Chapter 2
Radiation of an Accelerated Charge

Whatever the energy source and whatever the object, (but with the notable exception of neutrino emission that we will not consider further, and that of gravitational wave emission that we will discuss in Chap. 15), all objects lose energy, or expressed equivalently they cool, through the emission of electromagnetic radiation. The processes at the origin of the radiation differ widely, but all are based on the fact that accelerated electric charges radiate. What differs in the various emission processes is the force at the origin of the acceleration and hence the acceleration as a function of time, $a(t)$. We therefore start here by understanding how an accelerated charge radiates. The following chapters will take the results we obtain here and apply them to the different acceleration mechanisms we meet in astrophysical situations.

The path that leads from Maxwell's equations to the solution relevant in the case of the radiation field generated by an accelerated non-relativistic charge is somewhat arduous. It can be followed in your preferred electrodynamics course or in Feynman's lectures, and will not be given here. The essential insights can, however, be understood from an argument of J.J. Thomson as presented in Longair (1992).

2.1 Energy Loss by a Non-relativistic Accelerated Charge

Consider a charge at the origin of an inertial system at $t = 0$. Imagine then that the source is accelerated to a small velocity (small compared to the velocity of light c, this discussion is non-relativistic) $\Delta \mathrm{v}$ in a time interval Δt. Draw the electric field lines that result from this arrangement at a time t. At a distance large compared to the displacements $\Delta \mathrm{v} \times \Delta t$ of the charge, the field lines are radial and centred on the origin of the inertial system, because the signal that a perturbation has occurred to the charge has not yet had the time to reach there. At smaller distances, however, the lines are radial around the new position of the source. In between, the lines are connected in a non radial way in a small perturbed zone of width $c \times \Delta t$. Note that this presupposes that the solution describes a signal that moves with the velocity of light, one of the features that this simplified treatment does not demonstrate.

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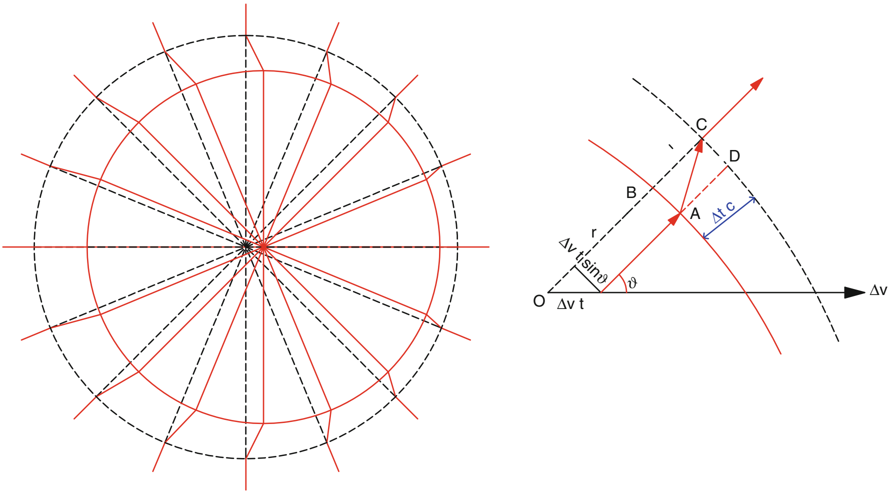

Fig. 2.1 Schematical view of the electric field lines at time t due to a charged particle accelerated to a velocity $\Delta \mathrm{v} \ll c$ in a time interval Δt (Adapted from Longair (1992))

Figure 2.1 gives the large picture and the detail of the perturbed field lines.

In this section we will denote electric fields by $\mathscr{E}$ in order to distinguish them from the energy, denoted E. You can read from Fig. 2.1 that the ratio of the tangential to the radial field line components in the perturbed zone is

$$\frac{\mathscr{E}_\theta}{\mathscr{E}_r} = \frac{\Delta \mathrm{v} \cdot t \sin\theta}{c \Delta t}. \tag{2.1}$$

The radial field is given by the Coulomb law

$$\mathscr{E}_r = \frac{e}{r^2}, \quad e \text{ in e.s.u., } r = ct. \tag{2.2}$$

You can therefore deduce the tangential field component and find

$$\mathscr{E}_\theta = e \cdot \frac{\Delta \mathrm{v}}{\Delta t} \sin\theta \frac{1}{cr^2} \cdot t \tag{2.3}$$

$$= e \frac{\ddot{r} \sin\theta}{c^2 r}. \tag{2.4}$$

Note that this field depends on the distance to the centre as r^{-1} rather than r^{-2}. This is a characteristics of the radiation field in the far zone. The only electrical field component that is relevant for radiation is that which is perpendicular to the direction of propagation, i.e. $\mathscr{E}_\theta$. It is the one we consider further here.

Introducing the electrical dipole moment $p = e \cdot r$, we write

$$\mathscr{E}_\theta = \frac{\ddot{p}\sin\theta}{c^2 r}. \tag{2.5}$$

We may now calculate the energy flux carried by this disturbance. The energy flux transported by electromagnet fields is given by the Poynting vector $\boldsymbol{S}$:

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}. \tag{2.6}$$

The magnetic field is equal and perpendicular to the electric field in electromagnetic radiation:

$$\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E} \tag{2.7}$$

Using (2.5) the energy loss in the direction θ in a solid angle $\mathrm{d}\Omega$, $\frac{\mathrm{d}E}{\mathrm{d}t}\mathrm{d}\Omega = |\boldsymbol{S}|r^2\,\mathrm{d}\Omega$, is therefore

$$\frac{\mathrm{d}E}{\mathrm{d}t}\mathrm{d}\Omega = \frac{c}{4\pi}\frac{|\ddot{p}|^2\sin^2\theta}{c^4 r^2}\cdot r^2\mathrm{d}\Omega. \tag{2.8}$$

In order to find the energy loss from the charge, one needs to integrate (2.8) over the solid angle $d\Omega$. The configuration is cylindrically symmetrical around the direction of the acceleration. The integration over one angle is therefore trivial and $d\Omega = 2\pi\sin\theta d\theta$. The result is

$$\left|\frac{\mathrm{d}E}{\mathrm{d}t}\right| = \frac{c}{4\pi}\frac{|\ddot{p}|^2}{c^4}\int_0^\pi 2\pi\sin^3\theta\,\mathrm{d}\theta = \frac{2}{3}\frac{|\ddot{p}|^2}{c^3} \tag{2.9}$$

This is the so-called Larmor formula. It is given here in Gaussian units and gives the energy carried by the electromagnetic radiation emitted by an accelerated charge as a function of this acceleration. The radiation is dipolar (see the $\sin^2\theta$ in (2.8)). The absolute value is there to remind us that the sign will be different whether one considers the energy loss from the charge or the gain in the radiation.

2.2 Spectrum of the Radiation

One may use the results we have obtained for the energy radiated by an accelerated charge to calculate the spectrum of the emitted radiation. This is done by considering the Fourier transform of the dipole, and calculating from there that of the electric field and of the energy flux. The Fourier transform of the dipole $p(t)$ is given by

$$p(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\hat{p}(\omega)\,\mathrm{d}\omega. \tag{2.10}$$

Remember that the Fourier transform of the second time derivative of a function is given by

$$\ddot{p}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t} \hat{p}(\omega)\, \mathrm{d}\omega. \tag{2.11}$$

Writing the definition of the transform of the electric field on one side and taking the Fourier transform of (2.5) on the other side, one obtains using (2.11)

$$\mathscr{E}_\theta(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \hat{\mathscr{E}}(\omega)\, \mathrm{d}\omega \tag{2.12}$$

$$= -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t} \hat{p}(\omega) \frac{\sin\theta}{c^2 r}\, \mathrm{d}\omega, \tag{2.13}$$

from which we read the following expression for the Fourier transform of the electric field

$$\hat{\mathscr{E}}(\omega) = -\omega^2 \hat{p}(\omega) \frac{\sin\theta}{c^2 r}. \tag{2.14}$$

Integrating the energy loss (2.9) over time one finds the energy that crosses a surface per surface element dA

$$\frac{\mathrm{d}E}{\mathrm{d}A} = \int_{-\infty}^{\infty} \text{energy flux} \cdot \mathrm{d}t = \int_{-\infty}^{\infty} \frac{c}{4\pi} \mathscr{E}^2(t)\, \mathrm{d}t \tag{2.15}$$

where we have used the fact that the energy flux is given by the Poynting vector (2.6).

From the theory of Fourier transforms we use

$$\int_{-\infty}^{\infty} \mathscr{E}^2(t)\, \mathrm{d}t = 2\pi \int_{-\infty}^{\infty} |\hat{\mathscr{E}}(\omega)|^2\, \mathrm{d}\omega = 4\pi \int_0^{\infty} |\hat{\mathscr{E}}(\omega)|^2\, \mathrm{d}\omega \tag{2.16}$$

and therefore

$$\frac{\mathrm{d}E}{\mathrm{d}A} = c \int_0^{\infty} |\hat{\mathscr{E}}(\omega)|^2\, \mathrm{d}\omega \tag{2.17}$$

giving finally the emitted spectrum

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \int c |\hat{\mathscr{E}}(\omega)|^2\, \mathrm{d}A \tag{2.18}$$

$$\overset{(2.14)}{=} \int c \frac{\omega^4 |\hat{p}(\omega) \sin\theta|^2}{c^4 r^2}\, \mathrm{d}A \tag{2.19}$$

$$= \frac{8\pi}{3} \frac{\omega^4}{c^3} |\hat{p}(\omega)|^2 \tag{2.20}$$

This shows that in a non-relativistic approximation (remember that we assumed Δv to be small compared to the velocity of light) the spectrum is proportional to the square of the Fourier transform of the dipole moment.

2.3 Radiation of a Relativistic Accelerated Particle

Not all the radiating charges that we will meet in this book are non-relativistic. We will see that, often, very high energy particles must be present in order to explain the observed radiation. It is therefore also necessary to know how relativistically-moving charges radiate. In order to approach this question we introduce the basic elements of special relativity, which we will use whenever appropriate. We do not give a presentation of special relativity here, rather, we recall those elements that we need for the derivation. We will introduce further elements as they become necessary in the following chapters.

In special relativity one considers the flat metric of four-dimensional space time

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2, \tag{2.21}$$

which describes the distance between two events in space time. This distance is invariant under the Lorentz transformations

$$t' = \gamma\left(t - \frac{\mathrm{v}}{c^2}x\right), x' = \gamma(x - \mathrm{v}t), y' = y, z' = z, \tag{2.22}$$

where $\gamma = \sqrt{1-\beta^2}^{-1}$ is the usual gamma factor, $\beta = \frac{\mathrm{v}}{c}$, and v is the relative velocity of the reference frames along the x-axis.

We next introduce the four-velocity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \tag{2.23}$$

which, as a small difference between coordinates, is a vector. Written explicitly

$$u^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \cdot c, \tag{2.24}$$

because

$$(\frac{\mathrm{d}\tau}{\mathrm{d}t})^2 = (\mathrm{d}t^2 - \frac{1}{c^2}\mathrm{d}\boldsymbol{x}^2)/\mathrm{d}t^2 = 1 - \frac{\mathrm{v}^2}{c^2} = \frac{1}{\gamma^2}. \tag{2.25}$$

Similarly

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \gamma \cdot \mathbf{v}, \tag{2.26}$$

as

$$(\frac{\mathrm{d}\tau}{\mathrm{d}\boldsymbol{x}})^2 = (\mathrm{d}t^2 - \frac{1}{c^2}\mathrm{d}\boldsymbol{x}^2)/\mathrm{d}\boldsymbol{x}^2 = (\frac{1}{\mathrm{v}^2} - \frac{1}{c^2})\boldsymbol{e}_\mathrm{v} = \frac{1}{\mathrm{v}^2}\frac{1}{\gamma^2}\boldsymbol{e}_\mathrm{v}, \tag{2.27}$$

where $\boldsymbol{e}_\mathrm{v}$ is a unit vector in the direction of the velocity. Note that we will write $\mathbf{v}$ to mean three-velocity, and in general bold italics vectors are three dimensional while bold upright vectors are four dimensional.

The acceleration is the second proper time derivative of the coordinates

$$a^{\mu} = \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau}, \tag{2.28}$$

which from (2.24) and (2.26) gives

$$a^0 = c \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \tag{2.29}$$

$$a^i = \frac{\mathrm{d}(\gamma \cdot \mathrm{v}^i)}{\mathrm{d}\tau}. \tag{2.30}$$

In order to generalise these results to describe the radiation of a non-relativistic accelerated charge to relativistic charges, we write (2.9) in an explicitly covariant form, i.e. in a form that is explicitly invariant under Lorentz transformations. In the system in which the particle is at rest, we have

$$\gamma = 1, \mathrm{d}\tau = \mathrm{d}t \Rightarrow a^0 = 0, a^i = \frac{\mathrm{d}\mathrm{v}^i}{\mathrm{d}t}. \tag{2.31}$$

In this system, the non-relativistic derivation of Sect. 2.1 is valid, and we know that the energy carried by the radiation field created by an accelerated charge per unit time $P = \frac{\mathrm{d}E}{\mathrm{d}t}$ is

$$P = \frac{2}{3} e^2 \frac{|\dot{\mathbf{v}}|^2}{c^3} = \frac{2}{3} \frac{e^2}{c^3} \mathbf{a} \cdot \mathbf{a}, \tag{2.32}$$

where $\mathbf{a} \cdot \mathbf{a}$ is the scalar product of the four-vector $\mathbf{a}$ with itself. To obtain the last equality in Eq. 2.32, we have used Eqs. 2.28–2.31. The formulation

$$P = \frac{2}{3} \frac{e^2}{c^3} \mathbf{a} \cdot \mathbf{a} \tag{2.33}$$

is manifestly covariant. In order to convince yourself of this, consider the transformations of the left part of the equality under Lorentz transformations

$$\text{Energy} \rightarrow \gamma \cdot \text{Energy} \tag{2.34}$$

$$\Delta t \rightarrow \gamma \cdot \Delta t \tag{2.35}$$

therefore

$$\frac{\mathrm{d}E}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}E}{\mathrm{d}t}. \tag{2.36}$$

This is clearly a scalar. The right side of the equality is a scalar product and thus also a scalar. The equality (2.33) is therefore valid in all systems of reference and corresponds to the relativistic generalisation of the Larmor formula (2.9) that we were seeking.

We can now write Eq. 2.33 in the system of the observer in which the particle is moving relativistically explicitly as

$$P = \frac{2}{3}\frac{e^2}{c^3}\left[c^2\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}(\gamma\cdot \mathrm{v})}{\mathrm{d}\tau}\right)^2\right] \tag{2.37}$$

Doing the algebra and using

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \frac{\gamma^3}{c^2}\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau}, \tag{2.38}$$

one obtains

$$P = \frac{2e^2}{3c^3}\gamma^6\left[-\left(\dot{\mathbf{v}}\cdot\frac{\mathbf{v}}{c}\right)^2 - \frac{|\dot{\mathbf{v}}|^2}{\gamma^2}\right] \tag{2.39}$$

and

$$|P| = \frac{2e^2}{3c^3}\gamma^6\left[a_{\|}^2 + \frac{1}{\gamma^2}a_{\perp}^2\right], \tag{2.40}$$

where we have introduced $(\mathbf{v}\cdot\dot{\mathbf{v}}) = \mathrm{v}\cdot a_{\|}$ and $|\mathbf{v}\times\dot{\mathbf{v}}| = \mathrm{v}\cdot a_{\perp}$, the components of the acceleration parallel and perpendicular to the velocity, and where we have used $|\dot{\mathrm{v}}|^2 = a_{\|}^2 + a_{\perp}^2$. The sign is naturally different if one considers the energy lost by the particles or that gained by the radiation field, and must be set accordingly. The power radiated by a relativistically-moving accelerated charge is finally expressed as

$$P = \frac{2e^2}{3c^3}\gamma^4(a_{\perp}^2 + \gamma^2 a_{\|}^2). \tag{2.41}$$

2.4 Relativistic Aberration

In order to understand the properties of the light observed from a relativistically-moving source it is still necessary to see how the geometry of the emission differs between the rest frame of the charge and that of the observer. Consider a source moving with a velocity v along the x-axis with respect to an observer and write with "′" the coordinate system in which the source is at rest, the "source system" in the following. Both systems are related by the Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - \mathrm{v}t) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \tfrac{\mathrm{v}}{c^2}x\right) \end{aligned} \tag{2.42}$$

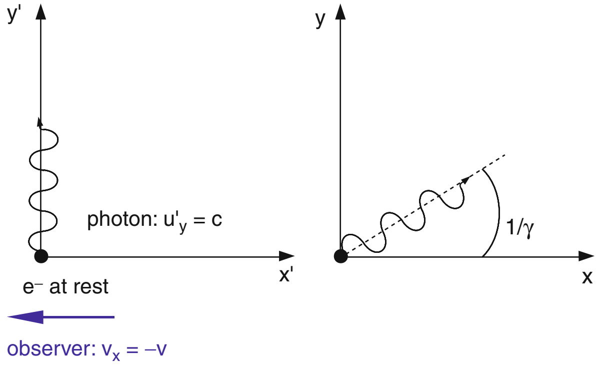

Fig. 2.2 The source at rest in the primed system of reference emits a photon along the y-axis, perpendicular to the velocity of the observer who sees the photon with an angle of $90° - 1/\gamma$ from the observer's y axis

Consider now a photon emitted by the source along a direction, say the y'-axis, perpendicular to the velocity $\mathbf{v}$. The photon moves with the speed of light: $u'_y = c$, while all other components vanish in the source rest frame, see Fig. 2.2.

The transformation of the velocities is given by

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma(\mathrm{d}x' + \mathrm{v}\mathrm{d}t')}{\gamma \mathrm{d}t'(1 + \frac{\mathrm{v}}{c^2}u'_x)} = \frac{u'_x + \mathrm{v}}{1 + \frac{\mathrm{v}u'_x}{c^2}} \tag{2.43}$$

$$u_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma \mathrm{d}t'(1 + \frac{\mathrm{v}}{c^2}u'_x)} = \frac{u'_y}{\gamma(1 + \frac{\mathrm{v}u'_x}{c^2})} \tag{2.44}$$

$$u_z = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z'}{\gamma \mathrm{d}t'(1 + \frac{\mathrm{v}}{c^2}u'_x)} = \frac{u'_z}{\gamma(1 + \frac{\mathrm{v}u'_x}{c^2})} \tag{2.45}$$

which for the special case of our photon simplifies to $u_x = \mathrm{v}$, $u_y = \frac{c}{\gamma}$ and $u_z = 0$. Consider now the direction under which the photon is observed in the observer's restframe

$$\tan\theta = \frac{u_y}{u_x} = \frac{1}{\gamma\beta}, \tag{2.46}$$

where, as usual $\beta = \frac{\mathrm{v}}{c}$.

We conclude from this analysis that all the photons emitted by a moving source in the forward half sphere are observed as coming from a cone of half opening angle $(\gamma\beta)^{-1}$ in the observer's rest frame. The change of apparent direction under which the photon is observed in both frames is called the relativistic aberration. The fact that all the photons emitted in a half sphere appear in a small cone leads to what is called beaming. The source appears much brighter to observers lying within the cone than to those located outside the cone.

We now have the tools needed to understand the radiation emitted by charges accelerated through different forces. In the following chapters we will deduce the properties of the radiation emitted by the different processes by estimating or calculating the second derivative of the dipole moment and deducing the efficiency of the process through the energy-loss formula of Larmor (2.9) or its relativistic generalisation (2.41) using beaming where appropriate (2.46). The source spectra will be calculated using Eq. (2.20).

2.5 Bibliography

Detailed derivations of the results presented here can be found in Jackson (1975) for the classical electrodynamics and in Rybicki and Lightman, (2004) and Longair (1992).

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