Chapter 2
The Main Methodology: Computing Control in Ownership Networks

The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it.

Fundamentally, we do not know why our theories work so well.

(E. Wigner in Wigner 1960)

In this chapter the mathematical bulk of the thesis is presented. The aim of having an exhaustive account of the methods results in the extensive scope of the chapter. As networks find their mathematical embodiment in adjacency matrices (see Appendix B), most of the formalism is comprised of linear-algebraic manipulations.

The reader who is mostly interested in the application of the methods, i.e., the empirical network analysis, can directly go to Chaps. 3 and 4. In Chap. 5 a network-evolution model is presented, shedding new light on the micro rules underlying the empirical properties. All these chapters are written to be self-supporting and provide a minimal introduction to the details of the methodology given in the following. Alternatively, a brief summary is found in Sects. 2.9 and 2.11. Chapter 6 also summarizes the results before discussing the relevance and implications of our findings.

2.1 Introduction

Given an adjacency matrix of an ownership network, what can be said about the distribution of control? The long answer to this question encompasses the following aspects:

1. the introduction of existing measures;
2. their extension and correction;
3. a reinterpretation and unification using network-theoretic notions.

J. B. Glattfelder, *Decoding Complexity*, Springer Theses,

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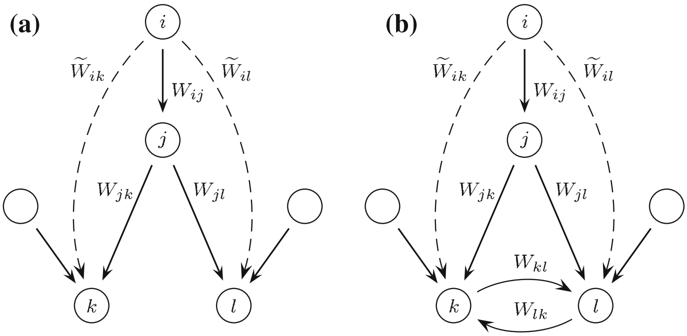

Fig. 2.1 **a** Company i has W_{ij} percent of direct ownership in company j and indirect ownership in companies k and l, through j; the computation of indirect ownership is straightforward as long as the network has no cycles; as an example, $\widetilde{W}_{ik} = W_{ij} W_{jk}$; **b** in the presence of cycles, it is necessary to account for the arising recursive paths of indirect ownership; the model of integrated ownership does this, see Sect. 2.2.1

In a nutshell, the answer is as follows: from the knowledge of the ownership relations the control associated with a shareholder can be estimated. This quantity then needs to be adapted to account for all the indirect paths in the network.

Recall from Sect. 1.1.3 that the percentage of ownership firm i has in company j is given by the entry W_{ij} in the adjacency matrix. The underlying value of the firms are denoted by v_j. In Chaps. 3 and 4, v_j is taken to be the market capitalization and the operating revenue, respectively. In the next section, a first try at assessing the impact of a network structure on ownership is presented.

2.2 Direct and Indirect Ownership

Figure 2.1 illustrates how chains of direct ownership lead to indirect paths of ownership. In the case of tree-like topologies, e.g., Fig. 2.1a, the indirect paths are multiples of the direct links comprising them. In the presence of cycles, this trivial procedure breaks down and the methodology introduced in the following replaces it.

2.2.1 A First Try: Group Value and Integrated Ownership

In Brioschi et al. (1989) and Brioschi and Paleari (1995), the authors propose a simple algebraic model of ownership structures that reflects the direct and indirect ownership relations. It is based on the input–output matrix methodology introduced to economics in Leontief (1966). The sum of all direct and indirect ownership shares a shareholder has in the equity capital of a firm is collectively called *integrated ownership*.

The authors analyzed a setting given by a single external shareholder owning shares in a cluster of firms in a business group, i.e., firms connected by cross-shareholdings. The authors derive two equations, one assigning values to the firms in the business group, and one for computing the integrated ownership shares attributed to this external owner.

Group Value

Let v be a column vector containing the intrinsic value[1] of the firms in the business group. The adjacency matrix W^G describes all the links between the group of firms connected by cross-shareholdings. In accordance with Eq. (1.2) the portfolio value p_i^G of firm i in the group is given by

$$p_i^G = \sum_{j \in \Gamma(i)} W_{ij}^G v_j. \tag{2.1}$$

The row vector of the direct ownership ties of the external shareholder is given by d. This shareholder's portfolio value is given by

$$p_{\text{ext}} = dv. \tag{2.2}$$

The so-called *group value* is defined as follows

$$v^G := W^G v^G + v. \tag{2.3}$$

In other words, the group value reflects the value or importance of a firm depending on its position in the business group, the group's interconnectedness and the distribution of v_j. In Sect. 2.6 we will reinterpret this quantity as a network centrality measure: a firm's value depends on the neighboring firms value plus an initial value. The solution is found to be

$$v^G = (I - W^G)^{-1} v, \tag{2.4}$$

where I denotes the identity matrix. The associated group value of the external shareholder can be computed directly as

$$v_{\text{ext}}^G := dv^G + v_{ext}, \tag{2.5}$$

where v_{ext} is the external shareholder's value.

Putting all these values into one vector, we get

[1] Brioschi et al. (1989) use the value of the net assets.

$$v^{G,\text{tot}} = \begin{pmatrix} d(I - W^G)^{-1}v + v_{ext} \\ (I - W^G)^{-1}v \end{pmatrix}. \tag{2.6}$$

Integrated Ownership

As mentioned, integrated ownership refers to the total of direct and indirect ownership relations. For the integrated ownership of the external shareholder, given by the row vector $\tilde{d}$, an equation similar to Eq. (2.3) holds

$$\tilde{d} = d + \tilde{d}W^G, \tag{2.7}$$

with the solution

$$\tilde{d} = d(I - W^G)^{-1}. \tag{2.8}$$

Note that $\tilde{d} = (I - [W^G]^t)^{-1}d^t$, where the t denotes the transposition operation. From Eqs. (2.4) and (2.8) the following duality relation can be derived, relating the group value of the external shareholder to its integrated ownership

$$dv^G = \tilde{d}v. \tag{2.9}$$

In Brioschi et al. (1989) this is interpreted as follows: the value of the external shareholder's direct group value portfolio dv^G is equivalent to the integrated portfolio of the underlying values $\tilde{d}v$. This means that the entanglement of ownership relations present in a business group, as seen by an external shareholder, can either be accounted for by considering all direct and indirect links and the firms original values v_j or by taking the direct portfolio using the group values v_j^G.

2.2.2 Application to Ownership Networks

What happens when the external shareholder is himself part of the business group. I.e., how can this model be generalize in the case of arbitrary ownership networks to apply to all nodes?

Note that Eq. (2.7) can be promoted to a matrix equation if we include the external shareholder in the analysis. Without loss of generality, the firms can be ordered in such a way that the adjacency matrix decomposes into the following blocks

$$W = \left(\begin{array}{c|c} 0 & d \\ \hline \vec{0} & W^G \end{array}\right), \tag{2.10}$$

where $\vec{0}$ denotes the column vector containing zeros.

For the group value, Eq. (2.3) simply becomes

$$v^G = Wv^G + v. \tag{2.11}$$

Observe that the dimension of v^G of the above equation is one larger than that of v^G of Eq. (2.3), as the external shareholder is now incorporated in the formalism. Furthermore, v^G of Eq. (2.11) is equivalent to Eq. (2.6),

$$v^G = v^{G,\text{tot}}. \tag{2.12}$$

The solution to Eq. (2.11) is

$$v^G = (I - W)^{-1} v. \tag{2.13}$$

For the integrated ownership, Eq. (2.7) is promoted to an operator equation

$$\widetilde{W} = W + \widetilde{W} W. \tag{2.14}$$

The matrix of integrated ownership $\widetilde{W}$ can be understood as a recursive computation of all the indirect paths plus all the direct ones in the network. The solution is given by

$$\widetilde{W} = (I - W)^{-1} W. \tag{2.15}$$

Observe that because for any matrix A

$$(I - A)^{-1} = I + A + A^2 + A^3 + \dots \tag{2.16}$$

the following equation holds

$$(I - W)^{-1} W = W (I - W)^{-1}. \tag{2.17}$$

Hence

$$\widetilde{W} = W + W\widetilde{W} \tag{2.18a}$$

$$= W + \widetilde{W} W. \tag{2.18b}$$

Or, as an example of Eq. (2.18a), in scalar form

$$\widetilde{W}_{ij} = W_{ij} + \sum_k W_{ik} \widetilde{W}_{kj}. \tag{2.19}$$

This symmetry seen in Eq. (2.18) in the definition of integrated ownership allows for two equivalent computations. Figure 2.2 gives an illustration of the first recursive definition. To exemplify, take a loop of four firms, where $1 \to 2$, $2 \to 3$, $3 \to 4$,

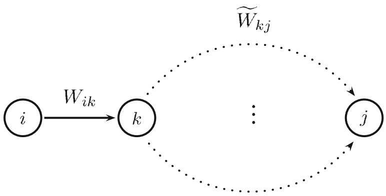

Fig. 2.2 A schematic illustration of integrated ownership as defined in Eq. (2.19): $\widetilde{W}_{ij}$ is composed of all direct links from firm i to j plus all the direct links (if existent) from i to its neighbors k, times the indirect paths of ownership from k to j

all links having the weight 0.5, and $4 \to 1$ with the weight 0.1. For the integrated ownership relation $\widetilde{W}_{11}$, Eq. (2.19) can be understood as the direct link $W_{11} = 0$ plus $\sum_k W_{1k}\widetilde{W}_{k1} = W_{12}\widetilde{W}_{21}$. In other words, the direct weighted link from $1 \to 2$ times all the indirect paths from $2 \to 1$. Numerically, $W_{12}\widetilde{W}_{21} = 0.5 \cdot 0.0253 = 0.0126$. Equation (2.18b) reinterprets $\widetilde{W}_{11}$ as the indirect paths from $1 \to 4$ times the direct path from $4 \to 1$: $\widetilde{W}_{14}W_{41} = 0.1266 \cdot 0.1 = 0.0126$.

For the matrix $(I - W)$ to be non-negative and non-singular, a sufficient condition is that the Perron-Frobenius root is smaller than one, $\lambda_{PF}(W) < 1$. A way to see this is by employing the Perron-Frobenius theorem, described in Appendix B.4: for any other eigenvalue λ of W, $|\lambda| < \lambda_{PF}$. Moreover, for the eigenvector v_{PF}, with $Wv_{PF} = \lambda_{PF}v_{PF}$, it holds that

$$(I - W)^{-1}v_{PF} = Iv_{PF} + Wv_{PF} + W^2v_{PF} + \cdots = \frac{1}{1 - \lambda_{PF}}v_{PF}, \tag{2.20}$$

which approaches infinity for $\lambda_{PF} \to 1$. Note that the last relation employed Eq. (2.16).

This condition is ensured by the following requirement: in each strongly connected component $\mathcal{S}$ there exists at least one node j such that $\sum_{i \in \mathcal{S}} W_{ij} < 1$. Hereafter the term "strongly connected component" will be referred to as SCC.[2] In an economics setting, this means that there exists no subset of k firms ($k = 1, \ldots, n$) that are entirely owned by the k firms themselves. A condition which is claimed to be always fulfilled in ownership networks (Brioschi et al. 1989).

The duality relation of Eq. (2.9) now reads

$$Wv^G = \widetilde{W}v. \tag{2.21}$$

As a result, the group value can be understood as

[2] A list of acronyms can be found in front matter of this book.

$$v^G = Wv^G + v \tag{2.22a}$$

$$= \widetilde{W}v + v. \tag{2.22b}$$

In other words, the group value of a firm is not only the sum of the direct ownership percentages in the neighboring companies times their group values plus the firm's own value, but, equivalently, the integrated ownership percentages times the underlying values plus the intrinsic value of the firm itself.

2.2.3 Example A

To illustrate all the above introduced concepts, consider the network given in Fig. 2.3. The group's adjacency matrix is

$$W^G = \begin{pmatrix} 0.0 & 0.3 & 0.2 \\ 0.4 & 0.0 & 0.8 \\ 0.4 & 0.7 & 0.0 \end{pmatrix}, \tag{2.23}$$

and the vector of direct ownership relations of the external shareholder is

$$d = \begin{pmatrix} 0.2 & 0.0 & 0.0 \end{pmatrix}. \tag{2.24}$$

Hence the adjacency matrix of the whole network is given by

$$W = \left(\begin{array}{c|ccc} 0.0 & 0.2 & 0.0 & 0.0 \\ \hline 0.0 & 0.0 & 0.3 & 0.2 \\ 0.0 & 0.4 & 0.0 & 0.8 \\ 0.0 & 0.4 & 0.7 & 0.0 \end{array}\right). \tag{2.25}$$

The matrix of integrated ownership is computed from Eq. (2.15) as

$$\widetilde{W} = (I - W)^{-1}W = \begin{pmatrix} 0.00 & 1.00 & 1.00 & 1.00 \\ 0.00 & 4.00 & 5.00 & 5.00 \\ 0.00 & 8.18 & 9.45 & 10.00 \\ 0.00 & 7.73 & 9.32 & 9.00 \end{pmatrix}. \tag{2.26}$$

The vector of group value is

$$v^G = \begin{pmatrix} 4.00 \\ 15.00 \\ 28.64 \\ 27.05 \end{pmatrix}, \tag{2.27}$$

if all $v_i = 1$. The duality relation reads

$$dv^G = 3 = \tilde{d}v. \tag{2.28}$$

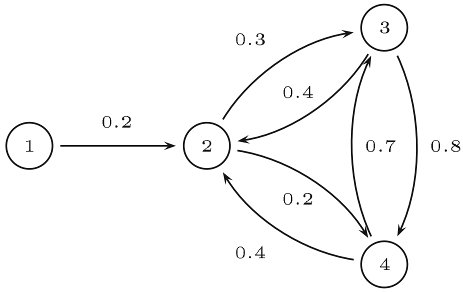

Fig. 2.3 Simple network example: the external shareholder 1 holds 20 % of the shares of firm 2, which, itself being part of a business group of interconnected companies, has cross-shareholdings in firms 3 and 4

2.2.4 Integrated Ownership: Refinements

Looking at the matrix given in Eq. (2.25), it is natural to consider the generalization where the external shareholder is also part of the business group. I.e., this shareholder also has incoming links. Formally, $W_{1j} \neq 0$ for a certain j.

It was, however, soon realized that the presence of self-loops (of any length) is generally problematic. As an example, if firm i owns shares of firm j which in turn owns shares in firm i, i owns a portion of itself. But this path of ownership is visited infinitely many times: i also holds shares of itself via the ownership link $i \rightarrow j \rightarrow i \rightarrow j \rightarrow i$, and so forth. This leads to a problem with the economic interpretation of the group value, which grows rapidly when the number of inter-firm cross-shareholdings grows. In effect, the computation overestimates the group value in the presence of SCCs in the network. This is very undesirable behavior and represents a big drawback of the model.

In the example of the last section, the network has many cross-shareholdings between firms 2, 3 and 4, as seen in Fig. 2.3. Looking at their group value given in Eq. (2.27), the smallest is found to be $v_2^G = 15$, although the total value of all three firms connected by cross-shareholdings is only $3 = \sum_i v_i$.

The problem originates from the recursive definition of integrated ownership, as seen in Eq. (2.18). Namely, the components $\widetilde{W}_{ii}$. A solution to this has been proposed in Baldone et al. (1998). The remedy is given by modifying the formalism to remove self-loops of firms connected through cross-shareholdings.

Following the clearer notation of Rohwer and Pötter (2005), Eq. (2.18b) is adapted as follows:

$$\widehat{W}_{ij} = W_{ij} + \sum_{k \neq i} \widehat{W}_{ik} W_{kj}. \tag{2.29}$$

This means that

$$\widehat{W}_{ij} \equiv \widetilde{W}_{ij} - \widetilde{W}_{ii} W_{ij}, \tag{2.30}$$

where $\widetilde{W}_{ii}$ represent all cycles of indirect ownership originating and ending in node i. Hence Eq. (2.29) removes incoming links of node i when computing its share of integrated ownership over j.

Note that the computations following below are easier to understand when the definition of Eq. (2.18b) is chosen, than when starting with Eq. (2.18a), although both equations are equivalent. In matrix notation, Eq. (2.29) can be manipulated to read

$$\widehat{W} = \left(I - \text{diag}(\widehat{W})\right) W + \widehat{W} W, \tag{2.31}$$

where $\text{diag}(A)$ is the matrix of the diagonal of the matrix A. In Baldone et al. (1998) the solution is found to be

$$\widehat{W} = \text{diag}(V)^{-1}(V - I), \tag{2.32}$$

defining the quantity

$$V := (I - W)^{-1}. \tag{2.33}$$

This can be re-expressed in scalar form as

$$\widehat{W}_{ij} = \frac{V_{ij}}{V_{ii}}; \quad i \neq j, \tag{2.34a}$$

$$\widehat{W}_{kk} = \frac{V_{kk} - 1}{V_{kk}}. \tag{2.34b}$$

Finally, it can derived that

$$V - I = (I + W + W^2 + \ldots) - I = (I + W + W^2 + \ldots)W = (I - W)^{-1} W = \widetilde{W}, \tag{2.35}$$

using Eq. (2.16).

2.2.5 Modifying the Group Value

The corrections introduced in the last section will impact the computation of the group value. Baldone et al. (1998) compute the following. Rearranging Eq. (2.31) to read

$$\left(I - \text{diag}(\widehat{W})\right)^{-1} \widehat{W} = W(I - W)^{-1}, \tag{2.36}$$

and multiplying with v, yields

$$\left(I - \mathrm{diag}(\widehat{W})\right)^{-1} \widehat{W} v = W(I - W)^{-1} v \tag{2.37a}$$
$$= W v^G \tag{2.37b}$$
$$= v^G - v. \tag{2.37c}$$

Recall the solution of v^G given in Eq. (2.13) and that from Eq. (2.11) the last relation of Eq. (2.37c) can be derived. Finally, Eq. (2.37) is found to be

$$v^G = \left(I - \mathrm{diag}(\widehat{W})\right)^{-1} \widehat{W} v + v. \tag{2.38}$$

By replacing

$$v = \left(I - \mathrm{diag}(\widehat{W})\right) \left(I - \mathrm{diag}(\widehat{W})\right)^{-1} v, \tag{2.39}$$

in Eq. (2.38) and rearranging terms, it can be seen that

$$v^G = \left(I - \mathrm{diag}(\widehat{W})\right)^{-1} \left(\widehat{W} - \mathrm{diag}(\widehat{W})\right) v + \left(I - \mathrm{diag}(\widehat{W})\right)^{-1} v \tag{2.40a}$$
$$= \left(I - \mathrm{diag}(\widehat{W})\right)^{-1} \left((\widehat{W} - \mathrm{diag}(\widehat{W}))v + v\right). \tag{2.40b}$$

Expressed in scalar form, this equation reads

$$v_k^G = \frac{1}{1 - \widehat{W}_{kk}} \left(\sum_{i=1}^{n} \widehat{W}_{ki} v_i - \widehat{W}_{kk} v_k + v_k \right) \tag{2.41a}$$
$$= \frac{1}{1 - \widehat{W}_{kk}} \left(\sum_{i=1, i \neq k}^{n} \widehat{W}_{ki} v_i + v_k \right). \tag{2.41b}$$

The economics interpretation given in Baldone et al. (1998) is as follows. The self-cycles of integrated ownership $\widehat{W}_{kk}$ are understood as referring to treasury shares. These are portions of shares that a company keeps in their own treasury.[3] If there are no treasury shares ($\widehat{W}_{kk} = 0$) and hence no loops back to k in the network, this is equivalent to the external shareholder case covered in Sects. 2.2.1 and 2.2.2. As mentioned, this means that the group value is the sum of the integrated ownership percentages times the values—or the direct ownership shares times the neighbors group values, cf. Eq. (2.22)—plus the underlying value of k.

If, however, $\widehat{W}_{kk} > 0$, the group value of firm k exceeds the sum of the (modified) integrated ownership percentages times the values plus v_k by the term $1/(1 - \widehat{W}_{kk})$,

[3] Treasury shares may have come from a repurchase of shares by the firm from shareholders or they may have never been issued to the public. These shares do not pay dividends and have no voting rights, see also Fig. 1.4.

seen in Eq. (2.41). The closer $\widehat{W}_{kk}$ is to one, the greater the group value and the bigger the divergence is.

It is imperative to note the peculiarities of this proposed solution and interpretation. First of all, although the authors in Baldone et al. (1998) correctly identify the problematic term and isolate it in Eq. (2.41), they still do not actually propose a correction to the group value that would result in smaller numerical v_k^G values in the case of cycles in the network. Secondly, in the interpretation of Eq. (2.41b), the modification of integrated ownership the authors propose is very mysterious, namely $\widehat{W} - \text{diag}(\widehat{W})$, as seen in Eq. (2.40b). Especially as a term $\text{diag}(\widehat{W})$ is already present in the definition of $\widehat{W}$, cf. Eq. (2.31).

Our first contributions will be to propose a straightforward correction to the group value in the case of self-loops and a clear interpretation thereof in the next sections. Note that the corrections to the integrated ownership proposed above are still being used, see for instance (Rohwer and Pötter 2005; Chapelle 2005). Indeed, even the uncorrected methodology is still in use (Almeida et al. 2007).

2.3 Introducing Network Value and Integrated Value

Building on the studies of Brioschi et al. (1989), Baldone et al. (1998) we present an extension of the formalism and remedy the shortcomings which have been unaddressed to this date (see the end of the last section for details). Moreover, there is a very elegant interplay between the notions of integrated ownership and group value which has never been explicitly pointed out in the literature.

2.3.1 Integrated Value

In analogy to the notion of the portfolio value of shareholders defined in Eq. (1.2), we define the *integrated value*

$$\tilde{\nu}^{\text{int}} := \widetilde{W} v = (I - W)^{-1} W v. \tag{2.42}$$

$\tilde{\nu}_i^{\text{int}}$ represents the portfolio value of shareholder i considering all its direct and indirect (i.e., integrated) paths of ownership in the network. Recall that the integrated ownership matrix obeys the operator equation

$$\widetilde{W}_{ij} = W_{ij} + \sum_k W_{ik} \widetilde{W}_{kj}. \tag{2.43}$$

as seen in Eq. (2.19) and Fig. 2.2. It should be noted, that in the spirit of the 3-level network analysis mentioned in Sect. 1.1.1, $\tilde{\nu}_i^{\text{int}}$ is a fully fledged Level 3 measure, incorporating all the available information of the complex network under study.

Observe that the integrated ownership value of Eq. (2.42) is also the solution to the following equation

$$\tilde{\nu}^{\text{int}} = W\tilde{\nu}^{\text{int}} + Wv, \tag{2.44}$$

or in scalar notation

$$\tilde{\nu}_i^{\text{int}} = \sum_k W_{ik}\tilde{\nu}_k^{\text{int}} + \sum_k W_{ik}v_k, \tag{2.45}$$

which can be interpreted as a centrality measure similarly to Eqs. (2.3) and (2.11), which are reminiscent of a Hubbell index centrality (see Sect. 2.6). In effect, the importance of node i reflected in $\tilde{\nu}_i^{\text{int}}$, is determined by the importance of its neighbors and the value of its neighbors. Alternatively, in an ownership setting, $\tilde{\nu}_i^{\text{int}}$ in Eq. (2.45) should be understood as the integrated value of i's neighbors plus i's portfolio value, given in Eq. (1.2). An additional interpretation in terms of network flow will be given in Sect. 2.4.

Although $\tilde{\nu}^{\text{int}}$ was implicitly used in the duality relation of Eq. (2.21), the context given in Eq. (2.44) was previously unobserved in the pertinent literature.

2.3.2 Network Value

In the rest of this thesis, the term *network value* will be used to replace the notion of group value introduced in Sect. 2.2.1. The change in naming reflects the fact that it is a general network measure and not necessarily constrained to the idea of business groups.

To generalize Eq. (2.11)

$$v^{\text{net}} := Wv^{\text{net}} + v, \tag{2.46}$$

and

$$v^{\text{net}} = (I - W)^{-1}v. \tag{2.47}$$

In an ownership context, Eq. (2.46) can be understood as computing i's network value as the direct portfolio of its neighbors network value plus i's own underlying value v_i.

2.3.3 The Whole Picture

For the duality relation of Eq. (2.21) one can easily see that the following relations hold, employing Eq. (2.17)

$$\widetilde{W}v = (I - W)^{-1}Wv = W(I - W)^{-1}v = Wv^{\text{net}}. \tag{2.48}$$

Finally, combining Eqs. (2.46) and (2.48) uncovers the novel connection between the two concepts

$$v^{\text{net}} = Wv^{\text{net}} + v = \widetilde{W}v + v = \tilde{\nu}^{\text{int}} + v. \tag{2.49}$$

In other words, the network value accounts for the overall value of an economic actor, given by its underlying value plus the value gained from the integrated value. Moreover, the integrated value reflects the value attained from the underlying values of all firms reachable by all direct and indirect paths of ownership. It is also equivalent to the network value of all directly owned firms.

2.3.4 The True Corrections

As mentioned at the end of Sect. 2.2.5, the corrections proposed in Baldone et al. (1998) have not been implemented correctly. In order to understand this, it is crucial to reformulate Eq. (2.31) appropriately. This is best done by introducing the *correction operator*

$$\mathcal{D} := \text{diag}\left((I - W)^{-1}\right)^{-1}, \tag{2.50}$$

or using Eq. (2.33)

$$\mathcal{D} = \text{diag}(V)^{-1}. \tag{2.51}$$

Recall that $\text{diag}(A)$ is defined as the matrix of the diagonal elements of the matrix A. The components of $\mathcal{D}$ are

$$\mathcal{D}_{kk} = \frac{1}{(I - W)^{-1}_{kk}}, \tag{2.52a}$$

$$\mathcal{D}_{ij} = 0, \quad i \neq j. \tag{2.52b}$$

To express $\widehat{W}$ in terms of $\mathcal{D}$ one can insert Eq. (2.51) directly into Eq. (2.32), recalling Eq. (2.35). Or, alternatively, one can start from the defining relation given in Eq. (2.31). One then must first derive the following algebraic identity using the properties of the diag() operation[4]

[4] $\text{diag}(\text{diag}(A)) = \text{diag}(A)$, $\text{diag}(A + B) = \text{diag}(A) + \text{diag}(B)$, and $\text{diag}(I) = I$.

$$\begin{aligned}\operatorname{diag}(\widehat{W}) &= \operatorname{diag}\left(\operatorname{diag}(V)^{-1}(V-I)\right) = \operatorname{diag}(V)^{-1}\operatorname{diag}(V-I) \\ &= \operatorname{diag}(V)^{-1}(\operatorname{diag}(V)-I) = I - \operatorname{diag}(V)^{-1} = I - \mathcal{D},\end{aligned} \tag{2.53}$$

in other words

$$\mathcal{D} = I - \operatorname{diag}(\widehat{W}). \tag{2.54}$$

Inserting this directly into Eq. (2.31) and solving for $\widehat{W}$ reveals the wanted relation. To summarize, both possibilities yield

$$\widehat{W} = \mathcal{D}W(I-W)^{-1} = \mathcal{D}(I-W)^{-1}W = \mathcal{D}\widetilde{W}. \tag{2.55}$$

It has now become apparent that the effect of the correction to the integrated ownership due to self-loops proposed by Baldone et al. (1998) in Eq. (2.29) is equivalent to the simple multiplication of $\mathcal{D}$ and $\widetilde{W}$.

This now allows us to define the corrected integrated value as

$$\hat{\nu}^{\text{int}} := \widehat{W}v = \mathcal{D}\widetilde{W}v = \mathcal{D}\tilde{\nu}^{\text{int}}, \tag{2.56}$$

recalling Eq. (2.42).

In a similar vein we introduce the missing correction to the network value. Starting from where Baldone et al. (1998) left off, namely Eq. (2.41b), we propose the following interpretation

$$v_k^{\text{net}} = \frac{1}{1-\widehat{W}_{kk}}\left(\sum_{i=1, i\neq k}^{n} \widehat{W}_{ki}v_i + v_k\right) \tag{2.57a}$$

$$= \frac{1}{1-\widehat{W}_{kk}}\left(\sum_{i=1}^{n} \widehat{W}_{ki}v_i + (1-\widehat{W}_{kk})v_k\right). \tag{2.57b}$$

Now Eq. (2.57b) can be re-expressed in matrix notation as

$$\begin{aligned}v^{\text{net}} &= \left(I - \operatorname{diag}(\widehat{W})\right)^{-1}\left(\widehat{W}v + \left(I - \operatorname{diag}(\widehat{W})\right)v\right) \\ &= \mathcal{D}^{-1}\left(\mathcal{D}\widetilde{W}v + \mathcal{D}v\right),\end{aligned} \tag{2.58}$$

employing Eq. (2.53). Hence a natural interpretation of the impact of the loop-correction on the network value is

$$\hat{v}^{\text{net}} := \mathcal{D}v^{\text{net}} = \mathcal{D}(\tilde{\nu}^{\text{int}} + v) = \mathcal{D}\widetilde{W}v + \mathcal{D}v = \widehat{W}v + \mathcal{D}v = \hat{\nu}^{\text{int}} + \mathcal{D}v. \tag{2.59}$$

The introduction of the modified network value $\hat{v}^{\text{net}} = \mathcal{D}v^{\text{net}}$ is in complete analogy to the corrected integrated value $\hat{\nu}^{\text{int}} = \mathcal{D}\tilde{\nu}^{\text{int}}$, seen in Eq. (2.56). To summarize,

Table 2.1 Summary of integrated value $\tilde{\nu}^{\text{int}}$, network value v^{net} and their correction $\hat{\nu}^{\text{int}}$ and $\hat{v}^{\text{net}}$

Definition	Solution	Self-loop-correction
$\tilde{\nu}^{\text{int}} = W(\nu^{\text{int}} + v)$	$\tilde{\nu}^{\text{int}} = \widetilde{W} v$	$\hat{\nu}^{\text{int}} = \mathcal{D}\widetilde{W} v$
$v^{\text{net}} = W v^{\text{net}} + v$	$v^{\text{net}} = \tilde{\nu}^{\text{int}} + v$	$\hat{v}^{\text{net}} = \hat{\nu}^{\text{int}} + \mathcal{D}v$

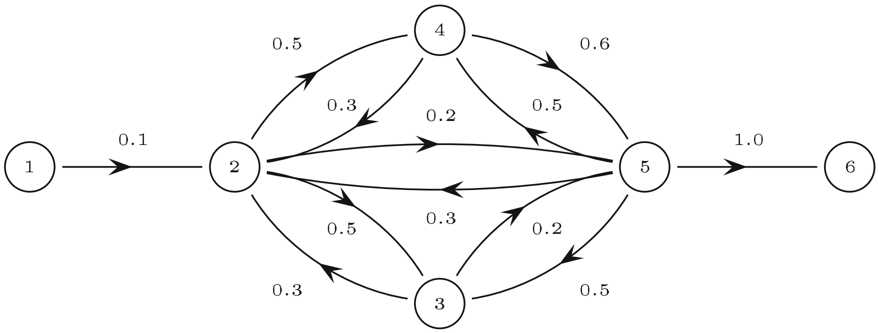

Fig. 2.4 Simple bow-tie network topology example with a high degree of interconnectedness of the firms in the strongly connected component (SCC)

the effect of removing the incoming links of a firm i in the analysis results in the underlying value v_i, the network value v_i^{net}, and the integrated value $\tilde{\nu}^{\text{int}}$ all being multiplied by a factor $\mathcal{D}_{ii} = 1 - \widehat{W}_{ii} = 1/(I - A)^{-1}_{ii}$:

$$\mathcal{D}v^{\text{net}} = \mathcal{D}\tilde{\nu}^{\text{int}} + \mathcal{D}v. \tag{2.60}$$

This also underlines the crucial role played by the correction operator $\mathcal{D}$, which, by incorporating all the effects of amending for self-loops, acts as a consistent measure to derive the corrected terms.

Table 2.1 summarizes all the important relations that have been derived so far. In the next section, a numerical example is presented.

2.3.5 Example B: And the Next Problem on the Horizon

Consider the network illustrated in Fig. 2.4. It is an example of a simple bow-tie network topology. The SCC is constructed in a way to highlight the problem of cross-shareholdings. Hence there are many cycles of indirect ownership originating and ending in each firm in the core of the bow-tie.

We assume the underlying value of each firm to be one, i.e., $v = (1, 1, 1, 1, 1, 1)^t$, where t denotes the transposition operation. This results in the network value and the integrated value to be

$$v^{\text{net}} = \begin{pmatrix} 6 \\ 50 \\ 27 \\ 49 \\ 55 \\ 1 \end{pmatrix}, \qquad \tilde{\nu}^{\text{int}} = \begin{pmatrix} 5 \\ 49 \\ 26 \\ 48 \\ 54 \\ 0 \end{pmatrix}. \tag{2.61}$$

So although the total value present in the network is $6 = \sum_i v_i$, firm 5 has an disproportionately large network value of $v_5^{\text{net}} = 55$.

Introducing the correction operator, one finds

$$\mathcal{D} = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.162 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.095 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.086 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.000 \end{pmatrix}, \tag{2.62}$$

allowing the corrected values to be computed as

$$\hat{v}^{\text{net}} = \begin{pmatrix} 6.000 \\ 5.000 \\ 4.378 \\ 4.667 \\ 4.714 \\ 1.000 \end{pmatrix}, \qquad \hat{\nu}^{\text{int}} = \begin{pmatrix} 5.000 \\ 4.900 \\ 4.216 \\ 4.571 \\ 4.629 \\ 0.000 \end{pmatrix}. \tag{2.63}$$

The correction reduces the values of the firms in the core of the bow-tie by approximately one order of magnitude. This confirms that $\hat{v}^{\text{net}}$ and $\tilde{\nu}^{\text{int}}$ are indeed the right measures to consider in the presence of SCCs in the network.

Unfortunately, this example highlights a further problem of the methodology. We will present two solutions to remedy this complication in Sect. 2.5. Before detailing the issue, it should be noted that the problem at hand was previously overlooked, because the ownership networks that were analyzed did not have a bow-tie structure and because the focus was not on the empirical analysis of control. It plagues both $\tilde{\nu}^{\text{int}}$ and $\hat{\nu}^{\text{int}}$.

In a nutshell, the problem can be described as follows: a single root node r, i.e., $k_r^{in} = 0$, connected to a SCC will be assigned an integrated value which is the sum of the underlying value of all the firms reachable by ownership paths from r. In the above example, $\tilde{\nu}_1^{\text{int}} = \hat{\nu}_1^{\text{int}} = 5$. This behavior is, however, independent of the percentage of ownership connecting r to the core, e.g., W_{12} in Fig. 2.4. This means that a company with no shareholders and an arbitrarily small share in a firm in the SCC (having no other external shareholders) still gets an integrated value totalling the underlying value of all firms it has integrated ownership in. This is obviously a very undesirable behavior. Note that if the SCC has multiple root-nodes connecting to it, the total underlying value gets distributed amongst them, also regardless of the link strength.

In order to fully understand this intricacy, a short digression into the theory of networks is necessary. In detail, the idea of a quantity flowing in the network gives an alternative interpretation of integrated value and network value. This change in point of view facilitates the understanding of the above mentioned problem.

2.4 A New Perspective: The Notion of Flow in Networks

Consider a directed and weighted network in which (i) a non-topological real value $v_j \geq 0$ can be assigned to the nodes (with the condition that $v_j > 0$ for at least all the leaf-nodes in the network, i.e., nodes with $k_i^{out} = 0$) and (ii) an edge from node i to j with weight W_{ij} implies that some of the value of j is transferred to i. In terms of a physical system, we think of the nodes as entities receiving material from the downstream nodes and transferring it to the upstream nodes, without dissipation, in proportion to the weights of the incoming links.

Assume that the nodes which are associated with a value v_j produce v_j units of mass (or energy) at time $t = 1$. Then the flow ϕ_i entering the node i from each node j at time t is the fraction W_{ij} of the mass produced directly by j plus the same fraction of the inflow of j:

$$\phi_i(t+1) = \sum_j W_{ij} v_j + \sum_j W_{ij} \phi_i(t). \tag{2.64}$$

where $\sum_i W_{ij} = 1$ for the nodes that have predecessors and $\sum_i W_{ij} = 0$ for the root-nodes (sinks). In matrix notation, at the steady state ($t \to \infty$), this yields

$$\phi = W(v + \phi). \tag{2.65}$$

The solution

$$\phi = (1 - W)^{-1} W v, \tag{2.66}$$

exists and is unique if $\lambda(W) < 1$. This condition is easily fulfilled in real networks as it requires that in each SCC $\mathcal{S}$ there exists at least one node j such that $\sum_{i \in \mathcal{S}} W_{ij} < 1$. Or, equivalently, the mass circulating in $\mathcal{S}$ is also flowing to some node outside of $\mathcal{S}$. Notice that this does not imply that mass is lost in the transfer. Indeed, the mass is conserved at all nodes except at the sinks. Some of the nodes only produce mass (all the leaf-nodes but possibly also other nodes) at time $t = 1$ and are thus sources, while the root-nodes accumulate the mass.

Note that it is straightforward to also define an equation for the evolution of the stock of mass (energy) present at each node. The convention used here implies that mass flows against the direction of the edges. The transported quantity is only created once at time $t = 1$ and the root nodes get self-loops assigned to them, so that no quantity is lost. For non-root nodes (i.e., $k_i^{in} > 0$) the stock never gets accumulated and is always passed on upstream. For the root nodes, or sinks, the value for the flow and the accumulated stock is equivalent.

The equation for the evolution of the stock s of mass present at each node can be derived as follows: for node i the stock at time $t+1$ is the value of the previous time step minus outflow plus inflow

$$s_i(t+1) := s_i(t) - \sum_k W_{ki} s_i(t) + \sum_j W_{ij} s_j(t), \tag{2.67}$$

or equivalently

$$s_i(t+1) = \begin{cases} s_i(t) + \sum_j W_{ij} s_j(t), & \text{if } i \text{ is a sink,} \\ \sum_j W_{ij} s_j(t), & \text{otherwise.} \end{cases} \tag{2.68}$$

In matrix notation the equation above reads

$$s(t+1) := \mathcal{T} s(t) = (\mathcal{S} + W) s(t), \tag{2.69}$$

where

$$\mathcal{S}_{ii} = 1, \quad \text{if } i \text{ is a sink,} \tag{2.70a}$$
$$\mathcal{S}_{ij} = 0, \quad \text{otherwise.} \tag{2.70b}$$

In effect, the diagonal matrix $\mathcal{S}$ assigns self-loops to sinks, conserving the mass or energy in the network.

As a result of Eq. (2.69)

$$s(n) = \mathcal{T}^n s(0). \tag{2.71}$$

Note that $\mathcal{T}^n$ describes paths of length n in the network. However, does Eq. (2.71) converge, meaning that after some time $\hat{t}$ the stock is unchanged: $s(\hat{t}+1) = s(\hat{t})$? To see that there exists a fixed point s_*

$$s_* = \mathcal{T} s_*, \tag{2.72}$$

consider the following. The ownership matrix W is per definition non-negative and column stochastic, i.e., $\sum_j W_{ij} \leq 1$ and, per construction, $W_{ii} = 0$. Hence $\mathcal{T}$ is also non-negative and column stochastic. This means that the Perron-Frobenius theorem, explained in Appendix B.4, holds and there exists a unit eigenvalue of $\mathcal{T}$: $\lambda = 1$. In other words

$$\mathcal{T} s_* = \lambda s_* = s_*. \tag{2.73}$$

2.4.1 Flow in Ownership Networks

So what does this all mean in the case of an ownership network? And what quantity can be seen as flowing along the links? The cash allowing an equity stake in a firm to be held flows in the direction of the edges. In contrast, the ownership of a firm's equity capital, i.e., the cash-flow rights, are transferred in the opposite direction, from the firm to its shareholders. The same is true for the payed dividends (and voting rights, see Sect. 2.7). See also Sect. 1.2.1.

Observe that Eq. (2.66) is equivalent to Eq. (2.42), uncovering the following interpretation

$$\phi \equiv \tilde{\nu}^{\text{int}}. \tag{2.74}$$

In other words, the integrated value $\tilde{\nu}_i^{\text{int}}$ in an ownership network corresponds to the inflow ϕ_i of the underlying units of value v_i in the steady state.

It is now possible to conceptually understand the problem mentioned at the end of Sect. 2.3.5. Since the integrated value of any node corresponds to the inflow over an infinite time, all the value ($v^{tot} = \sum_i v_i$) that is flowing in the network will ultimately accumulate in the root nodes. In the case of a single root node r connected to a SCC, as in the example given in Fig. 2.4, the total value of all the firms downstream of it will necessarily have to flow to it, regardless of the percentage of ownership with which the root node is connected to firms in the core: $\tilde{\nu}_r^{\text{int}} = \sum_{i \neq r} v_i$.

Finally, for the problem of the overestimation of integrated value, as mentioned in Sect. 2.2.4, the following should be noted. The more indirect self-cycles are present in the network, the longer the quantity will be circulating through the nodes connected by these paths. This explains the high numerical values of $\tilde{\nu}_i^{\text{int}}$ for nodes in SCCs. However, because $\tilde{\nu}^{\text{int}} = \phi$, these are actually the correct value to assign to such nodes in a physical network. Only in the context of ownership, this behavior is seen as pathological and $\hat{\nu}^{\text{int}}$ is introduced to alleviate this characteristic. It is therefore important to keep in mind, that although $\hat{\nu}^{\text{int}}$ has a more desirable behavior in the context of ownership networks, it has no correspondence to a physical system anymore.

2.5 The Final Corrections: Adjusting Network Value and Integrated Value for Bow-Tie Topologies

To summarize, all previous versions of network value and integrated value failed for ownership networks with bow-tie topologies:

1. firms in the SCC get assigned excessively high quantities;
2. firms with no shareholders accumulate the underlying value of the firms they have integrated ownership in.

As indicated at the end of Sect. 2.3.5, we will now present two related solutions to these problems. The first version will be an analytical derivation of the new quantities. The second solution is given by an algorithm.

For smaller networks, the analytical solution is easily implemented. However, if large networks need to be analyzed, none of the analytical measures are feasible, as already the computation of the inverse matrix $(I - W)$ becomes intractable. Hence the application of the algorithm is inevitable.

2.5.1 The Analytical Solution

Observe that the duality relation $\widetilde{W}v = Wv^{\text{net}}$, given in Eq. (2.48), is lost for the changes introduced by the correction operator $\mathcal{D}$, defined in Eqs. (2.50) or (2.52b). This is easily seen by

$$\hat{\nu}^{\text{int}} = \widehat{W}v = \mathcal{D}\widetilde{W}v = \mathcal{D}Wv^{\text{net}} \tag{2.75a}$$

$$\neq W\mathcal{D}v^{\text{net}} = W\hat{v}^{\text{net}}. \tag{2.75b}$$

Recall Table 2.1 for a summary of the definitions and relations. In effect, the non-commutative nature of the matrix multiplication, $\mathcal{D}W \neq W\mathcal{D}$, is responsible for the inequality $\widehat{W}v \neq W\hat{v}^{\text{net}}$. The result of this subtlety is that one can define two variants related to this correction. Next to $\hat{v}^{\text{net}} = \widehat{W}v + \mathcal{D}v = \mathcal{D}Wv^{\text{net}} + \mathcal{D}v$, cf. Eq. (2.59), the natural alternative is

$$\bar{v}^{\text{net}} := W\mathcal{D}v^{\text{net}} + \mathcal{D}v = W\hat{v}^{\text{net}} + \mathcal{D}v. \tag{2.76}$$

The following algebraic manipulations uncover the meaning of this equation. Replacing v^{net} with the relation given in Eq. (2.49)

$$\bar{v}^{\text{net}} = W\mathcal{D}\left(\widetilde{W}v + v\right) + \mathcal{D}v = W(\widehat{W} + \mathcal{D})v + \mathcal{D}v =: \overline{W}v + \mathcal{D}v. \tag{2.77}$$

This identifies an additional corrected integrated ownership matrix as

Table 2.2 Summary of $\tilde{\nu}^{\text{int}}$, υ^{net}, $\hat{\nu}^{\text{int}}$, $\hat{\upsilon}^{\text{net}}$, $\bar{\nu}^{\text{int}}$, $\bar{\upsilon}^{\text{net}}$, $\mathcal{D}$, $\widetilde{W}$, $\widehat{W}$, and $\overline{W}$

Definition	Solution	First correction	Second correction
$\tilde{\nu}^{\text{int}} = W(\nu^{\text{int}} + \upsilon)$	$\tilde{\nu}^{\text{int}} = \widetilde{W}\upsilon$	$\hat{\nu}^{\text{int}} = \widehat{W}\upsilon$	$\bar{\nu}^{\text{int}} = \overline{W}\upsilon$
$\upsilon^{\text{net}} = W\upsilon^{\text{net}} + \upsilon$	$\upsilon^{\text{net}} = \tilde{\nu}^{\text{int}} + \upsilon$	$\hat{\upsilon}^{\text{net}} = \hat{\nu}^{\text{int}} + \mathcal{D}\upsilon$	$\bar{\upsilon}^{\text{net}} = \bar{\nu}^{\text{int}} + \mathcal{D}\upsilon$
$\mathcal{D} = \text{diag}\left((I - W)^{-1}\right)^{-1}$,	$\widetilde{W} = (I - W)^{-1}W$,	$\widehat{W} = \mathcal{D}\widetilde{W}$,	$\overline{W} = W(\widehat{W} + \mathcal{D})$.

$$\overline{W} := WW^*, \tag{2.78}$$

with

$$W^* := \widehat{W} + \mathcal{D}, \tag{2.79}$$

or in scalar notation

$$W^*_{ij} = \begin{cases} 1, & i = j, \\ \widehat{W}_{ij}, & i \neq j. \end{cases} \tag{2.80}$$

In analogy to the previous sections, it is now straightforward to introduce the corresponding corrected integrated value

$$\bar{\nu}^{\text{int}} := \overline{W}\upsilon, \tag{2.81}$$

which identifies $\bar{\upsilon}^{\text{net}}$ as an additional corrected network value

$$\bar{\upsilon}^{\text{net}} = \bar{\nu}^{\text{int}} + \mathcal{D}\upsilon. \tag{2.82}$$

In Table 2.2 all the introduced concepts are summarized. Before we analyze the behavior of $\bar{\upsilon}^{\text{net}}$ and $\bar{\nu}^{\text{int}}$, we first introduce the corresponding algorithmic solution in the next section.

Finally, the following equations give all the equalities related to the various incarnations of network value and integrated value:

$$\upsilon^{\text{net}} = \nu^{\text{int}} + \upsilon = \widetilde{W}\upsilon + \upsilon, \tag{2.83}$$

$$\hat{\upsilon}^{\text{net}} = \hat{\nu}^{\text{int}} + \mathcal{D}\upsilon = \widehat{W}\upsilon + \mathcal{D}\upsilon = \mathcal{D}\widetilde{W}\upsilon + \mathcal{D}\upsilon = \mathcal{D}W\upsilon^{\text{net}} + \mathcal{D}\upsilon, \tag{2.84}$$

$$\bar{\upsilon}^{\text{net}} = \bar{\nu}^{\text{int}} + \mathcal{D}\upsilon = \overline{W}\upsilon + \mathcal{D}\upsilon = W\mathcal{D}\widetilde{W}\upsilon + W\mathcal{D}\upsilon + \mathcal{D}\upsilon = W\mathcal{D}\upsilon^{\text{net}} + \mathcal{D}\upsilon. \tag{2.85}$$

2.5.2 The Algorithmic Solution

We illustrate the algorithm for the computation of the network value. Then the integrated value can be obtained by deducting the underlying value. In order to calculate the network value for any specific node i, we extract the whole subnetwork that is downstream of a node i, including i. For this purpose, a breadth-first-search (BFS) returns the set of all nodes reachable from i, going in the direction of the links. Then, all the links among these nodes are obtained from the adjacency matrix of the entire network, except for the links pointing to i which are removed. This ensures that there are no cycles involving i present in the subnetwork. Let $B(i)$ denote the adjacency matrix of such a subnetwork, including i, extracted from the ownership matrix W. Without loss of generality, we can relabel the nodes so that $i = 1$. Since node 1 has now no incoming links, we can decompose $B = B(1)$ as follows:

$$B = \left(\begin{array}{c|c} 0 & d \\ \hline \vec{0} & B^{\text{sub}} \end{array} \right), \tag{2.86}$$

where d is the vector of all links originating from node 1, and B^{sub} is associated with the subgraph of the nodes downstream of i. This is similar to the decomposition given in Eq. (2.10) for the case with an external shareholder.

The underlying value of these nodes is given by the vector v^{sub}. By replacing the matrix B in the expression $v^{\text{net}} = \widetilde{W} v + v = W(I - W)^{-1} v + v$, cf. Eq. (2.49), and taking the first component we obtain:

$$v^{\text{net}}(1) = \left[B(I - B)^{-1} v \right]_1 + v_1 \tag{2.87a}$$

$$= \left[d(I^{\text{sub}} - B^{\text{sub}})^{-1} v^{\text{sub}} \right]_1 + v_1 =: \tilde{d} \cdot v^{\text{sub}} + v_1, \tag{2.87b}$$

where now $\tilde{\nu}^{\text{int}}(1) := \tilde{d} \cdot v^{\text{sub}}$, in analogy to the term in Eq. (2.8).

Notice that if node i has zero in-degree, this procedure yields the same result as the previous formula for the integrated ownership matrix of Eq. (2.15): $\tilde{B}_{(i,*)} = (0, \tilde{d}) = \widetilde{W}_{(i,*)}$. The notation $A_{(i,*)}$ for a matrix is understood as taking its i-th row. In Appendix D.1 it is formally shown that our calculation is in fact equivalent to the correction proposed by Baldone et al. (1998) to address the problems of the overestimation of network value in the case of ownership due to the presence of cycles.

However, the methods still suffer from the problem of root nodes accumulating all the value flowing in the network. To solve this issue, we adjust our algorithm to pay special attention to the IN-nodes of an SCC. We partition the bow-tie associated with this SCC into its components: the IN (to which we also add the T&T), the SCC itself, and the OUT. Then, we proceed in multiple steps to compute the network value for all parts in sequence. In this way, the value flows from the OUT, via the SCC to the IN. Finally, the integrated value of firm i is computed from the network value as

$$\tilde{\nu}^{\text{int}}(i) = v^{\text{net}}(i) - v_i. \tag{2.88}$$

In detail, our algorithm works as follows:

1. OUT: Compute the network value $v^{\text{net}}(i)$ for all the nodes in the OUT using Eq. (2.87b).
2. OUT → SCC: Identify the subset $\mathcal{S}1$ of nodes in the SCC pointing to nodes in the OUT, the latter subset denoted as $\mathcal{O}$. To account for the value entering the SCC from the OUT, compute the network value of these selected nodes by applying $v^{\text{net}}(s) = \sum_o W_{so} v^{\text{net}}(o) + v_s$ to them. This is an adaptation of Eq. (2.49), where s and o are labels of nodes in $\mathcal{S}1$ and $\mathcal{O}$, respectively. Note that we only needed to consider the direct links for this. This computation is also equivalent to applying Eq. (2.87b), which considers the downstream subnetworks of $\mathcal{S}1$, i.e., the whole OUT.
3. SCC: Employ Eq. (2.87b) to the SCC-nodes restricting the BFS to retrieve only nodes in the SCC itself. Note that for those SCC-nodes that were already considered in step 2, their network value is now taken as the intrinsic value in the computation. This means one first needs to assign $v_i \mapsto v^{\text{net}}(i) + v_i$.
4. SCC → IN: In this step we solve the problem of the root-nodes acquiring an exaggerated fraction of the network value. For the subset of IN-nodes $\mathcal{I}$ directly connected to some SCC-nodes $\mathcal{S}2$, we again apply $v^{\text{net}}(i) = \sum_s \mathcal{W}_{is} v^{\text{net}}(s) + v_i$, where i and s are labels of nodes in $\mathcal{I}$ and $\mathcal{S}2$, respectively. However, note that due to the cycles present in the SCC, this computation is not equivalent to Eq. (2.87b). In other words, the duality relation similar to Eq. (2.48), ensuring that the direct portfolio of group value is equivalent to the portfolio of the integrated underlying values, is violated due to the presence of self-loops[5]: $\sum_s \mathcal{W}_{is} v^{\text{net}}(s) \neq \nu^{\text{int}}(i)$. In this way only the direct share of network value over the SCC which is not owned by other SCC-nodes is transfered to the IN-nodes.
5. IN: Finally, use Eq. (2.87b) for assigning the network value to the nodes in the IN-subnetwork. In this case the BFS should not consider the SCC-nodes since their value has been already transfered to their first neighbors in the IN. However, it should retrieve the T&T departing from the IN. Again, for the IN-nodes treated in step 4, first assign $v_i \mapsto v^{\text{net}}(i) + v_i$.

Notice that if any part of the bow-tie structure contains additional smaller SCCs, these should be treated first, by applying steps two to four.

This dissection of the network into its bow-tie components also reduces the computational problems. Although we perform a BFS for each node and compute the inverse of the resulting adjacency matrix of the subnetwork as seen in Eq. (2.87b), the smaller sizes of the subnetworks allow faster computations.

To summarize, the algorithm computes the network value of firm i as $v^{\text{net}}(i)$. By deducting the underlying value, we retrieve the integrated value of i: $\tilde{\nu}^{\text{int}}(i)$.

The algorithm presented here is applied to the global network of transnational corporations in Chap. 4.

[5] This was also observed in Eq. (2.75).

2.5.3 Revisiting Example B: A Summary and Discussion

Coming back to the network example shown in Fig. 2.4, we now compute all the relevant expressions derived in this chapter and discuss the results. In the following, to avoid any confusion, the row vector of network values $v^{\text{net}}(i)$, computed from the algorithmic solution as detailed in Eq. (2.87), will be identified as

$$\overset{\circ}{v}{}_i^{\text{net}} := v^{\text{net}}(i). \tag{2.89}$$

Recall that the algorithmic computation of network value must be performed for each node separately. Hence the vector $\overset{\circ}{v}{}^{\text{net}}$ requires Eq. (2.87) to have been computed n times, if n is the length of $\overset{\circ}{v}{}^{\text{net}}$. Although this appears to be rather tedious, it actually makes the algorithm applicable for very large networks, as observed at the end of the last section.

Correspondingly, for the integrated value,

$$\overset{\circ}{\nu}{}_i^{\text{int}} := \tilde{\nu}^{\text{int}}(i) = \overset{\circ}{v}{}_i^{\text{net}} - v_i, \tag{2.90}$$

will denote the integrated value computed from the algorithm. Note that this relation stems from Eq. (2.87).

Recall that the underlying values are chosen to be

$$v = (1, 1, 1, 1, 1, 1)^t . \tag{2.91}$$

It follows that

$$\mathcal{D}v = (1.000, 0.100, 0.162, 0.095, 0.086, 1.000)^t , \tag{2.92}$$

using $\mathcal{D}$ from Eq. (2.62). As usual, t denotes the transposition operation.

The corresponding measures of network value are

$$v^{\text{net}} = \begin{pmatrix} 6 \\ 50 \\ 27 \\ 49 \\ 55 \\ 1 \end{pmatrix}, \quad \hat{v}^{\text{net}} = \begin{pmatrix} 6.000 \\ 5.000 \\ 4.378 \\ 4.667 \\ 4.714 \\ 1.000 \end{pmatrix}, \quad \overset{\circ}{v}{}^{\text{net}} = \begin{pmatrix} 1.500 \\ 5.000 \\ 4.378 \\ 4.667 \\ 4.714 \\ 1.000 \end{pmatrix}, \quad \bar{v}^{\text{net}} = \begin{pmatrix} 1.500 \\ 5.565 \\ 2.605 \\ 4.424 \\ 7.108 \\ 1.000 \end{pmatrix}. \tag{2.93}$$

The quantities of integrated value are

$$\tilde{\nu}^{\text{int}} = \begin{pmatrix} 5 \\ 49 \\ 26 \\ 48 \\ 54 \\ 0 \end{pmatrix}, \ \hat{\nu}^{\text{int}} = \begin{pmatrix} 5.000 \\ 4.900 \\ 4.216 \\ 4.571 \\ 4.629 \\ 0.000 \end{pmatrix}, \ \mathring{\nu}^{\text{int}} = \begin{pmatrix} 0.500 \\ 4.000 \\ 3.378 \\ 3.667 \\ 3.714 \\ 0.000 \end{pmatrix}, \ \bar{\nu}^{\text{int}} = \begin{pmatrix} 0.500 \\ 5.465 \\ 2.443 \\ 4.329 \\ 7.023 \\ 0.000 \end{pmatrix}. \quad (2.94)$$

A couple of remarks are in order. But first, to help clarify the discussion, let the nodes belonging to different components of the bow-tie topology be identified accordingly. The set of integer subscripts $\{\mathcal{I}\}$ denotes the IN-nodes. Of the set of IN-nodes, $\{\mathcal{R}\} \subset \{\mathcal{I}\}$ labels the actual root nodes. Nodes in the OUT section have indices $\{\mathcal{O}\}$. Moreover, $\{\mathcal{I}\} \cap \{\mathcal{O}\} = \emptyset$. The subscripts $\{\mathcal{S}\} := \{\mathcal{S}; \mathcal{S} \neq \mathcal{I} \wedge \mathcal{S} \neq \mathcal{O}\}$ denote the remaining nodes in the SCC (ignoring the T&T). The network value and integrated value can thus be symbolically dissected into these components. As an example, $\upsilon^{\text{net}} = \left(\upsilon^{\text{net}}_{\{\mathcal{I}\}}, \upsilon^{\text{net}}_{\{}\mathcal{S}\}, \upsilon^{\text{net}}_{\{\mathcal{O}\}}\right)^t$.

In the example seen in Fig. 2.4, $\{\mathcal{R}\} = \{\mathcal{I}\} = \{1\}$, $\{\mathcal{S}\} = \{2, 3, 4, 5\}$, and $\{\mathcal{O}\} = \{6\}$.

Root Nodes

The problem of root nodes accumulating the sum of the underlying value of downstream nodes as network vale, e.g., $\tilde{\nu}^{\text{int}}_{\mathcal{R}} = \hat{\nu}^{\text{int}}_{\mathcal{R}} = 5.0$, disappears for $\mathring{\nu}^{\text{int}}_{\mathcal{R}} = \bar{\nu}^{\text{int}}_{\mathcal{R}} = 0.5$. This concludes that the analytical and algorithmic solutions presented in this thesis remedy the lamented problem.

Observe also, that for a single root node the corresponding entry in the correction operator is one. Hence $[\mathcal{D}\upsilon]_{\mathcal{R}} = \upsilon_{\mathcal{R}}$. Recalling Eqs. (2.77)–(2.81), this means that $\upsilon^{\text{net}}_{\mathcal{R}} - \tilde{\nu}^{\text{int}}_{\mathcal{R}} = \hat{\upsilon}^{\text{net}}_{\mathcal{R}} - \hat{\nu}^{\text{int}}_{\mathcal{R}} = \mathring{\upsilon}^{\text{net}}_{\mathcal{R}} - \mathring{\nu}\nu^{\text{int}}_{\mathcal{R}} = \bar{\upsilon}^{\text{net}}_{\mathcal{R}} - \bar{\nu}^{\text{int}}_{\mathcal{R}} = [\mathcal{D}\upsilon]_{\mathcal{R}} = \upsilon_{\mathcal{R}}$.

Cycles

For the nodes in the SCC, the large values of $\upsilon^{\text{net}}_{\{\mathcal{S}\}}$ are decreased to $\hat{\upsilon}^{\text{net}}_{\{\mathcal{S}\}} = \mathring{\upsilon}^{\text{net}}_{\{\mathcal{S}\}}$ and $\bar{\upsilon}^{\text{net}}_{\{\mathcal{S}\}}$. As proven in Appendix D.1, the algorithmic network value is equivalent to the corrected one for root nodes. The quantities in $\bar{\upsilon}^{\text{net}}_{\{\mathcal{S}\}}$ are novel corrected network values. Because there is no straight-forward interpretation but only an analytical definition, cf. Eq. (2.77), mathematical consistency alone justifies the existence of this variant of network value. In essence, the original network value υ^{net} can be seen to be progressively transformed into the fully corrected form given by $\bar{\upsilon}^{\text{net}}$, with $\hat{\upsilon}^{\text{net}}$ and $\mathring{\upsilon}^{\text{net}}$ being the intermediate steps.

Note that although the actual numerical value sizes of $\bar{v}^{\text{net}}_{\{\mathcal{S}\}}$ are comparable to those of $\hat{v}^{\text{net}}_{\{\mathcal{S}\}}$, the order of its elements preserves the original order given in $v^{\text{net}}_{\{\mathcal{S}\}}$. Ordering these nodes by descending network value yields the labels 5, 2, 4, 3. This is not the case for $\hat{v}^{\text{net}}_{\{\mathcal{S}\}}$, where the same ordering gives: 2, 5, 4, 3.

For the integrated value, the correspondence of the network value variants in the SCC, i.e.,[6]

$$\hat{v}^{\text{net}}_{\{\mathcal{S}\}} = \hat{\nu}^{\text{int}}_{\{\mathcal{S}\}} + \mathcal{D}|_{\mathcal{S}}\, v_{\mathcal{S}} \tag{2.95a}$$

$$= \overset{\circ}{\nu}{}^{\text{int}}_{\{\mathcal{S}\}} + v_{\mathcal{S}} = \overset{\circ}{v}{}^{\text{net}}_{\{\mathcal{S}\}}, \tag{2.95b}$$

is not retained:

$$\hat{\nu}^{\text{int}}_{\{\mathcal{S}\}} \neq \overset{\circ}{\nu}{}^{\text{int}}_{\{\mathcal{S}\}}. \tag{2.96}$$

By employing the correction operator $\mathcal{D}$ in the computation of $\overset{\circ}{\nu}{}^{\text{int}}_{\{\mathcal{S}\}}$, the relationship would be restored, as a simple rearrangement of Eq. (2.95) reveals

$$\overset{\circ}{v}{}^{\text{net}}_{\{\mathcal{S}\}} - \mathcal{D}|_{\mathcal{S}}\, v_{\mathcal{S}} = \hat{\nu}^{\text{int}}_{\{\mathcal{S}\}}. \tag{2.97}$$

However, for the BFS algorithm, computing $\mathcal{D}$ from Eq. (2.51) would restrict the method's applicability to very large networks, as the inverse of the matrix $(I - W)$ is again the size of the whole network. This makes Eq. (2.97) unsuitable for the definition of integrated value in the BFS algorithm and we are left with the discrepancy[7]: $\hat{\nu}^{\text{int}}_{\{\mathcal{S}\}} - \overset{\circ}{\nu}{}^{\text{int}}_{\{\mathcal{S}\}}$. But overall, as the removal of incoming links to a node i (when constructing the subnetwork of nodes downstream, as the BFS algorithm does) only introduces a deviation to the integrated value and not the network value, this is a minor problem in any case. Especially as the maximal difference between $\hat{\nu}^{\text{int}}$ and $\overset{\circ}{\nu}{}^{\text{int}}$ is bounded. This can been seen as follows: by construction, it is true that

$$\overset{\circ}{\nu}{}^{\text{int}} = \hat{\nu}^{\text{int}} - \text{diag}(\widehat{W})v, \tag{2.98}$$

or in scalar notation

$$\overset{\circ}{\nu}{}^{\text{int}}_i = \hat{\nu}^{\text{int}}_i - \widehat{W}_{ii} v_i = \hat{\nu}^{\text{int}}_i - \frac{V_{ii} - 1}{V_{ii}} v_i, \tag{2.99}$$

recalling Eq. (2.34b). From Eqs. (2.33) and (2.16), noting that per definition $W_{ij} \in [0, 1]$, it follows that

$$V_{ii} = 1 + W_{ii} + [W^2]_{ii} + \cdots \geq 1. \tag{2.100}$$

[6] Recall Eqs. (2.59) and (2.88) or (2.87).

[7] It is not clear if this difference should be understood as an error in the computation or if simply $\overset{\circ}{\nu}{}^{\text{int}}$ is just another legitimate variant to the theme of integrated value.

Hence

$$\hat{\nu}_i^{\text{int}} - \mathring{\nu}_i^{\text{int}} = \left(1 - \frac{1}{V_{ii}}\right) v_i. \tag{2.101}$$

As $(1 - 1/V_{ii}) \in [0, 1]$, the maximal difference of $\mathring{\nu}_i^{\text{int}}$ and $\hat{\nu}_i^{\text{int}}$ is v_i, i.e., as big as or smaller than the underlying value of the node i itself.

A more rigorous derivation yields the exact quantification of the difference in the SCC. From Eq. (2.95) it can be derived that

$$\hat{\nu}_{\{\mathcal{S}\}}^{\text{int}} - \mathring{\nu}_{\{\mathcal{S}\}}^{\text{int}} = v_{\mathcal{S}} - \mathcal{D}|_{\mathcal{S}}\, v_{\mathcal{S}}. \tag{2.102}$$

A final observation is that the difference in the duality relation of Eq. (2.75) is tightly related to the above mentioned quantity:

$$|\widehat{W} v| - |W \hat{v}^{\text{net}}| = |v| - |\mathcal{D} v|, \tag{2.103}$$

where $|v| = \sum_i v_i$ is the norm of a vector v. It is an interesting fact that the failing of the duality relation (due to the implementation of the correction for cycles) can be expressed solely using the intrinsic value and the correction operator. In the example above, $|v| - |\mathcal{D}v| = 3.557$. Moreover, the difference between $\hat{\nu}^{\text{int}}$ and $\mathring{\nu}^{\text{int}}$ is identical to this value for SCC-nodes. This can be seen as justification that $\mathring{\nu}^{\text{int}}$ has an existence in its own right, as it emerges due to the correction.

This concludes the discussion of network value and integrated value in ownership networks, i.e., the different notions of value that can be derived from shareholding relations and a proxy for size or value of firms. The next question to be answered, in the quest to unveil the distribution of economic power worldwide, is: how to compute control from ownership relations? But before the concept of control is introduced in Sect. 2.7, in the next section the methodology presented so far is recast in a different context. It is straightforward to move away from the economic motivation and interpretation driving the above methods towards a general framing valid for generic complex and real-world networks.

2.6 The General Setting: The Notion of Centrality in Networks

In this chapter, the motivation and interpretation for the methodology was primarily given from an economics context. Only in Sect. 2.4 we generalized the concepts to generic networks and discovered the close correspondence between the integrated value and the notion of a quantity flowing in the network. Here we add another complementary point of view coming from centrality measures aiming at identifying the most important nodes of a certain network configuration.

The notion of centrality has a long history in social science as a structural attribute of nodes in a network, that depends on their position in the network (Hubbell 1965; Bonacich 1972; Freeman 1978). Centrality refers to the extent to which a network is centered around a single node. In a star network for example, the central node has the highest centrality, and all other nodes have minimum centrality.

Centrality is a fundamental concept in network analysis (Borgatti and Everett 2006). Recently, there has been a lot of work on centrality in networks in physics and biology (Freeman 2008) next to economics (Schweitzer et al. 2009). Most of the attention has been devoted to the feedback-type centrality which is discussed in the following.

This notions of centrality is based on the idea that a node is more central the more central its neighbors are themselves. The idea leads to a set of equations which need to be solved simultaneously. In general, this type of centrality is also categorized to as eigenvector centrality. The motto "the importance of a node depends on the importance of the neighboring nodes" can be quantified as

$$c_i = \sum_j A_{ij} c_j, \tag{2.104}$$

where A is the adjacency matrix of the graph and c_i denotes the centrality score of node i. In matrix notation

$$c = Ac. \tag{2.105}$$

A solution can be found if the equation is understood as an eigenvector equation

$$\lambda c = Ac. \tag{2.106}$$

The *Bonacich eigenvector centrality* reinterprets Eq. (2.106) in terms of centrality (Bonacich 1972)

$$c_i^B := \frac{1}{\lambda} \sum_j A_{ij} c_j^B, \tag{2.107}$$

with the solution

$$c^B = (\lambda I - A)^{-1} e, \tag{2.108}$$

e being a column vector of ones.

The *Hubbell index* is defined for weighted directed graphs (Hubbell 1965). The nodes can be thought to posses an intrinsic importance c^0, to which the importance from being connected to other nodes is added

$$c^H := Ac^H + c^0, \tag{2.109}$$

The solution is

$$c^H = (I - A)^{-1} c^0, \tag{2.110}$$

which exists if $I - A$ is invertible or equivalently, if there is no eigenvalue of A equal to one, $\lambda_i(A) \neq 1 \forall i$.

α-*Centrality* introduced in Bonacich and Lloyd (2001) is defined as

$$c^\alpha := \alpha A x + c^0, \tag{2.111}$$

the vector c^0 again assigning an initial centrality value and the solution is given by

$$c^\alpha = (I - \alpha A)^{-1} c^0. \tag{2.112}$$

An additional variant of eigenvector centrality is the $c(\alpha, \beta)$-*Centrality* introduced in Bonacich (1987)

$$c_i(\alpha, \beta) := \sum_j (\alpha + \beta c_j) A_{ij}, \tag{2.113}$$

with the solution

$$c(\alpha, \beta) = \alpha (I - \beta A)^{-1} A e, \tag{2.114}$$

e being the column vector of ones.

Comparing c^H and $c^{\alpha=1}$ with v^{net} of Eq. (2.47) uncovers a close similarity of these measures. Moreover, setting $e = v$ in Eq. (2.114), reveals that $c(1, 1)$ corresponds to $\tilde{\nu}^{\text{int}}$ given in Eq. (2.42). Consequently and in general, network value and integrated value should be understood as centrality scores giving the importance of nodes in the network. $\tilde{\nu}^{\text{int}}$ considers the centrality of the neighbors, while v^{net} employs $\tilde{\nu}^{\text{int}}$ plus an intrinsic centrality of the nodes themselves. It should be highlighted that the reinterpretation of network value and network control in terms of flow and centrality of generic networks is a novel contribution of this thesis.

Furthermore, we propose $\bar{v}^{\text{net}}$ and $\bar{\nu}^{\text{int}}$, cf. Eqs. (2.76) and (2.81), as new centrality measures for networks with bow-tie topologies. These novel quantities correct for self-cycles exaggerating the values and also solve the associated problem of root nodes accumulation. In a nutshell, $\widetilde{W}$ should be replaced as

$$\widetilde{W} = (I - W)^{-1} W \qquad \longmapsto \qquad \overline{W} = W \mathcal{D}(\widetilde{W} + I), \tag{2.115}$$

as seen from Table 2.2.

Two final remarks. Firstly, Borgatti (2005) discusses the relationship between centrality and flow. Secondly, in Chap. 5 we present a network-evolution model based on different centrality measures. There we compare and discuss integrated value and network value and compare them with Google's Pagerank centrality, also discussed in Appendix B.7.

2.7 Moving from Ownership to Control

Until now, the methods discussed in this chapter where directly related to ownership networks. The different centrality measures seen in the last section are interpreted as a proxy for the economic value associated with a corporation entangled in a web of ownership relations. But what does all of this have to do with control? Or even more fundamental, what is control in this context?

Ownership is an objective quantity given by the percentage of shares owned in a company. In detail, these percentages of ownership in the equity capital of a firm, also referred to as cash-flow rights, are associated with so-called voting rights. Such votes assigned to the holders can be exercised at shareholder's meetings. The more voting rights a shareholder has in a corporation, the greater the influence that can be exerted over the company, thus the higher the level of control. See also Sect. 1.2.1. There is a great freedom in how corporations are allowed to map cash-flow rights into voting rights assigned to the shareholders (e.g., nonvoting shares, dual classes of shares, multiple voting rights, golden shares, voting-right ceilings, etc.). As a consequence, control can only be estimated. Several models aiming at deriving control based on the knowledge of ownership have been proposed. In this section we discuss these and in the next section introduce a novel model of control.

As in the case of ownership, the presence of the network should also affect the computation of control. In the previous sections, the notion of integrated ownership[8] was introduced in Eq. (2.15). A similar shift from direct to integrated control will conclude the methodology in Sect. 2.9, giving rise to the main theme of this thesis: the flow of control. In an intermediary step, at the end of this section, an alternative method for the propagation of control in a network is first discussed.

In essence, the models of control take the adjacency matrix of the ownership network and transform it into a matrix reflecting the control relations, or symbolically

$$W \quad \longmapsto \quad \mathcal{C}, \tag{2.116}$$

where $\mathcal{C}$ now depends on the chosen model of control.

The "One-Share-One-Vote" Rule

Despite the mentioned freedom in how firms can issue voting rights to the shareholders on the basis of their cash-flow rights, empirical studies indicate that in many countries the corporations tend not to exploit all the opportunities allowed by national laws to skew voting rights. Instead, they adopt the so-called one-share-one-vote principle which states that ownership percentages yield identical percentages of voting rights (La Porta et al. 1999; The Deminor Group 2005; Goergen et al. 2005).

[8] Meaning on the basis of all direct and indirect paths in the network.

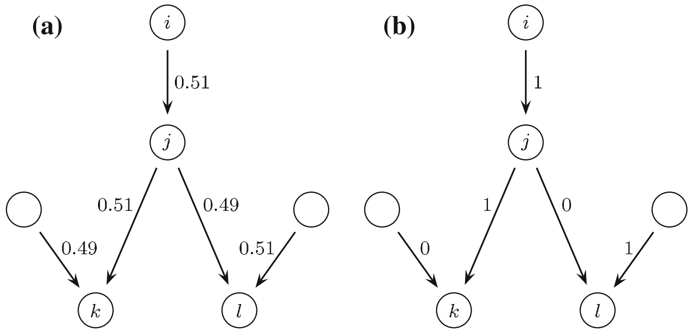

Fig. 2.5 The threshold model (TM) is a simple way of estimating direct control from direct ownership; if the percentages of ownership seen in (**a**) exceed a certain threshold, taken to be 50 % here, the shareholder gets assigned full control (**b**)

According to this linear model (LM), there is no deviation between ownership and control, thus the direct control matrix coincides with the direct ownership matrix and Eq. (2.116) reveals the simple relation

$$\mathcal{C}_{ij}^{LM} := W_{ij}. \tag{2.117}$$

The Threshold Model

The simple linear model of the last section overlooks one special trait of control, namely that it is often binary. As an example, holding over 50 % of the votes ensures that one has full or incontestable control. This feature is considered in the threshold model (TM), also referred to as majority model. Various values for the fixed threshold of absolute control have been proposed: 10–20 % (La Porta et al. 1999), next to a more conservative value of 50 % (Chapelle and Szafarz 2005).

For the control matrix $\mathcal{C}$ one finds

$$\mathcal{C}_{ij}^{TM} := \begin{cases} 1, \text{ if } W_{ij} > \vartheta, \\ 0, \text{ if } \exists k \neq i : W_{kj} > \vartheta, \\ W_{ij}, \text{ otherwise.} \end{cases} \tag{2.118}$$

where ϑ is the chosen threshold value. See Fig. 2.5 for an example illustrating the concept.

The Control Value

The notion of the shareholders portfolio value was introduced in Eq. (1.2) of Sect. 1.1.3. It is a measure of the value gained from the direct ownership relations.

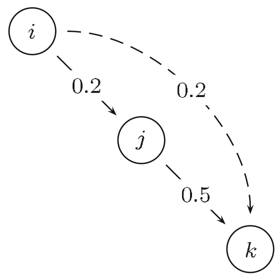

Fig. 2.6 Application of the weakest link model: the control of company i over company k is given by the minimum percentage of control along the path of indirect control, i.e., 20 %

In a similar vein, this measure can be easily extended to reflect and incorporate the notions of control. By replacing W_{ij} in Eq. (1.2) with a control matrix $\mathcal{C}_{ij}$, i.e., by symbolically applying Eq. (2.116), we introduce the *control value*:

$$c_i := \sum_{j=1}^{k_i^{out}} \mathcal{C}_{ij} v_j. \tag{2.119}$$

It is a measure of the economic value shareholder i can control considering its direct ownership shares.

Indirect Control

The two simple models presented above are examples of direct control estimations. However, how does control propagate along the links in a network?

The "weakest link" model proposes a measure of control which selects the weakest relation in a chain of control (Claessens and Djankov 2000). As seen in Fig. 2.6, if i owns 20 % of the votes of firm j and j owns 50 % of the votes of k, the weakest link rule assigns to i a control over k of 20 %, the minimum between 20 % and 50 %. The intuition is given by the fact that an outsider can gain control of firm k by acquiring a controlling stake of 20 % over firm j.

However, this methodology is not able to measure control in the case of complex ownership structures, such as cross-shareholdings. It is not clear how to adapt the method in light of the recursive nature of cross-shareholding relations. Moreover, for very long chains of indirect ownership, the weakest link model can overestimate the control the first firm in the chain has over the last one.

Another model for estimating control in networks is given by applying the integrated model to the control matrix $\mathcal{C}$. In the literature, however, the integrated model has nearly exclusively been applied to ownership adjacency matrices (Brioschi et al. 1989; Flath 1992; Baldone et al. 1998; Chapelle 2005). An exception being (Chapelle and Szafarz 2005), defining integrated control based on the TM. We can use Fig. 2.5 again as an example to highlight how this mechanism works. The shareholdings W_{ij}

get transformed into $\mathcal{C}_{ij}^{TM}$ according to Eq. (2.118). Then, for instance, the indirect control form $i \to k$ is simply given by the multiplication $\mathcal{C}_{ij}^{TM}\mathcal{C}_{jk}^{TM}$, in analogy to the example seen in Fig. 2.1a on p. 24. In other words, i has full control over k via j. In general, this means that, for example by ignoring the problems of cycles,

$$\widetilde{\mathcal{C}}^{TM} := (I - \mathcal{C}^{TM})^{-1}\mathcal{C}^{TM}, \tag{2.120}$$

employing Eq. (2.15). For the LM, the procedure is equivalent.

The notion of integrated control, in analogy to the integrated value defined in Eq. (2.42), is detailed in Sect. 2.9, after the introduction of of a new model of control in the next section.

2.8 The Relative Majority Model of Control

In the following, we introduce a new model for estimating control from ownership relations, extending the list of existing ones presented in the last section.

There is a very general problem plaguing the two models described in Sect. 2.7. Namely, that shareholders do not only act as individuals but can collaborate in shareholding coalitions that give rise to so-called voting blocks. The theory of political voting games in cooperative game theory has been applied to the problem of shareholder voting. There are four proposed ways to measure control in a relative manner.

The *fixed rule* simply classifies the degrees of control according to fixed thresholds of the leading shareholdings (Leech and Leahy 1991).

The *Herfindahl index*, or H-index, was originally used in economics as a standard measure of market concentration (Herfindahl 1959)

$$\mathcal{H} := \sum_i w_i^2, \tag{2.121}$$

where w_i are some sort of market shares. It has been employed as a measure of how concentrated or dispersed ownership is (where w_i are now the shareholdings of a specific firm) (Cubbin and Leech 1983; Demsetz and Lehn 1985; Leech 1988; Leech and Leahy 1991).

The so-called *power indices* were originally introduced as the Shapley-Shubik index (SS-index) by Shapley and Shubik (1954) and famously extended by Banzhaf (1965) to the Banzhaf index (B-index). They come from the notion of a weighted majority in cooperative games and measure the extent to which shareholders are pivotal to the success in potential voting pacts. Power indices measure the relative influence of shareholder over decision making. The SS-index considers orderings of N players (permutations) while the B-index counts coalitions. Both give essentially similar results and use a majority rule (Prigge 2007). They provide a continuous variable which is connected to the share in votes in a non-linear manner. Selected empirical studies were done by Leech (1988), Crama et al. (2003), Chen (2004), Prigge (2007).

However, the employment of these game theoretic power indices for measuring shareholder voting behavior has failed to find widespread acceptance in the corporate finance literature for estimating control. Mainly due to computational, inconsistency and conceptual issues. First of all, there is an ambiguity with the definition of "power" (Prigge 2007). Secondly, when voters have varying weights, the results of the main two power indices and their variants all yield different results (Leech 2002a,b). Thirdly, the stock of empirical studies is rather small, and the few results are inconclusive (Prigge 2007). Fourthly, for a large number of agents, the computational demands become challenging (Leech 2002a,b). And finally, the notion of integrated ownership is also ignored.

The so-called *degree of control*, or α, was introduced as a probabilistic voting model measuring the degree of control of a block of large shareholdings as the probability of it attracting majority support in a voting game (Cubbin and Leech 1983; Leech 1987a,b; Leech and Leahy 1991). Without going into details, the idea is as follows. Consider a shareholder i with ownership W_{ij} in the company j. Then the control of i depends not only on the value in absolute terms of W_{ij}, but also on how dispersed the remaining shares are (measured by the Herfindahl index). The more they tend to be dispersed, the higher the value of α. So even a shareholder with a small W_{ij} can obtain a high degree of control. The assumptions underlying this probabilistic voting model correspond to those behind the power indices (Cubbin and Leech 1983; Leech 1987a,b; Leech and Leahy 1991; Chen 2004). It relates to the SS-index by treating all permutations as equally likely and to the B-index by treating all coalitions as equiprobable (Leech 1987b). The degree of control is closely related to a measure of a priori voting power defined for weighted voting games (Leech 1988).

However, α suffers from drawbacks. It gives a minimum cutoff value of 0.5 (even for arbitrarily small shareholdings, see also Appendix D.2) and hence Eq. (1.1) is violated, meaning that it cannot be utilized in an integrated model. The computation of α can become intractable in situations with many shareholders.

Having listed these issues, we present a minimal list of requirements a reasonable model of control should fulfill:

1. Define a mapping from $F : (0, 1]^N \to (0, 1]^N$, for the N shareholding relations $\{W_{ij}\}$, where $F_1(\{W_{ij}\}), \dots, F_N(\{W_{ij}\})$ represent control and take on continuous values.
2. Be extendable to an integrated model.
3. Sum to one for each firm, as $\sum_j W_{ij}$ in principle does.
4. Emulate the behavior of α for large shareholders (coalitions and voting blocks).
5. Have an intuitive meaning of controlling power.
6. Be feasible to compute on large networks.

In the following section, we introduce our new model.

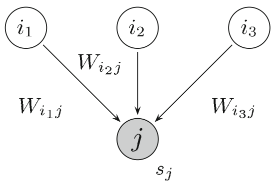

Fig. 2.7 Definition of the concentration index s_j, measuring the number of prominent incoming edges, respectively the effective number of shareholders of the company j; when all the weights are equal, then $s_j = k_j^{in}$, where k_j^{in} is the in-degree of vertex j; when one weight is overwhelmingly larger than the others, the concentration index approaches the value one, meaning that there exists a single dominant shareholder of j

2.8.1 Extending the Notions of Degree for Weighted Networks

Although the concepts about to be introduced here were motivated as being related to the separation of ownership and control in economics, they are best understood in the context of pure network theory.

In this paradigm, we substantiate the idea of the 3-Level network analysis mentioned in Sect. 1.1. To recall, complex real-world networks can be understood at three levels of resolution: the topological, with weighted and directed links, and by assigning non-topological sate variables to the nodes.

The following measures can be understood as Level 2 quantities, extending previous Level 1 network notions. Namely, the degree and strength, explained in Appendix B.2.

These quantities are used in Chap. 3 for the extraction of the backbone (see more in Sect. 3.3.2). Some empirical distributions are given in Figs. 3.2 and 3.3 of Sect. 3.3.3.

In-Degree

When there are no weights associated with the edges, we expect all edges to count the same. If weights have a large variance, some edges will be more important than others. A way of measuring the number of prominent incoming edges is to define the *concentration index* (Battiston 2004) as follows:

$$s_j := \frac{\left(\sum_{i=1}^{k_j^{in}} W_{ij}\right)^2}{\sum_{i=1}^{k_j^{in}} W_{ij}^2}. \tag{2.122}$$

If the equality in Eq. (1.1) holds, the numerator will be equal to one. Observe that this quantity is akin to the inverse of the Herfindahl index of Eq. (2.121). Notably, a similar measure has also been used in statistical physics as an order parameter (Derrida and Flyvbjerg 1986). A recent study (Serrano et al. 2009) employs a Herfindahl index in their backbone extraction method for weighted directed networks (where, however, the nodes hold no non-topological information). In the context of ownership networks, s_j is interpreted as the effective number of shareholders of the firm j, as explained in Fig. 2.7. Thus it can be understood as a measure of control from the point of view of a company.

Out-Degree

The second quantity to be introduced measures the number of important outgoing edges of the vertices. For a given vertex i, with a destination vertex j, we first define a measure which reflects the importance of i with respect to all vertices connecting to j:

$$H_{ij} := \frac{W_{ij}^2}{\sum_{l=1}^{k_j^{in}} W_{lj}^2}. \tag{2.123}$$

This quantity has values in the interval (0, 1]. For instance, if $H_{ij} \approx 1$ then i is by far the most important source vertex for the vertex j. For our ownership network, H_{ij} represents the *fraction of control* (Battiston 2004) shareholder i has on the company j. As shown in Fig. 2.8, this quantity is a way of measuring how important the outgoing edges of a node i are with respect to its neighbors' neighbors. For an interpretation of H_{ij} from an economics point of view, consult the following section.

From this, we then define the *control index*:

$$h_i := \sum_{j=1}^{k_i^{out}} H_{ij}. \tag{2.124}$$

Within the ownership network setting, h_i is interpreted as the effective number of stocks controlled by shareholder i. In essence, s and h replace the in- and out-degree in the case of weighted and directed networks.

2.8.2 Interpretation as a New Model of Control

The definition of the fraction of control H_{ij} given in Eq. (2.123), can be understood as yielding a new non-linear control model, that lies between the linear mapping given by the LM, cf. Eq. (2.117), and the digital threshold-driven TM, seen in Eq. (2.118). As this model assigns control based on the relative fraction of ownership shares that each shareholder has, it is called the *relative majority model of control*, or simply the

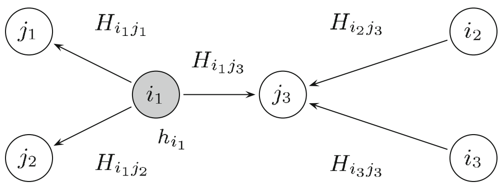

Fig. 2.8 The definition of the control index h_i, measuring the number of prominent outgoing edges; in the context of ownership networks this value represents the effective number of firms that are controlled by shareholder i; note that to obtain such a measure, we have to consider the fraction of control H_{ij}, which is a model of how ownership can be mapped to control (see the discussion in Sect. 2.8.2)

relative model (RM). In other words

$$\mathcal{C}_{ij}^{RM} := H_{ij}. \tag{2.125}$$

In summary, our quantity H_{ij} adheres to the small catalogue of desired features presented in the list on p. 56. It holds that $\sum_j H_{ij} = 1$, for all firms j. In effect, any shareholder gaining control will be offset by shareholders loosing control. As a result, this measure of control can also be used as an integrated model, by applying Eq. (2.15) to yield $\widetilde{H}_{ij}$. For large shareholders, the analytical expressions of H_{ij} and α share very similar behavior, as detailed in Appendix D.2. This means that to some extent our measure of control can take possible strategic alliances of shareholders into account without requiring the knowledge of data on voting blocks. There is an intuitive meaning of power associated with our model: how important is a shareholder with respect to all other shareholders, or what is the relative voting power of a shareholder considering the dispersion of the rest of the votes? We are able to compute $\widetilde{H}_{ij}$ for every shareholder in the sample without facing any computational restrictions. To summarize, the properties of our model make a sensible ranking of all shareholders according to their controlling power possible.

This concludes that the new measure of control merges crucial insights from the corporate finance literature and the game theoretic approach to voting while addressing their mentioned shortcomings. It should also be noted, that s_j represents the complementary of h_i: while the latter represents the control seen from the point of view of the shareholders, the former reflects the control seen by the firms.

2.9 Computing the Flow of Control in an Ownership Network

In this section we conclude the methodology chapter. To summarize, the existing notion of integrated ownership, considering all direct and indirect paths of ownership, was generalized for generic ownership networks: $\widetilde{W}$, given in Eq. (2.15). This allowed the introduction of the integrated (portfolio) value $\tilde{\nu}^{\text{int}}$, defined in Eq. (2.42). The associated notion of network value v^{net} reflects the integrated value of a firm plus its own intrinsic or underlying value, as seen in Eq. (2.49). These measures capture how the value of firms flow in a network of ownership relations, as described in Sect. 2.4.1.

It was shown in Sect. 2.2.4 that $\tilde{\nu}^{\text{int}}$ and v^{net} faced problems when applied to networks with a bow-tie structure, because of the presence of cross-shareholding relations in the SCC. In the literature $\widehat{W}$ was proposed as a remedy, seen in Eq. (2.32). We fully implemented this correction by introducing $\hat{\nu}^{\text{int}}$ and $\hat{v}^{\text{net}}$, given in Eqs. (2.56) and (2.59).

In a next step, an additional problem of these novel measures was identified, see Sect. 2.3.5. In Sect. 2.5 we proposed two solutions, an algorithmic and an analytical one. As a result, $\bar{\nu}^{\text{int}}$ and $\bar{v}^{\text{net}}$ were defined in Eqs. (2.76) and (2.81). As well as $\mathring{\nu}^{\text{int}}$ and $\mathring{v}^{\text{net}}$ given in Eqs. (2.90) and (2.89).

The context given by the economic nature of the concepts was extended by noting the relation to centrality measures in networks, as described in Sect. 2.6. And returning to an economics setting, the notion of control, which can be derived from the knowledge of the ownership relations, was introduced in Sect. 2.7. This resulted in the two definitions of the matrix of control $\mathcal{C}$, depending on the chosen model of control, seen in Eqs. (2.117) and (2.118).

We introduced a new model of relative direct control incorporating ideas originating in game theory in Sect. 2.8. It was noted how this new measure, defined in Eq. (2.123), can also be viewed as a pure network theoretic quantity extending the idea of degree for weighted and directed networks. In other words, it is a Level 2 measure.

Putting everything together, we arrive at a way to estimate corporate control in ownership networks. Parts of these methods were first published in Glattfelder and Battiston (2009) and Vitali et al. (2011).

Let $\mathcal{C}$ be a control matrix based on one of the three models of direct control: LM, TM and RM. I.e., $\mathcal{C} \in \{\mathcal{C}^{LM}, \mathcal{C}^{TM}, \mathcal{C}^{RM}\} = \{W, \mathcal{C}^{TM}, H\}$, recalling Eqs. (2.117), (2.118) and (2.125), respectively (2.123).

From this one can define *integrated control*, in analogy to integrated value, as

$$\varsigma^{\text{int}} := \dot{\mathcal{C}} v, \tag{2.126}$$

with v_i being the intrinsic value of the firm i and the symbol "˙" acting as a placeholder for "˜", "ˆ" or "¯", i.e., the chosen integrated model. Recall Table 2.2 for a summary of the corresponding definitions. In other words, integrated control measures the economic value a shareholder can control taking into account the network of firms in which it has direct and indirect shares (a Level 2 quantity). In addition, this last piece of the puzzle is in fact also a true Level 3 network measure. This means that it incorporates all the available information of the complex network under study: the weights and direction of links and (a proxy of) the intrinsic value or size firms, the non-topological state variable.

Finally, network value finds its correspondence in the so-called *network control*, defined as

$$c^{\text{net}} := \tilde{\varsigma}^{\text{int}} + v, \tag{2.127}$$

$$\hat{c}^{\text{net}} := \hat{\varsigma}^{\text{int}} + \mathcal{D}v, \tag{2.128}$$

$$\bar{c}^{\text{net}} := \bar{\varsigma}^{\text{int}} + \mathcal{D}v. \tag{2.129}$$

The network control of an economic actor is given by its intrinsic value plus the controlled value gained from the integrated control. For the algorithm described in Sect. 2.5.2, resulting in the integrated value and network value introduced in Eqs. (2.89) and (2.90) of Sect. 2.5.3, the corresponding analytical measures for control are

$$\overset{\circ}{\varsigma}^{\text{int}} = \overset{\circ}{c}^{\text{net}} - v, \tag{2.130}$$

where the algorithm computes $\overset{\circ}{c}^{\text{net}}$. The real-world meaning of these measures is discussed in Sect. 6.1, especially Sects. 6.1.8 and 6.1.3 offering an interpretation of integrated control in terms of potential power.

The knowledge of network control or integrated control can answer two questions:

1. Who are the most important economic actors in terms of control, ranked in descending order?
2. How is control distributed in the network?

In order to tackle the second question, it is necessary to find a way to measure the concentration of control. Such a concept is introduced in the following section.

2.10 Measuring the Concentration of Control

One last method needs to be introduced in order to round off this chapter. It is a general procedure for which the concentration of a random variable $X > 0$, drawn for all members of a given population, can be assessed. Figure 2.9 shows some possible examples of the distribution of X.

The methodology is similar to the construction of the Lorenz curve, uncovering the distribution of value in a market. In economics, the Lorenz curve gives a graphical representation of the cumulative distribution function (CDF) of a probability distribution. It is often used to represent income distributions, where the x-axis ranks the poorest x % of households and relates them to a percentage value of income on the y-axis.

In our version, we invert the ordering on the x-axis and rank the shareholders according to their importance, as measured by network or integrated control, and report the fraction they represent with respect to the whole set of shareholder. The y-axis shows the corresponding percentage of controlled market value. In detail, we relate the fraction of shareholders to the fraction of the total value they collectively represent.

In generic terms, the population, taken to be comprised of N individuals, is sorted by decreasing X_i values. Without loss of generality, the individuals are labelled with increasing indices. The total amount distributed in this population is given by

$$X_{tot} := \sum_{i=1}^{N} X_i. \tag{2.131}$$

The individual with the highest value of the random variable has X_1/X_{tot} percent of the total and represents $1/N$ percentage of the population. This corresponds to the first data point in the lower left-hand corner of the plots in Fig. 2.10. Similarly, the top right-hand corner of the diagrams represent 100 % of the population making up 100 % of the total. The concentration of X is thus defined as the set of data points (η, ϑ), with

$$\eta(n) := \frac{n}{N}, \tag{2.132}$$

and

$$\vartheta(n) := \frac{1}{X_{tot}} \sum_{i=1}^{n} X_n, \tag{2.133}$$

$n \in [1, N]$.

In order to understand what this method reveals, we compare some general probability distributions and the level of concentration they are associated with. We choose the following families of probability density functions (PDF)

$$\mathcal{P}_{\mathrm{sl}}(X; C, \alpha) := C X^{-\alpha}, \tag{2.134}$$

$$\mathcal{P}_{\exp}(X; \lambda) := \lambda e^{-\lambda X}, \tag{2.135}$$

$$\mathcal{P}_{\ln}(X; \mu, \sigma) := \frac{1}{X\sigma\sqrt{2\pi}} e^{-\frac{(\ln X - \mu)^2}{2\sigma^2}}. \tag{2.136}$$

In other words, P_{sl}, $P_{\ln}$ and $P_{\exp}$ describe scaling-law, log-normal and exponential distributions, respectively.

In Fig. 2.9, three scaling laws ($\alpha = 0.7$, $\alpha = 1.0$, $\alpha = 1.5$), one exponential ($\lambda = 0.02$) and two log-normal distributions ($(\mu, \sigma) = (3, 0.6)$, $(\mu, \sigma) = (9.1, 0.2.4)$)

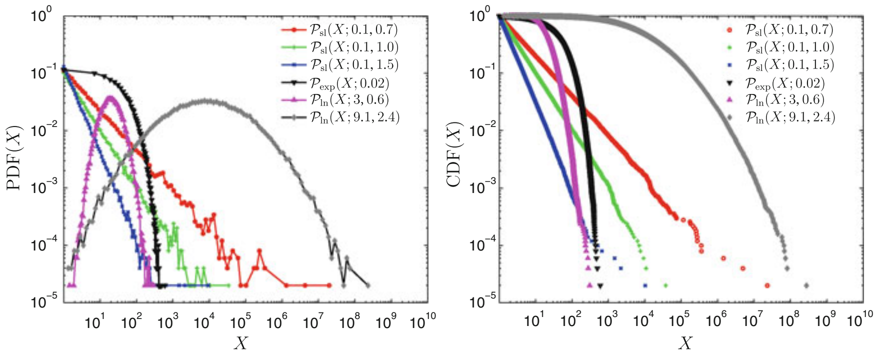

Fig. 2.9 Plotting the PDFs (*left*) and CDFs (*right*) of the probability distributions given in Eqs. (2.134)–(2.136) for a random variable X; the plots are in log-log scale

are shown. The corresponding concentration is seen in Fig. 2.10. The semi-log scale representation in the bottom panel reveals the clearest picture. The scaling-law distribution with $\alpha = 0.7$ yields the highest concentration. The most important individual has a very large fraction of over 60 % of the total. It is an interesting observation, that the two remaining scaling-law distributions result in much lower concentration. Surprisingly, a log-normal distribution with a wide range of X_i, as given by $(\mu, \sigma) = (9.1, 0.2.4)$, is more concentrated than the scaling law with $\alpha = 1.5$ for nearly the whole range.

To summarize, the measure of concentration we propose is not only sensitive to the tail of the probability distribution, but also the relative distribution of mid-range values matters.

The cross-country analysis of Chap. 3 employs a variant of the above described procedure, called cumulative control. The diagram is discussed in Sect. 3.4.1. In the empirical analysis of Chap. 4, Sect. 4.3.3 shows the control distribution for the global network of corporations.

2.11 A Brief Summary

In the existing literature, only the notions of integrated ownership ($\widetilde{W}$ and $\widehat{W}$) and network (or group) value v^{net} were introduced. The novel ideas presented in this thesis are the following. From an economics perspective:

1. the introduction of integrated value or control ($\tilde{\nu}^{\text{int}}, \hat{\nu}^{\text{int}}, \bar{\nu}^{\text{int}}, \tilde{\varsigma}^{\text{int}}, \hat{\varsigma}^{\text{int}}, \bar{\varsigma}^{\text{int}}, \mathring{\varsigma}^{\text{int}}$);
2. the idea of network control c^{net};
3. the corrected network value and control ($\hat{v}^{\text{net}}, \bar{v}^{\text{net}}, \hat{c}^{\text{net}}, \bar{c}^{\text{net}}, \mathring{c}^{\text{net}}$);
4. the relative model of control H_{ij};

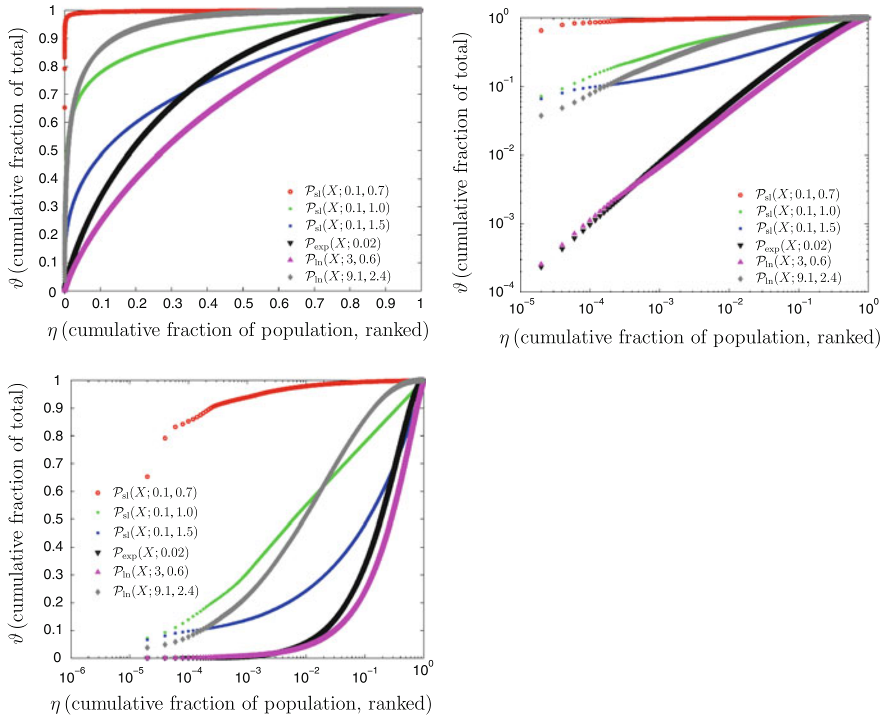

Fig. 2.10 Concentration of the random variable X resulting from the probability distributions shown in Fig. 2.9; the three plots are all identical, but displayed with different scales: (*top left panel*) linear plot, (*top right panel*) log-log plot and (*bottom panel*) semi-log plot; the construction of these curves is similar to the Lorenz curve used in economics, as described in the main text

5. the method to measure the concentration of a random variable;
6. the connection between network value or control, integrated value or control and the underlying value, e.g, $v^{\text{net}} = \tilde{\nu}^{\text{int}} + v$;
7. the identification of the correction operator $\mathcal{D}$.

From a complex-networks perspective:

1. the generalization of the methodology in terms of flow;
2. the interpretation of the methods as centrality measures;
3. the explanation of integrated value or control as true Level 3 quantities, incorporating the weights and direction of links next to non-topological state variables.

Having set aside all the required tools in order to estimate the flow of control in ownership networks, in Chaps. 3 and 4, these methods will be applied in two different empirical studies. The analysis uncovers important patterns and unveils the structural organization of ownership networks.

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