Chapter 2
Basics of Finite Groups

We start by introducing the basics of group theory, considering in particular finite groups. For pedagogical purposes, we shall use several theorems without proof, although proofs of useful theorems are given in Appendix A. (See also, e.g., [1–6].) On the other hand, we shall present several examples in order to obtain a clear understanding of these basic theorems.

A group G is a set with a product satisfying the following properties:

1. **Closure**
   If a and b are elements of the group G, then $c = ab$ is also an element of G.
2. **Associativity**
   $(ab)c = a(bc)$ for all $a, b, c \in G$.
3. **Identity**
   The group G includes an identity element e, which satisfies $ae = ea = a$ for any element $a \in G$.
4. **Inverse**
   The group G includes an inverse element a^{-1} for any element $a \in G$, such that $aa^{-1} = a^{-1}a = e$.

Let us present some simple examples.

Example (Cyclic Group Z_N) Discrete rotations of a complex plane form a group. Let us denote the $\exp[2\pi \mathrm{i}/N]$ rotation by a. Then the $\exp[2\pi \mathrm{i} m/N]$ rotation for $m =$ integer can be written a^m. The multiplication rule is defined such that $a^m a^n = a^{m+n}$. The operator a^N corresponds to the identity, $a^N = e$, and the inverse of a^m is obtained as a^{N-m}. Thus, the set

$$\{e, a, a^2, \ldots, a^{N-1}\} \tag{2.1}$$

forms a group. Its closure and associativity should be obvious. This group is called the cyclic group Z_N.

H. Ishimori et al., *An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists*, Lecture Notes in Physics 858,
DOI 10.1007/978-3-642-30805-5_2,

Example (S_3 and S_N) All possible permutations among three objects, (x_1, x_2, x_3), form a group denoted by S_3. There are six permutations:

$$\begin{aligned}
e &: (x_1, x_2, x_3) \to (x_1, x_2, x_3), \\
a_1 &: (x_1, x_2, x_3) \to (x_2, x_1, x_3), \\
a_2 &: (x_1, x_2, x_3) \to (x_3, x_2, x_1), \\
a_3 &: (x_1, x_2, x_3) \to (x_1, x_3, x_2), \\
a_4 &: (x_1, x_2, x_3) \to (x_3, x_1, x_2), \\
a_5 &: (x_1, x_2, x_3) \to (x_2, x_3, x_1).
\end{aligned} \tag{2.2}$$

The element e is clearly the identity. Their products form a closed algebra, e.g.,

$$\begin{aligned}
a_1 a_2 &: (x_1, x_2, x_3) \to (x_2, x_3, x_1), \\
a_2 a_1 &: (x_1, x_2, x_3) \to (x_3, x_1, x_2), \\
a_4 a_2 &: (x_1, x_2, x_3) \to (x_1, x_3, x_2),
\end{aligned} \tag{2.3}$$

whence

$$a_1 a_2 = a_5, \qquad a_2 a_1 = a_4, \qquad a_4 a_2 = a_2 a_1 a_2 = a_3. \tag{2.4}$$

It is straightforward to check the closure rule for other products, as well as associativity and the presence of an inverse for each element.

Using the multiplication rules, one can write all six elements in terms of two proper elements and their products. For example, by defining $a_1 = a, a_2 = b$, all elements are written as

$$\{e, a, b, ab, ba, bab\}. \tag{2.5}$$

Note that $aba = bab$. The group S_3 is the symmetry group of an equilateral triangle, as shown in Fig. 2.1. The elements a and ab correspond to a reflection and the $2\pi/3$ rotation, respectively.

Similarly, all possible permutations among N objects x_i with $i = 1, \ldots, N$,

$$(x_1, \ldots, x_N) \to (x_{i_1}, \ldots, x_{i_N}), \tag{2.6}$$

form a group. This is denoted by S_N and contains $N!$ elements. It is often called the symmetric group.

The *order* of a group G is the number of elements in G. Obviously, the order of a finite group is finite. For example, the order of the group Z_N is N, while the order of the group S_N is $N!$.

A group G is said to be *Abelian* if all its elements commute with each other, i.e., $ab = ba$ for any elements a and b in G. If not all pairs of elements satisfy commutativity, the group is said to be *non-Abelian*. The group Z_N is Abelian, but S_3 and S_N ($N \geq 3$) are non-Abelian. For example, for S_3, we see that $a_1 a_2 \neq a_2 a_1$ in the above notation (2.2).

If a subset H of a group G is also a group, H is said to be a *subgroup* of G. The order of the subgroup H is always a divisor of the order of G. This is *Lagrange's theorem* (see Appendix A). If a subgroup N of G satisfies $g^{-1}Ng = N$ for any element $g \in G$, the subgroup N is called a *normal subgroup* or an *invariant subgroup*.

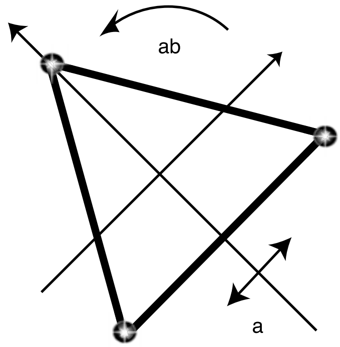

Fig. 2.1 The S_3 symmetry of an equilateral triangle

Any subgroup H and normal subgroup N of G satisfy $HN = NH$, where HN denotes

$$\{h_i n_j | h_i \in H, n_j \in N\}, \tag{2.7}$$

and NH has a similar meaning. Furthermore, HN is a subgroup of G.

Example For example, the three elements $\{e, ab, ba\}$ form a subgroup of S_3. Indeed, these elements correspond to even permutations, while the other elements, $\{a, b, bab\}$, correspond to odd permutations. This subgroup is nothing but the Z_3 group, because $(ab)^2 = ba$ and $(ab)^3 = e$. Lagrange's theorem implies that the order of any subgroup of S_3 must be equal to 1, 2, or 3, because the order of S_3 is 6 $(= 3 \times 2)$. The subgroup of order 3 corresponds to the above Z_3. In addition, the S_3 group includes three subgroups of order 2, viz., $\{e, a\}$, $\{e, b\}$, and $\{e, bab\}$. These subgroups are Z_2 groups. Furthermore, it can be shown that the above Z_3 is a normal subgroup of S_3.

When $a^h = e$ for an element $a \in G$ and h is the smallest positive integer for which this is so, the number h is called the *order* of a. The elements $\{e, a, a^2, \ldots, a^{h-1}\}$ form a subgroup, which is the Abelian group Z_h of order h.

The elements $g^{-1}ag$ for $g \in G$ are called elements conjugate to the element a. The set containing all elements conjugate to an element a of G, i.e., $\{g^{-1}ag, \forall g \in G\}$, is called a *conjugacy class*. All elements in a conjugacy class have the same order since

$$\left(gag^{-1}\right)^h = ga\left(g^{-1}g\right)a\left(g^{-1}g\right)\cdots ag^{-1} = ga^h g^{-1} = geg^{-1} = e. \tag{2.8}$$

The conjugacy class containing the identity e consists of the single element e.

Example All the elements of S_3 are classified into three conjugacy classes:

$$C_1 : \{e\}, \qquad C_2 : \{ab, ba\}, \qquad C_3 : \{a, b, bab\}. \tag{2.9}$$

Here, the subscript n of C_n denotes the number of elements in the conjugacy class C_n.

We consider two groups, G and G', and a map f of G into G'. This map is *homomorphic* if and only if it preserves the multiplicative structure, that is,

$$f(a)f(b) = f(ab), \tag{2.10}$$

for all $a, b \in G$. Furthermore, the map is *isomorphic* when it is a one-to-one correspondence.

A *representation* D of G is a homomorphic map of elements of G onto matrices $D(g)$ for $g \in G$. The representation matrices then satisfy $D(a)D(b) = D(c)$ if $ab = c$ for $a, b, c \in G$. The vector space V on which the representation matrices act is called a *representation space*, with $D(g)_{ij}v_j$, $(j = 1, \dots, n)$, for $v \in V$ with components v_j relative to some basis $\{e_1, \dots, e_n\}$. The dimension n of the vector space V is called the *dimension* of the representation.

A subspace in the representation space is said to be an *invariant subspace* if, for any vector v in the subspace and any element $g \in G$, $D(g)_{ij}v_j$ also corresponds to a vector in the same subspace. If there is an invariant subspace, such a representation is said to be *reducible*. In contrast, a representation is *irreducible* if it has no invariant subspace. In particular, a representation is said to be *completely reducible* if, for every $g \in G$, $D(g)$ can be written in the following block diagonal form:

$$\begin{pmatrix} D_1(g) & 0 & & \\ 0 & D_2(g) & & \\ & & \ddots & \\ & & & D_r(g) \end{pmatrix}, \tag{2.11}$$

where each $D_\alpha(g)$ is irreducible for $\alpha = 1, \dots, r$. We then say that the reducible representation $D(g)$ is the direct sum of the $D_\alpha(g)$:

$$\bigoplus_{\alpha=1}^{r} D_\alpha(g). \tag{2.12}$$

Every (reducible) representation of a finite group is completely reducible. Furthermore, every representation of a finite group is equivalent to a unitary representation (see Appendix A). The simplest (irreducible) representation is just $D(g) = 1$ for all elements g, that is, a trivial 1D representation or *singlet*. The matrix representations satisfy the following orthogonality relation:

$$\sum_{g \in G} D_\alpha(g)_{i\ell} D_\beta\left(g^{-1}\right)_{mj} = \frac{N_G}{d_\alpha} \delta_{\alpha\beta}\delta_{ij}\delta_{\ell m}, \tag{2.13}$$

where N_G is the order of G and d_α is the dimension of $D_\alpha(g)$ for each α (see Appendix A).

The *character* $\chi_D(g)$ of a representation $D(g)$ is the trace of the representation matrix:

$$\chi_D(g) = \operatorname{tr} D(g) = \sum_{i=1}^{d_\alpha} D(g)_{ii}. \tag{2.14}$$

The elements conjugate to a have the same character because of the following property of the trace:

$$\operatorname{tr} D\left(g^{-1}ag\right) = \operatorname{tr}\left[D\left(g^{-1}\right)D(a)D(g)\right] = \operatorname{tr} D(a). \tag{2.15}$$

That is, the characters are constant in a conjugacy class. The characters satisfy the following orthogonality relation:

$$\sum_{g\in G} \chi_{D_\alpha}(g)^* \chi_{D_\beta}(g) = N_G \delta_{\alpha\beta}, \tag{2.16}$$

where N_G denotes the order of a group G (see Appendix A). That is, the characters of different irreducible representations are orthogonal and different from each other. Furthermore, it can be shown that *the number of irreducible representations must be equal to the number of conjugacy classes* (see Appendix A). In addition, they satisfy the following orthogonality relation:

$$\sum_{\alpha} \chi_{D_\alpha}(g_i)^* \chi_{D_\alpha}(g_j) = \frac{N_G}{n_i} \delta_{C_i C_j}, \tag{2.17}$$

where C_i denotes the conjugacy class of g_i and n_i denotes the number of elements in the conjugacy class C_i (see Appendix A). The right-hand side is equal to N_G/n_i if g_i and g_j belong to the same conjugacy class, and otherwise it must vanish. A trivial singlet, $D(g) = 1$ for any $g \in G$, must always be included. Thus, the corresponding character satisfies $\chi_1(g) = 1$ for any $g \in G$.

Suppose that there are m_n n-dimensional irreducible representations, that is, with $D(g)$ represented by $(n \times n)$ matrices. The identity e is always represented by the $(n \times n)$ identity matrix. Clearly, the character $\chi_{D_\alpha}(C_1)$ for the conjugacy class $C_1 = \{e\}$ is just $\chi_{D_\alpha}(C_1) = n$ for an n-dimensional representation. The orthogonality relation (2.17) then requires

$$\sum_{\alpha} \left[\chi_\alpha(C_1)\right]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = N_G, \tag{2.18}$$

where $m_n \geq 0$. Furthermore, m_n must satisfy

$$\sum_n m_n = \text{number of conjugacy classes}, \tag{2.19}$$

because the number of irreducible representations is equal to the number of conjugacy classes. Equations (2.18) and (2.19), together with (2.16) and (2.17), will often be used in the following sections to determine characters.

Example Let us study the irreducible representations of S_3. The number of irreducible representations must be equal to three, because there are three conjugacy classes. We assume that there are m_n n-dimensional representations, that is, with $D(g)$ represented by $(n \times n)$ matrices. Here, m_n must satisfy $\sum_n m_n = 3$. Furthermore, the orthogonality relation (2.18) requires

$$\sum_{\alpha} \left[\chi_\alpha(C_1)\right]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 6, \tag{2.20}$$

Table 2.1 Characters of S_3 representations

	h	χ_1	$\chi_{1'}$	χ_2
C_1	1	1	1	2
C_2	3	1	1	−1
C_3	2	1	−1	0

where $m_n \geq 0$. This equation has only two possible solutions, $(m_1, m_2) = (2, 1)$ and $(6, 0)$, but only the former $(m_1, m_2) = (2, 1)$ satisfies $m_1 + m_2 = 3$. Thus, irreducible representations of S_3 include two singlets **1** and **1**$'$, and a doublet **2**. We denote their characters by $\chi_1(g)$, $\chi_{1'}(g)$, and $\chi_2(g)$, respectively. Clearly, $\chi_1(C_1) = \chi_{1'}(C_1) = 1$ and $\chi_2(C_1) = 2$. Furthermore, one of the singlet representations corresponds to a trivial singlet, that is, $\chi_1(C_2) = \chi_1(C_3) = 1$.

The characters, which are not fixed at this stage, are $\chi_{1'}(C_2)$, $\chi_{1'}(C_3)$, $\chi_2(C_2)$, and $\chi_2(C_3)$. Now let us determine them. For a non-trivial singlet **1**$'$, the representation matrices are nothing but the characters, $\chi_{1'}(C_2)$ and $\chi_{1'}(C_3)$. They must satisfy

$$\left[\chi_{1'}(C_2)\right]^3 = 1, \qquad \left[\chi_{1'}(C_3)\right]^2 = 1. \tag{2.21}$$

Thus, $\chi_{1'}(C_2)$ is one of 1, ω, and ω^2, where $\omega = \exp[2\pi \mathrm{i}/3]$, and $\chi_{1'}(C_3)$ is 1 or -1. On top of that, the orthogonality relation (2.16) requires

$$\sum_g \chi_1(g)\chi_{1'}(g) = 1 + 2\chi_{1'}(C_2) + 3\chi_{1'}(C_3) = 0. \tag{2.22}$$

Its unique solution is $\chi_{1'}(C_2) = 1$ and $\chi_{1'}(C_3) = -1$. Furthermore, the orthogonality relations (2.16) and (2.17) require

$$\sum_g \chi_1(g)\chi_2(g) = 2 + 2\chi_2(C_2) + 3\chi_2(C_3) = 0, \tag{2.23}$$

$$\sum_\alpha \chi_\alpha(C_1)^* \chi_\alpha(C_2) = 1 + \chi_{1'}(C_2) + 2\chi_2(C_2) = 0. \tag{2.24}$$

Their solution is $\chi_2(C_2) = -1$ and $\chi_2(C_3) = 0$. These results are shown in Table 2.1.

Next, we figure out the representation matrices $D(g)$ of S_3 using the character Table 2.1. For singlets, their characters are nothing but representation matrices. Let us consider representation matrices $D(g)$ for the doublet, where $D(g)$ are (2×2) unitary matrices. Obviously, $D_2(e)$ is the (2×2) identity matrix. Since $\chi_2(C_3) = 0$, we can diagonalize one element of the conjugacy class C_3. Here we choose, e.g., a in C_3, as the diagonal element:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.25}$$

The other elements in C_3, as well as those in C_2, are non-diagonal matrices. Recalling that $b^2 = e$, we can write

$$b = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \qquad bab = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{2.26}$$

Then, we can write elements in C_2 as

$$ab = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad ba = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.27}$$

Recall that the trace of elements in C_2 is equal to -1, whence $\cos\theta = -1/2$, that is, $\theta = 2\pi/3, 4\pi/3$. When we choose $\theta = 4\pi/3$, we obtain the matrix representation of S_3 as

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
$$ab = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad ba = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad bab = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}. \tag{2.28}$$

We can construct a larger group from two or more groups G_i, by means of certain products. A rather simple one is the *direct product*. We consider, e.g., two groups G_1 and G_2. Their direct product is denoted $G_1 \times G_2$, and its multiplication rule is defined as

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2), \tag{2.29}$$

for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.

The *semi-direct product* is a less trivial product between two groups G_1 and G_2, and it is defined such that

$$(a_1, a_2)(b_1, b_2) = \big(a_1 f_{a_2}(b_1), a_2 b_2\big), \tag{2.30}$$

for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$, where $f_{a_2}(b_1)$ denotes a homomorphic map from G_2 to the automorphisms of G_1. This semi-direct product is denoted by $G_1 \rtimes_f G_2$.

We consider the group G with a subgroup H and a normal subgroup N, whose elements are denoted h_i and n_j, respectively. When $G = NH = HN$ and $N \cap H = \{e\}$, the semi-direct product $N \rtimes_f H$ is isomorphic to G, $G \simeq N \rtimes_f H$, where we use the map f defined by

$$f_{h_i}(n_j) = h_i n_j (h_i)^{-1}. \tag{2.31}$$

For the notation of the semi-direct product, we will often omit f and denote it simply by $N \rtimes H$.

Example Let us study the semi-direct product $Z_3 \rtimes Z_2$. Here we denote the Z_3 and Z_2 generators by c and h, i.e., $c^3 = e$ and $h^2 = e$. In this case, (2.31) can be written

$$hch^{-1} = c^m, \tag{2.32}$$

where $m \neq 0$, because all the elements of Z_3 can be written c^m (the case $m = 0$ being inconsistent). When $m = 1$, the above relation is trivial and leads simply to the direct product $Z_3 \times Z_2$. Thus, only the case with $m = 2$ is non-trivial, i.e.,

$$hch^{-1} = c^2. \tag{2.33}$$

Indeed, this algebra is isomorphic to S_3, and h and c are identified with a and ab, respectively. Similarly, we can consider $Z_n \rtimes Z_m$. When we denote the Z_n and Z_m generators by a and b, respectively, they satisfy

$$a^n = b^m = e, \qquad bab^{-1} = a^k, \tag{2.34}$$

where $k \neq 0$, although the case with $k = 1$ leads to the direct product $Z_n \times Z_m$.

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http://www.springer.com/978-3-642-30804-8

An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists

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2012, XII, 283 p. 8 illus., Softcover

ISBN: 978-3-642-30804-8