

$$t \mapsto \mu^{-1}t,\ \mathbf{x} \mapsto \mu^{\mathbf{G}}\mathbf{x},\ \text{where}\mathbf{G} = \alpha\mathbf{S},\ \alpha = \frac{1}{\mathrm{q}-1}. \tag{1.7}$$

It is likewise easy to see that if a system of differential equations is semi-quasihomogeneous—i.e. if its right side is a semi-quasihomogeneous vector field—then under the action of the group (1.7) it will assume the form

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t) + \mathbf{f}^{*}(\mathbf{x}, t, \mu). \tag{1.8}$$

Here $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is a quasihomogeneous vector field, chosen with positive or with negative faces for the Newton polytopes, and $\mathbf{f}^{*}(\mathbf{x}, t, \mu)$ represents a formal power series in μ^{β}, $\beta \in \mathbb{R}\backslash\{0\}$, $\alpha = |\beta|$ with zero free term. If $\beta > 0$, then $\mathbf{f}(\mathbf{x}, t)$ is positive semi-quasihomogeneous, and if $\beta < 0$ the vector field considered is negative semi-quasihomogeneous.

Setting $\mu = 0$ in (1.8) in the positive semi-quasihomogeneous case and $\mu = \infty$ in the negative semi-quasihomogeneous case, we obtain a "*truncated*" or a "*model*" system, as it is also called:

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t). \tag{1.9}$$

We note that the reasoning introduced above can be used for the determination of quasihomogeneous and semi-quasihomogeneous systems. If we aren't interested in the specific numerical values of the powers of a quasihomogeneous system, then its qualitative structure is completely determined by the matrix of the transformation $\mathbf{G}$.

In essence, in order to know the parameters of a semi-quasihomogeneous system, it is sufficient to determine $\beta \neq 0$ and $\mathbf{G}$. For this, it is even possible to avoid the rather burdensome requirement $\mathrm{q} \neq 1$. In the sequel we will as before designate the chosen quasihomogeneous truncation by $\mathbf{f}_{\mathrm{q}}$, whereby q signifies the *quasihomogeneity* property rather than the homogeneity degree.

We now introduce a more general definition of quasihomogeneity and semi-quasihomogeneity of vector fields that allows us to avoid both the concept of the degree of quasihomogeneity and the use of techniques that are associated with Newton polytopes.

We consider an $(n+1)$–dimensional *Fuchsian* system of differential equations of the form

$$\mu \frac{d\mathbf{x}}{d\mu} = \mathbf{G}\mathbf{x}, \quad \mu \frac{dt}{d\mu} = -t, \tag{1.10}$$

where $\mathbf{G}$ is some real matrix and where the flow will be denoted by

$$t \mapsto \mu^{-1} t, \quad \mapsto \mu^{\mathbf{G}}\mathbf{x}. \tag{1.11}$$

We recall that the right hand side of a linear Fuchsian system has singularities in the form of simple poles in the independent variable (in the present instance, in μ).

Definition 1.1.5. The vector field $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is quasihomogeneous if the corresponding system of differential equations is invariant with respect to the action of the phase flow (1.11) of the Fuchsian system (1.10).

This definition is equivalent to Definition 1.1.2 if we set

$$\mathbf{G} = \alpha\mathbf{S}, \;\; \lambda = \mu^{\alpha}, \;\; \textbf{where } \alpha = 1/(\mathrm{q}-1).$$

The definition of semi-quasihomogeneity can then be reformulated in the following way.

Definition 1.1.6. The system of differential equations (1.3) is called *semi-quasihomogeneous* if, under the action of the flow (1.11), its right hand side is transformed into (1.8), where $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ is some quasihomogeneous vector field in the sense of Definition 1.1.5 and where $\mathbf{f}^*(\mathbf{x}, t, \mu)$ is a formal power series with respect to μ^{β}, $\beta \in R \setminus \{0\}$, without a free term. If $\beta > 0$, the system (1.3) will be called *positive semi-quasihomogeneous*; it is called *negative semi-quasihomogeneous* for $\beta < 0$.

We now look at a few simple examples.

Example 1.1.1. The system of ordinary differential equations in the plane

$$\dot{x} = y, \quad \dot{y} = x^2$$

is quasihomogeneous of degree $\mathrm{q} = 2$ with exponents $s_x = 2$, $s_y = 3$. In fact, an arbitrary system of the form

$$\dot{x} = y + f(x, y), \quad \dot{y} = x^2 + g(x, y),$$

where f contains only quadratic or higher terms and g only cubic or higher terms, is semi-quasihomogeneous.

Example 1.1.2. The system of differential equations

$$\dot{x} = (x^2 + y^2)(ax - by), \quad \dot{y} = (x^2 + y^2)(ay + bx)$$

is clearly (quasi-)homogeneous of degree q $= 3$ with exponents $s_x = s_y = 1$. It is, moreover, invariant with respect to the actions of the phase flows of the following family of Fuchsian systems

$$\mu \frac{dx}{d\mu} = \frac{1}{2}x + \delta y, \ \mu \frac{dy}{d\mu} = \frac{1}{2}y - \delta x, \ \mu \frac{dt}{d\mu} = -t.$$

Therefore this system is quasihomogeneous in accordance with Definition 1.1.5 Furthermore, an arbitrary system of the form

$$\dot{x} = (\rho + f(\rho))(ax - by), \quad \dot{y} = (\rho + f(\rho))(ay + bx),$$

where $\rho = x^2 + y^2$ and $f(\rho) = o(\rho)$ as $\rho \to 0$, is semi-quasihomogeneous.

Example 1.1.3. The Emden-Fowler equation [14], which describes the expansion process of a polytropic gas,

$$t\ddot{x} + 2\dot{x} - atx^p = 0, \quad p \geq 2,$$

offers the simplest example of a nonautonomous quasihomogeneous system.

This equation can be rewritten as a system of two equations

$$\dot{x} = t^{-2}y, \quad \dot{y} = at^2x^p,$$

which are quasihomogeneous of degree q $= p$, with exponents $s_x = 2$, $s_y = 3 - p$.

The need to extend the concepts of quasihomogeneity and semi-quasihomogeneity is dictated by the following circumstance. Situations are frequently encountered where the truncation chosen with the aid of the group of quasihomogeneous coordinate dilations does not have solutions with the desired asymptotic properties. Thus for the choice of "significant" truncations we resort to a procedure whose algorithm is rather fully described in the book by A.D. Bryuno [32]. One of the most important facets of this algorithm is the application to (1.1) of a "birational" transformation that is defined on some cone with singular point, resulting in the possibility of selecting, from the full system, a quasihomogeneous subsystem that has one or another necessary property. The very introduction of the definition extends a priori the best understood truncations. In analyzing planar systems in the neighborhood of a critical point we likewise frequently resort to certain other procedures, specifically to a σ–process or a procedure for blowing up singularities (see the book [7] and also the survey [3]).

We will now examine more closely the quasihomogeneous truncation (1.9) for the original system of differential equations (1.3). Due to quasihomogeneity, in a sufficiently general setting (1.10) has a particular solution in the form of a "quasihomogeneous ray"

$$\mathbf{x}^{\gamma}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{x}_0^{\gamma}, \tag{1.12}$$

where $\gamma = \pm 1$ and $\mathbf{x}_0^{\gamma}$ is a nonzero real vector. Below we will clarify the concept "sufficiently general setting".

If (1.9) has a particular solution of the form (1.12), then the vector $\mathbf{x}_0^{\gamma}$ must satisfy the following algebraic system of equations:

$$-\gamma \mathbf{G}\mathbf{x}_0^{\gamma} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}, \gamma). \tag{1.13}$$

We then say that the vector $\mathbf{x}_0^{\gamma}$ is an *eigenvector* of the quasihomogeneous vector field $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ of the induced quasihomogeneous system $\mathbf{G}$.

The fundamental result of this section amounts to the fact that the system (1.3) has a particular solution that, in a certain sense, is reminiscent of the asymptotic particular solution (1.12) of the truncated system (1.9).

The following assertion generalizes a theorem in [60].

Theorem 1.1.2. *Suppose that system (1.3) is semi-quasihomogeneous and that there exists a nonzero vector* $\mathbf{x}_0^{\gamma} \in \mathbb{R}^n$ *and a number* $\gamma = \pm 1$ *such that the equality (1.13) holds. Then the system (1.3) has a particular solution whose principal component has the asymptotic* $(\gamma t)^{\mathbf{G}}\mathbf{x}_0^{\gamma}$ *as* $t^{\chi} \to \gamma \times \infty$*, where* $\chi = \operatorname{sign} \beta$ *is the "semi-quasihomogeneity" index.*

Before proving the above theorem, we discuss its hypothesis. Demonstrating the existence of particular asymptotic solutions for the complete system reduces to the search for eigenvectors of the truncated system. With this interpretation, the hypothesis of the theorem recalls the Lyapunov hypothesis for the existence of exponential solutions obtained from the first approximation system. In our case, this role of the linearized system is played by the quasihomogeneous truncation obtained. The particular solution (1.12) of the truncated system will correspond to the exponential particular solution of the system of first approximation, whose existence in itself implies the existence of a particular solution of the complete system with exponential asymptotic.

In the Lyapunov case the determination of eigenvectors reduces to a well known problem in linear algebra. But in the nonlinear case under investigation the search for such eigenvectors may turn out not to be an easy matter. Nonetheless, their existence in a number of instances follows from rather simple geometric considerations.

From Definition 1.1.5 we have the identity

$$\mathbf{f}_{\mathrm{q}}(\mu^{\mathbf{G}}\mathbf{x}, \mu^{-1}t) = \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t). \tag{1.14}$$

Staying with the simple case where the truncated system (1.9) is autonomous: $\mathbf{f}_q(\mathbf{x}, t) \equiv \mathbf{f}_q(\mathbf{x})$, where $\mathbf{x} = \mathbf{0}$ is a critical point of this vector field and the matrix $\mathbf{G}$ is nondegenerate. The vector $\mathbf{x}_0^\gamma$ will be sought in the following form:

$$\mathbf{x}_0^\gamma = \mu^{\mathbf{G}} \mathbf{e}^\gamma,$$

where μ is a positive number and $\mathbf{e}^\gamma \in \mathbb{R}^n$ is a unit vector, i.e. $\|\mathbf{e}^\gamma\| = 1$.

Using (1.14), we rewrite Eq. (1.13) in the form

$$\mathbf{G}^{-1}\mathbf{f}_q(\mathbf{e}^\gamma) = -\gamma\mu^{-1}\mathbf{e}^\gamma. \tag{1.15}$$

Let $\mathbf{x} = \mathbf{0}$ be a critical point of the vector field $\mathbf{f}_q$. We consider the vector field

$$\mathbf{g}(\mathbf{x}) = \mathbf{G}^{-1}\mathbf{f}_q(\mathbf{x}).$$

The following lemma amounts to a "quasihomogeneous version" of assertions from [59, 62, 64, 111].

Lemma 1.1.1. *Let* $\mathbf{x} = \mathbf{0}$ *be the unique singular point of the autonomous quasihomogeneous vector field* $\mathbf{f}_q : \mathbb{R}^n \to \mathbb{R}^n$. *We have:*

1. *If the index* i *of the vector field* $\mathbf{g}$ *at the point* $\mathbf{x} = \mathbf{0}$ *is even, then* $\mathbf{f}_q$ *has eigenvectors with both positive and negative eigenvalues* γ*,*
2. *If the dimension* n *of phase space is odd, then* $\mathbf{f}_q$ *has at least one eigenvector with either a positive or with a negative eigenvalue.*

Proof. To prove the first assertion, we consider the Gauss map

$$\boldsymbol{\Gamma}(\mathbf{P}) = \frac{\mathbf{g}(\mathbf{P})}{\|\mathbf{g}(\mathbf{P})\|}, \quad \mathbf{P} \in S^{n-1}$$

of the unit sphere S^{n-1} to itself. The index of the vector field $\mathbf{g}$ at the point $\mathbf{x} = 0$ is equal to the degree of the map $\boldsymbol{\Gamma}$. Consequently the degree of $\boldsymbol{\Gamma}$ is different from $(-1)^{n-1}$ and therefore $\boldsymbol{\Gamma}$ has a fixed point [139, 191]. This means that there exists a vector $\mathbf{e}^- \in \mathbb{R}^\mathbf{n}$ such that

$$\mathbf{g}(\mathbf{e}^-) = \mu^{-1}\mathbf{e}^-, \quad \mu = \|\mathbf{g}(\mathbf{e}^-)\|^{-1},$$

i.e., (1.15) holds with $\gamma = -1$.

In order to further show that the given map also has the antipodal point ($\gamma = +1$), we need to look at the antipodal map

$$\boldsymbol{\Gamma}_-(\mathbf{P}) = -\boldsymbol{\Gamma}(\mathbf{P}),$$

whose degree is also even.

We thus observe that the degree Γ will always be even, provided that each element of $\Im\Gamma$ has an even number of inverse images.

In order to prove the conclusion of item 2, it suffices to examine the tangent vector field

$$\mathbf{v}(\mathbf{P}) = \mathbf{g}(\mathbf{P}) - \langle \mathbf{g}(\mathbf{P}), \mathbf{P} \rangle \mathbf{P}, \quad \mathbf{P} \in S^{n-1}$$

on the sphere S^{n-l} of even dimension (here and in the sequel the symbol $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$), on which there does not exist a smooth nonzero vector field [139, 191]. We can thus find a vector $\mathbf{e}^\gamma \in \mathbb{R}^n$ such that

$$\mathbf{g}(\mathbf{e}^\gamma) = -\gamma\mu^{-1}\mathbf{e}^\gamma, \ \mu = \left| \langle \mathbf{g}(\mathbf{e}^\gamma), \mathbf{e}^\gamma \rangle \right|^{-1}, \ \gamma = -\mathrm{sign}\langle \mathbf{g}(\mathbf{e}^\gamma), \mathbf{e}^\gamma \rangle.$$

Inasmuch as the vector $\mathbf{g}(\mathbf{e}^\gamma)$ is parallel to $\mathbf{e}^\gamma$ and $\mathbf{x} = \mathbf{0}$ is the unique singular point of the vector field $\mathbf{g}$, we have $\mu^{-1} \neq 0$.

The lemma is thus established.

The index i of a vector field $\mathbf{g}$ will be even if, for instance, $\mathbf{f}_\mathrm{q}$ is a homogeneous vector field of even degree with an isolated singular point. Thus a system with a quadratic right hand side *in general* has linear asymptotic solutions. In order to clarify this situation, we consider a simple example.

Example 1.1.4. The conditions for the existence of nontrivial particular solutions of the form

$$x^\gamma(t) = (\gamma t)^{-1} x_0^\gamma, \quad y^\gamma(t) = (\gamma t)^{-1} y_0^\gamma$$

for a system of differential equations in the plane

$$\dot{x} = P_1(x, y) = a_1 x^2 + b_1 xy + c_1 y^2, \ \dot{y} = P_2(x, y) = a_2 x^2 + b_2 xy + c_2 y^2$$

may be expressed, for example, in the form of the inequalities:

$$\begin{aligned} \Delta &= a_1^2 c_2^2 + c_1^2 a_2^2 + a_1 c_1 b_2^2 + b_1^2 a_2 c_2 - \\ &\quad - a_1 b_1 b_2 c_2 - b_1 c_1 a_2 b_2 - 2 a_1 c_1 a_2 c_2 \neq 0 \\ \delta &= (a_1^2 + a_2^2)(c_1^2 + c_2^2) \neq 0. \end{aligned}$$

Since, for fixed y, the resultant of the polynomials $P_1(x, y)$ and $P_2(x, y)$, as functions of x, will equal $y^4 \Delta$, we have that the inequality $\Delta \neq 0$ is equivalent to the condition that $P_1(x, y)$ and $P_2(x, y)$, with $y \neq 0$, don't have (even a complex!) common root [187]. If $a_1^2 + a_2^2 \neq 0$, then the polynomials $P_1(x, 0)$ and $P_2(x, 0)$ are simultaneously zero only for $x = 0$. Analogously we consider the case of fixed x. Therefore, by the inequalities established above, the origin $x = y = 0$ will be an isolated critical point of the system considered. It is obvious that, in the six-dimensional parameter space, the dimension of the set for which $\Delta = 0$ or $\delta = 0$ equals five, and thus has measure zero.

Proof of Theorem 1.1.2.

First step. Construction of a formal solution.

We seek a formal solution for the system of differential equations (1.3) in the form of a series

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k(\ln(\gamma t))(\gamma t)^{-k\beta}, \tag{1.16}$$

where the $\mathbf{x}_k$ are polynomial functions of $\ln(\gamma t)$.

We should thus note that the series (1.16) is analogous to series that are used in applying Frobenius's method to the solution of linear systems of differential equations in the neighborhood of a regular singular point [42].

We note the analog of formulas (1.2) and (1.16). The sum in (1.16) is obtained from the corresponding sum in (1.2) with the aid of a logarithmic substitution in time, $t \mapsto \ln(\gamma t)$, i.e. the principles for the construction of solutions of a strictly nonlinear system are analogous to those used in Lyapunov's first method.

We note that if the semi-quasihomogeneity index χ equals $+1$, then the powers of t under the summation sign in (1.16) will be negative; they will be positive in the opposite case.

We show that such a formal particular solution exists. We use the fact that the right hand side of (1.3) can be developed in a formal series in the "quasihomogeneous" forms

$$\mathbf{f}(\mathbf{x}, t) = \sum_{m=0} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x}, t).$$

The following identity holds, generalizing (1.14):

$$\mathbf{f}_{\mathrm{q}+\chi m}(\mu^{\mathbf{G}}\mathbf{x}, \mu^{-1}t) = \mu^{\mathbf{G}+(1+\beta m)\mathbf{E}}\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x}, t). \tag{1.17}$$

Using (1.17), we make a substitution of the dependent and independent variables:

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(s), \quad s = (\gamma t)^{-\beta},$$

whereby the original system of equations (1.3) takes the form

$$-\gamma\beta s\mathbf{y}' = \boldsymbol{\gamma}\mathbf{G}\mathbf{y} + \sum_{m=0} s^m \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}, \gamma), \tag{1.18}$$

where the prime indicates differentiation with respect to the new independent variable s, and the formal solution (1.16) is converted to

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{x}_k(-1/\beta \ln s)s^k. \tag{1.19}$$

We substitute (1.19) into (1.18) and equate coefficients of s^k. Supposing that the first coefficient $\mathbf{x}_0$ is fixed, for $k = 0$ we obtain

$$-\boldsymbol{\gamma}\mathbf{G}\mathbf{x}_0 = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0, \gamma).$$

Therefore the existence of the coefficient $\mathbf{x}_0 = \mathbf{x}_0^\gamma$ of the series (1.19) is guaranteed by the conditions of the theorem (see (1.13)). For $k \geq 1$ we have the following system of equations:

$$\frac{d\mathbf{x}_k}{d\tau} - \mathbf{K}_k \mathbf{x}_k = \mathbf{\Phi}_k(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}), \tag{1.20}$$

where the $\mathbf{\Phi}_k$ are certain polynomial vector functions of their arguments, where $\tau = -1/\beta \ln s = \ln(\gamma t)$, $\mathbf{K}_k = k\beta \mathbf{E} + \mathbf{K}$, and where

$$\mathbf{K} = \mathbf{G} + \gamma d_{\mathbf{x}} \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma, \gamma)$$

is the so-called *Kovalevsky matrix* [197].

If it is assumed that all coefficients up to the k–th have been found as certain polynomials in τ, then Φ_k is represented in terms of some known polynomials in τ. The system obtained can be regarded as a system of ordinary differential equations with constant coefficients and polynomial right hand side which, as is known, always has a polynomial particular solution $\mathbf{x}_k(\tau)$, whose degree equals $N_k + S_k$, where N_k is the degree of Φ_k as a polynomial in τ and where S_k is the multiplicity of zero as an eigenvalue of $\mathbf{K}_k$. In this way, the determination of all the coefficients of the series (1.19) can be realized by induction. The formal construction of particular asymptotic solutions for (1.3) is thus complete.

Generally speaking, the coefficients $\mathbf{x}_k(\tau)$ are not determined uniquely, but only within the addition of polynomial functions belonging to the kernel of the differential operator $\frac{d}{d\tau} - \mathbf{K}_k$. Therefore, at each step, we obtain some family of polynomial solutions of (1.20), dependent on S_k arbitrary constants. For this reason, our algorithm generally yields not just one particular solution of (1.3), but a whole manifold of such formal particular solutions.

We note that the expansion (1.16) won't contain powers of the logarithmic "time" γt in two instances: (a) if, among the eigenvalues of the Kovalevsky matrix $\mathbf{K}$, there is no number of the form $-k\beta$, $k \in \mathbb{N}$, and (b) such a number exists, but the projections of the vectors $\mathbf{\Phi}_k$ (which by assumption don't depend on τ) onto the kernel of the operator with matrix $\mathbf{K}_k$ are necessarily zero. The arithmetic properties of the eigenvalues of the Kovalevsky matrix play a role in testing systems of differential equations by the ARS-test. In the literature these eigenvalues are also called *resonances* [1] or *Kovalevsky indices* [197].

We conclude, finally, by looking at some properties of the eigenvalues of the matrix $\mathbf{K}$.

Lemma 1.1.2. *If the truncated system is autonomous, then* -1 *must belong to the spectrum* $\mathbf{K}$.

Proof. Differentiating (1.14) with respect to μ and setting $\mu = 1$, we obtain:

$$d\mathbf{f}_{\mathrm{q}}(\mathbf{x})\mathbf{G}\mathbf{x} = (\mathbf{G} + \mathbf{E})\mathbf{f}_{\mathrm{q}}(\mathbf{x}). \tag{1.21}$$

We denote the vector $\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma})$ by $\mathbf{p}$. Then, using the identity (1.21) and Eq. (1.13), we find

$$\mathbf{K}\mathbf{p} = \mathbf{G}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}) - d\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma})\mathbf{G}\mathbf{x}_0^{\gamma} = -\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}) = -\mathbf{p}.$$

The lemma is thus proved.

This result does not hold in the nonautonomous case.

Example 1.1.5. We return to the Emden-Fowler equation (see Example 1.1.3). The system considered has, for even p, the obvious solution

$$x(t) = x_0 t^{-2\beta}, \quad y(t) = -2\beta x_0 t^{(p-3)\beta},$$
$$\beta = 1/(\rho - 1), \quad x_0 = \left(\frac{2(3-p)}{a(p-1)^2}\right)^{\beta}.$$

The eigenvalues of the Kovalevsky matrix that correspond to this solution are

$$\rho_{1,2} = \frac{\beta}{2}\left(5 - p \pm \sqrt{1 + 16p - 7p^2}\right).$$

It is clear that $\rho_{1,2}$ doesn't reduce to -1 for any choice of p.

Thus, in the autonomous case, the presence of logarithms in the corresponding asymptotic solutions, when (1.3) is positive semi-quasihomogeneous in the sense of Definition 1.1.4, represents the general case, since for $k = \mathrm{q} - 1$ we have degeneracy of the matrix $\mathbf{K}_k$.

Remark 1.1.1. The properties considered for the eigenvalues of the Kovalevsky matrix clearly don't change if we should likewise consider complex solutions $\mathbf{x}_0^{\gamma}$ of system (1.13), which accordingly reduces to the series (1.16) with complex coefficients.

The ensuing step in the proof of Theorem 1.1.2 should consist of a proof of convergence of (1.14) or of an asymptotic analysis of its partial sums. These questions are quite profound and so we dedicate a separate section to their discussion.

1.2 Problems of Convergence

In the preceding section we proved, under the hypothesis of Theorem 1.1.2, that Eq. (1.3) has a formal particular solution in the form of series (1.16). If we should succeed in proving convergence of these series on some time interval—or be able to show that they are asymptotic expansions of some solution $\mathbf{x}(t)$ of class $\mathbf{C}^{\infty}[T, +\infty)$ in the positive semi-quasihomogeneous case, or of class $\mathbf{C}^{\infty}(0, T^{-1}]$ in the negative semi-quasihomogenous case (with positive γ), where T is a sufficiently large positive number—then Theorem 1.1.2 would be established. Thus we come to the inevitable

Second step. Proof of the existence of a particular solution of system (1.3) with asymptotic expansion (1.16).

The proof of convergence or divergence of the series (1.16) turns out to be a rather difficult task. For instance, the unproved assertions in the previously cited paper of G.V. Kamenkov [93] are concerned with just these convergence questions. We should likewise note that it makes sense to talk about convergence only in the analytic case, where the series (1.4) representing the right side of the system under consideration converges over some complex domain. The standard method of proof in similar situations is by using majorants, which always involves elaborate computations. For the moment we will avoid the question of convergence of (1.16) and prove that these series approximate some smooth solution of the system (1.3) with the required asymptotic properties. The results introduced below originally appeared in the paper [115].

We first put (1.3) into the form (1.18) and change the "time scale":

$$s = \varepsilon\xi, \quad 0 < \varepsilon \ll 1.$$

As a result, this system is rewritten in the form

$$-\gamma\beta\xi\frac{d\mathbf{y}}{d\xi} = \gamma\mathbf{G}\mathbf{y} + \sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{q+\chi m}(\mathbf{y}, \gamma). \tag{1.22}$$

If, for the construction of a formal solution of (1.3), it is merely required that the right sides (1.3) be represented as a formal power series (1.4), then we consequently only require that the right sides of (1.18) and the relation (1.22) be functions of class $\mathbf{C}^\infty$, at least in some small neighborhood of the point $s = 0$, $\mathbf{y} = \mathbf{x}_0^\gamma$.

For $\varepsilon = 0$, this system reduces to a truncated system corresponding to (1.9):

$$-\beta\xi\frac{d\mathbf{y}}{d\xi} = \mathbf{G}\mathbf{y} + \gamma\mathbf{f}_q(\mathbf{y}, \gamma),$$

which has the particular solution $\mathbf{y}_0(\xi) = \mathbf{x}_0^\gamma$, corresponding to the "quasihomogeneous ray" (1.12).

After the transformation described, the K–th partial sum of the series (1.19) takes the form:

$$\mathbf{y}_K^\varepsilon(\xi) = \sum_{k=0}^{K}\varepsilon^k\mathbf{x}_k\left(-\frac{1}{\beta}\ln(\varepsilon\xi)\right)\xi^k,$$

from which it is obvious that, as $\varepsilon \to +0$, this sum converges to $\mathbf{y}_0(\xi) = \mathbf{x}_0$ uniformly on the interval $[0, 1]$.

Let $K \in \mathbb{N}$ be large enough so that $-\beta K < \Re\rho_i$, $i = 1, \ldots, n$, where the ρ_i are the eigenvalues of the Kovalevsky matrix $\mathbf{K}$.

We will look for a particular solution of (1.22) of the form

$$\mathbf{y}(\xi) = \mathbf{y}_K^\varepsilon(\xi) + \mathbf{z}(\xi)$$

for sufficiently small $\varepsilon > 0$ on the interval $[0, 1]$, with initial condition $\mathbf{y}(+0) = \mathbf{0}$, where $\mathbf{z}(\boldsymbol{\xi})$ has the asymptotic $\mathbf{z}(\xi) = O(\xi^{K+\delta})$ as $\xi \to +0$ and where $\delta > 0$ is fixed but sufficiently small.

We write (1.22) in the form of an equation on a Banach space:

$$\begin{gathered}\Phi(\varepsilon, \mathbf{z}) = \mathbf{0}, \\ \Phi(\varepsilon, \mathbf{z}) = \beta\xi \tfrac{d}{d\xi}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \mathbf{G}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \\ +\gamma \textstyle\sum_{m=0} \varepsilon^m \xi^m \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}, \gamma).\end{gathered} \tag{1.23}$$

We regard $\Phi(\varepsilon, z)$ as a mapping

$$\Phi\colon (0, \varepsilon_0) \times \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta},$$

where:

$\mathfrak{B}_{1,\Delta}$ is the Banach space of vector functions $\mathbf{z} : [0, 1] \to \mathbb{R}^n$ that are continuous on $[0, 1]$ along with their first derivatives, and for which the norm

$$\|z\|_{1,\Delta} = \sup_{[0,1]} \xi^{-\Delta}(\|z(\xi)\| + \xi\|z'(\xi)\|)$$

is finite (here the prime indicates differentiation with respect to ξ),

and where

$\mathfrak{B}_{0,\Delta}$ is the Banach space of vector functions $u\colon [0, 1] \to \mathbb{R}^n$ that are continuous on $[0, 1]$ and for which the norm

$$\|u\|_{0,\Delta} = \sup_{[0,1]} \xi^{-\Delta}\|u(\xi)\|,$$

is finite, where $\Delta = K + \delta$.

We note several properties of the map Φ:

(a) $\Phi(0, \mathbf{0}) = \beta\xi \frac{d}{d\xi} y_0(\xi) + G y_0(\xi) + \gamma f_{\mathrm{q}}(y_0(\xi), \gamma) = 0$,
(b) Φ is continuous for $\varepsilon, \mathbf{z}$ in $(0, \varepsilon_0) \times \mathfrak{U}_{1,\Delta}$, where $\mathfrak{U}_{1,\Delta}$ is some neighborhood of zero in $\mathfrak{B}_{1,\Delta}$,
(c) Φ is strongly differentiable with respect to $\mathbf{z}$ on$(0, \varepsilon_0) \times \mathfrak{U}_{1,\Delta}$, and its Frechet derivative:

$$\nabla_{\mathbf{z}}\Phi(\varepsilon, \mathbf{z})\mathbf{h} = \beta\xi \tfrac{d}{d\xi}\mathbf{h} + \mathbf{G}\mathbf{h} + \gamma \textstyle\sum_{m=0} \varepsilon^m \xi^m d\mathbf{f}_{\mathrm{q}+\boldsymbol{\chi}\mathbf{m}}(\mathbf{y}_{\mathbf{J}}^{\boldsymbol{\varepsilon}} + \mathbf{z}, \boldsymbol{\gamma})\mathbf{h}, \\ \mathbf{h} \in \mathfrak{B}_{\mathbf{1},\boldsymbol{\Delta}}$$

is a bounded operator, continuously dependent on ε, z.

(d) The assertions (a), (b), (c) are rather obvious. The following assertion is less trivial.

Lemma 1.2.1. *The operator* $\nabla_{\mathbf{z}}\Phi(0,\mathbf{0})\colon \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}$

$$\nabla_{\mathbf{z}}\Phi(0,\mathbf{0}) = \beta\xi\frac{d}{d\xi} + \mathbf{K}$$

has a bounded inverse.

Proof. We prove the existence of a unique particular solution of the system of differential equations

$$\beta\xi\frac{dz}{d\xi} + \mathbf{Kz} = \mathbf{u}, \quad \mathbf{u} \in \mathfrak{B}_{0,\Delta}, \tag{1.24}$$

with initial condition $\mathbf{z}(+0) = 0$, that satisfies the inequality

$$\|\mathbf{z}\|_{1,\Delta} \le C\|\mathbf{u}\|_{0,\Delta}, \tag{1.25}$$

where the constant $C > 0$ is independent of $\mathbf{u} \in \mathfrak{B}_{0,\Delta}$.

Since the space $\mathbb{R}^n$ decomposes into the direct sum of Jordan subspaces invariant under the linear operator with matrix $\mathbf{K}$, the bound (1.25) suffices for proof in the particular case where $\mathbf{K}$ is a complex Jordan matrix with eigenvalue ρ, $\operatorname{Re}\rho > -\beta\Delta$.

Setting $\widetilde{\rho} = \beta^{-1}\rho$, $\widetilde{\mathbf{u}} = \beta^{-1}\mathbf{u}$, we transform system (1.24) to scalar form:

$$\begin{aligned} &\xi\tfrac{dz^i}{d\xi} + \widetilde{\rho}z^i + z^{i+1} = \widetilde{u}^i(\xi), \quad i = 1,\ldots,n-1 \\ &\xi\tfrac{dz^n}{d\xi} + \widetilde{\rho}z^n = \widetilde{u}^n(\xi). \end{aligned} \tag{1.26}$$

For the initial conditions $z^1(+0) = \ldots = z^n(+0) = 0$, the solution of system (1.26) assumes the following form:

$$\begin{aligned} z^n(\xi) &= \xi^{-\widetilde{\rho}}\int_0^\xi \eta^{\widetilde{\rho}-1}\widetilde{u}^n(\eta)d\eta, \\ z^i(\xi) &= \xi^{-\widetilde{\rho}}\int_0^\xi \eta^{\widetilde{\rho}-1}(\widetilde{u}^i(\eta) - z^{i+1}(\eta))d\eta, \quad i = 1,\ldots,n-1. \end{aligned}$$

The constructed solution, of course, belongs to the space $\mathfrak{B}_{1,\Delta}$. Because $\operatorname{Re}\widetilde{\rho} > -\Delta$, the following bounds hold:

$$\begin{aligned} \|z^n\|_{1,\Delta} &\le (1 + (1 + |\widetilde{\rho}|))(\Re\widetilde{\rho} + \Delta)^{-1}\|\widetilde{u}^n\|_{0,\Delta}, \\ \|z^i\|_{1,\Delta} &\le (1 + (1 + |\widetilde{\rho}|))(\Re\widetilde{\rho} + \Delta)^{-1}\|\widetilde{u}^i - z^{i+1}\|_{0,\Delta}, \\ &\qquad i = 1,\ldots,n-1. \end{aligned}$$

Noting that $\|\cdot\|_{0,\Delta} \le \|\cdot\|_{1,\Delta}$ and making recursive estimates in each coordinate $z^i(\xi)$, we obtain the inequality (1.25), from which follows the assertion on the existence of a bounded inverse for the operator $\nabla_z\Phi(0,0)$.

The lemma is proved.

Thus all the hypothesis of the abstract theorem on implicit functions [94] is fulfilled so that, for arbitrary $\varepsilon \in (0,\varepsilon_0)$, where $\varepsilon_0 > 0$ is sufficiently small, the

Eq. (1.23) will have a solution in the space $\mathfrak{B}_{1,\Delta}$ that is continuously dependent on ε. Progressing to the variables $\mathbf{y}, s$, we obtain that the differential equation (1.18) has a particular solution $\mathbf{y}(s)$ of class $\mathbf{C}^1[0, \varepsilon]$ with asymptotic

$$\mathbf{y}(s) = \sum_{k=0}^{K} \mathbf{x}_k(-\frac{1}{\beta} \ln s)s^k + o(s^K).$$

In fact, inasmuch as the right side of (1.18) is a smooth vector function, we have that $\mathbf{y} \in \mathbf{C}^\infty[0, \varepsilon]$. Returning to the original variables $\mathbf{x}, t$, we obtain the required smooth existence on $[T, +\infty)$ or on $(0, T^{-1}]$, $T = \varepsilon_0^{-1/\beta}$ (in case $\gamma > 0$), depending on the sign of semi-quasihomogeneity, of a particular solution of the original system with the prescribed principal asymptotic component.

Theorem 1.1.2 is proved.

Remark 1.2.1. The application we have just observed of the implicit function theorem does not permit us to make any claims about the convergence of the series (1.19) nor, consequently, of (1.16).The method of majorant estimates recalled above is perhaps quickest in providing a positive answer to the convergence question for a series given in some neighborhood of $s = 0$. We will examine below a substantially more complicated convergence problem for series constructed in the complex domain.

In our particular situation it can be asserted that the series in question converges, provided that the following requirements are met:

1. The right sides of (1.3) are complex analytic functions on a domain containing the desired solution. Thus the series on the right side of (1.18) represents some holomorphic vector function $\mathbf{y}$ over a neighborhood of $\mathbf{x}_0^\gamma$, for sufficiently small s, $|s| < s_0$.
2. The coefficients of the series (1.19), and consequently those of (1.16), don't depend on the logarithms of the corresponding variables.

This last fact is based on the circumstance that the logarithms in the expansion (1.19) can appear only after the initial steps.

For the proof, it is unavoidable to have to somewhat modify the reasoning introduced above in connection with the implicit function theorem, and precisely to "narrow down" the domain of definition of the map $\mathbf{\Phi}\colon (0, \varepsilon_0) \times \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}$, replacing the spaces $\mathfrak{B}_{1,\Delta}$, $\mathfrak{B}_{0,\Delta}$ by the spaces $\mathfrak{E}_{1,K}$, $\mathfrak{E}_{0,K}$, where

$\mathfrak{E}_{1,K}$ is the Banach space of vector functions $\mathbf{z}\colon \mathcal{K}_1 \to \mathbb{C}^n$, holomorphic on the open unit disk $\mathcal{K}_1 = \{\xi \in \mathbb{C}, |\xi| < 1\}$, continuous on the boundary along with their first derivatives, real on the real axis ($\mathbf{z}(\overline{\boldsymbol{\xi}}) = \overline{\mathbf{z}}(\boldsymbol{\xi})$) and having at the center $\xi = 0$ of the disk a zero of order $K + 1$. In connection with the norm of $\mathfrak{E}_{1,K}$, we consider the expression

$$\|\mathbf{z}\|_{1,K} = \sup_{|\xi|<1} \xi^{-(K+1)}(\|\mathbf{z}(\boldsymbol{\xi})\| + \boldsymbol{\xi}\|\mathbf{z}'(\boldsymbol{\xi})\|),$$

where the prime once again denotes differentiation with respect to ξ.
$\mathfrak{E}_{0,K}$ is the Banach space of vector functions $\mathbf{u}: \mathcal{K}_1 \to \mathbb{C}^n$ holomorphic on the open unit disk $\mathcal{K}_1$, continuous on its boundary, real on the real axis $(\mathbf{u}(\overline{\boldsymbol{\xi}}) = \overline{\mathbf{u}}(\boldsymbol{\xi}))$ and having at the center $\xi = 0$ of the disk a zero of order $K + 1$. In connection with the norm of $\mathfrak{E}_{0,K}$ we consider the expression

$$\|\mathbf{u}\|_{0,K} = \sup_{|\xi|\le 1} \xi^{-(K+1)}\|\mathbf{u}(\boldsymbol{\xi})\|.$$

Subsequent to contracting the domain of definition and the set of values of the mapping $\boldsymbol{\Phi}$, it is likewise possible to apply the implicit function theorem by the strategy already considered. We need only somewhat revise point (d) of the proof.

Lemma 1.2.2. *The Frechet derivative* $\nabla_{\mathbf{z}}\boldsymbol{\Phi}(0, \mathbf{0}): \mathfrak{E}_{1,K} \to \mathfrak{E}_{0,K}$ *has a bounded inverse.*

Proof. We observe that the operator $\nabla_{\mathbf{z}}\Phi(0, \mathbf{0})$ is bounded and we consider a system of differential equations of type (1.24):

$$\beta\xi\frac{d\mathbf{z}}{d\xi} + \mathbf{K}\mathbf{z} = \mathbf{u}, \quad \mathbf{u} \in \mathfrak{E}_{0,K}. \tag{1.27}$$

We expand the function $\mathbf{u}(\xi)$ in a Taylor series

$$\mathbf{u}(\xi) = \sum_{k=K+1}^{\infty} \mathbf{u}_k \xi^k.$$

The solution of (1.27) will likewise be sought in the form of a Taylor series:

$$\mathbf{z}(\xi) = \sum_{k=K+1}^{\infty} \mathbf{z}_k \xi^k.$$

The coefficients of the two series are connected by this relation:

$$\mathbf{u}_k = \mathbf{K}_k\mathbf{z}_k, \quad \mathbf{K}_k = k\beta\mathbf{E} + \mathbf{K}.$$

By virtue of satisfying the inequalities

$$-\beta K < \Re\rho_i, i = 1, \dots, n$$

for arbitrary $k \ge K + 1$, the matrix $\mathbf{K}_k$ is nonsingular, and furthermore has, for large k, the asymptotic estimate

$$\|\mathbf{K}_k\|^{-1} = O(k^{-1}).$$

From this, after application of Cauchy's theorem, it follows that the Taylor series of the functions $\mathbf{z}(\xi)$ and $\mathbf{z}'(\xi)$ have the same radius of convergence as the series for $\mathbf{u}(\xi)$. Therefore the operator $\nabla_{\mathbf{z}}\mathbf{\Phi}(0,\mathbf{0})$ maps the space $\mathfrak{E}_{1,K}$ one-to-one onto the space $\mathfrak{E}_{0,K}$. In consequence of this, by Banach's theorem on the inverse operator [94], $(\nabla_{\mathbf{z}}\mathbf{\Phi}(0,\mathbf{0}))^{-1}$ is bounded.

The lemma is proved.

It is consequently possible to apply the implicit function theorem, which proves the convergence of the series (1.19) as a Taylor series, representing some function holomorphic on $|s| < \varepsilon_0$.

The nonapplicability in general of the procedure described is connected with the fact that the Riemann surface of $\ln t$ is not compact, so that there does not exist a reasonable Banach space of functions that are holomorphic on this surface.

In the general case the logarithm in series (1.16) is "indestructible". However, below we formulate simple conditions that are sufficient for the existence of some "uniformizing" time substitution, subsequent to which the existence of a formal solution can be represented in the form of an ordinary Taylor series. The idea for such a substitution is due to the American mathematician S.D. Taliaferro [182]. The proof of convergence of the series repeats almost exactly the proof introduced above. Therefore the series (1.16), constructed in "real time", also converges.

Theorem 1.2.1. *Suppose that system (1.3) is autonomous, semi-quasihomogeneous in the sense of Definition 1.1.4 and satisfies all the hypothesis of Theorem 1.1.2. Suppose too that the right side terms of system (1.18) are holomorphic on a neighborhood of* $s = 0$, $\mathbf{y} = \mathbf{x}_{\mathbf{0}}^{\gamma}$. *If the number* -1 *is the unique solution of the characteristic equation* $\det(\mathbf{K} - \rho\mathbf{E}) = 0$ *of the form* $\rho = -k\beta$, $k \in \mathbb{N}$, *then there exists a particular solution* $\mathbf{x}(t)$ *of system (1.3) with asymptotic expansion (1.16) such that* $s^{-\mathbf{S}}\mathbf{x}(t(s))$ *is a vector function holomorphic on the domain* $|s| < \varepsilon_0$, *where* $\varepsilon_0 > 0$ *is sufficiently small,* $\mathbf{S} = (\mathrm{q}-1)\mathbf{G}$, *and* $t(s) = \gamma(s^{1-\mathrm{q}} - \alpha\beta^{-1}\ln s)$, α *being some real parameter.*

In the paper [115], in which this result appears, there are some errors and misprints.

Proof. The function $t(s)$ is the inverse of the solution of the differential equation

$$\dot{s} = -\gamma\beta\frac{s^{\mathrm{q}}}{(1+\alpha s^{\mathrm{q}-1})}, \tag{1.28}$$

satisfying the condition $s(\gamma \times \infty) = 0$.

We make a change of dependent variable $\mathbf{x}(t) = s^{\mathbf{S}}\mathbf{y}(s)$ and a change of independent variable $t \mapsto s$, determined by condition (1.28). Subsequent to this, system (1.3) assumes the form

$$-\gamma\beta s\mathbf{y}' = \gamma\mathbf{G}\mathbf{y} + (1+\alpha s^{\mathrm{q}-1})\sum_{m=0} s^m\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}). \tag{1.29}$$

For the autonomous case with $\alpha = 0$, this system transforms into (1.18). The conformity of systems (1.18) and (1.29) still depends on the solutions of Eq. (1.28) being subject to the condition $s(\gamma \times \infty) = 0$ and having the asymptotic $s(t) \sim (\gamma t)^{-\beta}$.

We will look for a formal particular solution of (1.29) in the form of an ordinary Taylor series

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{y}_k s^k. \tag{1.30}$$

We substitute (1.30) into (1.29) and equate coefficients of like powers of s. For the zero-th power of s we obtain

$$-\mathbf{G}\mathbf{y}_0 = \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0),$$

whereby $\mathbf{y}_0 = \mathbf{x}_0^{\gamma}$.

For the k–th power of s, $k < \mathrm{q} - 1$, we have the equations

$$\mathbf{K}_k \mathbf{y}_k = \mathbf{\Phi}_k(\mathbf{y}_0, \ldots, \mathbf{y}_{k-1}), \quad \mathbf{K}_k = k\beta\mathbf{E} + \mathbf{K}, \tag{1.31}$$

where the quantity Φ_k depends polynomially on its arguments and doesn't depend, for the moment, on the parameter α, which still remains to be determined.

Since for $k \neq \mathrm{q} - 1$ the matrix $\mathbf{K}_k$ is nonsingular, we have that the coefficients $\mathbf{y}_k$ are uniquely determined by the formula

$$\mathbf{y}_k = \mathbf{K}_k^{-1} \mathbf{\Phi}_k(\mathbf{y}_0, \ldots, \mathbf{y}_{k-1}).$$

For $k = \mathrm{q} - 1$ we have:

$$\mathbf{K}_{\mathrm{q}-1} \mathbf{y}_{\mathrm{q}-1} = \alpha \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) + \mathbf{\Phi}_{\mathrm{q}-1}(\mathbf{y}_0, \ldots, \mathbf{y}_{\mathrm{q}-2}), \quad \mathbf{K}_{\mathrm{q}-1} = \mathbf{K} + \mathbf{E}. \tag{1.32}$$

We note that $\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) = \mathbf{p}$, where $\mathbf{p}$, is an eigenvector of the Kovalevsky matrix $\mathbf{K}$ with eigenvalue $\rho = -1$.

We expand $\mathbf{y}_{\mathrm{q}-1}$, $\mathbf{\Phi}_{\mathrm{q}-1}$ into a sum of components, each belonging, respectively, to the eigenspace of the matrix $\mathbf{K}$ generated by the vector $\mathbf{p}$ and to its orthogonal complement:

$$\mathbf{y}_{\mathrm{q}-1} = y_{\mathrm{q}-1}\mathbf{p} + \mathbf{y}_{\mathrm{q}-1}^{\perp}, \; \mathbf{\Phi}_{\mathrm{q}-1} = \phi_{\mathrm{q}-1}\mathbf{p} + \mathbf{\Phi}_{\mathrm{q}-1}^{\perp}.$$

Since the matrix $\mathbf{K}_{\mathrm{q}-1}$ is nonsingular on the invariant subspace orthogonal to the vector $\mathbf{p}$, we have

$$\mathbf{y}_{\mathrm{q}-1}^{\perp} = \mathbf{K}_{\mathrm{q}-1}^{-1} \mathbf{\Phi}_{\mathrm{q}-1}^{\perp}.$$

Setting $\alpha = -\phi_{\mathrm{q}-1}$, we finally satisfy Eq. (1.32). The number $y_{\mathrm{q}-1}$ may now be chosen arbitrarily.

For $k > \mathrm{q} - 1$, the equations for determining $\mathbf{y}_k$ likewise have the form (1.31), where the quantities Φ_k depend on the parameter α determined above. These equations, analogous to those preceding, are easily solved for the $\mathbf{y}_k$ due to the nonsingularity of the matrix $\mathbf{K}_k$.

We have thus shown that Eq. (1.29) has a particular formal solution in the form of a Taylor series (1.30). We now prove that (1.30) is the Taylor series of a function holomorphic on the disk $|s| < \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small.

The proof repeats almost exactly the content of Remark 1.2.1. After the substitution $s = \varepsilon\xi$, $0 < \varepsilon \ll 1$, system (1.29) assumes the form

$$-\gamma\boldsymbol{\beta}\xi\frac{d\mathbf{y}}{d\xi} = \gamma\mathbf{G}\mathbf{y} + (1 + \alpha\varepsilon^{\mathrm{q}-1}\xi^{\mathrm{q}-1})\sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}). \tag{1.33}$$

We let $\mathbf{y}_K^\varepsilon$ denote the partial sum

$$\mathbf{y}_K^\varepsilon(\xi) = \sum_{k=0}^{K}\varepsilon^k\mathbf{y}_k\xi^k,$$

where $-\boldsymbol{\beta}K < \operatorname{Re}\rho_i$, $i = 1, \dots, n$, and we will seek a particular solution (1.33) in the form

$$\mathbf{y}(\xi) = \mathbf{y}_K^\xi(\xi) + \mathbf{z}(\xi),$$

where $\mathbf{z}(\xi)$ is some function holomorphic on the disk $|\xi| < 1$ having a zero of order $K + 1$ at the point $\xi = 0$.

To prove the existence of such a solution it suffices to apply the implicit function theorem [94] to the Banach space equation

$$\Phi(\varepsilon, \mathbf{z}) = \mathbf{0},$$
$$\Phi\colon (0, \varepsilon_0) \times \mathfrak{E}_{1,k} \to \mathfrak{E}_{0,k},$$

where

$$\Phi(\varepsilon, \mathbf{z}) = \gamma\beta\xi\tfrac{d}{d\xi}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \mathbf{G}(\mathbf{y}_K^\varepsilon + \mathbf{z}) +$$
$$+\gamma(1 + \alpha\varepsilon^{\mathrm{q}-1}\xi^{\mathrm{q}-1})\sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}).$$

Equation (1.28) has a particular solution $s(t)$, $s(+\infty) = 0$, given by the function inverse to

$$t(s) = \gamma(s^{1-\mathrm{q}} - \alpha\boldsymbol{\beta}^{-1}\ln s).$$

Let $\mathcal{R}$ be the Riemann surface of the function $s(t)$. The system of equations (1.29) has a holomorphic particular solution $\mathbf{y}(s)$, so that the vector function $\mathbf{y}(s(t))$ is holomorphic on the encompassing region $|t| > \varepsilon_0^{-1/\beta}$ of the Riemann surface $\mathcal{R}$, on which (1.19) provides an asymptotic solution.

The theorem is proved.

Thus in this section we have completed the proof of Theorem 1.1.2, which has important applications to the theory of the stability of motion. We have the following assertion:

Theorem 1.2.2. *Let* $\mathbf{x} = \mathbf{0}$ *be a critical point of the system (1.3) and let the system (1.3) be autonomous. Suppose too that the system (1.3) is positive* *semi-quasihomogeneous with respect to the quasihomogeneous structure given by the matrix* $\mathbf{G}$*, whose eigenvalues have positive real parts. If there exists a vector* $\mathbf{x}_0^- \in \mathbb{R}^n$*,* $\mathbf{x}_0^- \neq \mathbf{0}$ *such that*

$$\mathbf{G}\mathbf{x}_0^- = \mathbf{f}_\mathrm{q}(\mathbf{x}_0^-)$$

(i.e. $\gamma = -1$*), then the critical point* $\mathbf{x} = \mathbf{0}$ *is unstable.*

The proof follows immediately from the existence of a particular solution $t \mapsto x(t)$ such that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$.

At the present time, this is the most general result we have connecting stable equilibrium in the total system to stable equilibrium in the truncated (so-called *model*) system. Here we actually prove that the existence of a particular solution of the type of an increasing quasihomogeneous ray for the model system implies, by itself, the unstable equilibrium of the total system. Until recently this assertion was merely a hypothesis whose proof was attempted by many authors, although basically only for "semihomogeneous" systems. In this connection it is worth mentioning the results of Kamenkov [93] already cited, whose proofs contain a number of lacunae, as was mentioned. In the book [100] there is given a rather simple proof of this assertion for the case of an attracting ray. The more general case where the ray is hyperbolic is much more complicated. The corresponding result is proved in the article [170] with the help of sophisticated topological techniques. (The terms "attracting ray" and "hyperbolic ray" are connected with the distribution of the eigenvalues of the Kovalevsky matrix over the complex plane.) Theorem 1.2.2 proves the indicated hypothesis completely.

In his classical study [133], A.M. Lyapunov considered the more general problem of the stability of solutions with respect to a given function of the state of the system. We will show how the approach we have developed can be applied in this more general situation [190]. To this end we examine the smooth system (1.1) with critical point $\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} = \mathbf{0}$ is an equilibrium position. We then have its stability with respect to a smooth (i.e. infinitely differentiable) function $Q(\mathbf{x})$, where it is assumed that $Q(\mathbf{0}) = 0$.

We introduce a new system of equations

$$\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x}), \tag{1.34}$$

obtained from (1.1) by time reversal.

Lemma 1.2.3. *Suppose that the system (1.34) admits a solution* $t \mapsto \mathbf{x}(t)$ *such that 1)*$\mathbf{x}(t) \to \mathbf{0}$ *as* $t \to +\infty$ *and, for all* t*, 2)* $q(t) = Q(\mathbf{x}(t)) \neq 0$*. Then the equilibrium point* $\mathbf{x} = \mathbf{0}$ *of the system (1.1) is stable with respect to the function* Q*.*

In fact, in this case there is a solution $t \mapsto \mathbf{x}(-t)$, which asymptotically "exits" from the equilibrium state: $q(t) \to 0$ as $t \to -\infty$. Consequently, along this solution the function Q changes continuously from zero to appreciable finite values. But this then signifies instability with respect to Q.

An asymptotic solution of Eq. (1.34) can be sought in the form of a series of a certain form. Let $\mathbf{A}$ be the Jacobian matrix of some vector field $\mathbf{f}$ at zero. If this matrix has a positive real eigenvalue β, then system (1.34) admits an asymptotic solution in the form of the series (1.2). Substituting this series into the Maclaurin series of the function Q, we again obtain a series in powers of $\exp(-\beta t)$, whose coefficients are polynomials in t. If at least one of the coefficients of this series is distinct from zero, then (by Lemma 1.2.3) the equilibrium of system (1.1) is unstable with respect to the function Q.

This observation can be generalized. The necessary condition for the stability of the equilibrium point $\mathbf{x} = \mathbf{0}$ for system (1.1) with respect to the function Q is implied by the constancy of this function on the unstable manifold of system (1.1). The last property is verified constructively by means of an iterative method for constructing Lyapunov series that represent asymptotic solutions (as $t \to -\infty$) of system (1.1).

In degenerate cases, asymptotic solutions of system (1.34) can be sought in the form of the series (1.16). The conditions for the existence of such solutions is given by Theorem 1.2.1. The next assertion generalizes Theorem 1.2.2, but prior to this we give a generalized definition of quasihomogeneous and semi-quasihomogeneous functions.

Let $\mathbf{G}$ be some real matrix from (1.10) yielding a quasihomogeneous structure on $\mathbb{R}^n[\mathbf{x}]$.

Definition 1.2.1. The function $Q : \mathbb{R}^n \to \mathbb{R}$ is called *quasihomogeneous* of degree m, provided that

$$Q(\mu^{\mathbf{G}}\mathbf{x}) = \mu^m Q(\mathbf{x}) \tag{1.35}$$

for all $\mu > 0$.

If $G = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, then the relation (1.35) takes the following explicit form:

$$Q(\mu^{\alpha_1} x_1, \ldots, \mu^{\alpha_n} x_n) = \mu^m Q(x_1, \ldots, x_n).$$

For $\alpha_1 = \ldots = \alpha_n = 1$ we have an ordinary homogeneous function of degree m.

Differentiating (1.35) with respect to μ and setting $\mu = 1$, we get a generalized *Euler identity*

$$\left\langle \frac{\partial Q}{\partial \mathbf{x}}, \mathbf{G} \right\rangle = mQ.$$

Definition 1.2.2. The function Q is said to be positive (negative) semi-quasihomogeneous if it can be represented in the form $Q_m(\mathbf{x}) + \tilde{Q}(\mathbf{x})$, where Q_m is a quasihomogeneous function of degree m and

$$\mu^{-m} \tilde{Q}(\mu^{\mathbf{G}}\mathbf{x}) \to 0$$

as $\mu \to 0$ ($\mu \to +\infty$).

Theorem 1.2.3. *Let all the conditions for Theorem 1.2.2 be satisfied for system (1.34) and let Q be a smooth positive semi-quasihomogeneous function, where*

$$Q_m(\mathbf{x}_0) \neq 0.$$

Then the equilibrium point $x = 0$ of the original system (1.1) is unstable with respect to the function Q.

This assertion is a simple consequence of Theorem 1.1.2 and Lemma 1.2.3.

We now suppose that system (1.34) admits an asymptotic solution in the form of the series (1.16). Then the substitution of this series into the Maclaurin series of the infinitely differentiable function Q gives us an expansion of the function $t \mapsto q(t)$ as a series with a convenient form (with reciprocal degrees t^{β} and coefficients that are polynomials in "logarithmic time" with constant coefficients). If at least one coefficient of this formal series is different from zero, then the trivial equilibrium point of system (1.1) is unstable with respect to the function Q. We will return to these issues in Chap. 3.

1.3 Exponential Methods for Finding Nonexponential Solutions

In the preceding two sections we have explained how to construct particular solutions of differential equations, whose principal asymptotic parts were determined by the quasihomogeneous structure of the chosen truncation. The algorithms introduced allow effective construction of solutions in the form of series. However, it is obvious even at first glance that the given algorithm is far from giving all solutions with the required asymptotic. As series of type (1.2) don't exhaust all solutions of exponential type, so also the series (1.16) don't describe all solutions with a generalized power asymptotic. In the quasilinear case, all exponential solutions lie on the stable and unstable manifolds $W^{(s)}$, $W^{(u)}$. For observing the "strong nonlinearity", we establish results related to the Hadamard-Perron theorem, i.e. we attempt to apply techniques that are typical in searching for solutions with exponential asymptotic.

In this section we alter somewhat the conditions that are imposed on the right hand side of (1.3). Let (1.3) either be autonomous or suppose that the right hand side is a bounded function of t over the entire real line.

We begin with the autonomous case. We have

Theorem 1.3.1. *Suppose that the right hand side of (1.3) doesn't explicitly depend on t. Suppose that all the hypothesis of Theorem 1.1.2 is fulfilled and that l eigenvalues of the Kovalevsky matrix $\mathbf{K}$ have real parts whose sign agrees with the sign of the quantity $-\beta$, at the same time as the real parts of the remaining* *eigenvalues are either zero or have the opposite sign. Then (1.3) has an l–parameter family of particular solutions of the form*

$$\mathbf{x}(\mathbf{c}, t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^\gamma + o(1)) \ \ as \ \ t^\chi \to \gamma \times \infty,$$

where $\mathbf{c} \in \mathbb{R}^l$ is a vector of parameters.

This result is a consequence of the more general Theorem 1.3.3 that we formulate and prove below.

There is still another question connected with the search for asymptotic solutions using the algorithms described above. In Sect. 1.1 we sought particular solutions of the truncated quasihomogeneous system (1.9) in the form (1.12). However, it is unknown whether these exhaust all possible solutions of a quasihomogeneous system having the generalized power asymptotic (1.12). It might happen that none of the possible quasihomogeneous truncations has a particular solution in the form of a quasihomogeneous ray, while the total system has an asymptotic solution. This leads us to consider the question of existence of particular solutions of truncated type (1.9) in "nonstationary" form, more general than (1.12), and precisely

$$\mathbf{x}^\gamma(t) = (\gamma t)^{-\mathbf{G}}\mathbf{x}_0^\gamma(\gamma t), \tag{1.36}$$

where $\mathbf{x}_0^\gamma(\cdot)$ is an infinitely differentiable vector function bounded on the positive half-line. We show, with some rather weak restrictions, that these generate solutions of the total system with analogous asymptotic. We likewise extend the class of systems studied by considering nonautonomous systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1.37}$$

where the components of the right sides admit expansion in formal power series:

$$f^j = \sum_{i_1,\ldots,i_n} f^j_{i_1,\ldots,i_n}(t)(x^1)^{i_1}\ldots(x^n)^{i_n}, \tag{1.38}$$

with coefficients $f^j_{i_1,\ldots,i_n}(t)$ that are smooth $\mathbb{R}[t]$ functions that are uniformly bounded over the entire axis.

Next, let system (1.37), *where the time t on the right sides is regarded as a parameter*, be semi-quasihomogeneous in the sense of Definition 1.1.6 with respect to a quasihomogeneous structure induced by some matrix $\mathbf{G}$. We will consider the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_\mathrm{q}(\mathbf{x}, t) \tag{1.39}$$

and look for its particular solutions in the form (1.36).

Making the logarithmic substitution $\tau = \ln(\gamma t)$ for the independent variable, we find that in the "new time" the vector function $\mathbf{x}_0^\gamma(\cdot)$ is a particular solution of the system of differential equations

$$\frac{d\mathbf{x}_0^\gamma}{d\tau} = \mathbf{G}\mathbf{x}_0^\gamma + \gamma\mathbf{f}_\mathrm{q}(\mathbf{x}_0^\gamma, \gamma e^\tau). \tag{1.40}$$

Linearizing (1.40) in the neighborhood of some solution $\mathbf{x}_0^\gamma(\cdot)$, we obtain the linear system

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{K}(\tau)\mathbf{u}, \tag{1.41}$$

where

$$\mathbf{K}(\tau) = \mathbf{G} + \gamma d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma(\gamma e^\tau), \gamma e^\tau),$$

with nonautonomous Kovalevsky matrix $\mathbf{K}(\tau)$, whose components—in view of the boundedness of $\mathbf{x}_0^\gamma$ and of the coefficients in expansion (1.38)—are smooth and bounded over all values of τ.

We have the following assertion, which is analogous to Lemma 1.1.2.

Lemma 1.3.1. *If the system (1.39) is autonomous, then the system (1.41) has the particular solution*

$$\mathbf{u}_0(\tau) = e^{-\tau}\mathbf{p}(\tau), \qquad \mathbf{p}(\tau) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma(\gamma e^\tau)).$$

Proof. Indeed, from the definition of the nonautonomous Kovalevsky matrix, using system (1.40) and Eq. (1.21), we obtain

$$\begin{aligned}\tfrac{d\mathbf{u}_0}{d\tau} &= e^\tau\left(-\mathbf{p} + d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma)\tfrac{d\mathbf{x}_0^\gamma}{d\tau}\right) = \\ &= e^{-\tau}(-\mathbf{p} + d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma)(\mathbf{G}\mathbf{x}_0^\gamma + \gamma\mathbf{p})) = \\ &= e^{-\tau}(\mathbf{G} + \gamma d_{\mathbf{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^\gamma))\mathbf{p} = \mathbf{K}\mathbf{u}_0.\end{aligned}$$

The lemma is proved.

If the bounded vector function $p(\tau)$ doesn't tend to zero as $\tau \to \to \pm\infty$, then the characteristic exponent of the solution $\mathbf{u}_0(\tau)$ of (1.41) equals -1, so that generally -1 belongs to the full spectrum of the linear system (1.41)

We recall briefly several concepts from the theory of the asymptotic behavior of solutions of nonautonomous linear systems of type (1.41). For closer acquaintance with this subject, we recommend the corresponding sections of the book [44].

The *right characteristic exponents* of a scalar or vector valued function $\mathbf{u}(\tau)$ of arbitrary dimension are given by the quantities

$$r^+ = \kappa^+[\mathbf{u}(\tau)] = \limsup_{\tau\to+\infty} \tau^{-1}\ln\|\mathbf{u}(\tau)\|,$$

the corresponding *left characteristic exponents* being given by

$$r^- = \kappa^-[\mathbf{u}(\tau)] = \kappa^+[\mathbf{u}(-\tau)].$$

In what follows, unless otherwise stipulated, we will be dealing with "right" exponents, which determine the asymptotic behavior of a function as $\tau \to +\infty$, whereby the transition to "left" (the case $\tau \to -\infty$) is effected by the change of independent variable $\tau \mapsto -\tau$.

We note some properties of the quantities that have been introduced.

For any vector function $\mathbf{u}(\tau)$ and any constant τ_0,

$$\kappa[\mathbf{u}(\tau-\tau_0)] = \kappa[\mathbf{u}(\tau)].$$

Let $\mathbf{u}_{(1)}(\tau), \mathbf{u}_{(2)}(\tau)$ be two vector functions with finite characteristic exponents. The characteristic exponents of their linear combinations $c_1\mathbf{u}_{(1)}(\tau) + c_2\mathbf{u}_{(2)}(\tau)$ and of their general "composition" $\mathbf{B}(\mathbf{u}_{(1)}(\tau), \mathbf{u}_{(2)}(\tau))$, where $\mathbf{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is some bilinear vector function, possess the following properties:

$$\kappa[c_1\mathbf{u}_{(1)}(\tau) + c_2\mathbf{u}_{(2)}(\tau)] = \max\left(\kappa[\mathbf{u}_{(1)}(\tau)], \kappa[\mathbf{u}_{(2)}(\tau)]\right)$$
$$\kappa[\mathbf{B}(\mathbf{u}_{(1)}(\tau), \mathbf{u}_{(2)}(\tau))] \le \kappa[\mathbf{u}_{(1)}(\tau)] + \kappa[\mathbf{u}_{(2)}(\tau)].$$

If $\mathbf{u}(\tau)$ is a vector function with finite nonnegative characteristic exponents, then

$$\kappa\left[\int_0^\tau \mathbf{u}(\xi)d\xi\right] \le \kappa[\mathbf{u}(\tau)];$$

but if a characteristic exponent of $\mathbf{u}(\tau)$ is negative, then

$$\kappa\left[\int_\tau^{+\infty} \mathbf{u}(\xi)d\xi\right] \le \kappa[\mathbf{u}(\tau)].$$

Furthermore, let $\mathbf{U}(\tau) = (u^i_j(\tau))^n_{i,j=1}$ be some matrix with smooth bounded components. Its characteristic exponents are given by

$$\kappa\left[\mathbf{U}(\tau)\right] = \max_{i,j}\left(\kappa\left[u^j_i(\tau)\right]\right) = \max_{i,j}\left(\limsup_{\tau\to+\infty}\tau^{-1}\ln|u^i_j(\tau)|\right).$$

The *full spectrum* of the linear system (1.41) is defined to be the set of quantities $\left\{r_i = \kappa\left[\mathbf{u}_{(i)}(\tau)\right]\right\}^n_{i=1}$, where $\left\{\mathbf{u}_{(i)}(\tau)\right\}^n_{i=1}$ is some fundamental system of its solutions. If system (1.41) is autonomous, then $r_i = \operatorname{Re}\rho_i$, where the ρ_i are the roots of the characteristic equation $\det(\mathbf{K} - \rho\mathbf{E}) = 0$.

It is clear that the full spectrum of the system (1.41) depends on the choice of a fundamental system of solutions. For any fundamental system of solutions, the so-called Lyapunov inequality [133] applies:

$$\sum_{i=1}^n r_i \ge \limsup_{\tau\to+\infty}\tau^{-1}\int_0^\tau \operatorname{Tr} K(\xi)d\xi, \tag{1.42}$$

where $\operatorname{Tr} K$ denotes the trace of the matrix $\mathbf{K}$.

There exist fundamental systems of solutions, called *normal*, for which the sum of the characteristic exponents is maximal [44].

For any normal system of solutions $\{\mathbf{u}_i(\tau)\}_{i=1}^n$, the *irregularity measure* is defined as the quantity

$$\sigma = \sum_{i=1}^{n} r_i - \liminf_{\tau\to+\infty} \tau^{-1} \int_0^{\tau} \mathrm{Tr}\, K(\xi) d\xi,$$

which is nonnegative by virtue of the Lyapunov inequality (1.42).

The system (1.41) is called *proper* if, for some normal system of its solutions, the irregularity measure σ equals zero, in which case strict equality holds in Lyapunov's inequality (1.42).

The definitions and concepts introduced above are concerned with the asymptotic behavior of solutions of the system (1.41) as $\tau \to +\infty$. In order to introduce analogous characteristics for the case $\tau \to -\infty$, we need to consider the system with time reversal:

$$\frac{d\mathbf{u}}{d\tau} = -\mathbf{K}(-\tau)\mathbf{u}. \tag{1.43}$$

It will be shown below that the task of looking for particular solutions of the system (1.37) that have nonexponential asymptotic can be reduced to the investigation of some quasilinear system, whose linear part has the form (1.41), i.e. by the application of "exponential" methods. We will formulate some results that generalize known theorems of V.I. Zubov [203].

We first prove an analog of Theorem 1.1.2 for system (1.37).

Theorem 1.3.2. *Let the quasihomogeneous truncation (1.39) of the semi-quasihomogeneous system (1.37) have a particular solution of form (1.36) and let the irregularity measure of system (1.41) for the case* $\beta > 0$ *(or the irregularity measure of system (1.43) for the case* $\beta < 0$*) satisfy the inequality*

$$\sigma < \frac{|\beta|}{2}. \tag{1.44}$$

Then (1.37) has a particular solution whose principal part has asymptotic

$$(\gamma t)^{-\mathbf{G}} \mathbf{x}_0^{\gamma}(\gamma t)$$

as $t^{\chi} \to \gamma \times \infty$.

The proof is divided into two parts, just as in the preceding section.

First step. Construction of a formal solution.

We first construct a formal solution for system (1.37) in the form of the series

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}, \tag{1.45}$$

where $\mathbf{x}_k(\tau)$, $\tau = \ln(\gamma t)$ are some vector valued functions on the entire number line which satisfy the inequality

$$\chi\kappa^{\chi}[\mathbf{x}_k(\tau)] \leq (2k-1)\sigma. \tag{1.46}$$

The symbol κ^{χ} here denotes the characteristic exponent κ^+ or κ^- in accordance with the sign of the semi-quasihomogeneity.

Since inequality (1.44) is satisfied, the inequality (1.46) guarantees that the desired solution, at least formally, has the required asymptotic.

We change the dependent and independent variables $\mathbf{x} \mapsto \mathbf{u}$, $t \mapsto \tau$,

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^{\gamma}(\gamma t) + \mathbf{u}(\gamma)),\ \ \tau = \ln(\gamma t),$$

subsequent to which the system (1.37) is rewritten

$$\mathbf{u}' = \mathbf{K}(\tau)\mathbf{u} + \phi(\mathbf{u}, \tau) + \psi(\mathbf{u},\tau), \tag{1.47}$$

where the prime indicates differentiation with respect to the new "time" τ, where

$$\phi(\mathbf{u}, \tau) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma} + \mathbf{u}, \gamma e^{\tau}) - \mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}, \gamma e^{\tau}) - d_{\mathrm{x}}\mathbf{f}_{\mathrm{q}}(\mathbf{x}_0^{\gamma}, \gamma e^{\tau})\mathbf{u}.$$

It is clear that $\phi(\mathbf{u}, \tau)$ is a bounded vector function τ for all fixed finite $\mathbf{u}$ and that, moreover, $\phi(\mathbf{u}, \tau) = O(\|\mathbf{u}\|^2)$ as $\mathbf{u} \to \mathbf{0}$ uniformly in τ.

The vector function $\psi(\mathbf{u}, \tau)$ admits a formal expansion in the series

$$\psi(\mathbf{u}, \tau) = \sum_{m-1} e^{-m\beta\tau}\mathbf{f}_{\mathrm{q}+m\chi}(\mathbf{x}_0^{\gamma} + \mathbf{u}, \gamma e^{\tau}).$$

As before, we assume that the right sides of (1.37) are smooth, at least over some domain containing the desired solution. Therefore $\psi(\mathbf{u}, \tau) = O(e^{-\beta\tau})$ as $\tau \to \chi \times \infty$ for each fixed $\mathbf{u}$.

We have the following

Lemma 1.3.2. *Under the assumptions that were made above concerning properties of the matrix* $\mathbf{K}(\tau)$ *and the vector functions* $\phi(\mathbf{u}, \tau)$ *and* $\psi(\mathbf{u}, \tau)$*, the system of equations (1.47) has a formal particular solution of the form*

$$\mathbf{u}(\tau) = \sum_{k=1}^{\infty} \mathbf{x}_k(\tau)e^{-k\beta\tau}, \tag{1.48}$$

where the characteristic exponents of the coefficients satisfy the inequality (1.46).

We restrict ourselves to examining the case $\chi = +1$; the opposite case is reduced to this one by means of the "logarithmic time" transformation: $\tau \mapsto -\tau$.

We substitute (1.48) into (1.47) and equate coefficients of corresponding powers $e^{-\beta\tau}$. The coefficients $\mathbf{x}_k(\tau)$ of the series (1.48) are found inductively. We assume that $\mathbf{x}_1(\tau), \dots, \mathbf{x}_{k-1}(\tau)$ have been found and that their characteristic exponents satisfy inequality (1.46) We write the differential equation for determining $\mathbf{x}_k(\tau)$:

$$\mathbf{x}_k' - \mathbf{K}(\tau)\mathbf{x}_k = \Phi_k(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \tau), \tag{1.49}$$

where $\mathbf{K}_k(\tau) = k\beta\mathbf{E} + \mathbf{K}(\boldsymbol{\tau})$ and the Φ_k are certain vector functions that are bounded in τ and are polynomially dependent on $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$.

Using the properties of the characteristic exponents of a linear combination and a generalized product, we can prove that the characteristic exponents of the vector functions Φ_k, after the substitution into them of $\mathbf{x}_1(\tau), \dots, \mathbf{x}_{k-1}(\tau)$, satisfy the inequalities

$$\kappa\,[\Phi_k(\tau)] = \kappa\,[\Phi_k(\mathbf{x}_1(\tau), \dots, \mathbf{x}_{k-1}(\tau), \tau)] \le (2k-2)\sigma. \tag{1.50}$$

The index "+" will henceforth be dropped in order to shorten the notation. We prove that the system of differential equations (1.49) has a particular solution for which the characteristic exponents satisfy inequality (1.46).

Let $\mathbf{U}(\tau)$ be the fundamental matrix of the system (1.41), normalized by the condition $\mathbf{U}(0) = \mathbf{E}$. Let $r_1, \dots, r_n$ be the full spectrum of the system (1.41). We consider the diagonal matrix $\mathbf{R} = \operatorname{diag}(r_1, \dots, r_n)$.

We perform the substitutions

$$\mathbf{x}_k = e^{k\beta\tau}\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{y}_k,$$
$$\Psi_k(\tau) = e^{-k\beta\tau}\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)\Phi_k(\tau).$$

In the new variables Eq. (1.49) takes on the form

$$\mathbf{y}_k' - \mathbf{R}\mathbf{y}_k = \Psi_k(\tau). \tag{1.51}$$

In conformity with the bound (1.50) and the general properties of characteristic exponents,

$$\kappa[\Psi_\tau(\tau)] \le -k\beta + (2k-2)\sigma + \kappa[\exp(\mathbf{R}_\tau)\mathbf{U}^{-1}(\tau)].$$

Using the formula for the inverse matrix, we have

$$\mathbf{U}^{-1}(\tau) = (\Delta^{-1}(\tau)\Delta_i^j(\tau))_{i,j=1}^n,$$

where $\Delta(\tau) = \det\mathbf{U}(\tau)$, and the $\Delta_j^i(\tau)$ are the cofactors of the elements $u_j^i(\tau)$.

Using $\Delta(0) = 1$ and the Ostrogradski-Liouville formula, we will have

$$\Delta(\tau) = e^{\int_0^\tau \operatorname{Tr}\mathbf{K}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi}}.$$

We recall that the columns of the fundamental matrix $\mathbf{U}(\tau)$ are vector functions belonging to a fundamental system of solutions $\left\{\mathbf{u}_{(i)}(\tau)\right\}_{i=1}^{n}$, so that

$$\kappa\left[\exp\left(\mathbf{R}\tau\right)\mathbf{U}^{-1}\left(\tau\right)\right] = \max_{i,j}\left(\kappa\left[e^{r_i\tau}\Delta_i^j\left(\tau\right)e^{-\int_0^\tau \mathrm{Tr}\,\mathbf{K}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi}}\right]\right) \leq$$
$$\leq \max_{i,j}\left(r_i + \sum_{l=1}^{n} r_l - r_i - \underline{\lim}_{\tau\to+\infty}\tau^{-1}\int_0^\tau \mathrm{Tr}\,\mathbf{K}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi}\right) = \sigma.$$

Consequently,

$$\kappa[\Psi_k(\tau)] \leq -k\beta + (2k-1)\sigma.$$

We rewrite (1.51) in coordinate form

$$(y_k^i)' - r_i y_k^i = \Psi_k^i(\tau),\ i = 1,\ldots,n,$$

and construct a particular solution of this system by the following recipe:

$$y_k^i(\tau) = -e^{r_i\tau}\int_\tau^{+\infty} e^{-r_i\xi}\Psi_k^i(\xi)d\xi$$

for those i for which $\kappa\left[e^{-r_i\tau}\Psi_k^i(\tau)\right] < 0$, and

$$y_k^i(\tau) = e^{r_i\tau}\left(c_k^i + \int_0^\tau e^{-r_i\xi}\Psi_k^i(\xi)d\xi\right)$$

for those i for which $\kappa\left[e^{-r_i\tau}\Psi_k^i(\tau)\right] \geq 0$, where $c_k^i \in R$ are free parameters that can be chosen arbitrarily if the characteristic exponent appearing in the integral formula is nonnegative, and that we set equal to zero in the opposite case.

From the given formulas it is clear that the characteristic exponent of the solution of (1.51), constructed by the given method, satisfies the bound

$$\kappa[\mathbf{y}_k(\tau)] \leq -k\beta + (2k-1)\sigma.$$

We next compute the characteristic exponents of $\mathbf{U}(\tau)\exp\left(-\mathbf{R}\tau\right)$:

$$\kappa[\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)] = \max_{i,j}\left(\kappa\left[u_j^i(\tau)e^{-r_i\tau}\right]\right) = 0.$$

Therefore $\mathbf{x}_k = e^{k\beta\tau}\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{y}_k$ has a characteristic exponent that satisfies the inequality (1.46).

Thus $\mathbf{x}_k(\tau)$ is the desired solution of (1.49) and the construction of the series (1.48) has been accomplished.

Lemma 1.3.2 is proved.

We have, moreover, proved that the original system (1.37) has a formal solution of the form (1.45).

Second step. We now pass to the construction of the actual solution.

Lemma 1.3.3. *Let the parameter β in the right side of (1.47) be positive. If the conditions of Lemma 1.3.2 are satisfied, then the system of equations (1.47) has a particular solution of the form*

$$\mathbf{u}(\tau) = \mathbf{u}_K(\tau) + \mathbf{v}(\tau),$$

where $\mathbf{u}_K(\tau)$ is the K-th partial sum of the series (1.48) and $\mathbf{v}(\tau)$ has asymptotic

$$\mathbf{v}(\tau) = o\left(e^{-K\beta\tau}\right) \text{ as } \tau \to +\infty,$$

where $b = \beta - 2\sigma$ and K is sufficiently large.

Proof. We write a system of differential equations for $\mathbf{v}$:

$$\mathbf{v}' = \mathbf{K}(\tau)\mathbf{v} + \theta(\mathbf{v}, \tau). \tag{1.52}$$

and introduce the notation

$$\theta(\mathbf{v}, \tau) = -\mathbf{u}'_K(\tau) + \mathbf{K}(\tau)\mathbf{u}_K(\tau) + \phi(\mathbf{u}_K(\tau) + \mathbf{v}, \tau) + \psi(\mathbf{u}_k(\tau) + \mathbf{v}, \tau),$$

where K is chosen so that the inequality $Kb > -r_i$ is satisfied for $i = 1, \ldots, n$.

Using the substitution

$$\mathbf{v} = \mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{w},$$

we dispose of the nonautonominity in the linear portion of (1.52), after which (1.52) takes on the form

$$\mathbf{w}' = \mathbf{R}\mathbf{w} + \widehat{\theta}(\mathbf{w}, \tau), \tag{1.53}$$

where

$$\hat{\theta}(\mathbf{w}, \tau) = \exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)\theta(\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)\mathbf{w}, \tau).$$

We will show that the system of equations (1.53) has a particular solution $\mathbf{w}(\tau)$ that is determined on some half line $[T, +\infty)$, where $T > 0$ is sufficiently large, with asymptotic $\mathbf{w}(\tau) = O(e^{-\Delta\tau})$ as $\tau \to +\infty$, where $\Delta = Kb + \delta$ and $\delta > 0$ is sufficiently small.

Under the assumptions made, we can assert that $\widehat{\theta}(\mathbf{w}(\tau), \tau)$ generally has a higher order of decay than $\mathbf{w}(\tau)$. Because the characteristic exponent of the matrix $\mathbf{U}(\tau)\exp(-\mathbf{R}\tau)$ equals zero, we can write $\mathbf{v}(\tau) = O(e^{-(Kb+\frac{\delta}{2})\tau})$.

We will estimate $\theta(\mathbf{v}, \tau)$ using the mean value theorem:

$$\|\theta(\mathbf{v}, \tau)\| \le \|\theta(\mathbf{0}, \tau)\| + \sup_{\zeta\in[0,1]} \|d_{\mathrm{v}}\theta(\zeta\mathbf{v}, \tau)\| \, \|\mathbf{v}\|.$$

We note that each term of the series (1.48) clearly has the asymptotic

$$\mathbf{x}_k(\tau)e^{-k\beta\tau} = O(e^{-(kb+\sigma-\delta/2)\tau})$$

for arbitrary $\delta > 0$. Therefore

$$\theta(\mathbf{0}, \tau) = -\mathbf{u}'_K(\tau) + \mathbf{K}(\tau)\mathbf{u}_K(\tau) + \phi(\mathbf{u}_K(\tau), \tau) + \psi(\mathbf{u}_K(\tau), \tau).$$

We will estimate the quantity

$$d_{\mathbf{u}}\theta(\mathbf{v}, \tau) = d_{\mathbf{u}}\phi(\mathbf{u}_K(\tau) + \mathbf{v}, \tau) + d_{\mathbf{u}}\psi(\mathbf{u}_K(\tau) + \mathbf{v}, \tau),$$

whose order with respect to **u**, like its minimum, is quadratic over the space of vector functions ϕ. Therefore $d_{\mathbf{u}}\phi(\mathbf{u}, \tau) = O(\|\mathbf{u}\|)$ and the asymptotic of this quantity is determined by the asymptotic of the first term of the series (1.48), whereby we may write

$$d_{\mathbf{u}}\phi(\mathbf{u}_k(\tau) + \mathbf{v}(\tau), \tau) = O(e^{-(\beta-\sigma-\delta/2)\tau}) = O(e^{-(b+\sigma-\delta/2)\tau}).$$

Over the space of vector functions ψ we have

$$d_{\mathbf{u}}\psi(\mathbf{u}_K(\tau) + \mathbf{v}(\tau), \tau) = O(e^{-\beta\tau}) = O(e^{-(b+2\sigma)\tau}).$$

The last asymptotics of the estimate are uniform in **v**, in some small neighborhood of $\mathbf{v} = \mathbf{0}$, with respect to the standard norm on $\mathbb{R}^n$. Therefore

$$\theta(\mathbf{v}(\tau), \tau) = O(e^{-((K+1)b+\sigma-\delta/2)\tau})$$

for small $\delta > 0$.

Earlier we showed that the characteristic exponent of the matrix

$$\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)$$

equals σ, i.e. $\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau) = O(e^{(\sigma+\delta/2)\tau})$. It is therefore immediate that $\widehat{\theta}(\mathbf{w}(\tau), \tau)$ has the asymptotic

$$\widehat{\theta}(\mathbf{w}(\tau), \tau) = O(e^{-((K+1)b-\delta)\tau}).$$

Since the vector function $\widehat{\theta}(\mathbf{w}, \tau)$ is continuous with respect to all its arguments jointly, we may assume that $\widehat{\theta}$ establishes some mapping Θ of a neighborhood $U_{0,\Delta}$ of zero of the normed space $H_{0,\Delta}$ into itself, where:

$H_{0,\Delta}$ is the Banach space of vector functions $\mathbf{w}[\mathrm{T}, +\infty] \rightarrow \mathbb{R}^n$, continuous on the closed half-line $[\mathrm{T}, +\infty)$, for which the finite norm is

$$\|\mathbf{w}\|_{0,\Delta} = \sup_{[\mathrm{T},+y]} e^{\Delta\tau}\|\mathbf{w}(\tau)\|.$$

We consider too the much more restricted space $H_{1,\Delta}$:

$H_{1,\Delta}$ is the Banach space of vector functions $\mathbf{w} : [\mathrm{T}, +y]R^n$, continuous on the closed half-line $[\mathrm{T}, +y)$ along with their first derivatives, for which the finite norm is

$$\|\mathbf{w}\|_{1,\Delta} = \sup_{[\mathrm{T},+y]} e^{\Delta\tau}(\|\mathbf{w}(\tau)\| + \|\mathbf{w}'(\tau)\|),$$

and we likewise consider the bounded linear operator $\mathbf{L}$: $H_{1,\Delta} \to H_{0,\Delta}$, given by the formula

$$\mathbf{L} = \frac{d}{d\tau} - \mathbf{R}.$$

Lemma 1.3.4. *The operator* $\mathbf{L}$ *has a bounded inverse, whose norm does not depend on* T.

Proof. We consider the system of linear differential equations

$$\frac{d\mathbf{w}}{d\tau} - \mathbf{R}\mathbf{w} = \mathbf{h}, \quad \mathbf{h} \in H_{0,\Delta} \tag{1.54}$$

and find the unique particular solution of this system that is continuously differentiable on the half-line $[\mathrm{T}, +y)$ and satisfies the boundary condition $\mathbf{w}(+y) = \mathbf{0}$ as well as the inequality

$$\|\mathbf{w}\|_{1,\Delta} \leq C\|\mathbf{h}\|_{0,\Delta}, \tag{1.55}$$

where the constant $C > 0$ depends neither on $\mathbf{h} \in H_{0,\Delta}$ nor on the quantity $\mathrm{T} > 0$. The greatest lower bound of all such $C > 0$ will be the norm of the operator $\mathbf{L}^{-1}$.

The desired solution of the system (1.54) now assumes the form

$$\mathbf{w}(\tau) = -\exp(\mathbf{R}\tau)\int_{\tau}^{+y} \exp(\mathbf{R}\xi)\mathbf{h}(\xi)d\xi.$$

This solution satisfies the necessary condition for smoothness and has the required asymptotic. Because it satisfies the inequality $\Delta > -r_i$ for arbitrary i, the improper integral in these variables converges. Estimating separately in each coordinate, we obtain an explicit expression for the constant C in (1.55):

$$C = \max_i(1 + (|r_i| + 1)(\Delta + r_i)^{-1}).$$

The lemma is proved.

We rewrite the system of differential equations (1.53) in the form

$$\mathbf{w} = \mathcal{F}(\mathbf{w}), \tag{1.56}$$

where $\mathcal{F} = \mathbf{L}^{-1}\Theta$ maps the space $H_{0,\Delta}$ into itself.

We show that $\mathcal{F}$ is contractive on some small neighborhood $U_{0,\Delta}$ of zero of the space $H_{0,\Delta}$. For this we first note that, for large $\mathrm{T} > 0$, the inclusion $\mathcal{F}(U_{0,\Delta}) \subset U_{0,\Delta}$ holds.

In fact, we almost literally repeat the reasoning used for the estimate of $\widehat{\theta}(\tau, \mathbf{w}(\tau))$, obtaining the estimate

$$\|\Theta(\mathbf{w})\|_{0,\Delta} = O(e^{-b\mathrm{T}}) \text{ as } \mathrm{T} \to +y,$$

from which it follows that there exists a constant $L > 0$ such that

$$\|\mathcal{F}(\mathbf{w})\|_{(0,\Delta)} \leq \|\mathcal{F}(\mathbf{w})\|_{(1,\Delta)} \leq LCe^{-b\mathrm{T}},$$

where C is the norm of the operator $\mathbf{L}^{-1}$.

We estimate the difference:

$$\Theta(\mathbf{w}^{(1)}) - \Theta(\mathbf{w}^{(2)}) = \exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)(\theta(\mathbf{v}^{(1)}, \tau) - \theta(\mathbf{v}^{(2)}, \tau))$$

for $\mathbf{w}^{(1)},\ \mathbf{w}^{(2)} \in U_{0,\Delta}$.

From the mean value theorem,

$$\begin{aligned} &\|\theta(\mathbf{v}^{(1)}, \tau) - \theta(\mathbf{v}^{(2)}, \tau)\| \leq \\ &\leq \sup_{\zeta\in[0,1]} \|d_{\mathrm{v}}\theta(\mathbf{v}^{(1)} + \zeta(\mathbf{v}^{(2)} - \mathbf{v}^{(1)}), \tau)\| \, \|\mathbf{v}^{(2)} - \mathbf{v}^{(1)}\|. \end{aligned}$$

Because of the smallness—measured by the standard norm on $\mathbb{R}^n$—of the linear combination $\mathbf{v}^{(1)} + \zeta(\mathbf{v}^{(2)} - \mathbf{v}^{(1)})$, the *supremum* appearing in the above formula is a quantity with asymptotic $O(e^{-(b+\sigma-\delta/2)\tau})$ (see the argument for the corresponding order of $d_{\mathrm{v}}\theta$, given above). The norm of the matrix $\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)$ is a quantity of order $O(e^{(\sigma+\delta/2)\tau})$. Finally,

$$\|\mathbf{v}^{(2)}(\tau) - \mathbf{v}^{(1)}(\tau)\| \leq M(\tau)\|\mathbf{w}^{(2)}(\tau) - \mathbf{w}^{(1)}(\tau)\|,$$

where $M(\tau) = O(e^{\frac{\delta}{2}\tau})$ is some quantity dependent only on the "time" τ.

We can therefore assert that there is a constant $N > 0$, independent of τ, such that

$$\|\Theta(\mathbf{w}^{(1)}) - \Theta(\mathbf{w}^{(2)})\|_{0,\Delta} \leq Ne^{-(b-\frac{3\delta}{2})\mathrm{T}}\|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\|_{0,\Delta},$$

from which the following estimate is immediate:

$$\begin{aligned} \|\mathcal{F}(\mathbf{w}^{(1)}) - \mathcal{F}(\mathbf{w}^{(2)})\|_{0,\Delta} &\leq \|\mathcal{F}(\mathbf{w}^{(1)}) - \mathcal{F}(\mathbf{w}^{(2)})\|_{1,\Delta} \leq \\ &\leq CNe^{-(b-\frac{3\delta}{2})\mathrm{T}}\|\mathbf{w}^{(1)} - \mathbf{w}^{(2)}\|_{0,\Delta}. \end{aligned}$$

From this it is clear that, for large $\mathrm{T} > 0$, the mapping $\mathcal{F}$ is contractive. Applying the Caciopoli-Banach principle [94] to Eq. (1.56), we obtain that the equation has a solution in $U_{0,\Delta}$, i.e. the mapping $\mathcal{F}$ has a fixed point. Inasmuch as the set of values of $\mathcal{F}$ in general lies in the space $H_{1,\Delta}$, the solution $\mathbf{w}(\tau)$ of (1.56) belongs to class $\mathbf{C}^1[\mathrm{T}, +y)$ and thus is a solution of the system of differential equations (1.53) with asymptotic $\mathbf{w}(\tau) = O(e^{-\Delta\tau})$. Since the right sides of (1.53) are infinitely differentiable vector functions, we have that $\mathbf{w}(\tau) \in \mathbf{C}^y[\mathrm{T}, +y)$. Returning to the variable $\mathbf{v}$, we obtain that $\mathbf{w}(\tau) = o(e^{-Kb\tau})$.

Lemma 1.3.3 is proved.

To complete the proof of Theorem 1.3.2, it is now necessary to return to the original variables $\mathbf{x}, t$. Thus the original system of equations (1.37) has an infinitely differentiable solution with the required asymptotic.

The theorem is proved.

Theorem 1.3.2, which we have just now proved in general, guarantees but one solution with the asymptotic that interests us. This solution can be produced constructively in the form of a series where, for finding the coefficients, we need at each step to solve a nonautonomous linear system of differential equations. In spite of this obvious advantage, the theorem doesn't even allow us to estimate the "size" of the set of such solutions, although the algorithm for constructing the series (1.45) indicates that the desired solution may depend on some number of free parameters. Below we establish a result, related to the Hadamard-Perron theorem [137], which guarantees the existence of some l–parameter family of solutions with the required asymptotic.

Theorem 1.3.3. *Suppose that the truncated system of equations (1.39) has a particular solution of form (1.36) such that, for $\beta > 0$, the full spectrum of system (1.41) and, for $\beta < 0$, the full spectrum of system (1.43), contains l negative characteristic exponents, the remaining ones being positive or zero, and such that the irregularity measure σ of the corresponding system satisfies the inequality*

$$\sigma < \min(\frac{|\beta|}{2}, -R), \tag{1.57}$$

where $R = \max(r_i : r_i < 0)$. Then (1.37) has an l–parameter family of particular solutions of the form

$$\mathbf{x}(\mathbf{c}, t) = (\gamma t)^{-\mathbf{G}}(\mathbf{x}_0^{\gamma}(\gamma t) + o(1)) \text{ as } t^{\chi} \to \gamma \times \inf,$$

where $\mathbf{c} \in R^l$ is a vector of parameters.

The proof is in many ways conceptually similar to the proof of the preceding theorem. For the original system we take (1.47) and, as in the proof of the preceding theorem, we only look at the positive semi-quasihomogeneous case ($\chi = +1$). We actually only need prove the following assertion.

Lemma 1.3.5. *Suppose that on the right side of (1.47) the parameter β is positive, that the full spectrum of system (1.41) contains l negative characteristic exponents, the remaining ones being positive or zero, and that the irregularity measure σ of (1.41) satisfies the inequality (1.57). Then (1.47) has an l–parameter family of solutions that tend to $\mathbf{u} = \mathbf{0}$ as $\tau \to +\inf$.*

Proof. We change the dependent and independent variables

$$\mathbf{u}(\tau) = \varepsilon \mathbf{v}(\tau), \quad \tau_\varepsilon = \tau + 2\beta^{-1} \ln \varepsilon,$$

after which system (1.47) assumes the form

$$\mathbf{v}' = \mathbf{K}_\varepsilon(\tau_\varepsilon)\mathbf{v} + \varepsilon(\phi(\mathbf{v}, \tau_\varepsilon, \varepsilon) + \psi(\mathbf{v}, \tau_\varepsilon, \varepsilon)), \tag{1.58}$$

where $\varepsilon > 0$ is some small parameter and where the prime now indicates differentiation with respect to the "new time" τ_ε, and the vector functions ϕ, ψ appearing on the right side of (1.58) are expressed by means of the preceding in the following form:

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} (\mathbf{v}, \tau_\varepsilon, \varepsilon) = \varepsilon^{-2} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} (\varepsilon \mathbf{v}, \tau + 2\beta^{-1} \ln \varepsilon).$$

We let $\theta(\mathbf{v}, \tau_\varepsilon, \varepsilon)$ denote the sum $\Phi(\mathbf{v}, \tau_\varepsilon, \varepsilon) + \Psi(\mathbf{v}, \tau_\varepsilon, \varepsilon)$. It is clear that this vector function is continuous in ε.

The matrix of the linear part of (1.58) is expressed by means of the Kovalevsky matrix in the following form:

$$\mathbf{K}_\varepsilon(\tau_\varepsilon) = \mathbf{K}(\tau_\varepsilon - 2\beta^{-1} \ln \varepsilon).$$

It is easy to see that the matrix $\mathbf{K}_\varepsilon(\tau_\varepsilon)$ gives rise to a linear system with exactly the same asymptotic properties as are enjoyed by system (1.41). For brevity we now drop the index ε in the variable τ_ε.

With the aid of the change of variables

$$\mathbf{v} = \mathbf{U}_\varepsilon(\tau) \exp(-\mathbf{R}\tau)\mathbf{w},$$

where $\mathbf{U}_\varepsilon(\tau)$ is the fundamental matrix of the system of linear differential equations with matrix $\mathbf{K}_\varepsilon(\tau)$, we get rid of nonautonominity in the linear part. As a result, we obtain a system resembling (1.53):

$$\mathbf{w}' = \mathbf{R}\mathbf{w} + \varepsilon\widehat{\theta}(\mathbf{w}, \tau, \varepsilon), \tag{1.59}$$

where

$$\widehat{\theta}(\mathbf{w}, \tau, \varepsilon) = \exp(\mathbf{R}\tau)\mathbf{U}_\varepsilon^{-1}(\tau)\theta(\mathbf{U}_\varepsilon(\tau) \exp(-\mathbf{R}\tau)\mathbf{w}, \tau, \varepsilon).$$

The phase space of system (1.59) decomposes into a direct sum

$$\mathbb{R}^n = E^{(s)} \oplus E^{(u,c)}$$

of subspaces invariant under the operator $\mathbf{R}$ and such that the spectrum of the restriction $\mathbf{R}|_{E^{(s)}} = \mathbf{R}^{(s)}$ is negative and the spectrum of $\mathbf{R}|_{E^{(u,c)}} = \mathbf{R}^{(u,c)}$ is nonnegative. The projections of $\mathbf{w}$ and $\widehat{\theta}$ onto the subspaces $E^{(s)}$ and $E^{(u,c)}$ are denoted, respectively, by $\mathbf{w}^{(s)}$, $\mathbf{w}^{(u,c)}$ and $\widehat{\theta}^{(s)}$, $\widehat{\theta}^{(u,c)}$.

We write the system of differential equations (1.59) in the form of a system of integral equations:

$$\begin{aligned} \mathbf{w}^{(s)} &= \varepsilon \exp\left(\mathbf{R}^{(s)}\tau\right) \left(\mathbf{c} + \int_0^\tau \exp\left(-\mathbf{R}^{(s)}\xi\right) \widehat{\theta}^{(s)}(\mathbf{w}, \xi, \varepsilon\right) d\xi \\ \mathbf{w}^{(u,c)} &= \varepsilon \exp\left(\mathbf{R}^{(u,c)}\tau\right) \left(\mathbf{c} + \int_\tau^\infty \exp\left(-\mathbf{R}^{(u,c)}\xi\right) \widehat{\theta}^{(u,c)}(\mathbf{w}, \xi, \varepsilon\right) d\xi, \end{aligned} \tag{1.60}$$

where $\mathbf{c} \in \mathbb{R}^l = E^{(s)}$ and $\|\mathbf{c}\| \leq 1$ is a vector of free parameters.

From (1.60) it is clear that the desired solution $\mathbf{w}(\tau)$ will satisfy the conditions $\mathbf{w}^{(s)}(0) = \mathbf{c}$, $\mathbf{w}^{(u,c)}(+\infty) = \mathbf{0}$. We will additionally require that $\mathbf{w}(\tau) = o(1)$ as $\tau \to +\infty$.

The problem of finding a solution for (1.60) may be regarded as the problem of finding a fixed point

$$\mathbf{w} = q_\varepsilon(\mathbf{w}), \tag{1.61}$$

where q_ε is the mapping of some small neighborhood $U_{0,\Delta}$ of the space $H_{0,\Delta}$into itself that was considered above, where now $\Delta = \sigma + 2\delta$, $\delta > 0$ is sufficiently small and where the vector functions $\mathbf{w}(\tau)$ are determined on the closed half-line $[0, +\infty)$.

Again Θ_ε denotes the mapping of $U_{0,\Delta}$ into the space $H_{0,\Delta}$ induced by the vector function $\widehat{\theta}(\mathbf{w}, \tau, \varepsilon)$.

Since

$$\mathbf{w}(\tau) = O(e^{-(\sigma+2\delta)\tau}) \text{ as } \tau \to +\inf,$$

it is clear that $\mathbf{v}(\tau) = = O(e^{-(\sigma+\frac{3\delta}{2})\tau})$, whence

$$\phi(\mathbf{v}, \tau, \varepsilon) = O(e^{-(2\sigma+3\delta)\tau}) \text{ and } \psi(\mathbf{v}, \tau, \varepsilon) = O(e^{-\beta\tau}).$$

Because of the inequality (1.57), it is possible to choose δ small enough so that the following estimate holds:

$$\theta(\mathbf{v}, \tau, \varepsilon) = O(e^{-(2\sigma+3\delta)\tau}).$$

We can therefore use the asymptotic estimates obtained previously for the norm of the matrix $\exp(\mathbf{R}\tau)\mathbf{U}_\varepsilon^{-1}(\tau)$:

$$\widehat{\theta}(\mathbf{v}, \tau, \varepsilon) = O(e^{-(\sigma+\frac{5\delta}{2})\tau}).$$

These estimates are uniform in $\varepsilon, \mathbf{w}$ for small ε and for $\mathbf{w}$ in some small neighborhood $U_{0,\Delta}$ of the space $H_{0,\Delta}$. Thus there exists a constant $L > 0$ such that

$$\|\Theta_\varepsilon(\mathbf{w})\|_{0,\Delta} \leq L.$$

The mapping q_ε can be rewritten in the form $q_\varepsilon = \varepsilon \mathbf{P}\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}}$, where $\mathbf{P}$ is some linear mapping on the space $H_{0,\Delta}$, induced by the integral transformation of (1.60) and applied to $\widehat{\theta}(\mathbf{v}, \boldsymbol{\tau}_{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon})$.

Lemma 1.3.6. *The mapping* $\mathbf{P}$ *is continuous.*

Proof. We consider two arbitrary vector functions $\mathbf{h}^{(1)}, \mathbf{h}^{(2)} \in H_{0,\Delta}$ and set $\mathbf{w}^{(1)} = \mathbf{P}\mathbf{h}^{(1)}$, $\mathbf{w}^{(2)} = \mathbf{P}\mathbf{h}^{(2)}$. Since the Eq. (1.57) is satisfied, we can choose δ small enough so that the inequality $\Delta = \sigma + 2\delta < -R$ holds. Therefore

$$\begin{aligned}
&\left| \left(w^{(1)}\right)^i (\tau) - \left(w^{(2)}\right)^i (\tau) \right| \leq \left(e^{r_i\tau} \int_0^\tau e^{-(r_i+\Delta)\xi} d\xi \right) \left\| \mathbf{h}^{(1)} - \mathbf{h}^{(2)} \right\|_{0,\Delta} = \\
&= \left(-(\Delta + r_i)^{-1} e^{\Delta\tau} \left(1 - e^{(r_i+\Delta)\tau}\right)\right) \left\| \mathbf{h}^{(1)} - \mathbf{h}^{(2)} \right\|_{0,\Delta} \leq \\
&\leq e^{-\Delta\tau} \left|\Delta + r_i\right|^{-1} \left\| \mathbf{h}^{(1)} - \mathbf{h}^{(2)} \right\|_{0,\Delta}
\end{aligned}$$

for the i–th component, corresponding to the subspace $E^{(s)}$.

Then for the component corresponding to $E^{(u,c)}$ we have

$$\left|\left(w^{(1)}\right)^i(\tau)-\left(w^{(2)}\right)^i(\tau)\right|\le\left(e^{r_i\tau}\int_\tau^\infty e^{-(r_i+\Delta)\xi}d\xi\right)\left\|\mathbf{h}^{(1)}-\mathbf{h}^{(2)}\right\|_{0,\Delta}=$$
$$=e^{-\Delta\tau}\left(\Delta+r_i\right)^{-1}\left\|\mathbf{h}^{(1)}-\mathbf{h}^{(2)}\right\|_{0,\Delta}$$

From these inequalities it follows that there exists a constant $C>0$ such that

$$\left\|\mathbf{w}^{(1)}-\mathbf{w}^{(2)}\right\|_{0,\Delta}\le C\left\|\mathbf{h}^{(1)}-\mathbf{h}^{(2)}\right\|_{0,\Delta}.$$

The lemma is proved.

Consequently—since $\exp(\mathbf{R}^{(s)}\tau)\mathbf{c}\in H_{0,\Delta}$—the composition $\mathbf{P}\Theta_\varepsilon$ is bounded on $U_{0,\Delta}$. Since the mapping q_ε has the form $q_\varepsilon=\varepsilon\mathbf{P}\boldsymbol{\Theta}_{\boldsymbol{\varepsilon}}$, a suitable choice of ε can be made so that $q_\varepsilon(U_{0,\Delta})\subset U_{0,\Delta}$.

We prove that q_ε is a contraction on $U_{0,\Delta}$ for small $\varepsilon>0$. We consider arbitrary functions $\mathbf{w}^{(1)},\mathbf{w}^{(2)}\in U_{0,\Delta}$ and estimate the difference in the norm:

$$\Theta_\varepsilon(\mathbf{w}^{(1)})-\Theta(\mathbf{w}^{(2)})\le\widehat{\theta}(\mathbf{v}^{(1)},\tau,\varepsilon)-\widehat{\theta}(\mathbf{v}^{(2)},\tau,\varepsilon).$$

Using the mean value theorem,

$$\left\|\theta(\mathbf{v}^{(1)},\tau,\varepsilon)-\theta(\mathbf{v}^{(2)},\tau,\varepsilon)\right\|\le$$
$$\le\sup_{\xi\in[0,1]}\left\|d_\nu\theta\left(\nu^{(1)}+\zeta\left(\nu^{(2)}-\nu^{(1)}\right),\tau,\varepsilon\right)\right\|\left\|\nu^{(2)}-\nu^{(1)}\right\|.$$

In view of the smallness under the standard norm on $\mathbb{R}^n$ of the linear combination $\mathbf{v}^{(1)}+\zeta(\mathbf{v}^{(2)}-\mathbf{v}^{(1)})$ and the fact that the inequality (1.57) is satisfied, an upper bound of the matrix $d_{\mathbf{v}}\boldsymbol{\theta}$ has the asymptotic $O(e^{-(\sigma+3\Delta/2)\tau})$. The norm of the matrix $\exp(\mathbf{R}\tau)\mathbf{U}^{-1}(\tau)$ is a quantity of order $O(e^{(\sigma+3\Delta/2)\tau})$. Consequently, by studying the difference between the asymptotics of the quantities $\mathbf{v}(\tau)$ and $\mathbf{w}(\tau)$, we can write

$$\|\boldsymbol{\Theta}_\varepsilon(\mathbf{w}^{(1)})-\boldsymbol{\Theta}_\varepsilon(\mathbf{w}^{(2)})\|_{0,\Delta}\le N\|\mathbf{w}^{(1)}-\mathbf{w}^{(2)}\|_{0,\Delta},$$

where $N>0$ is some constant not depending on $\mathbf{w}^{(1)},\mathbf{w}^{(2)}\in\mathfrak{U}_{0,\Delta}$, whence follows at once the estimate

$$\|\mathcal{F}_\varepsilon(\mathbf{w}^{(1)})-\mathcal{F}_\varepsilon(\mathbf{w}^{(2)})\|_{0,\Delta}\le\varepsilon CN\|\mathbf{w}^{(1)}-\mathbf{w}^{(2)}\|_{0,\Delta}.$$

With a suitable choice of $\varepsilon>0$ we can obtain contractiveness for the mapping q_ε. From the Caccioppoli-Banach principle [137], applied to Eq. (1.61), we obtain that this equation has a solution in $\mathfrak{U}_{0,\Delta}$, i.e. the mapping $\mathcal{F}_\varepsilon$ has a fixed point and the system of integral equations (1.60) has a solution with asymptotic $\mathbf{w}(\tau)=O(e^{-\Delta\tau})$. Since in general the operator $\mathbf{P}$ increases the order of smoothness by one, the solution $\mathbf{w}(\tau)$ of (1.61) belongs to class $\mathbf{C}^1[0,+\infty)$, i.e. it constitutes a solution to the system of differential equations (1.59). In all of this we are not speaking of an isolated solution, but of a family of solutions tending to zero as $\tau \to +\infty$.

Lemma 1.3.5 is proved.

Returning to the original variables $\mathbf{x}, t$ we obtain that the original system of differential equations (1.37) has an l–parameter family of particular solutions with the required asymptotic.

Theorem 1.3.3 is proved.

We finish this section with a series of remarks.

Remark 1.3.1. The classical Hadamard-Perron theorem [137] asserts that exponential trajectories "sweep out" some set that has, in the neighborhood of a critical point, the structure of a smooth manifold. Even in the autonomous case, the set of solutions with generalized power asymptotic—whose existence is guaranteed by the theorem just proved—don't form a smooth manifold.

Remark 1.3.2. In proving the preceding theorem, we in fact proved that, in the new "logarithmic" time τ, the "perturbed" $\mathbf{u}(\tau)$, which generates a solution (1.36) of the truncated system, has asymptotic $\mathbf{u}(\tau) = O\left(e^{-\left(\sigma + \frac{3}{2}\delta\right)\tau}\right)$ for arbitrarily small $\delta > 0$, which implies formally that, with a reduced irregularity measure σ, this asymptotic worsens and that, in the case of the regularity of the system of first approximation, (1.41) can only guarantee the very weak asymptotic $O\left(e^{-\left(\frac{3}{2}\delta\right)\tau}\right)$ for $\mathbf{u}(\tau)$. Nonetheless, by carrying out a very similar proof for the case of a regular system (1.41), it can be shown that $\mathbf{u}(\tau)$ has an asymptotic of a much higher order, specifically $O\left(e^{-(\beta-\delta)\tau}\right)$.

Remark 1.3.3. It can happen that some components of the solution $\mathbf{x}^{\gamma}(t)$ generated turn out to equal zero. Then, in these components the principal terms of the asymptotic will be determined in the autonomous case by the eigenvalues of the Kovalevsky matrix. We will come upon a similar situation when we consider Example 1.4.5 of the following section.

Remark 1.3.4. From the very beginning of this section we have assumed that the right side of system (1.37) is a semi-quasihomogeneous vector field, provided the time appearing in them is regarded as a parameter (i.e. t in the right sides of (1.37) isn't changed under the action of the group (1.11)). However, practically all the results obtained remain valid even in much more general situations. Let the right side of (1.37) have the following form:

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}_q(\mathbf{x}, t) + \mathbf{f}^{*}(\mathbf{x}, t) + \mathbf{f}^{**}(\mathbf{x}, t),$$

where $\mathbf{f}_q(\mathbf{x}, t)$ is a quasihomogeneous vector field, $\mathbf{f}^{*}(\mathbf{x}, t)$ transforms, under the action of the group (1.11), into a power series in μ^{β} without free term, and where $\mathbf{f}^{**}(\mathbf{x}, t)$ becomes a quantity of order $O\left(\mu^{M\beta}\right)$ for some sufficiently large M (as before, the time t in the right hand side does not change). In this, the result of the transformation of $\mathbf{f}^{**}(\mathbf{x}, t)$ can depend on μ in a rather complicated way (it can, for example, contain logarithms and periodic or quasiperiodic functions of μ).

The presence of the $\mathbf{f}^{**}(\mathbf{x}, t)$ term can hinder attaining the construction of a formal series (1.45). However, this does not prevent the proof of existence of a particular solution of the original system with required asymptotic.

1.4 Examples

We now apply the methods that have been developed in the preceding sections to some concrete examples.

Example 1.4.1. Following the article [60], we investigate the problem of the stability of a critical point and the existence of asymptotic solutions of a multiple-dimensional smooth system of differential equations, whose linear part represents a Jordan decomposition with zero diagonal (Lyapunov's problem).

We write this system of differential equations in the following way:

$$\dot{x}^i = x^{i+1} + \dots, \quad i = 1, \dots, n-1, \quad \dot{x}^n = a(x^1)^2 + \dots, \tag{1.62}$$

where the dots in the equations indicate the presence of nonlinear terms, of which only the monomial $a(x^1)^2$ remains in the final equation. It's easy to see that the chosen system is quasihomogeneous of degree $\mathrm{q} = 2$ in the sense of Definition 1.2.1, with quasihomogeneity indices determined by the integral diagonal matrix $\mathbf{S} = \operatorname{diag}(n, n+1, \dots, 2n-1)$. It can be shown that the quasihomogeneous system under consideration is generated by the positive faces of the Newton polytope for the system (1.62), so that the system (1.62) is positive semi-quasihomogeneous ($\chi = +1$). If the coefficient $a \neq 0$, then the quasihomogeneous truncation has the particular asymptotic solution

$$\mathbf{x}^-(t) = (-t)^{-\mathbf{S}}\mathbf{x}_0^-, \quad \text{or} \quad x^{-i}(t) = \frac{x_0^{-i}}{(-t)^{n+i-1}}, \quad i = 1, \dots, n,$$

where

$$x_0^{-i} = \frac{(2n-1)!(n+i-2)!}{a((n-1)!)^2}, \quad i = 1, \dots, n,$$

from which it follows that the full system has the particular asymptotic solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$. This fact indicates the instability of the critical point considered. For the two-dimensional system ($n = 2$) the result obtained represents Lyapunov's theorem [133].

It is possible to show that the truncated system also has a "positive" particular solution $\mathbf{x}^+(t) = t^{-\mathbf{S}}\mathbf{x}_0^+$—which implies as well the existence of an asymptotic solution of the full system that approaches the critical point as $t \to +\infty$. This solution of the truncated system can be found immediately. On the other hand, its existence follows from part (a) of Lemma 1.1.1. Inasmuch as the right sides of the truncated system are invariant with respect to the substitution $x^1 \mapsto -x^1$, the degree of the Gauss map Γ is even, so that Γ has a fixed point for which the antipode is also a fixed point.

Suppose that in the system (1.62) there is the additional degeneracy ($a = 0$). The two-dimensional case was investigated in detail by Lyapunov [133], so that we will concentrate on the case $n \geq 3$.

We rewrite system (1.62) in the following form:

$$\begin{aligned} \dot{x}^i &= x^{i+1} + \dots, \quad i = 1, \dots, n-2 \\ \dot{x}^{n-1} &= x^n + b(x^1)^2 + \dots, \quad \dot{x}^n = 2cx^1x^2 + \dots \end{aligned} \tag{1.63}$$

Here the dots in the equations again indicate the presence of nonlinear terms, from which the monomial $b(x^1)^2$ is singled out in the penultimate equation, and $2cx^1x^2$ is singled out in the last equation. The system chosen is quasihomogeneous of degree q $= 2$ with matrix of indices $\mathbf{S} = \operatorname{diag}(n-1, \dots, 2n-2)$. If $b + c \neq 0$, then this system has a particular asymptotic solution of the form

$$\mathbf{x}^-(t) = (-t)^{-\mathbf{S}}\mathbf{x}_0^-, \quad \text{or} \quad x^{-i}(t) = \frac{x_0^{-i}}{(-t)^{n+i-2}}, \quad i = 1, \dots, n,$$

where

$$x_0^{-i} = \frac{(2n-3)!(n+i-3)!}{(b+c)((n-2)!)^2}, \quad i = 1, \dots, n-1,$$

$$x_0^{-n} = c\left(\frac{(2n-3)!}{(b+c)(n-2)!}\right)^2.$$

It is easy to show that the quasihomogeneous truncation is determined by the positive faces of the Newton polytope for system (1.63), so that (1.63) has an asymptotic solution $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$, which guarantees instability. It is also possible to show the existence of a solution tending toward $\mathbf{x} = \mathbf{0}$ as $t \to +\infty$.

We note that in both cases the solutions that are asymptotic as $t \to +\infty$ can be written in the following form:

$$\mathbf{x}(t) = t^{-\mathbf{S}} \sum_{k=0}^{\infty} \mathbf{x}_k(\ln t) t^{-k}.$$

Here the $\mathbf{x}_k(\cdot)$ are certain vector polynomials.

Example 1.4.2. The so-called logistical system of equations offers the simplest example of a situation where $\chi = -1$:

$$\dot{N}^i = N^i \left(k_i + b_i^{-1} \sum_{p=1}^{n} a_p^i N^p\right), \quad i = 1, \dots, n. \tag{1.64}$$

With the help of system (1.64) we can describe the interactions of diverse population types in an ecosystem. Here $N^i(t)$ is the number of individuals in the population of i–th type at time t, $(a^i_p)^n_{i,p=1}$ is a constant matrix (if $a^i_p > 0$, then this indicates that the i–th species is growing on account of the p–th species; in the opposite case the i–th species decreases on account of the p–th species), k_i is the difference between the birth and death rates of the i–th species when left to itself, the $b_i > 0$ are parameters characterizing the fact that the reproduction of one of the "predators" is associated with death in one or more of the prey populations. It makes sense to consider a real system of type (1.64) only in the first orthant ($N^i \geq 0$, $i = 1, \ldots, n$). The properties of solutions of system (1.64) were first considered by Volterra [189], and the system (1.64) was later used for modeling other important applied problems. It was originally proposed in [189] that the matrix (a^i_p) be skew-symmetric; at the present time there are other models, where this requirement has been removed. For example, in the article [95], where system (1.64) was applied to the analysis of the dynamics of competing subsystems of production (job systems), it was proposed that off-diagonal elements be positive: $a^i_p > 0$, $i, p = 1, \ldots, n$, $i \neq p$, and the diagonal elements be negative: ($a^i_i < 0$).

Removing the linear terms from (1.64), we get a quadratic homogeneous truncation ($\mathbf{S} = \mathbf{E}, \mathrm{q} = 2$),

$$\dot{N}^i = b_i^{-1} N^i \sum_{p=1}^{n} a^i_p N^p, \quad i = 1, \ldots, n, \tag{1.65}$$

which is clearly selected by the negative faces of the Newton polytope for system (1.64).

System (1.65) has a particular solution of ray type,

$$\mathbf{N}^+(t) = t^{-1}\mathbf{N}_0^+, \quad \mathbf{N} = (N^1, \ldots, N^n),$$

if the algebraic system of linear equations

$$\sum_{p=1}^{n} a^i_p N_0^{+p} = -b_i$$

is solved for $\mathbf{N}_0^+ = (N_0^{+1}, \ldots, N_0^{+n})$.

Then the full system of equations has a particular solution with asymptotic decomposition

$$\mathbf{N}(t) = t^{-1} \sum_{k=0}^{\infty} \mathbf{N}_k (\ln t) t^k.$$

In order that the given particular solution be positive, it is sufficient that there be a positive solution for the above linear system.

The particular solution we have found has the asymptotic $\mathbf{N}(t) = O(t^{-1})$ as $t \to +0$. Since the right side of the chosen truncation is invariant with respect to the substitution $\mathbf{N} \mapsto -\mathbf{N}$, there likewise exists a particular solution with asymptotic $\mathbf{N}(t) = O((-t)^{-1})$ as $t \to -0$.

Example 1.4.3. Rössler's system [153]. We will find particular solutions with nonexponential asymptotic for a nonlinear system for which a chaotic attractor had previously been detected numerically. We will show that there are logarithmic terms in the corresponding expansions of these solutions, i.e. that the given system doesn't pass the Painlevé test.

This third order system has the form

$$\dot{x} = -(y + z), \quad \dot{y} = x + ay, \quad \dot{z} = a + xz - bz, \tag{1.66}$$

where a, b are real parameters.

There are many ways of choosing a quasihomogeneous truncated system for (1.66) and thus also of constructing nonexponential asymptotic solutions of the full system.

We subject system (1.66) to the action of the quasihomogeneous group of dilations of type (1.7):

$$x \mapsto \mu^{g_x} x, \quad y \mapsto \mu^{g_y} y, \quad z \mapsto \mu^{g_z} z, \quad t \mapsto \mu^{-1} t,$$

upon which this system (1.66) assumes the form

$$\begin{aligned}
\dot{x} &= -\mu^{g_z - g_x - 1} y - \mu^{g_z - g_x - 1} z, \\
\dot{y} &= \mu^{g_x - g_y - 1} x + a\mu^{-1} z, \\
\dot{z} &= a\mu^{-g_z - 1} + \mu^{g_x - 1} xz - b\mu^{-1} z.
\end{aligned}$$

The three numbers g_x, g_y, g_z are the diagonal elements of the diagonal matrix $\mathbf{G}$. It is clear that $\chi = -1$.

We restrict ourselves to cases where the elements of the matrix $\mathbf{G}$ are integers. In order to preserve the unique nonlinear term in the quasihomogeneous truncation, we must set $g_x = 1$. Then the possible values of the other two indices will be: $g_y \in \{0, 1, 2\}$, $g_z \in \{-1, 0, 1, 2\}$. Upon calculation of all principal possible combinations we enumerate the nontrivial particular solutions of "ray" type:

$$\begin{array}{lll}
\text{(a)} & \mathbf{G} = \operatorname{diag}(1, 0, -1), & x = 0,\ y = \eta,\ z = at, \\
\text{(b)} & \mathbf{G} = \operatorname{diag}(1, 0, 2), & x = 0,\ y = \eta,\ z = 0, \\
\text{(c)} & \mathbf{G} = \operatorname{diag}(1, 1, -1), & x = 0,\ y = 0,\ z = at, \\
\text{(d)} & \mathbf{G} = \operatorname{diag}(1, 1, 0), & x = 0,\ y = 0,\ z = \zeta, \\
\text{(e)} & \mathbf{G} = \operatorname{diag}(1, 1, 2), & x = -2t^{-1},\ y = 0,\ z = -2t^{-2}, \\
\text{(f)} & \mathbf{G} = \operatorname{diag}(1, 2, -1), & x = 0,\ y = 0,\ z = at, \\
\text{(g)} & \mathbf{G} = \operatorname{diag}(1, 2, 2), & x = -2t^{-1},\ y = 0,\ z = -2t^{-2}.
\end{array}$$

Cases (b) and (d) correspond to particular solutions of the full system (1.66), represented by the series

$$x(t) = \sum_{k=0}^{\infty} x_k t^k, \quad y(t) = \sum_{k=0}^{\infty} y_k t^k, \quad z(t) = \sum_{k=0}^{\infty} z_k t^k,$$

where the coefficients x_k, y_k, z_k don't depend on $\ln t$ and are polynomial functions of three free parameters ξ, η, ζ, and where we can take $\xi = x_0$, $\eta = y_0$, $\zeta = z_0$.

Cases (a), (c) and (f) correspond to particular solutions representing subfamilies of the considered family for $\zeta = 0$, which implies that $z_1 = a$.

We can also assert that the series constructed converge for arbitrary finite ξ, η, ζ in some small complex neighborhood of $t = 0$. This assertion is a simple consequence of Cauchy's theorem on the holomorphic dependence on time and initial conditions (see e.g. [42]).

Cases (e) and (g) yield particular solutions with the asymptotic expansions

$$\begin{aligned} x(t) &= t^{-1} \sum_{k=0}^{\infty} x_k (\ln t) t^k, \\ y(t) &= t^{-2} \sum_{k=0}^{\infty} y_k (\ln t) t^k, \\ z(t) &= t^{-2} \sum_{k=0}^{\infty} z_k (\ln t) t^k, \end{aligned}$$

where $x_0 = -2$, $y_0 = 0$, $z_0 = -2$ and where the logarithmic terms are unavoidable.

This indicates that Rössler's system doesn't have the Painlevé property and provides indirect confirmation of its chaotic nature.

Example 1.4.4. Following [116], we consider Hill's problem [80]. It is written as a Hamiltonian system of equations, whose Hamiltonian function is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + p_x y - p_y x - x^2 + \frac{1}{2} 12 y^2 - (x^2 + y^2)^{-1/2}. \tag{1.67}$$

The corresponding differential equations

$$\begin{aligned} \dot{p}_x &= p_y + 2x - x(x^2 + y^2)^{-\frac{3}{2}}, & \dot{x} &= p_x + y, \\ \dot{p}_y &= -p_x - y - y(x^2 + y^2)^{-\frac{3}{2}}, & \dot{y} &= p_y - x \end{aligned} \tag{1.68}$$

describe the planar motion of a satellite of small mass, e.g. a moon, in the gravitational field of two bodies, the mass of one of which is small in comparison with the mass of the other. for example the earth and the sun. A detailed statement of the problem can be found in the monograph [46]. It is interesting to note that, as was shown by Spring and Waldvogel [179], Eq. (1.68) also approximately describes the joint motion of two satellites of a massive attractive body in close orbit.

We introduce the new auxiliary variable $s = (x^2 + y^2)^{-1/2}$, converting (1.68) into a polynomial system of differential equations of the fifth order:

$$\begin{aligned} \dot{p}_x &= p_y + 2x - xs^3, \quad \dot{x} = p_x + y, \\ \dot{p}_y &= -p_x - y - ys^3, \quad \dot{y} = p_y - x, \\ \dot{s} &= -xp_x s^3 - yp_y s^3. \end{aligned} \tag{1.69}$$

Introducing for the phase variables the quasihomogeneous scale

$$\begin{aligned} p_x \mapsto \mu^{g_{p_x}} p_x, \quad x \mapsto \mu^{g_x}, \quad p_y \mapsto \mu^{g_{p_y}} p_y, \\ y \mapsto \mu^{g_y}, \quad s \mapsto \mu^{g_s} s, \quad t \mapsto \mu^{-1} t, \end{aligned}$$

we find various truncations.

If we choose

$$g_{p_x} = -2/3, \quad g_x = -5/3, \quad g_{p_y} = 1/3, \quad g_y = -2/3, \quad g_s = 2/3,$$

then, under the action of the transformation indicated, the system of equations (1.69) takes the form

$$\begin{aligned} \dot{p}_x &= \ p_y + 2\mu^{-2}x - xs^3, \qquad \dot{x} = p_x + y, \\ \dot{p}_y &= -\mu^{-2}p_x - \mu^{-2}y - ys^3, \ \dot{y} = p_y - \mu^{-2}x, \\ \dot{s} &= -\mu^{-2}xp_x s^3 - yp_y s^3. \end{aligned} \tag{1.70}$$

From this it is clear that, with respect to the "scale" introduced, the system (1.69) is negative semi-quasihomogeneous. Setting $\mu = \infty$, we get a truncated quasihomogeneous system having the particular solution

$$p_x = p_{x_0} t^{2/3}, \ x = x_0 t^{5/3}, \ p_y = p_{y_0} t^{-1/3}, \ y = y_0 t^{2/3}, \ s = s_0 t^{-2/3},$$

where

$$\begin{aligned} p_{x_0} = \pm\tfrac{2}{3}\left(\tfrac{9}{2}\right)^{1/3}, \quad x_0 = \pm\left(\tfrac{9}{2}\right)^{1/3}, \\ p_{y_0} = \pm\tfrac{2}{3}\left(\tfrac{9}{2}\right)^{1/3}, \quad y_0 = \pm\left(\tfrac{9}{2}\right)^{1/3}, \quad s_0 = \left(\tfrac{2}{9}\right)^{1/3}. \end{aligned}$$

In accordance with Theorem 1.1.2, the Hamiltonian system of equations with Hamiltonian (1.67) has a particular solution admitting the asymptotic expansion

$$\begin{aligned} p_x(t) &= t^{2/3} \textstyle\sum_{k=0}^{\infty} p_{x_k} t^{k/3}, \qquad x(t) = t^{5/3} \sum_{k=0}^{\infty} x_k t^{k/3}, \\ p_y(t) &= t^{-1/3} \textstyle\sum_{k=0}^{\infty} p_{y_k} t^{k/3}, \quad y(t) = t^{2/3} \sum_{k=0}^{\infty} y_k t^{k/3}. \end{aligned} \tag{1.71}$$

But if we choose

$$g_{p_x} = 1/3, \quad g_x = -2/3, \quad g_{p_y} = -2/3, \quad g_y = -5/3, \quad g_s = 2/3,$$

then (1.69) transforms to

$$\begin{aligned} \dot{p}_x &= \mu^{-2} p_y + 2\mu^{-2} x - x s^3, & \dot{x} &= p_x + \mu^{-2} y, \\ \dot{p}_y &= -p_x - \mu^{-2} y - y s^3, & \dot{y} &= p_y - x, \\ \dot{s} &= -x p_x s^3 - \mu^{-2} y p_y s^3. \end{aligned} \tag{1.72}$$

For $\mu = \infty$ we get a quasihomogeneous truncated system with the particular solution

$$p_x = p_{x_0} t^{-1/3}, \quad x = x_0 t^{2/3}, \quad p_y = p_{y_0} t^{2/3}, \quad y = y_0 t^{5/3}, \quad s = s_0 t^{-2/3},$$

where now

$$\begin{gathered} p_{x_0} = \pm \tfrac{2}{3} \left(\tfrac{9}{2}\right)^{1/3}, \quad x_0 = \pm \left(\tfrac{9}{2}\right)^{1/3}, \\ p_{y_0} = \mp \tfrac{2}{3} \left(\tfrac{9}{2}\right)^{1/3}, \quad y_0 = \mp \left(\tfrac{9}{2}\right)^{1/3}, \quad s_0 = \left(\tfrac{2}{9}\right)^{1/3}. \end{gathered}$$

Consequently, the full system of equations (1.68) has a particular solution with asymptotic expansions

$$\begin{aligned} p_x(t) &= t^{-1/3} \sum_{k=0}^{\infty} p_{x_k} t^{k/3}, & x(t) &= t^{2/3} \sum_{k=0}^{\infty} x_k t^{k/3}, \\ p_y(t) &= t^{2/3} \sum_{k=0}^{\infty} p_{y_k} t^{k/3}, & y(t) &= t^{5/3} \sum_{k=0}^{\infty} y_k t^{k/3}. \end{aligned} \tag{1.73}$$

Application of the algorithm in its general form for constructing formal asymptotics of the type (1.14) is explained in Sect. 1.1 and provides polynomial dependency on $\ln t$ for the coefficients of the expansions (1.71) and (1.73). But straightforward (although tedious) calculations show that p_{x_k}, x_k, p_{y_k}, y_k are constant. This happens for the following reasons. Inasmuch as the system (1.69) is negative semi-quasihomogeneous, so the logarithms may appear in stages, the k-th of which "resonates" with the positive eigenvalues of the Kovalevsky matrix, i.e. $-k\beta = \rho_i$, where the ρ_i are roots of the characteristic equation

$$\det(\mathbf{K} - \rho \mathbf{E}) = 0.$$

The Kovalevsky exponents for all the cases considered are as follows:

$$\rho_{1,2} = -1, \quad \rho_{3,4} = -4/3, \quad \rho_5 = 2/3,$$

where here $\beta = -1/3$. This indicates that the logarithms can't appear before the second stage. But from Eqs. (1.70) and (1.72) it is clear that the nonzero free terms $\mathbf{\Phi}_k$ in the equations for determining the coefficients appear only at the sixth stage.

The trajectories for Hill's problem corresponding to the particular solutions (1.68), with asymptotic expansions (1.71) and (1.73), are so-called collision trajectories. On these trajectories there occur collisions of the less massive body and the satellite (Earth and Moon) after a finite time (which depends on the chosen reference frame as $t \to 0$). The algorithm explained in Sect. 1.1 allows for the recurrent determination of the coefficients of these expansions and thus for the constructive determination of these collision trajectories in real time. Previously, in considering collisions in Hill's problem, the equations of motion were subjected to regularizing changes of variable that had been introduced by G.D. Birkhoff [16]. A contemporary treatment of this problem is given in the monograph [8].

Example 1.4.5. Generalized Henon-Heiles [78] system. This example is considered in detail in the articles [36, 37].

We consider the motion of a mechanical system with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + Dx^2y - \frac{C}{3}y^3. \tag{1.74}$$

The corresponding system of equations has the form

$$\begin{aligned} \dot{p}_x &= -x - 2Dxy, & \dot{x} &= p_x, \\ \dot{p}_y &= -y - Dx^2 + Cy^2, & \dot{y} &= p_y. \end{aligned} \tag{1.75}$$

We subject system (1.75) to the natural quasihomogeneous group of transformations

$$p_x \mapsto \mu^{g_x+1}p_x,\ x \mapsto \mu^{g_x},\ p_y \mapsto \mu^{g_y+1}p_y,\ y \mapsto \mu^{g_y},\ t \mapsto \mu^{-1}t,$$

after which (1.80) assumes the form

$$\begin{aligned} \dot{p}_x &= -\mu^{-1}x - 2\mu^{g_y-2}Dxy, & \dot{x} &= p_x, \\ \dot{p}_y &= -\mu^{-1}x - \mu^{2g_x-g_y-2}Dx^2 + \mu^{g_y-2}Cy^2, & \dot{y} &= p_y. \end{aligned} \tag{1.76}$$

From (1.76) it is clear that $\chi = 1$, so we seek the asymptotics as $t \to 0$.

To keep the nonlinear term in the first equation of (1.76) we need to set $g_y = 2$. The choices for g_x can lead to quite different variants. If our goal is to keep all the nonlinear terms in (1.76), then we need to set $g_x = 2$. Setting $\mu = \infty$, we obtain a truncated system with the particular solution:

$$p_x = p_{x_0}t^{-3}, \quad x = x_0t^{-2}, \quad p_y = p_{y_0}t^{-3}, \quad y = y_0t^{-2},$$

where

$$p_{x_0} = -2x_0,\ x_0 = \pm\frac{3}{D}(2+\delta^{-1})^{1/2},\ p_{y_0} = -2y_0,\ y_0 = -\frac{3}{D},\ \delta = \frac{D}{C}.$$

The parameter $\delta \in \mathbb{R}$ will play an essential role in what follows.

We investigate the case where the expansions of solutions of (1.75), whose existence is guaranteed by Theorem 1.1.2, do not contain logarithmic time, so that they have the form

$$p_x(t) = t^{-3} \sum_{k=0}^{\infty} p_{x_k} t^k, \quad x(t) = t^{-2} \sum_{k=0}^{\infty} x_k t^k,$$
$$p_y(t) = t^{-3} \sum_{k=0}^{\infty} p_{y_k} t^k, \quad y(t) = t^{-2} \sum_{k=0}^{\infty} y_k t^k, \tag{1.77}$$

where $p_{x_k} = (k-2)x_k$, $p_{y_k} = (k-2)y_k$.

We note that in view of the invertibility of the system of equations with Hamiltonian (1.74), all the coefficients in (1.77) with odd indices reduce to zero.

It is not difficult to compute the eigenvalues of the Kovalevsky matrix, which in the case considered equal

$$\rho_1 = -1, \quad \rho_2 = 6, \quad \rho_{3,4} = \frac{5}{2} \pm \frac{1}{2} \left(1 - 24(1 + \delta^{-1})\right)^{1/2}.$$

Thus logarithms can appear either after the third stage ($k = 6$) or from the resonances

$$\frac{5}{2} + \frac{1}{2} \left(1 - 24(1 + \delta^{-1})\right)^{1/2} = 2l = k. \tag{1.78}$$

Calculations that were carried out with the aid of symbolic computation show that, with $k = 6$, logarithms do not appear. Therefore, in the absence of the resonances (1.78), the expansions (1.77) actually do not contain logarithmic time.

If we don't require keeping the term quadratic in x in the second equation of (1.76), in selecting the truncation, then we must choose $g_x < 2$. In order that the system (1.75) remain semi-quasihomogeneous in the sense of the usual definition it is also necessary that g_x be rational.

The analysis carried out in the paper [36] shows that the system of equations (1.75) has a particular solution with principal terms that have asymptotic expansions

$$p_x(t) \sim p_{x_0} t^{\Delta_\pm(\delta)-1}, \; x(t) \sim x_0 t^{\Delta_\pm(\delta)}, \; p_y(t) \sim p_{y_0} t^{-3}, \; y(t) \sim y_0 t^{-2}$$

as $t \to 0$, where $p_{x_0} = \Delta_\pm(\delta)c$, $x_0 = c$, $p_{y_0} = \frac{12}{C}$, $y_0 = \frac{6}{C}$, where c is some arbitrary constant and $\Delta_\pm(\delta) = \frac{1}{2} \pm \frac{1}{2}(1 - 48\delta)^{1/2}$. The quantities $\Delta_\pm(\delta)$ can of course be irrational and, for $\delta > \frac{1}{48}$, even complex. This indicates that this kind of solution can't be constructed with the aid of the algorithm of Theorem 1.1.2, nor indeed with any quasihomogeneous truncation.

What is the nature of the solution constructed in [36]? For arbitrary rational $g_x < 2$, the truncation

$$\dot{p}_x = -2Dxy, \quad \dot{x} = p_x, \quad \dot{p}_y = Cy^2, \quad \dot{y} = p_y$$

of the system has the particular solution

$$p_x \equiv 0, \quad x \equiv 0, \quad p_y = -\frac{12t^{-3}}{C}, \quad y = \frac{6t^{-2}}{C},$$

generated by the given quasihomogeneous structure.

We easily convince ourselves that the Kovalevsky exponents are here equal to

$$\rho_1 = -1, \quad \rho_2 = 6, \quad \rho_{3,4} = \Delta_\pm(\delta) + g_x.$$

Subsequent application of Theorem 1.3.1 makes it possible to establish existence of particular solutions of (1.75) with the indicated asymptotic.

Example 1.4.6. The Painlevé equations [68,71,87] can contain examples of nonautonomous semi-quasihomogeneous systems. It is claimed that a differential equation doesn't have movable critical points if the critical points of its solutions don't fill some region in the complex plane. Painlevé and Gambier classified equations of the form

$$\ddot{x} = R(\dot{x}, x, t),$$

not having moving singularities, where R is a rational function of x, $\dot{x}$, with coefficients meromorphic in t. Equations satisfying these requirements are often called equations of class P. A list of 50 equations was found such that each equation of class P can be obtained from one of the equations in the list with the aid of a certain holomorphic diffeomorphism satisfying some supplementary properties that we won't discuss here. Of these 50 equations, 44 are integrable by quadratures or can be transformed to an equation of the type

$$Q(\dot{x}, x, t) = 0,$$

where Q is a polynomial in $\dot{x}, x$ with meromorphic coefficients. The remaining six are called Painlevé equations. We enumerate them:

$$\text{I.} \qquad \ddot{x} = 6x^2 + t \tag{1.79}$$

$$\text{II.} \qquad \ddot{x} = 2x^3 + tx + a \tag{1.80}$$

$$\text{III.} \qquad \ddot{x} = \dot{x}^2 x^{-1} + e^t(ax^2 + b) + e^{2t}(cx^3 + dx^{-1}) \tag{1.81}$$

$$\text{IV.} \qquad \ddot{x} = \frac{1}{2}\dot{x}^2 x^{-1} + \frac{3}{2}x^3 + 4tx^2 + 2(t^2 - a)x + bx^{-1} \tag{1.82}$$

$$\text{V.} \qquad \ddot{x} = \dot{x}^2\left(\frac{1}{2x} + \frac{1}{x-1}\right) - \frac{\dot{x}}{t} + \frac{(x-1)^2}{t^2}\left(ax + \frac{b}{x}\right) + {} + c\frac{x}{t} + d\frac{x(x+1)}{x-1} \tag{1.83}$$

$$\text{VI.} \qquad \ddot{x} = \frac{\dot{x}^2}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right) - \dot{x}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right) + {} + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left(a + b\frac{1}{x^2} + c\frac{t-1}{(x-1)^2} + d\frac{t(t-1)}{(x-t)^2}\right) \tag{1.84}$$

All solutions of the first four Painlevé equations are meromorphic functions. The solutions of the fifth have logarithmic branch points at $t = 0, t = \infty$, as do those of the sixth at: $t = 0, t = 1, t = \infty$.

We investigate the form of the expansions of solutions of the Painlevé equations, obtained with the aid of the algorithms described above.

Written in the form of systems of two equations of first order,

$$\dot{x} = y, \quad \dot{y} = 6x^2 + t,$$
$$\dot{x} = y, \quad \dot{y} = 2x^3 + tx + a,$$

the first and second Painlevé equations (1.79) and (1.80) are negative semi-quasihomogeneous with respect to the structure generated by the corresponding matrices $\mathbf{G} = \operatorname{diag}(2,3)$ and $\mathbf{G} = \operatorname{diag}(1,2)$. The parameter β in both cases can be taken equal to -1.

Their quasihomogeneous truncations

$$\dot{x} = y, \quad \dot{y} = 6x^2,$$
$$\dot{x} = y, \quad \dot{y} = 2x^3$$

have the obvious solutions

$$x = \frac{1}{t^2}, \quad y = -\frac{2}{t^3},$$
$$x = \frac{1}{t}, \quad y = -\frac{1}{t^2}.$$

The eigenvalues of the Kovalevsky matrix equal -1, 6 for the first Painlevé equation and -1, 4 for the second. A detailed analysis shows that logarithms don't appear in the corresponding steps, which is confirmed by the general theory. The corresponding Laurent expansions for the solutions will have the form

$$x(t) = t^{-2} \sum_{k=0}^{\infty} x_k t^k$$

for Eq. (1.79) and the form

$$x(t) = t^{-1} \sum_{k=0}^{\infty} x_k t^k$$

for Eq. (1.80).

By introducing the auxiliary variables $y = \dot{x}$, $z = x^{-1}$, the third and fourth Painlevé equations (1.81) and (1.82) are transformed into semi-quasihomogeneous systems of three equations:

$$\dot{x} = y, \quad \dot{y} = y^2 z + e^t(ax^2 + b) + e^{2t}(cx^3 - dz), \qquad \dot{z} = -yz^2,$$
$$\dot{x} = y, \quad \dot{y} = \frac{1}{2}y^2 z + \frac{3}{2}x^3 + 4tx^2 + 2(t^2 - a)x + bz, \quad \dot{z} = -yz^2.$$

The corresponding matrices **G** then have form $\mathrm{diag}(1, 2, -1)$, and the parameter β has the value -1. The truncated systems in the two cases are analogous. We can write them as

$$\dot{x} = y, \quad \dot{y} = Ay^2 z + Bx^3, \quad \dot{z} = -yz^2, \tag{1.85}$$

where $A = 1$, $B = c$ for the third Painlevé equation, $A = 1/2$, $B = 3/2$ for the fourth.

System (1.85) has a particular solution in the form of the quasihomogeneous ray

$$x = \left(\frac{2-A}{B}\right)^{1/2} t^{-1},\ y = -\left(\frac{2-A}{B}\right)^{1/2} t^{-2},\ z = \left(\frac{2-A}{B}\right)^{1/2} t.$$

The eigenvalues of the Kovalevsky matrix that correspond to this solution equal $-1,\ 1,\ -2(A-2)$. Therefore, for the third and fourth Painlevé equation, they respectively equal $-1,\ 1,\ 2$ and $-1,\ 1,\ 3$.

Very complex computations, expedited with the aid of symbolic computation, show that logarithms don't appear at any of the corresponding stages, so that the desired solutions of Eqs. (1.81) and (1.82) can be represented by Laurent series of the form

$$x(t) = t^{-1} \sum_{k=0}^{\infty} x_k t^k,$$

consistent with their meromorphic nature.

However, it turns out that the fifth and sixth Painlevé equations (1.83) and (1.84) can't be represented as semi-quasihomogeneous systems without a supplementary "trick".

First consider the fifth Painlevé equation. In (1.83) we perform a logarithmic change of variable $\tau = \ln t$. Then, after the introduction of the auxiliary variables $y = x'$, $z = x^{-1}$ already described, where the prime denotes differentiation with respect to the new independent variable τ, (1.83) is written as a system of three equations:

$$\begin{aligned} x' &= y, \\ y' &= y^2 \frac{z(z-3)}{2(z-1)} + (x-1)^2(ax+bz) + ce^{\tau}x + de^{2\tau}\frac{x(z+1)}{(z-1)}, \\ z' &= -yz^2. \end{aligned}$$

This system is semi-quasihomogeneous: $\mathbf{G} = \mathrm{diag}\,(\mathbf{1}, \mathbf{2}, -\mathbf{1})$, $\beta = -1$. The corresponding truncated system once again has the form (1.83) if, for the role of independent variable, we choose the logarithmic time τ. In this, $A = 3/2$, $B = a$, so that the Kovalevsky exponents are $-1,\ 1,\ 1$. Consequently logarithms can appear in the desired solution only in the first stage.

An analysis done with the aid of symbolic computation showed that this doesn't happen, so that the desired solution of the fifth Painlevé equation (1.83) can be expanded in a Laurent series in τ:

$$x(\tau) = \tau^{-1} \sum_{k=0}^{\infty} x_k \tau^k,$$

or, after reverting to the original independent variable t:

$$x(t) = \ln^{-1} t \sum_{k=0}^{\infty} x_k \ln^k t,$$

which shows that $t = 0$ is a logarithmic movable singularity.

In order to investigate the sixth Painlevé equation (1.84), we perform a logarithmic change of time with two singularities: $\tau = \ln(t(t-1))$. We further introduce the auxiliary variables $y = x'$, $z = x^{-1}$, where the prime denotes differentiation with respect to the new "time" τ. Equation (1.84) then takes the form of the first order system

$$\begin{aligned}
x' &= y,\\
y' &= y^2 \frac{z(\phi z^2 - 2z(\phi+1) + 3)}{2(z-1)(\phi z - 1)} + \frac{e^\tau y}{2\phi - 1}\left(\frac{2}{2\phi - 1} - \frac{z}{\phi z - 1}\right) +\\
&\quad + \frac{x(x-1)(x-\phi)}{(2\phi-1)^2}\left(a + b\phi z^2 + c\frac{z^2(\phi-1)}{(z-1)^2} + d\frac{z^2\phi(\phi-1)}{(\phi z-1)^2}\right),\\
z' &= -yz^2.
\end{aligned}$$

where, for convenience, we introduce the notation

$$\phi = \phi(\tau) = \frac{1}{2}\left(1 \pm \sqrt{1 + 4e^\tau}\right).$$

The system obtained is negative semi-quasihomogeneous and the matrix which gives the quasihomogeneous scale again has the form $\mathbf{G} = \operatorname{diag}(1, 2, -1)$, $\beta = -1$. The truncated system has the form (1.84), where we have taken the new time τ as the independent variable, and where $A = 3/2$, $B = 4a/5$. Just as with the fifth Painlevé equation, the Kovalevsky exponents equal $-1,\ 1, 1$, so that logarithms can appear only in the first step.

For this case too, with the aid of symbolic computation, a system of equations was obtained for finding the first coefficients, and it was discovered that logarithms don't appear in their solution. Consequently, the desired solution can be constructed in the form of an ordinary Taylor series

$$x(\tau) = \tau^{-1} \sum_{k=0}^{\infty} x_k \tau^k,$$

from which we obtain the expansion with respect to the original independent variable t:

$$x(t) = \ln^{-1}(t(t-1)) \sum_{k=0}^{\infty} x_k \ln^k(t(t-1)).$$

Consequently, $t = 0$ and $t = 1$ represent logarithmic movable singularities. In order to investigate the character of the singularities of the solutions at infinity, we make the substitution $t \mapsto \frac{1}{t}$.

Example 1.4.7. Finally, we discuss V.V. Ten's hypothesis concerning the stability of isolated equilibrium states of dynamical systems with invariant measure in a space of odd dimension. Let

$$\dot{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n, \tag{1.86}$$

be an autonomous system of differential equations admitting an invariant measure with smooth density:

$$\operatorname{div}(\rho \mathbf{f}) = 0. \tag{1.87}$$

Let $\mathbf{x} = \mathbf{0}$ be an equilibrium position: $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. V.V. Ten proposed that if n is odd and if the equilibrium position $\mathbf{x} = \mathbf{0}$ is isolated, then it is Lyapunov stable. This hypothesis has important consequences: all isolated equilibrium points in a stationary fluid flow in three-dimensional Euclidean space would then be stable.

In a typical situation, system (1.86) has the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + O(|\mathbf{x}|), \quad \det \mathbf{A} \neq 0. \tag{1.88}$$

We rewrite the "continuity equation" (1.87):

$$\dot{\nu} = -\operatorname{div} \mathbf{f}, \quad \nu = \ln \rho.$$

Setting $\mathbf{x} = \mathbf{0}$ in this equation, we obtain the equality $\operatorname{tr} A = 0$. Consequently, the sum of all eigenvalues of the matrix $\mathbf{A}$ is equal to zero. Then at least one of the eigenvalues lies in the right half-plane. In fact, in the opposite case the spectrum of $\mathbf{A}$ would be distributed over the imaginary axis. But the sum of all its eigenvalues is zero and n is odd, so zero must then be an eigenvalue (since for a real matrix the eigenvalues come in conjugate pairs). On the other hand, this contradicts the assumption of a nonsingular matrix. Consequently, by Lyapunov's theorem, $\mathbf{x} = \mathbf{0}$ is a stable equilibrium position for the system (1.88).

It can be proved analogously that nondegenerate periodic trajectories of a dynamical system with invariant measure in an *even-dimensional* space are always stable. Recall that a periodic orbit is called *nondegenerate* if its multipliers are distinct from 1. This observation is also correct for nondegenerate *reducible* invariant tori of odd codimension, filled with conditionally periodic trajectories.

As proved in [119], assertions about stability may be extended to positive semi-quasihomogeneous systems of differential equations:

$$\mathbf{f}_m + \sum_{\alpha > m} \mathbf{f}_\alpha,$$

where $\mathbf{f}_k$ is a quasihomogeneous field of degree k with one and the same quasi-homogeneity matrix $\mathbf{G}$. The only additional condition is that $\mathbf{x} = \mathbf{0}$ is the *unique* equilibrium point of the quasihomogeneous field $\mathbf{f}_m$.

In fact, inasmuch as n is odd and $\mathbf{x} = \mathbf{0}$ is the only zero of the quasihomogeneous field $\mathbf{f}_m$, there is (in agreement with Lemma 1.1.1) a nonzero vector $\mathbf{z}$ satisfying one of the equations

$$\mathbf{f}_m(\mathbf{z}) = -\mathbf{G}\mathbf{z} \quad \text{or} \quad \mathbf{f}_m(\mathbf{z}) = \mathbf{G}\mathbf{z}.$$

By Theorem 1.1.2, Eq. (1.86) in this case admits a solution with the asymptotic

$$Zt^{-G} \quad \text{or} \quad Z(-t)^{-G},$$

respectively, as $t \to +\infty$ or $t \to -\infty$. This solution tends to the equilibrium point $\mathbf{x} = \mathbf{0}$ as $t \to \pm\infty$. If there is a solution of the second type ("departing from the point $\mathbf{x} = \mathbf{0}$"), then obviously the equilibrium point is unstable. It remains to consider the case where there is a solution asymptotic to the equilibrium point as $t \to +\infty$.

We make use of an assertion that is interesting in itself.

Lemma 1.4.1. *Suppose that system (1.86) has an invariant measure and admits a nontrivial solution $t \mapsto x(t)$ that tends toward zero as $t \to +\infty$. Then the equilibrium point $\mathbf{x} = \mathbf{0}$ is unstable.*

In fact, let the point $x_0 = x(0)$ lie in some ε_0-neighborhood of zero. For arbitrary $\varepsilon > 0$ there is a small neighborhood U_ε of the point x_0 which, under the action of the phase flow over a certain time, appears in its entirety in an ε-neighborhood of the point $\mathbf{x} = \mathbf{0}$. However, by the ergodic theorem of Schwarzschild-Littlewood [173], almost all points distributed initially over U_ϵ leave the ϵ_0-neighborhood of zero. This proves the instability of the equilibrium point, inasmuch as each exiting trajectory intersects the ε-neighborhood of zero.

In the article [119] it is proved that Ten's hypothesis is not true, even for an *infinitely smooth* vector field $\mathbf{f}$, without some supplementary assumption about nonsingularity: in the construction of the counterexample there is a Maclaurin series of this vector field that is zero. But it is not impossible that Ten's hypothesis will turn out to be correct in the *analytic* case.

1.5 Group Theoretical Interpretation

Before beginning the exposition of new material, we return to the content of the first section. Our goal there was the construction of particular solutions of certain systems of differential equations with the help of particular solutions of a so-called truncated or model system. The choice of a truncation is dictated by some "scale" generated by the quasihomogeneous group of transformations (1.11) on extended phase space. The truncated system obtained is invariant under the given transformation group, and the "supporting" solution used lies on some orbit of this group. For the system of differential equations—whose right sides are represented as a series (1.4)—the quasihomogeneous scale introduced is rather natural. The question however arises about the possibility of using this or that one-parameter group of transformations of extended phase space for constructing such a scale. Another rather important problem consists of successively appending particular solutions to the truncated system obtained in this way until there is solution of the full system. It is to these questions that we devote the present section.

The material set forth below contains several facts from the theory of symmetry groups of ordinary differential equations. In order to obtain a fuller acquaintance with this material we recommend the monographs [145, 146].

We consider a smooth autonomous system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1.89}$$

and some one-parameter semigroup $\mathbf{X}$ of transformations from Ω to itself:

$$\mathbf{X}\colon \Omega[0, \chi \times \infty) \to \Omega, \;\; \chi = \pm 1, \;\; \mathbf{X}(\cdot, 0) = \mathrm{id}.$$

Let the smooth vector field $\mathbf{g}(\mathbf{x})$ be the infinitesimal generator, and let σ be the parameter of this semigroup. The parameter σ belongs either to a right or left half-line depending on the sign. We consider the system of differential equations

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{g}(\mathbf{x}). \tag{1.90}$$

that generates the given semigroup.

In general, we restrict the domain of variation of the parameter σ to some small finite interval $(-\sigma_0, +\sigma_0)$; then, according to the theorem on existence and uniqueness of solutions to the system (1.90), $\mathbf{X}$ will be a group. However, in the sequel we will likewise be interested in infinite values of the parameter σ, so that here it is more logical to speak about *semigroups* of transformations, inasmuch as it may happen that various elements of $\mathbf{X}$ will not have inverses (we suppose a priori that solutions exist, at least on half-lines). For the simplicity of the presentation we will henceforth ignore this fact and speak about groups of transformations. Likewise for simplicity, we will assume that the group considered acts globally on the domain Ω.

Definition 1.5.1. We say that $\mathbf{X}$ is a *generalized symmetry group* of the system (1.89) if, after the substitution

$$\mathbf{x} = \mathbf{X}(\mathbf{y}, \boldsymbol{\sigma}), \tag{1.91}$$

the system (1.89) assumes the form

$$\dot{\mathbf{y}} = \boldsymbol{\phi}(\mathbf{y}, \boldsymbol{\sigma})\mathbf{f}(\mathbf{y}), \tag{1.92}$$

where $\phi\colon \Omega \times [0, \chi \times \infty) \to R^+$ is some positive function. If $\mathbf{X} = \mathrm{id}$ for $\sigma = 0$, then $\phi(\mathbf{y}, \mathbf{0}) = \mathbf{1}$. If $\phi(\mathbf{y}, \boldsymbol{\sigma}) \equiv \mathbf{1}$ for arbitrary $\mathbf{y}$, $\boldsymbol{\sigma}$, then we will say that $\mathbf{X}$ is the *symmetry group* of (1.89).

If $\mathbf{X}$ is a symmetry group, then the corresponding transformations take solutions of system (1.89) to solutions of the very same system. But if $\mathbf{X}$ is an extended symmetry group, then $\mathbf{X}$ only leaves a family of trajectories invariant, and the parameterization of solutions changes [145, 146]. We prove one auxiliary result.

Lemma 1.5.1. *Let* $\mathbf{X}$ *be the extended symmetry group of system (1.89) that is generated by system (1.92). Then for arbitrary* $\in \boldsymbol{\Omega}$ *the following equation holds:*

$$[\mathbf{f}, \mathbf{g}](\mathbf{x}) = -\frac{\partial \phi}{\partial \sigma}(\mathbf{x}, 0)\mathbf{f}(\mathbf{x}), \tag{1.93}$$

where

$$[\mathbf{f}, \mathbf{g}] = (d\,\mathbf{g})\mathbf{f} - (d\,\mathbf{f})\mathbf{g}$$

and where the brackets are those for the Lie algebra of a smooth vector field.

Proof. We apply transformation (1.91) to system (1.89). Since

$$\dot{\mathbf{x}} = d_y \mathbf{X}(\mathbf{y}, \sigma)\dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}),$$

it follows from (1.92) that

$$\phi(\mathbf{y}, \sigma) d_y \mathbf{X}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}) \tag{1.94}$$

(for brevity, here and in the sequel we write $\mathbf{x}$ instead of $\mathbf{X}(\mathbf{y},\boldsymbol{\sigma})$).

In the sequel we will need the following assertion from the general theory of differential equations (see e.g. [42]).

Lemma 1.5.2. *Let* $\mathbf{U}(\mathbf{y},\boldsymbol{\sigma})$ *be the fundamental matrix of the linear system of differential equations*

$$\frac{d\mathbf{u}}{d\sigma} = d\,\mathbf{g}(\mathbf{x})\mathbf{u},$$

normalized by the condition $\mathbf{U}(\mathbf{y}, 0) = \mathbf{E}$*. Then the following identity holds:*

$$d_y \mathbf{X}(\mathbf{y}, \sigma) = \mathbf{U}(\mathbf{y}, \sigma). \tag{1.95}$$

We continue the proof of Lemma 1.5.1. We substitute (1.95) into (1.94) and differentiate with respect to σ. As a result we obtain the following identity:

$$\phi(\mathbf{y}, \sigma) d\,\mathbf{g}(\mathbf{x})\mathbf{U}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{y}) + \frac{\partial \phi}{\partial \sigma}(\mathbf{y}, \sigma)\mathbf{U}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{y}) = d\,\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x}).$$

Since $\mathbf{x} \to \mathbf{y}$ as $\sigma \to 0$, passing to the limit we get

$$d\,\mathbf{g}(\mathbf{y})\mathbf{f}(\mathbf{y}) + \frac{\partial \phi}{\partial \sigma}(\mathbf{y}, 0)\mathbf{f}(\mathbf{y}) = d\,\mathbf{f}(\mathbf{y})\mathbf{g}(\mathbf{y}).$$

Since, for fixed σ, the transformation $\mathbf{x}$ establishes a diffeomorphism of the domain Ω, returning to the variable $\mathbf{x}$ we obtain the relation (1.93).

The lemma is proved.

The converse also holds:

Lemma 1.5.3. *Suppose there exists a smooth function* $\psi\colon \Omega \to R$ *such that the identity*

$$[\mathbf{f}, \mathbf{g}](\mathbf{x}) = -\psi(\mathbf{x})\mathbf{f}(\mathbf{x}) \tag{1.96}$$

holds. Then $\mathbf{X}$ *(the phase flow of the vector field* $\mathbf{g}$*) is the extended symmetry group of system (1.89).*

Suppose that, under the action of the group $\mathbf{X}$, system (1.89) assumes the form

$$\dot{\mathbf{y}} = \phi(\mathbf{y}, \sigma)\widehat{\mathbf{f}}(\mathbf{y}, \sigma).$$

If we set

$$\phi(\mathbf{y}, \sigma) = \int_0^\sigma \psi(\mathbf{x})d\sigma,$$

then

$$\widehat{\mathbf{f}}(\mathbf{y}, \sigma) = \left(-\int_0^\sigma \psi(\mathbf{x})d\sigma\right)\mathbf{U}^{-1}(\mathbf{y}, \sigma)\mathbf{f}(\mathbf{x}). \tag{1.97}$$

We have to show that $\widehat{\mathbf{f}}(\mathbf{y}, \sigma) = \mathbf{f}(\mathbf{y})$ holds for arbitrary σ.

It follows at once from (1.97) that $\widehat{\mathbf{f}}(\mathbf{y}, 0) = \mathbf{f}(\mathbf{y})$. Differentiating (1.97) with respect to σ and using (1.96) and the formula for the differentiated inverse matrix, we obtain

$$\frac{\partial\widehat{\mathbf{f}}}{\partial\sigma}(\mathbf{y}, \sigma) = \exp\left(-\int_0^\sigma\right)\mathbf{U}^{-1}(\mathbf{y}, \sigma)(-d\mathbf{g}(x)\mathbf{f}(\mathbf{x}) - \\ -\psi(\mathbf{x})\mathbf{f}(\mathbf{x}) + d\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})) = 0,$$

from which the required assertion follows.

The lemma is proved.

We mention yet another technical result.

Lemma 1.5.4. *Let* $\mathbf{X}$ *be the extended symmetry group of system (1.89). After the change of variables*

$$\mathbf{x} = \mathbf{X}(\mathbf{y}, t) \tag{1.98}$$

system (1.89) assumes the form

$$\dot{\mathbf{y}} = \phi(\mathbf{y}, t)\mathbf{f}(\mathbf{y}) - \mathbf{g}(\mathbf{y}).$$

If $\mathbf{X}$ *is the symmetry group, then (1.89) can be rewritten in the form*

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) - \mathbf{g}(\mathbf{y}). \tag{1.99}$$

Proof. For brevity of notation we will, as before, set $\mathbf{x} = \mathbf{X}(\mathbf{y}, t)$. Then the identity

$$\dot{\mathbf{x}} = d_y \mathbf{X}(\mathbf{y}, t)\dot{\mathbf{y}} + \frac{\partial \mathbf{X}}{\partial t}(\mathbf{y}, t) = \mathbf{f}(\mathbf{x}).$$

holds.

Suppose that after the transformation (1.98), system (1.89) assumes the form

$$\dot{\mathbf{y}} = \mathbf{U}^{-1}(\mathbf{y}, t)\,(\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})) = \phi(\mathbf{y}, t)\left(\mathbf{f}(\mathbf{y}) - \tilde{\mathbf{f}}(\mathbf{y}, t)\right).$$

It is clear that for $t = 0$, we have $\tilde{\mathbf{f}}(\mathbf{y}, 0) = \mathbf{g}(\mathbf{y})$. We will compute the time derivative of $\tilde{\mathbf{f}}(\mathbf{y}, t)$. Using the formula for the differentiated inverse matrix, we obtain

$$\frac{\partial \tilde{\mathbf{f}}}{\partial t}(\mathbf{y}, t) = -\mathbf{U}^{-1}(\mathbf{y}, t)\,(d\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x}) - d\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})) = 0.$$

The lemma is proved.

Let the symmetry group of system (1.89) generate some solution of this system, i.e. suppose that there exists a $\mathbf{y}_0 \in \Omega$ such that $\mathbf{X}(\mathbf{y}_0, t)$ is a particular solution of (1.89). Then, for the autonomous system (1.99), $\mathbf{y} = \mathbf{y}_0$ will be a critical point, i.e.

$$\mathbf{f}_q(\mathbf{y}_0) = \mathbf{g}(\mathbf{y}_0). \tag{1.100}$$

In order to preserve the unity of the presentation, vector fields having group symmetry will be denoted $\mathbf{f}_{\mathrm{q}}$. As before, we can observe the analog for the Kovalevsky matrix

$$\mathbf{K} = d\mathbf{f}_q(\mathbf{y}_0) - d\mathbf{g}(\mathbf{y}_0). \tag{1.101}$$

Lemma 1.5.5. *Zero is always an eigenvalue of the matrix (1.101).*

Proof. Consider the vector $\mathbf{p} = \mathbf{f}_q(\mathbf{y}_0)$. Using the identity $[\mathbf{f}_q, \mathbf{g}] \equiv 0$ and the equality (1.100), we get

$$\mathbf{K}\mathbf{p} - d\mathbf{f}_q(\mathbf{y}_0)\mathbf{f}_q(\mathbf{y}_0) - d\mathbf{g}(\mathbf{y}_0)\mathbf{f}_q(\mathbf{y}_0) = d\mathbf{f}_q(\mathbf{y}_0)\left(\mathbf{f}_q(\mathbf{y}_0) - \mathbf{g}(\mathbf{y}_0)\right) = 0.$$

The lemma is proved.

Definition 1.5.2. We say that $\mathbf{X}$ is an *exponentially-asymptotic symmetry group* of system (1.89) if the right side of (1.89) can be expanded in a sum

$$\mathbf{f}(\mathbf{x}) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x})$$

such that, after the substitution (1.91), system (1.89) takes on the form

$$\dot{\mathbf{y}} = \sum_{m=0}^{\infty} e^{-m\beta\sigma}\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}), \tag{1.102}$$

where $\mathrm{sign}\beta = \chi$ is analogous to the sign of semi-quasihomogeneity.

It is clear that the vector functions must satisfy the relation

$$[\mathbf{f}_{q+\chi m}, \mathbf{g}](\mathbf{x}) = m\beta \mathbf{f}_{q+\chi m}(\mathbf{x}). \tag{1.103}$$

Formally, for $\sigma \to \chi\times\infty$ and the substitution of $\mathbf{y}$ for $\mathbf{x}$, the system of differential equations (1.102) transforms to the "truncated" system

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}).$$

It is clear that after the substitution (1.98), system (1.89) assumes the form

$$\dot{\mathbf{y}} = \mathbf{f}_q(\mathbf{y}) - \mathbf{g}(\mathbf{y}) + \sum_{m=1} e^{-m\beta t}\mathbf{f}_{q+\chi m}(\mathbf{y}). \tag{1.104}$$

If the truncated system has the particular solution $\mathbf{X}(\mathbf{y}_0, t)$, discussed above, then taking $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$, we rewrite the system (1.104) in the form used in the Lemmas 1.3.2, 1.3.3, and 1.3.5:

$$\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{y}, t), \tag{1.105}$$

where

$$\begin{gathered}\boldsymbol{\phi}(\mathbf{u}) = \mathbf{f}_q(\mathbf{y}_0 + \mathbf{u}) - \mathbf{g}(\mathbf{y}_0 + \mathbf{u}) - \mathbf{K}\mathbf{u} - \mathbf{f}_q(\mathbf{y}_0) + \mathbf{g}(\mathbf{y}_0),\\ \boldsymbol{\psi}(\mathbf{u}, t) = \textstyle\sum_{m=1} e^{-m\beta t}\mathbf{f}_{q+\chi m}(\mathbf{y}_0 + \mathbf{u}).\end{gathered}$$

Using the method developed in the preceding sections, we can assert that (1.105) always has a smooth particular solution that can be represented in the form of an exponential series

$$\mathbf{u}(t) = \sum_{k=1}^{\infty} \mathbf{u}_k(t)e^{-k\beta t}, \tag{1.106}$$

where the $\mathbf{u}_k$ are polynomial vector functions, which may be taken as constants if the matrix $\mathbf{K}$ doesn't have eigenvalues of the form $-k\beta$, $k \in \mathbb{N}$.

If all the coefficients $\mathbf{u}_k$ are constant, then in the analytic case, using the abstract implicit function theorem [94], it is possible to prove convergence for the series (1.106).

But if, in addition, the matrix $\mathbf{K}$ has l eigenvalues whose real parts have signs that coincide with the sign $-\beta$, and if the sign of the remaining real parts is opposite, then (1.105) possesses an l–parameter family of particular solutions tending to $\mathbf{u} = 0$ as $t \to \chi \times \infty$.

Thus we have established the following result.

Theorem 1.5.1. *Let system (1.89) have an exponentially-asymptotic group of symmetries, generating a particular solution $\mathbf{X}(\mathbf{y}_0, t)$ of the truncated system for some $y_0 \in \Omega$. Then the full system (1.89) has a particular solution of the form*

$$\mathbf{x}(t) = \mathbf{X}(\mathbf{y}_0 + o(1), t) \quad as \quad t \to \chi \times \infty.$$

Moreover, there exists an l–parameter family of such solutions, provided that the characteristic equation

$$\det(\mathbf{K} - \rho\mathbf{E}) = 0$$

has l roots for which the signs of the real parts coincide with the sign of $-\beta$, the real part of each of the remaining roots being zero or having the opposite sign.

Example 1.5.1. The classical Lyapunov first method can be given a group theoretic interpretation. Indeed, consider an autonomous system of differential equations for which $\mathbf{x} = \mathbf{0}$ is a critical point:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{m=1}^{\infty} \mathbf{f}_{m+1}(\mathbf{x}), \tag{1.107}$$

where $\mathbf{A}$ is a real matrix and $\mathbf{f}_{m+1}$ is a homogeneous vector function of degree $m+1$.

As our first example, we consider the simplest system of linear differential equations

$$\frac{d\mathbf{x}}{d\sigma} = -\beta\mathbf{x}, \tag{1.108}$$

where β is some real number.

It is easily seen that system (1.108) generates a homogeneous group of dilations of the phase space $\mathbb{R}^n$, being the group of symmetries of the truncated linear system ($\mathbf{f}_{\mathrm{q}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$); and, for the full system, the designated group will be an exponentially–asymptotic symmetry group. If the number $-\beta$ is an eigenvalue of the matrix $\mathbf{A}$, then the truncated system will have a particular solution of the form $\mathbf{x} = e^{-\beta t}\mathbf{y}_0$ ($\mathbf{y}_0$ is a nonzero eigenvector of the matrix $\mathbf{A}$), situated on an orbit of this group. From Theorem 1.5.1 it then follows that system (1.107) has a particular solution in the form of the series

$$\mathbf{x}(t) = e^{-\beta t} \sum_{k=0}^{\infty} \mathbf{x}_k(t) e^{-k\beta t},$$

where $\mathbf{x}_k(t)$ is some polynomial function of the time t and $\mathbf{x}_0 \equiv \mathbf{y}_0$.

The existence of a family of solutions, converging to the exponential "supporting" solution $\mathbf{x} = e^{-\beta t}\mathbf{y}_0$ as $t \to \chi \times \infty$, depends on the number of eigenvalues of the matrix $\mathbf{K} = \mathbf{A} + \beta\mathbf{E}$ with positive (or negative) real part.

The method considered likewise makes it possible to construct solutions with *exponential asymptotic*, whose presence cannot be predicted by the *classical theory*.

Example 1.5.2. We consider the system of two differential equations

$$\dot{x} = 2x + ax^2 y, \quad \dot{y} = -3x^3 y^3 + bxy^2. \tag{1.109}$$

The phase flow of the linear system

$$\frac{dx}{d\sigma} = 2x, \quad \frac{dy}{d\sigma} = -3y$$

generates an exponentially-asymptotic symmetry group of (1.109), with $\beta = 1$.

In fact, introducing the notations

$$\mathbf{g} = (2x, -3y), \quad \mathbf{f}_{\mathrm{q}} = (2x, -3x^3y^3), \quad \mathbf{f}_{\mathrm{q}+1} = (ax^2y, bxy^2),$$

we obtain commutation relations of the type (1.103):

$$[\mathbf{f}_{\mathrm{q}}, \mathbf{g}] = (0, 0), \quad [\mathbf{f}_{\mathrm{q}+1}, \mathbf{g}] = \mathbf{f}_{\mathrm{q}+1}.$$

The truncated system

$$\dot{x} = 2x, \quad \dot{y} = -3x^3y^3$$

has a one-parameter family of particular solutions

$$x_0(c, t) = ce^{2t}, \quad y_0(c, t) = c^{-3/2}e^{-3t}, \quad c \in \mathbb{R}^+.$$

It is easy to compute the set of eigenvalues of the matrix $\mathbf{K}$ corresponding to the given family of solutions: $0, -12$.

Using Theorem 1.5.1 we obtain that the system of equations (1.109) has a one-parameter family of particular solutions, each represented as a series

$$x(c, t) = e^{2t} \sum_{k=0}^{\infty} x_k(t)e^{-kt}, \quad y(c, t) = e^{-3t} \sum_{k=0}^{\infty} y_k(t)e^{-kt},$$

where $x_0 = c$, $y_0 = c^{-3/2}$, and where nonconstant polynomials in t can appear only at the twelfth stage.

It is noteworthy that, as $t \to +\infty$, the solutions from the family constructed behave just like solutions of a linear hyperbolic system with characteristic exponents $2, -3$, although the linearized system only predicts the existence of solutions with asymptotic $\sim e^{2t}$ as $t \to -\infty$. The form of these solutions is obvious: the line $y = 0$ is a stable invariant manifold. On this solution manifold the solutions of (1.109) have the form

$$c(c_*, t) = c_*e^{2t}, \quad y(c_*, t) = 0.$$

Here c_* is the parameter for the given family of solutions.

Until now we have considered only the autonomous case and transformation groups whose action was applied to systems of differential equations under study and did not affect the independent variable.

We now consider a nonautonomous system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1.110}$$

that is given on some domain Ω of the extended phase space $\mathbb{R}^{n+1}$.

We can apply the method developed above to the nonautonomous system (1.110), represented as an autonomous system in extended phase space:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \varphi), \quad \dot{\varphi} = 1.$$

But the nonautonomous case has its very own peculiarities. In order to indicate them, we again construct the necessary theory.

Let some one-parameter transformation semigroup act on Ω:

$$(\mathbf{X}, T)\colon \Omega \times [0, \chi \times \infty) \to \Omega, \quad \chi = \pm 1, \quad (\mathbf{X}, T)(\cdot, \cdot, 0) = \mathrm{id},$$

yielding the system of differential equations

$$\frac{d\mathbf{x}}{d\sigma} = \mathbf{g}(\mathbf{x}, t), \quad \frac{dt}{d\sigma} = h(\mathbf{x}, t). \tag{1.111}$$

It is further assumed that the smooth function $h(\mathbf{x}, t)$ doesn't vanish on the domain Ω.

We consider a change of dependent and independent variables generated by the system of differential equations (1.111)

$$\mathbf{x} = \mathbf{X}(\mathbf{y}, \tau, \sigma), \quad t = T(\mathbf{y}, \tau, \sigma), \quad \sigma = \mathrm{const.} \tag{1.112}$$

Suppose that, under the action of the substitution (1.112), system (1.110) assumes the form

$$\mathbf{y}' = \widehat{\mathbf{f}}(\mathbf{y}, \tau, \sigma), \tag{1.113}$$

where the prime denotes differentiation with respect to the new independent variable τ. It is clear that $\widehat{\mathbf{f}}(\mathbf{y}, \tau, 0) = \mathbf{f}(\mathbf{y}, \tau)$.

Lemma 1.5.6. *In order that the system (1.110) be transformed by the action of the substitution (1.113) into the form (1.113), it is necessary and sufficient that, for arbitrary* $(\mathbf{x}, t) \in \Omega$, *the following equation be satisfied:*

$$(L^{\mathbf{g}}\mathbf{f})(\mathbf{x}, t) + \mathbf{d}(\mathbf{x}, t) - h(\mathbf{x}, t)\frac{\partial \mathbf{f}}{\partial t}(\mathbf{x}, t) - (D_t^{\mathbf{f}} h)(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t) = 0, \tag{1.114}$$

where $L^{\mathbf{g}} = \frac{\partial \mathbf{g}}{\partial t} + [\cdot, \mathbf{g}]$ *is the Lie operator,* $D_t^{\mathbf{f}} = \frac{\partial}{\partial t} + \langle d_{\mathbf{x}}, \mathbf{f}\rangle$ *is the total derivative operator with respect to time by virtue of the system of equations (1.110), and where* $\mathbf{d}\colon \Omega \to \mathbb{R}^n$ *is a smooth vector function satisfying*

$$\mathbf{d}(\mathbf{x}, t) = \frac{\partial \widehat{\mathbf{f}}}{\partial \sigma}(\mathbf{x}, t, 0). \tag{1.115}$$

Proof. We first compute the differentials of the left and right sides of (1.112):

$$\begin{aligned} d\mathbf{x} &= d_y\mathbf{X}(\mathbf{y},\tau,\sigma)\,d\mathbf{y} + \tfrac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,\sigma)\,d\tau + \tfrac{\partial \mathbf{X}}{\partial \sigma}(\mathbf{y},\tau,\sigma)\,d\sigma, \\ dt &= \langle d_y T(\mathbf{y},\tau,\sigma),\, d\mathbf{y} \tfrac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)\,d\tau + \tfrac{\partial T}{\partial \sigma}(\mathbf{y},\tau,\sigma)\,d\sigma. \end{aligned} \tag{1.116}$$

Since the parameter σ is fixed and the differential relations $d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\,dt$, $d\mathbf{y} = \widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma)\,d\tau$ hold, from (1.116) we obtain the equation

$$\begin{aligned} &d_\mathbf{y}\mathbf{X}(\mathbf{y},\tau,\sigma)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) + \tfrac{\partial \mathbf{X}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = \\ &\qquad = \left\{\left\langle d_\mathbf{y} T(\mathbf{y},\tau,\sigma), \widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma)\right\rangle + \tfrac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)\right\} \mathbf{f}(\mathbf{x}, t), \end{aligned} \tag{1.117}$$

where (as earlier) the relations $\mathbf{x} = \mathbf{X}(\mathbf{y},\tau,\sigma)$, $t = T(\mathbf{y},\tau,\sigma)$ are satisfied.

In the remainder of this section, for notational brevity in multiplying vector objects of various dimensions, we will use the standard notation of matrix algebra. Here vector fields and vector functions $\mathbf{f}, \mathbf{g}, \mathbf{X}$, and their derivatives with respect to the "time-like" variables t, τ, σ, are vectors (i.e. $n \times 1$ matrices), and the differentials of the scalar functions h, T with respect to the spatial variables $\mathbf{x}, \mathbf{y}$ are covectors (i.e. $1 \times n$ matrices). All scalar quantities are considered to be 1×1 matrices.

According to Lemma 1.5.2, the matrix $d_\mathbf{y}\mathbf{X}(\mathbf{y},\tau,\sigma)$, the vector $\frac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,\sigma)$, the covector $d_\mathbf{y} T(\mathbf{y},\tau,\sigma)$ and the function $\frac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)$ satisfy the matrix differential equation

$$\frac{\partial}{\partial \sigma}\begin{pmatrix} d_\mathbf{y}\mathbf{X} & \frac{\partial}{\partial \tau}\mathbf{X} \\ d_\mathbf{y} T & \frac{\partial}{\partial \tau} T \end{pmatrix} = \begin{pmatrix} d_\mathbf{x}\mathbf{g}(\mathbf{x},t) & -\frac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t) \\ d_\mathbf{x} h(\mathbf{x},t) & \frac{\partial}{\partial t} h(\mathbf{x},t) \end{pmatrix} \begin{pmatrix} d_\mathbf{y}\mathbf{X} & \frac{\partial}{\partial \tau}\mathbf{X} \\ d_\mathbf{y} T & \frac{\partial}{\partial \tau} T \end{pmatrix} \tag{1.118}$$

and the initial conditions

$$\begin{aligned} d_\mathbf{y}\mathbf{X}(\mathbf{y},\tau,0) &= \mathbf{E}, & \tfrac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,0) &= 0, \\ d_\mathbf{y} T(\mathbf{y},\tau,0) &= 0, & \tfrac{\partial T}{\partial \tau}(\mathbf{y},\tau,0) &= 1. \end{aligned} \tag{1.119}$$

Differentiating relation (1.117) with respect to the variable σ, using (1.118) and (1.119), passing to the limit as $\sigma \to 0$ and letting $\mathbf{x} \to \mathbf{y}$, $t \to \tau$, we obtain

$$\begin{aligned} &d_\mathbf{y}\mathbf{g}(\mathbf{y},\tau)\mathbf{f}(\mathbf{y},\tau) + \tfrac{\partial \widehat{\mathbf{f}}}{\partial \sigma}(\mathbf{y},\tau,0) + \tfrac{\partial \mathbf{g}}{\partial \tau}(\mathbf{y},\tau) = \\ &= \mathbf{f}(\mathbf{y},\tau)\left\{d_\mathbf{y} h(\mathbf{y},\tau)\mathbf{f}(\mathbf{y},\tau) + \tfrac{\partial h}{\partial \tau}(\mathbf{y},\tau)\right\} + \\ &+ d_\mathbf{y}\mathbf{f}(\mathbf{y},\tau)\mathbf{g}(\mathbf{y},\tau) + h(\mathbf{y},\tau)\tfrac{\partial \mathbf{f}}{\partial \tau}(\mathbf{y},\tau). \end{aligned}$$

Returning to the original phase variables $\mathbf{x}, t$ and introducing the notation (1.115), we get Eq. (1.114). Necessity is proved.

Now suppose that Eq. (1.114) is satisfied. We then show that the substitution (1.112) transforms system (1.110) into the form (1.111), with fulfillment of (1.115).

We write (1.117) in the following form:

$$\mathbf{U}(\mathbf{y},\tau,\sigma)\widehat{\mathbf{f}}(\mathbf{y},\tau,0) = \mathbf{q}(\mathbf{y},\tau,\sigma),$$

with the notations

$$\mathbf{U}(\mathbf{y},\tau,\sigma) = d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma),$$
$$\mathbf{q}(\mathbf{y},\tau,\sigma) = -\tfrac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,\sigma) + \mathbf{f}(\mathbf{x},t)\tfrac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma).$$

From (1.119) it follows that $\mathbf{U}(\mathbf{y},\tau,0) = \mathbf{E}$, whereby the matrix $\mathbf{U}(\mathbf{y},\tau,\sigma)$ is invertible for small values of σ. That is

$$\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\mathbf{q}(\mathbf{y},\tau,\sigma),$$

from which it follows that $\widehat{\mathbf{f}}(\mathbf{y},\tau,0) = \mathbf{f}(\mathbf{y},\tau)$.

We compute the derivative

$$\tfrac{\partial \widehat{\mathbf{f}}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\times \times\left(\tfrac{\partial \mathbf{U}}{\partial \sigma}(\mathbf{y},\tau,\sigma)\mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\mathbf{q}(\mathbf{y},\tau,\sigma) - \tfrac{\partial \mathbf{q}}{\partial \sigma}(\mathbf{y},\tau,\sigma)\right). \tag{1.120}$$

Then the following assertion holds:

Lemma 1.5.7. *The matrix* $\mathbf{U}(\mathbf{y},\tau,\sigma)$ *and the vector* $\mathbf{q}(\mathbf{y},\tau,\sigma)$ *satisfy the matrix nonhomogeneous differential equations*

$$\frac{\partial \mathbf{U}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = \mathbf{A}(\mathbf{x},t)\mathbf{U}(\mathbf{y},\tau,\sigma) - \mathbf{d}(\mathbf{x},t)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma), \tag{1.121}$$

$$\frac{\partial \mathbf{q}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = \mathbf{A}(\mathbf{x},t)\mathbf{q}(\mathbf{y},\tau,\sigma) - \mathbf{d}(\mathbf{x},t)\frac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma), \tag{1.122}$$

where

$$\mathbf{A}(\mathbf{x},t) = d_{\mathbf{x}}\mathbf{g}(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{x}}h(\mathbf{x},t).$$

Proof. Using the matrix differential equation (1.118), we obtain

$$\tfrac{\partial \mathbf{U}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = (d_{\mathbf{x}}\mathbf{g}(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{x}}h(\mathbf{x},t))\, d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma) + \left(\tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t) - d_{\mathbf{x}}\mathbf{f}(\mathbf{x},t)\mathbf{g}(\mathbf{x},t) - \tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t)h(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)\tfrac{\partial h}{\partial t}(\mathbf{x},t)\right) d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma).$$

Using Eq. (1.114), we arrive at (1.121). Equation (1.122) is obtained analogously. In fact,

$$\tfrac{\partial \mathbf{q}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = -(d_{\mathbf{x}}\mathbf{g}(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)d_{\mathbf{x}}h(\mathbf{x},t))\,\tfrac{\partial}{\partial \tau}\mathbf{X}(\mathbf{y},\tau,\sigma) - \left(\tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t) - d_{\mathbf{x}}\mathbf{f}(\mathbf{x},t)\mathbf{g}(\mathbf{x},t) - \tfrac{\partial}{\partial t}\mathbf{g}(\mathbf{x},t)h(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)\tfrac{\partial h}{\partial t}(\mathbf{x},t)\right)\tfrac{\partial}{\partial \tau}T(\mathbf{y},\tau,\sigma).$$

Using (1.114) once more, we obtain (1.122).

Lemma 1.5.7 is proved.

We continue the proof of Lemma 1.5.6. Substituting Eq. (1.121) and (1.122) into (1.120):

$$\frac{\partial \widehat{\mathbf{f}}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\times$$
$$\times\left(\mathbf{d}(\mathbf{x},t)d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma)\mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\mathbf{q}(\mathbf{y},\tau,\sigma) + \mathbf{d}(\mathbf{x},t)\tfrac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)\right).$$

Passing to the limit as $\sigma \to 0$, we obtain the relation (1.115).

The lemma is proved.

In the change of variables (1.112), the parameter σ was fixed. We now fix τ and consider the transformation $(\mathbf{x}, t) \mapsto (\mathbf{y}, \sigma)$ of extended phase space:

$$\mathbf{x} = \mathbf{X}(\mathbf{y},\tau,\sigma), \quad = T(\mathbf{y},\tau,\sigma), \quad \tau = \text{const}. \tag{1.123}$$

Lemma 1.5.8. *Under the action of transformation (1.123) system (1.57) assumes the form*

$$\frac{d\mathbf{y}}{d\sigma} = h(\mathbf{y},\tau)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) - \mathbf{g}(\mathbf{y},\tau), \tag{1.124}$$

where τ now takes on the role of parameter.

Proof. Let $d\mathbf{y} = \widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma)\,d\sigma$. Since τ is fixed, (1.116) yields the equation

$$\begin{aligned} &d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) + \tfrac{\partial \mathbf{X}}{\partial \sigma}(\mathbf{y},\tau,\sigma) = \\ &\quad = \mathbf{f}(\mathbf{x},t)\left\{d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma)\widetilde{\mathbf{f}}(\mathbf{y},\tau,\sigma) + \tfrac{\partial T}{\partial \sigma}(\mathbf{y},\tau,\sigma)\right\}. \end{aligned} \tag{1.125}$$

Equation (1.125) can be rewritten in the form

$$\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\left(-\frac{\partial \mathbf{X}}{\partial \sigma}(\mathbf{y},\tau,\sigma) + \mathbf{f}(\mathbf{x},t)\frac{\partial T}{\partial \sigma}(\mathbf{y},\tau,\sigma)\right).$$

We use the local invertibility of the change of variables (1.112). The one-parameter (semi)group of transformations

$$\mathbf{y} = \mathbf{X}(\mathbf{x},\tau,-\sigma), \quad \tau = T(\mathbf{y},\tau,\sigma) \tag{1.126}$$

represents the phase flow of the system of differential equations

$$\frac{d\mathbf{y}}{d\sigma} = -\mathbf{g}(\mathbf{y},\tau), \quad \frac{d\tau}{d\sigma} = -h(\mathbf{y},\tau). \tag{1.127}$$

Substituting (1.126) into (1.112), we obtain an identity for an arbitrary triple $(\mathbf{x},\tau,\sigma) \in \Omega \times (-\sigma_0, +\sigma_0)$. Differentiating this identity with respect to σ and using (1.127), we obtain

$$0 = -d_{\mathbf{y}}\mathbf{X}(\mathbf{y},\tau,\sigma)\mathbf{g}(\mathbf{y},\tau) - \tfrac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,\sigma)h(\mathbf{y},\tau) + \tfrac{\partial \mathbf{X}}{\partial \sigma}(\mathbf{y},\tau,\sigma),$$
$$0 = -d_{\mathbf{y}}T(\mathbf{y},\tau,\sigma)\mathbf{g}(\mathbf{y},\tau) - \tfrac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)h(\mathbf{y},\tau) + \tfrac{\partial \mathbf{X}}{\partial \sigma}(\mathbf{y},\tau,\sigma).$$

Then, since

$$\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) = \mathbf{U}^{-1}(\mathbf{y},\tau,\sigma)\left(-\frac{\partial \mathbf{X}}{\partial \tau}(\mathbf{y},\tau,\sigma) + \mathbf{f}(\mathbf{x},\tau)\frac{\partial T}{\partial \tau}(\mathbf{y},\tau,\sigma)\right),$$

we easily obtain

$$\widetilde{\mathbf{f}}(\mathbf{y},\tau,\sigma) = h(\mathbf{y},\tau)\widehat{\mathbf{f}}(\mathbf{y},\tau,\sigma) - \mathbf{g}(\mathbf{y},\tau).$$

The lemma is proved.

We now return to the study of symmetry groups for nonautonomous systems of differential equations.

Definition 1.5.3. We say that $(\mathbf{X}, T)$ is a *extended symmetry group* of system (1.110) if (1.110) is invariant under the action of the transformation (1.112).

The introduction of the additional term "extended" in the definition is necessary in order to distinguish symmetry groups of autonomous systems—acting on generalized phase space—from symmetry groups not affecting the independent variable.

From Lemma 1.5.6 it follows that, in order that $(\mathbf{X}, T)$ be an extended symmetry group for (1.110), it is necessary and sufficient that, for arbitrary $(\mathbf{x}, t) \in \Omega$, the following relation be satisfied:

$$\left(L^{\mathbf{g}}\mathbf{f}\right)(\mathbf{x},t) - h(\mathbf{x},t)\frac{\partial \mathbf{f}}{\partial t}(\mathbf{x},t) - \left(D^{\mathbf{f}}h\right)(\mathbf{x},t)\mathbf{f}(\mathbf{x},t) = 0. \tag{1.128}$$

If $(\mathbf{X}, T)$ is the extended symmetry group of system (1.110) then, after the substitution (1.123), this system assumes the form

$$\frac{d\mathbf{y}}{d\sigma} = h(\mathbf{y},\tau)\mathbf{f}(\mathbf{y},\tau) - \mathbf{g}(\mathbf{y},\tau), \tag{1.129}$$

where τ is here some fixed parameter, i.e. under the action of the transformation (1.123) the resulting system becomes autonomous.

Let the system considered have some particular solution, situated on an orbit of its extended symmetry group. This implies the existence of $(\mathbf{y}_0, \tau_0) \in \Omega$ such that the given parametric vector function $\mathbf{x} = \mathbf{X}(\mathbf{y}_0, \tau_0, \sigma)$, $t = (\mathbf{y}_0, \tau_0, \sigma)$ will be a particular solution of (1.110). Such a solution will be smooth provided that, as was stipulated above, the function $h(\mathbf{x}, t)$ is not transformed into zero. Then for $\tau = \tau_0$ the system of equations (1.129) has a critical point $\mathbf{y} = \mathbf{y}_0$, so that

$$h(\mathbf{y}_0,\tau_0)\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0,\tau_0) = \mathbf{g}(\mathbf{y}_0,\tau_0). \tag{1.130}$$

Further, as was done earlier, the right sides of systems of equations possessing extended symmetry groups will be assigned an index "q".

An analog of the Kovalevsky matrix will serve as the matrix of a linear system of differential equations with constant coefficients, obtained from (1.129) by linearization in the neighborhood of $\mathbf{y} = \mathbf{y}_0$:

$$\mathbf{K} = h(\mathbf{y}_0, \tau_0) d_{\mathbf{y}} \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0, \tau_0) + \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0, \tau_0) d_{\mathbf{y}} h(\mathbf{y}_0, \tau_0) - d_{\mathbf{y}} \mathbf{g}(\mathbf{y}_0, \tau_0). \tag{1.131}$$

We have the following

Lemma 1.5.9. *Let* $\dfrac{\partial \mathbf{f}_{\mathrm{q}}}{\partial t} \equiv 0$, $\dfrac{\partial \mathbf{g}}{\partial t} \equiv 0$. *Then the number* $-\frac{\partial h}{\partial \tau}(\mathbf{y}_0, \tau_0)$ *is an eigenvalue of the matrix (1.131).*

Proof. We consider the vector $\mathbf{p} = \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0)$. Using Eqs. (1.128) and (1.130), we compute

$$\begin{aligned} \mathbf{K}\mathbf{p} = d_{\mathbf{y}} \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) \left(h(\mathbf{y}_0, \tau_0) \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) - \mathbf{g}(\mathbf{y}_0) \right) - \\ -\tfrac{\partial h}{\partial \tau}(\mathbf{y}_0, \tau_0) \mathbf{f}_{\mathrm{q}}(\mathbf{y}_0) = -\tfrac{\partial h}{\partial \tau}(\mathbf{y}_0, \tau_0)\mathbf{p}. \end{aligned}$$

The lemma is proved.

Definition 1.5.4. We say that $(\mathbf{X}, T)$ is an *exponentially-asymptotic generalized symmetry group* of system (1.110) if the right side of (1.110) can be expanded in a sum

$$\mathbf{f}(\mathbf{x}, t) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x}, t)$$

such that, after the substitution (1.112), system (1.110) assumes the form

$$\mathbf{y}' = \sum_{m=0}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}, \tau). \tag{1.132}$$

If a system possesses an exponentially-asymptotic generalized symmetry group then, in accordance with Lemma 1.5.6, the vector function $\mathbf{f}(\mathbf{x}, t)$ satisfies a system of partial differential equations of type (1.114). Since the truncations $\mathbf{f}_{\mathrm{q}}(\mathbf{x}, t)$ satisfy system (1.128), the system of equations for the perturbations can be written in the form

$$\begin{aligned} \sum_{m=1}^{\infty} \Bigg\{ h(\mathbf{x}, t) \frac{\partial \mathbf{f}_{\mathrm{q}+\chi m}}{\partial t}(\mathbf{x}, t) - \left(\mathbf{f}_{\mathrm{q}+\chi m}, \mathbf{g} \right)(\mathbf{x}, t) + \\ + \left(m\beta + \tfrac{\partial h}{\partial t}(\mathbf{x}, t) + \langle d_{\mathbf{x}} h, \mathbf{f} \rangle (\mathbf{x}, t) \right) \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x}, t) \Bigg\} = 0. \end{aligned} \tag{1.133}$$

It is clear (as it was earlier) that, with convergence of the parameter σ to $\chi \times \infty$ and the formal substitution $(\mathbf{y}, \tau)$ into $(\mathbf{x}, t)$, system (1.132) goes over to the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}, t).$$

The transformation of generalized phase space (1.123) reduces a system possessing an exponentially asymptotic generalized symmetry group to the form

$$\frac{d\mathbf{y}}{d\sigma} = h(\mathbf{y}, \tau) \sum_{m=0}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}, \tau) - \mathbf{g}(\mathbf{y}, \tau). \tag{1.134}$$

But, if the truncated system has a particular solution that is situated on an orbit of the group considered, then (1.134), with the aid of the "perturbing" substitution $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$, may be rewritten in the customary form

$$\frac{d\mathbf{u}}{d\sigma} = \mathbf{K}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u}) + \boldsymbol{\psi}(\mathbf{u}, \sigma), \tag{1.135}$$

where

$$\boldsymbol{\phi}(\mathbf{u}) = h(\mathbf{y}_0 + \mathbf{u}, \tau_0)\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0 + \mathbf{u}, \tau_0) - \mathbf{g}(\mathbf{y}_0 + \mathbf{u}, \tau_0) - \\ -\mathbf{K}\mathbf{u} - h(\mathbf{y}_0, \tau_0)\mathbf{f}_{\mathrm{q}}(\mathbf{y}_0, \tau_0) - \mathbf{g}(\mathbf{y}_0, \tau_0),$$

and

$$\boldsymbol{\psi}(\mathbf{u}, \sigma) = h(\mathbf{y}_0 + \mathbf{u}, \tau_0) \sum_{m=1}^{\infty} e^{-m\beta\sigma} \mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{y}_0 + \mathbf{u}, \tau_0).$$

We need to remember that the right side of (1.135) depends on τ_0 only as a parameter and that for this reason this dependency is not reflected in (1.135).

In this section we have already considered the asymptotic properties of solutions of system (1.135) since, within substitution of σ for t, it coincides with system (1.105). These properties by themselves imply the following assertion about the properties of particular solutions of the original system.

Theorem 1.5.2. *Let system (1.110) possess an exponentially-asymptotic generalized symmetry group, on whose orbit there is a particular solution of the truncated system, having in parametric form the aspect*

$$\mathbf{x} = \mathbf{X}(\mathbf{y}_0, \tau_0, \sigma), \quad t = T(\mathbf{y}_0, \tau_0, \sigma),$$

for some $(\mathbf{y}_0, \tau_0) \in \Omega$. *Then the full system (1.110) has a particular solution of the form*

$$\mathbf{x}(\sigma) = \mathbf{X}\left(\mathbf{y}_0 + \mathbf{u}(\sigma), \tau_0, \sigma\right), \quad t(\sigma) = T\left(\mathbf{y}_0 + \mathbf{u}(\sigma), \tau_0, \sigma\right),$$

where $\mathbf{u} = o(1)$ *as* $\sigma \to \chi \times \infty$.

We can furthermore assert that there exists an l–parameter family of such solutions if the characteristic equation

$$\det(\mathbf{K} - \rho\mathbf{E}) = 0$$

has l roots, the signs of whose real parts coincide with the sign of $-\beta$, and the real part of each of the remaining roots is either zero or has the opposite sign.

We consider an example that illustrates the theorem.

Example 1.5.3. Let a relativistic particle, whose rest mass equals unity, complete a straight line motion along the Ox axis under the action of the force $f(x, t)$. The equation of motion of this particle has the form [128]

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1 - c^{-2}\dot{x}^2}}\right) = f(x, t), \tag{1.136}$$

where c is the speed of light.

It is known that, in the absence of exterior force fields ($f(x,t) \equiv 0$), Eq. (1.136) is invariant under the one-dimensional subgroup of the Lorentz group, being a group of hyperbolic rotations. The orbits of this group are "pseudospherical" in the Minkowski metric $(ds)^2 = c^2(dt)^2 - (dx)^2$ and have the form

$$\begin{aligned} x &= y\cosh\sigma + c\tau\sinh\sigma, \\ t &= c^{-1}y\sinh\sigma + \tau\cosh\sigma, \quad \sigma \in (-\infty, +\infty). \end{aligned} \tag{1.137}$$

Using the extension of the group action operation [146], it is also not difficult to compute the law of the speed change:

$$\dot{x} = c\,\frac{y' + c\sigma}{y'\sigma + c}, \tag{1.138}$$

where the prime denotes differentiation with respect to the new "time" τ.

From (1.138) it is clear that, as $\sigma \to \pm\infty$, the speed of the particle tends toward the speed of light.

But, in the absence of exterior fields, the unique trajectory situated on the orbits of this group corresponds to the singular solution (1.136) $x = ct$. We first find a nontrivial force field $f(x,t)$ such that Eq. (1.110) will be invariant with respect to the designated subgroup of the Lorentz group and will possess particular (nonsingular) solutions lying on its orbit. This in fact would indicate the acceleration toward the speed of light as $t \to \infty$.

We introduce the notation $u = \dot{x}$, $v = y'$ and write a system of three first order differential equations for which formulas (1.137) and (1.138) would give the general solution:

$$\frac{dx}{d\sigma} = ct, \quad \frac{du}{d\sigma} = c^{-1}(c^2 - u^2), \quad \frac{dt}{d\sigma} = c^{-1}x.$$

We rewrite the second order Eq. (1.136) under investigation as an equivalent system of two first order equations

$$\dot{x} = u, \quad \dot{u} = (1 - c^{-2}u^2)^{3/2} f(x,t).$$

Introducing the notations

$$\begin{gathered} \mathbf{x} = (x, u), \quad \mathbf{f} = \left(u, (1 - c^{-2}u^2)^{3/2} f(x,t)\right), \\ \mathbf{g} = \left(ct, c^{-1}(c^2 - u^2)\right), \quad h = c^{-1}x, \end{gathered}$$

we attempt to solve the system of equations (1.128).

It is easily observed that the structure of the equations considered is such that

$$\left(D_t^{\mathbf{f}} h\right)(\mathbf{x}, t) = c^{-1}u$$

does not depend on the concrete form of $f(x,t)$, thanks to which Eq. (1.128) becomes linear.

This system consists of two equations, the first satisfying an identity and the second leading to a linear equation in the first order partial derivatives of the function $f(x,t)$:

$$c^2 t \frac{\partial f}{\partial x}(x,t) + x \frac{\partial f}{\partial t}(x,t) = 0,$$

which has the obvious solution

$$f_{\mathrm{q}}(x,t) = \phi(x^2 - c^2 t^2),$$

where ϕ is an arbitrary smooth function.

It is interesting to note that the function f_{q} we have found is an invariant under the group of hyperbolic rotations. This function is ascribed the index "q", indicating invariance of the Eq. (1.136) with respect to hyperbolic rotations.

We consider the problem of existence of point trajectories, situated on the orbits of the group considered, such that the speed of the point after finite time does *not* attain the speed of light. For this it is necessary to solve a system of algebraic equations of type (1.130). After a few calculations we arrive at the equation

$$\xi_0 \phi(\xi_0^2) = c^2,$$

where we have introduced the notation $\xi_0 = y_0 \sqrt{1 - c^{-2} v_0^2}$. This equation has real roots, at least for functions ϕ that are positive and nondecreasing on $[0, +\infty)$.

The initial moment of time τ_0 is determined here by the formula

$$\tau_0 = c^{-2} y_0 v_0.$$

So, as was emphasized earlier, the trajectories with the determined initial conditions accelerate to the speed of light as $t \to \infty$.

We now pose the following question: which additional perturbing forces allow the existence of trajectories with analogous asymptotic properties?

We expand the function f in the formal series

$$f(x,t) = \sum_{m=0} f_{\mathrm{q}+\chi m}(x,t),$$

such that, subsequent to the transformation (1.137), the terms of Eq. (1.136) that corresponding to terms of this series are multiplied by $e^{-m\beta\sigma}$.

In order to find the functions $f_{\mathrm{q}+\chi m}$, it is necessary to solve a system analogous to (1.133). Thanks to the rather simple structure of the original Eq. (1.136), the corresponding system becomes linear and can furthermore be decomposed into a countable number of independent subsystems. The left and right sides of the first equations are identically zero and the second is reduced to the form

$$c^2 t \frac{\partial}{\partial x} f_{\mathrm{q}+\chi m}(x,t) + x \frac{\partial}{\partial t} f_{\mathrm{q}+\chi m}(x,t) - m\beta c f_{\mathrm{q}+\chi m}(x,t) = 0.$$

It is easy to see that the solutions of this equation will be function of the form

$$f_{\mathrm{q}+\chi m}(x,t) = \psi(x^2 - c^2t^2)\left(a_+(x+ct)^{m\beta} + a_-(x-ct)^{-m\beta}\right),$$

where ψ is an arbitrary function and a_+, a_- are arbitrary constants.

Thus the equation of motion of a particle in the constructed force field will have a particular solution of the form

$$\begin{aligned} x(\sigma) &= (y_0 + z(\sigma))\cosh\sigma + c\tau_0\sinh\sigma, \\ t(\sigma) &= c^{-1}\,(y_0 + z(\sigma))\sinh\sigma + \tau_0\cosh\sigma, \end{aligned}$$

where the function $z(\sigma)$ can be expanded in a series

$$z(\sigma) = \sum_{k=1}^{\infty} z_k(\sigma)e^{-k\beta\sigma}.$$

This solution clearly possesses the very same asymptotic properties as the corresponding particular solution of the truncated system.

We note that the problem just solved represents the simplest example of an inverse problem in relativistic mechanics.

The systems of Eqs. (1.128) and (1.133) that must be satisfied by the right sides of systems of ordinary equations possessing a given generalized symmetry group, or exponentially-asymptotic generalized symmetry group in the nonautonomous case, are substantially more complicated than the corresponding systems (1.103) obtained for the autonomous case, in part because their nonlinearity makes it impossible in general to obtain a chain of independent systems of equations for each component $\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x},t)$. In the preceding example such a "decomposition" was in fact realized, but only thanks to the highly special form of the right side of the equation under consideration. In the general case the possibility of such a decomposition occurs only under the following rather rigid requirements.

Let the function $h(\mathbf{x},t)$ be given as a product

$$h(\mathbf{x},t) = \mu(t)H(\mathbf{x},t),$$

where $\mu(t)$ is some smooth function of time t and $H(\mathbf{x},t)$ is the first integral (possibly trivial) of the (full) system considered. In this case the vector function $\mathbf{f}_{q+\chi m}(\mathbf{x},t)$ satisfying the chain of equations

$$\begin{aligned} &\left[\mathbf{f}_{\mathrm{q}+\chi m}, \mathbf{g}\right](\mathbf{x},t) - \mu(t)H(\mathbf{x},t)\tfrac{\partial \mathbf{f}_{\mathrm{q}+\chi m}}{\partial t}(\mathbf{x},t) + \\ &+ (\mu(t)H(\mathbf{x},t) + m\beta)\,\mathbf{f}_{\mathrm{q}+\chi m}(\mathbf{x},t) = 0, \quad m = 1,2,\ldots, \end{aligned} \tag{1.139}$$

likewise satisfies system (1.133).

However, this approach assumes an a priori knowledge of one of the integrals of the full system, which as a rule isn't available. So, in the rare exceptional case, under the action of an generalized symmetry group, time is transformed independent of the phase variables, i.e. it is happens that $H(\mathbf{x}, t) \equiv 0$. If in this the required vector functions $\mathbf{f}_{q+\chi m}$ are considered independent of time, then μ automatically becomes a linear function of t, $\mu = \nu t + \mu_0$, and (1.139) is rewritten in the form

$$\left[\mathbf{f}_{q+\chi m}, \mathbf{g}\right](\mathbf{x}) = (\mu\beta + \nu)\mathbf{f}_{q+\chi m}(\mathbf{x}) = 0, \quad m = 1, 2, \ldots \tag{1.140}$$

This shows that $\mathbf{g}$ generates a generalized symmetry group for each vector of the field $\mathbf{f}_{q+\chi m}$.

We note the connection of the theory of exponentially-asymptotic generalized symmetry groups with semi-quasihomogeneous systems:

Example 1.5.4. The method of constructing particular solutions of semi-quasihomogeneous systems investigated in the preceding section can be interpreted from the group-theoretical viewpoint. So again we consider some semi-quasihomogeneous system (1.3). If in the Fuchsian system of equations (1.10) we make the obvious change of variable $\sigma = -\ln \mu$, this system is transformed to the constant coefficient system of linear equations

$$\frac{d\mathbf{x}}{d\sigma} = -\mathbf{G}\mathbf{x}, \quad \frac{dt}{d\sigma} = t. \tag{1.141}$$

It is of course understood that (1.141) generates the generalized symmetry group of the truncated system (1.9) and the exponentially-asymptotic generalized symmetry group of the full system (1.3). Particular solutions of the truncated system, lying on an orbit of this group, should thus be sought in the form

$$\mathbf{x} = \exp(-\mathbf{G}\sigma)\mathbf{y}_0, \quad t = e^{\sigma}\tau_0,$$

which is equivalent to

$$\mathbf{x} = (t/\tau_0)^{-\mathbf{G}}\, y_0.$$

From (1.130) it follows that the quantities $\mathbf{y}_0$ and τ_0 must satisfy the equality

$$\tau_0 \mathbf{f}_q(\mathbf{y}_0, \tau_0) = -\mathbf{G}\mathbf{y}_0, \tag{1.142}$$

analogous to (1.13).

A substitution of the type (1.123), generating the system of linear equations (1.141), converts system (1.3) to the system

$$\frac{d\mathbf{y}}{d\sigma} = \mathbf{G}\mathbf{y} + \tau \sum_{m=0} e^{-m\beta\sigma}\mathbf{f}_{q+\chi m}(\mathbf{y}, \tau). \tag{1.143}$$

From this it is partly evident that, by satisfying (1.142) for some fixed $(\mathbf{y}_0, \tau_0) \in \Omega$, the matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \mathbf{G} + \tau_0 d_{\mathbf{y}} \mathbf{f}_{\mathrm{q}}(y_0, \tau_0)$$

and coincides with the Kovalevsky matrix if we set $\tau_0 = \gamma$, $\mathbf{y}_0 = \mathbf{x}_0^{\gamma}$.

System (1.143) has a particular solution $\mathbf{y}(\sigma)$, represented in the form of a series

$$\mathbf{y}(\sigma) = \sum_{k=0}^{\infty} \mathbf{y}_k(\sigma) e^{-k\beta\sigma},$$

analogous to (1.19).

This means that the original system (1.3) has a solution in the form of a series (1.16).

With the help of the theory of exponentially-asymptotic symmetry groups it is likewise possible to construct solutions converging to singular points with unbounded increase or decrease of the independent variable and represented in the form of hybrid Lyapunov series [133], containing exponentials, and of series of type (1.16).

Example 1.5.5. We consider the two-dimensional system of equations

$$\dot{x} = -\left(1 + \sum_{m=2}^{\infty} a_m y^m\right) x, \quad \dot{y} = -y^2. \tag{1.144}$$

It has an obvious family of solutions converging to $x = y = 0$ as $t \to +\infty$:

$$\begin{aligned} x(t) &= ce^{-t} \prod_{m=2}^{\infty} \exp\left(\tfrac{a_m}{m-1} t^{1-m}\right) = ce^{-t}\left(1 + \textstyle\sum_{k=1}^{\infty} x_k t^{-k}\right), \\ y(t) &= t^{-1}, \end{aligned} \tag{1.145}$$

where the coefficients x_k depend polynomially on the a_m.

We consider the group of transformations of three-dimensional phase space, generating the following system of equations:

$$\frac{dx}{d\sigma} = -tx, \ \frac{dy}{d\sigma} = -y, \ \frac{dt}{d\sigma} = t. \tag{1.146}$$

System (1.146) is easily interpreted. The flow it generates has the form:

$$x = \exp\left(\tau(1 - e^{\sigma})\right)\xi, \quad y = e^{-\sigma}\eta, \quad t = e^{\sigma}\tau. \tag{1.147}$$

We consider the change of variables (1.147), which converts (1.144) into the form

$$\xi' = -\left(1 + \sum_{m=2} a - m e^{(1-m)\sigma} \eta^m\right)\xi, \ \ \eta' = -\eta^2,$$

where $(\cdot)' = \frac{d}{d\tau}$.

Consequently, the flow (1.147) of the system of differential equations (1.146) is exponentially asymptotic to the symmetry group of system (1.144). The truncated system

$$\dot{x} = -x,\ \dot{y} = -y^2$$

has the one-parameter family of particular solutions

$$x = ce^{\tau_0 - t},\ y = y_0(\frac{t}{\tau_0})^{-1},\ y_0 = \tau_0^{-1},$$

where c is the parameter of the family and τ_0 is fixed, lying on the orbit of the group (1.147).

Thus, in agreement with Theorem 1.5.2, the family of particular solutions (1.145) for system (1.144) generates the group (1.147).

More details about the analysis of hybrid series containing negative powers of e^t and t are left for Chap. 3.

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