Chapter 2
Maxwell’s Equations, Photons and the Density of States

In this chapter we consider Maxwell’s equations and what they reveal about the propagation of light in vacuum and in matter. We introduce the concept of photons and present their density of states. Since the density of states is a rather important property in general and not only for photons, we approach this quantity in a rather general way. We will use the density of states later also for other (quasi-) particles including systems of reduced dimensionality. In addition, we introduce the occupation probability of these states for various groups of particles.

It should be noted, that we shall approach the concept of photons on an elementary level only, in correspondence with the concept of this book. We do not delve into present research topics on photon physics itself like photon-correlation and -statistics, squeezed light, photon anti-bunching, entangled photon states, etc., but give some introductory references for those interested in these fields [02L1, 98B3, 99P1, 99S1, 02G1, 02Y1, 05G1, 09S1]. Some interesting considerations of the second order correlation function \( g^{(2)}(\tau) \) and its differences for incandescent light, laser light and light from a single photon source can be found in [09A1, 09W1].

Einstein, who obtained the Nobel prize for physics in 1921 for the explanation of the photo-electric effect (not for the theory of relativity!), once stated: “Was das Licht sei, das weiß ich nicht” (What the light might be, I do not know). So there still seems to be ample place for research in these fields.

2.1 Maxwell’s Equations

Maxwell’s equations ([65M1] and [98B4, 10C1] for a modern discussion) can be written in different ways. We use here the macroscopic Maxwell’s equations in their differential form. Throughout this book the internationally recommended system of units known as SI (système international) is used. These equations are given in
their general form in (2.1a–f), where bold characters symbolize vectors and normal characters scalar quantities.

\[ \nabla \cdot \mathbf{D} = \rho , \quad \nabla \cdot \mathbf{B} = 0 , \quad (2.1a,b) \]

\[ \nabla \times \mathbf{E} = -\dot{\mathbf{B}} , \quad \nabla \times \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}} , \quad (2.1c,d) \]

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} , \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} . \quad (2.1e,f) \]

The various symbols have the following meanings and units:

\( \mathbf{E} = \) electric field strength; 1 \( \text{V/m} = 1 \text{ m kg s}^{-3} \text{A}^{-1} \)

\( \mathbf{D} = \) electric displacement; 1 \( \text{A s/m}^2 = 1 \text{ C/m}^2 \)

\( \mathbf{H} = \) magnetic field strength; 1 \( \text{A/m} \)

\( \mathbf{B} = \) magnetic induction or magnetic flux density; 1 \( \text{Wb/m}^2 = 1 \text{ T} \)

\( \rho = \) charge density; 1 \( \text{A s/m}^3 = 1 \text{ C/m}^3 \)

\( \mathbf{j} = \) electrical current density; 1 \( \text{A/m}^2 \)

\( \mathbf{P} = \) polarization density of a medium, i.e., electric dipole moment per unit volume; 1 \( \text{A s/m}^2 \)

\( \mathbf{M} = \) magnetization density of the medium, i.e., magnetic dipole moment per unit volume\(^1\); 1 \( \text{V s/m}^2 \)

\( \varepsilon_0 \approx 8.859 \times 10^{-12} \text{ A s/V m} \) is the permittivity of vacuum

\( \mu_0 = 4\pi \times 10^{-7} \text{ V s/A m} \) is the permeability of vacuum

\( \nabla = \) Nabla-operator, in Cartesian coordinates \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \)

\( \dot{} = \partial/\partial t \) i.e., a dot means differentiation with respect to time.

The applications of \( \nabla \) to scalar or vector fields are usually denoted by

\[ \nabla \cdot f(\mathbf{r}) = \text{grad } f , \]

\[ \nabla \cdot \mathbf{A}(\mathbf{R}) = \text{div } \mathbf{A} , \]

\[ \nabla \times \mathbf{A}(\mathbf{r}) = \text{curl } \mathbf{A} , \]

and the Laplace operator \( \Delta \) is defined as

\[ \Delta \equiv \nabla^2 . \]

If \( \Delta \) is applied to a scalar field \( \rho \) we obtain in cartesian coordinates

\[ \Delta \rho = \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} \quad (2.2) \]

\(^1\)Some authors prefer to use \( \mathbf{M}' = \mathbf{M} \mu_0^{-1} \) and thus \( \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}') \). We prefer (2.1e, f) for symmetry arguments [93R1].
Application to a vector field $E$ results in

$$\Delta E = \begin{pmatrix} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \\ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \end{pmatrix}. \quad (2.3)$$

Further rules for the use of $\nabla$ and of $\Delta$ and their representations in other than Cartesian coordinates (polar or cylindrical coordinates) are found in compilations of mathematical formulae [84A1, 91B1, 92S1].

To complete classical electro-magnetism, we have to add the equation for the Lorentz force $F$ exerted by the electric and magnetic fields on a charge $q$ moving with a velocity $v$

$$F = qE + qv \times B \quad (2.1g)$$

Equations 2.1a, b show that free electric charges $\rho$ are the sources of the electric displacement and that the magnetic induction is source-free. Equations 2.1c, d demonstrate how temporally varying magnetic and electric fields generate each other. In addition, the $\mathbf{H}$ field can be created by a macroscopic current density $\mathbf{j}$. Equations 2.1e, f are the material equations in their general form. From them we learn that the electric displacement is given by the sum of electric field and polarization, while the magnetic flux density is given by the sum of magnetic field and magnetization. Some authors prefer not to differentiate between $\mathbf{H}$ and $\mathbf{B}$. This leads to difficulties, as can be easily seen from the fact that $\mathbf{B}$ is source-free (2.1b) but $\mathbf{H}$ is not, as follows from the inspection of the fields of every simple permanent magnet.

By applying $\nabla \cdot$ to (2.1d) we obtain the continuity equation for the electric charges

$$\text{div } \mathbf{j} = -\frac{\partial}{\partial t} \rho, \quad (2.4)$$

which corresponds to the conservation law of the electric charge in a closed system.

The integral forms of (2.1) can be obtained from the differential forms by integration and the use of the laws of Gauss or Stokes resulting in

$$\int \rho(r) \, dV = \oint \mathbf{D} \cdot d\mathbf{f} \quad (2.5a)$$

$$-\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{f} = \oint \mathbf{E} \cdot ds \quad (2.5b)$$

where $dV$, $d\mathbf{f}$ and $ds$ give infinitesimal elements of volume, surface or area and line, respectively.
In their microscopic form, Maxwell’s equations contain all charges as sources of the electric field $E_{\text{micro}}$ including all electrons, protons bound in atoms as $\rho_{\text{bound}}$ and not only the free space charges $\rho$. By analogy, not only the microscopic current density $j$ has to be used as a source of $H_{\text{micro}}$ but all spins with their magnetic moments and all $l \neq 0$ orbits of charged particles have to be included as “bound” current density $j_{\text{bound}}$. The transition to macroscopic quantities can then be performed by averaging over small volumes (larger than an atom but smaller than the wavelength of light) and replacing $\rho_{\text{bound}}$ by $-\nabla \cdot P$ and $j_{\text{bound}}$ by $\dot{P} + \text{curl} \, M(T, t)/\mu_0$. For more details see [98B1, 98D1] or Chap. 27.

Concerning the units, some theoreticians still prefer the so-called c g s (cm, g, second) system. Though it has only marginal differences in mechanics to the SI system, which is based on the units $1\,\text{m}$, $1\,\text{kg}$, $1\,\text{s}$, $1\,\text{A}$, $1\,\text{K}$, $1\,\text{mol}$ and $1\,\text{cd}$, the c g s system produces strange units in electrodynamics like the electrostatic units (esu), which contain square roots of mass and are therefore unphysical and even ill-defined. For conversion tables see [96L1].

### 2.2 Electromagnetic Radiation in Vacuum

In vacuum the following conditions are fulfilled

$$ P = 0; \quad M = 0; \quad \rho = 0; \quad j = 0. \quad (2.6) $$

With the help of (2.1e, f) this simplifies (2.1c, d) to

$$ \nabla \times E = -\mu_0 \dot{H} \quad \text{and} \quad \nabla \times H = \varepsilon_0 \dot{E} \quad (2.7a,b) $$

Applying $\nabla \times$ to (2.7a) and $\partial / \partial t$ to (2.7b) yields

$$ \nabla \times (\nabla \times E) = -\mu_0 \nabla \times \dot{H} \quad \text{and} \quad \nabla \times \dot{H} = \varepsilon_0 \ddot{E} . \quad (2.8) $$

From (2.8) we find with the help of the properties of the $\nabla$ operator

$$ -\mu_0 \varepsilon_0 \ddot{E} = \nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E . \quad (2.9) $$

With (2.6), (2.3) and (2.1a) we see that

$$ \nabla E = 0 \quad (2.10) $$

and (2.9) reduces to the usual wave equation, written here for the electric field

$$ \nabla^2 E - \mu_0 \varepsilon_0 \ddot{E} = 0 . \quad (2.11a) $$
For monochromatic plane waves (2.14) the general wave (2.11a) can be simplified to the vectorial Helmholtz equation (2.11b)

\[(\nabla^2 + k^2)E(\vec{r}) = 0\]  

(2.11b)

or to a scalar Helmholtz equation for situations, where the vector character is of no importance.

An analogous equation can be obtained for the magnetic field strength. Solutions of this equation are all waves of the form

\[E(\vec{r}, t) = E_0 f(k \vec{r} - \omega t).\]  

(2.12)

\(E_0\) is the amplitude, \(f\) is an arbitrary function the second derivative of which exists. As can be shown by inserting the ansatz (2.12) into (2.11a,b) the wave vector \(k\) and the angular frequency \(\omega\) obey the relation

\[\frac{\omega}{k} = \left(\frac{1}{\mu_0 \varepsilon_0}\right)^{1/2} = c \quad \text{with} \quad k = |k| = 2\pi/\lambda_v.\]  

(2.13)

In the following we use for simplicity only the term “frequency” instead of “angular frequency” for \(\omega = 2\pi/T\) where \(T\) is the temporal period of the oscillation.

In (2.13), \(c\) is the vacuum speed of light and \(\lambda_v\) is the wavelength in vacuum. From all possible solutions of the form (2.12) we shall concentrate in the following on the most simple ones, namely on plane harmonic waves, which can be written as

\[E(\vec{r}, t) = E_0 \exp[i(k \vec{r} - \omega t)].\]  

(2.14)

For all waves (not only those in vacuum), the phase and group velocities \(v_{\text{ph}}\) and \(v_g\) are given by

\[v_{\text{ph}} = \frac{\omega}{k}; \quad v_g = \frac{\partial \omega}{\partial k} = \text{grad}_k \omega,\]  

(2.15)

where \(v_{\text{ph}}\) gives the velocity with which a certain phase propagates, (e.g., a maximum of a monochromatic wave) while \(v_g\) gives the speed of the center of mass of a wave packet with middle frequency \(\omega\) and covering a small frequency interval \(d\omega\) as shown schematically in Fig. 2.1a, b, respectively. The formulas (2.15) are of general validity. The \(\text{grad}_k\) on the r.h.s. of (2.15) means a differentiation with respect to \(k\); in the sense of \(\nabla_k = (\partial/\partial k_x, \partial/\partial k_y, \partial/\partial k_z)\) and has to be used instead of the more simple expression \(\partial \omega/\partial k\) in anisotropic media. For the special case of electromagnetic radiation in vacuum we find from (2.13) and (2.15)

\[v_{\text{ph}} = v_g = c = (\mu_0 \varepsilon_0)^{-1/2}.\]  

(2.16)
Now we want to see what constraints are imposed by Maxwell’s equations on the various quantities such as $E_0$ and $k$. Inserting (2.12) or (2.14) into (2.10) gives

$$\nabla \cdot E = iE_0 \cdot k \exp[i(kr - \omega t)] = 0. \quad (2.17)$$

This means that

$$E_0 \perp k \quad (2.18)$$

or, in other words, the electromagnetic wave is transverse in $E$. What can we learn from Maxwell’s equations for the other fields? From (2.7) we have for plane waves

$$H = (\omega \mu_0)^{-1} k \times E = H_0 \exp[i(kr - \omega t)] \quad (2.19a)$$

with

$$H_0 = (\omega \mu_0)^{-1} k \times E_0. \quad (2.19b)$$

Furthermore we have with (2.1e, f) and (2.6)

$$D = D_0 \exp[i(kr - \omega t)] = \varepsilon_0 E_0 \exp[i(kr - \omega t)], \quad (2.19c)$$

$$B = B_0 \exp[i(kr - \omega t)] = \omega^{-1} k \times E_0 \exp[i(kr - \omega t)]. \quad (2.19d)$$
2.2 Electromagnetic Radiation in Vacuum

The electromagnetic wave is, according to (2.19b), also transverse in \( B \) and the electric and magnetic fields are perpendicular to each other, that is, we have in general

\[
D \perp k \perp B \perp D .
\]  

(2.19e)

In vacuum and isotropic media one has in addition

\[
E \parallel D \quad \text{and} \quad H \parallel B .
\]  

(2.19f)

As we shall see later in connection with (2.17), (2.43), and (2.44) one has in matter usually transverse waves, which obey (2.19e) but additionally, longitudinal waves exist under certain conditions.

The momentum density \( \mathbf{\Pi} \) of the electromagnetic field is given by [93R1]

\[
\mathbf{\Pi} = D \times B , \quad \mathbf{\Pi} \parallel k
\]  

(2.20)

and the energy flux density by the Poynting vector \( S \)

\[
S = E \times H
\]  

(2.21)

with \( S \parallel \mathbf{\Pi} \) in vacuum and isotropic materials.

\( S \) is a rapidly oscillating function of space and time. The average value \( \langle S \rangle \) is usually called the intensity \( I \) or the energy flux density. The intensity is proportional to the amplitude squared for all harmonic waves. For the plane monochromatic waves treated here, we obtain

\[
\langle S \rangle = \frac{1}{2} |E_0 \times H_0| = \frac{1}{2} \frac{1}{c \mu_0} E_0^2 = \frac{1}{2} \frac{c}{\mu_0} B_0^2 = \frac{1}{2} c \mu_0 H_0^2 .
\]  

(2.22)

Equations 2.20 and 2.21 are also valid in matter.

Since a plane electromagnetic wave is a transverse vector wave (2.19), it can be polarized (see e.g. Sect. 3.1). There are two basis sets, namely either two orthogonally and linearly polarized waves or two right and left circularly polarized ones. Both sets of bases can be transformed into each other in the sense that the superposition of a left and right circular polarized wave with equal \( \omega, k \) and amplitude \( E \) gives with relative phase shifts of 0 or \( \pm \pi / 2 \) results in two orthogonal, linearly polarized waves, while the superposition of two linear polarized waves again with equal \( \omega, k \) and amplitude \( E_0 \) and with phase shifts of \( \pm \pi / 4 \) results in a right and left circularly polarized wave.

In a similar way, all types of elliptically polarized waves can be created possibly with different amplitudes of the two partial waves and phase shifts different from \( \pm \pi / 4 \) or \( \pi / 2 \).

For details including the formal description of the polarization state by the Jones vector and the action of optical devices by the Jones matrix for completely polarized light and by the Stokes parameters and the Mueller matrix for partly polarized light see e.g. Ref. [07M1, 07S1] of Chap. 1 or [07B1].
2.3 Electromagnetic Radiation in Matter; Linear Optics

Now we treat Maxwell’s equations in matter. Doing so we have in principle to use the equations in their general form \((2.1)\). However, we will still make in several steps some assumptions which are reasonable for semiconductors: we assume first that there are no macroscopic free space charges i.e. \(\rho = 0\). Then a treatment of \((2.1)\) in analogy with \((2.7)\)–\((2.11)\) results in

\[
\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \ddot{\mathbf{E}} = \mu_0 \ddot{\mathbf{P}} + \mu_0 \dot{\mathbf{j}} + \nabla \times \mathbf{M},
\]

\((2.23)\)

Actually, there also occurs in the derivation of \((2.23)\) a term \(\nabla (\nabla \mathbf{P})\) or \(\nabla (\nabla \mathbf{E})\). This term vanishes for transverse waves and is therefore neglected otherwise.

This equation is the inhomogeneous analogue of \((2.11)\) telling us that the sources of an electro-magnetic radiation field can be

- A dipole moment \(\mathbf{p}\) or a polarization \(\mathbf{P}\) with a non-vanishing second time derivative
- A temporally varying current density
- The curl of a temporally varying magnetization

Again a similar equation can be obtained for the magnetic field. We continue now with the application of further simplifications and assume that we have a nonmagnetic material, i.e., that the third term on the r.h.s. of \((2.23)\) vanishes. Actually, all matter has some diamagnetism. But this is a rather small effect of the order of \(10^{-6}\) so it can be neglected for our purposes. Paramagnetic and especially ferromagnetic contributions can be significantly larger for low frequencies. However, even these contributions diminish rapidly for higher frequencies. Consequently the assumption of a nonmagnetic material is a good approximation over a wide range of the electromagnetic spectrum starting in the IR even for ferromagnetic materials. Furthermore, the more common semiconductors are not ferro-, ferri- or antiferromagnetic and have only a small concentration of paramagnetic centres which may be seen in electron paramagnetic resonance (EPR), but which have negligible influence on the optical properties. The only exceptions are semiconductors which contain a considerable amount of e.g., Mn or Fe ions as does \(\text{Zn}_{1-y}\text{Mn}_y\text{Se}\). We refer the reader to \([88D1,91O1,92G1,92Y1,94D1,94G1,96H1,98O1,03D1]\) or to Sect. 16.1 and references therein for this class of materials.

The current term \(\mathbf{j}\) in \((2.1d)\) deserves some more consideration. The current is driven by the electric field

\[
\mathbf{j} = \sigma \mathbf{E},
\]

\((2.24)\)

where \(\sigma\) is the conductivity. For intrinsic or weakly doped semiconductors, the carrier density is small and consequently \(\sigma\) is as well. Then the following inequality holds

\[
|\mathbf{j}| = |\sigma \mathbf{E}| \ll |\mathbf{D}|.
\]

\((2.25)\)
In the following we will consider this case and neglect $j$ in (2.1d). For heavily
doped semiconductors (2.25) is no longer valid and $\sigma$ will have some influence on
the optical properties at least in the infrared (IR). We come back to this situation in
connection with plasmons in Chaps. 10 and 12.

The basic material equation still left in comparison with the vacuum case is
now (2.1e) $D = \varepsilon_0 E + P$.

If we proceed with this equation again in the manner of (2.7)–(2.11) the result is

$$\nabla^2 E - \mu_0 \varepsilon_0 \ddot{E} = \mu_0 \ddot{P}$$

Equation 2.26 states the well-known fact that every dipole $p$ and every polarization $P$
with a non vanishing second derivative in time radiates an electromagnetic
wave.

As long as we have no detailed knowledge about the relationships between $D$, $E$
and $P$ we cannot go beyond (2.26). Now we make a very important assumption.
We assume a linear relationship between $P$ and $E$:

$$\frac{1}{\varepsilon_0} P = \chi E$$

(2.27a)
or

$$D = \varepsilon_0 (1 + \chi) E = \varepsilon_0 E$$

(2.27b)

with

$$\varepsilon = \chi + 1.$$  

(2.27c)

This linear relation is the reason why everything that is treated in the following
Chaps. 3–18 is called linear optics. A linear relation is what one usually assumes
between two physical quantities as long as one does not have more precise
information. In principle we can also consider (2.27a) as an expansion of $P(E)$
in a power series in $E$ which is truncated after the linear term. We come back to
this aspect in Chap. 19.\footnote{A constant term in this power expansion such as $P = P_0 + \chi E$ would describe a spontaneous polarization of matter which occurs e.g., in pyro- or ferro-electric materials. With arguments similar to the ones given for ferromagnetics we can neglect such phenomena in the discussion of the optical properties of semiconductors.} The quantities $\varepsilon$ and $\chi$ are called the dielectric function and the susceptibility, respectively. They can be considered as linear response functions [93S1, 98B1, 98D1].

Both quantities depend on the frequency $\omega$ and on the wave-vector $k$, and they
both have a real and an imaginary part as shown for $\varepsilon$.

$$\varepsilon = \varepsilon(\omega, k); \quad \chi = \chi(\omega, k) = \varepsilon(\omega, k) - 1,$$

$$\varepsilon(\omega, k) = \varepsilon_1(\omega, k) + i\varepsilon_2(\omega, k).$$  

(2.28)

(2.29)
The frequency dependence is dominant and will be treated first in Chaps. 3–4. We drop the \( k \) dependence for the moment but come back to it in connection with spatial dispersion in Chap. 5. In Chap. 6 we discuss the properties of \( \varepsilon \) as a function of frequency and wave vector or as a function of time and space.

The value of \( \varepsilon(\omega) \) for \( \omega \simeq 0 \) is usually called the dielectric constant.

In general \( \varepsilon \) and \( \chi \) are tensors. For simplicity we shall consider them to be scalar quantities if not stated otherwise, e.g., in connection with birefringence in Sect. 3.1.7.

Using the linear relations of (2.27) we can transform (2.23) into

\[
\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \varepsilon(\omega) \mathbf{\dot{E}} = 0,
\]

where we assumed also that \( \varepsilon(\omega) \) is spatially constant on a length scale of the order of the wavelength of light. Deviations of this assumption are treated in Sects. 17.2–17.4.

If magnetic properties are to be included, a corresponding linear approach would lead to

\[
\nabla^2 \mathbf{E} - \mu_0 \mu(\omega) \varepsilon_0 \varepsilon(\omega) \mathbf{\dot{E}} = 0,
\]

where \( \mu(\omega) \) is the magnetic permeability. As outlined above we have in the visible for most semiconductors \( \mu(\omega) \simeq 1 \).

As for (2.12) the solutions of (2.30) are again all functions of the type

\[
\mathbf{E} = E_0 f(\mathbf{k} \mathbf{r} - \omega t),
\]

or for our present purposes, i.e. again for the case of plane harmonic waves

\[
\mathbf{E} = E_0 \exp[i(\mathbf{k} \mathbf{r} - \omega t)].
\]

The relationship between \( k \) and \( \omega \) is however now significantly different from (2.13). It follows again from inserting the ansatz (2.31) or (2.32) into (2.30) and now reads compared to (2.13):

\[
\frac{c^2 k^2}{\omega^2} = \varepsilon(\omega).
\]

This relation appears in Chap. 5 again under the name “polariton equation”. It can also be written in other forms:

\[
k = \frac{\omega}{c} \varepsilon^{1/2}(\omega) = \frac{2\pi}{\lambda_v} \varepsilon^{1/2}(\omega) = k_v \varepsilon^{1/2}(\omega),
\]

where \( \lambda_v \) and \( k_v \) refer to the vacuum values of the light wave.
For the square root of $\varepsilon$ we introduce for simplicity a new quantity $\tilde{n}(\omega)$ which we call the complex index of refraction

$$\tilde{n}(\omega) = n(\omega) + i\kappa(\omega) = \varepsilon^{1/2}(\omega).$$

The (2.13) and (2.33)–(2.35) can be interpreted in the following way. In vacuum an electromagnetic wave propagates with a wave vector $k_v$ which is real and given by (2.13). In matter, light propagates with a wave vector $k$ which can be a complex quantity given by (2.34), or, with the help of (2.35), by

$$k = \frac{\omega}{c} \tilde{n}(\omega) = \frac{\omega}{c} n(\omega) + i \frac{\omega}{c} \kappa(\omega) = \frac{2\pi}{\lambda_v} \tilde{n}(\omega) = k_v \tilde{n}.$$  

We should notice that $k$ is for complex $\tilde{n}$ not simply $|k|$ since $|k|$ is always a positive, real quantity. Here $k$ means just neglecting the vector character of $k$ but $k$ can still be a real, imaginary or complex quantity according to (2.36). The direction of the real part of $k$, which describes the oscillatory part of the wave, is still parallel to $D \times B$ as in (2.20).

Writing the plane wave explicitly we have:

$$E_0 \exp[i(kr - \omega t)] = E_0 \exp\left\{i \left[\frac{\omega}{c} n(\omega) \tilde{k} r - \omega t\right]\right\} \exp\left[-\frac{\omega}{c} \kappa(\omega) \tilde{k} r\right],$$

where $\tilde{k}$ is the unit vector in the direction of $k$, i.e., in the direction of propagation.

Obviously $n(\omega)$ describes the oscillatory spatial propagation of light in matter; it is often called the refractive index in connection with Snell’s law of refraction. This means that the wavelength $\lambda$ in a medium is connected with the wavelength $\lambda_v$ in vacuum by

$$\lambda = \lambda_v n^{-1}(\omega).$$

In (2.37) $\kappa(\omega)$ describes a damping of the wave in the direction of propagation. This effect is usually called absorption or, more precisely, extinction. We give the precise meaning of these two quantities in Sect. 3.1.5. Here we compare (2.37) with the well-known law of absorption for the light intensity $I$ of a parallel beam propagating in $z$-direction

$$I(z) = I(z = 0) e^{-\alpha z}$$

3On the basis of (2.30b) one obtains $\tilde{n}(\omega) = \pm (\varepsilon(\omega) \mu(\omega))^{1/2}$. The $+$ sign has to be used for all usual material, for which $\mu(\omega) > 0$ holds. Recently a new class of artificially designed materials has been invented, the so-called left-handed or meta materials. They have in some spectral range $\mu(\omega) < 0$ and $\varepsilon(\omega) < 0$. In this case the $-$ sign has to be used, resulting in rather unexpected phenomena. Though this topic leads beyond semiconductor optics, we give a few words and references in Sect. 17.5.
with $I$ following from (2.22) to be now

$$I = \frac{1}{2} \frac{n(\omega)}{c\mu_0} E_0^2 = \frac{1}{2} \varepsilon_0 c n(\omega) E_0^2$$

(2.39b)

where $\alpha(\omega)$ is usually called the absorption coefficient, especially in Anglo-Saxon literature. In German literature $\alpha(\omega)$ is also known as “Absorptionskonstante” (absorption constant) and dimensionless quantities proportional to $k(\omega)$ are called “Absorptionskoeffizient” or “Absorptionsindex” (absorption coefficient or absorption index). So some care has to be taken regarding what is meant by one or the other of the above terms.

Bearing in mind that the intensity is still proportional to the amplitude squared (2.24), a comparison between (2.37) and (2.39) yields

$$\alpha(\omega) = \frac{2\omega}{c} \kappa(\omega) = \frac{4\pi}{\lambda_v} \kappa(\omega).$$

(2.40)

The phase velocity of light in a medium is now given by (2.15)

$$v_{ph} = \frac{\omega}{\text{Re}\{\kappa\}} = c n^{-1}(\omega).$$

(2.41)

For the group velocity we can get rather complicated dependencies originating from

$$v_g = \frac{\partial \omega}{\partial k}.$$  

(2.42)

We return to this aspect later.

### 2.4 Transverse, Longitudinal and Surface Waves

The only solution of (2.9) for light in vacuum is a transverse electromagnetic wave (2.19). This solution exists for light in matter as well. However (2.9) now with the use of (2.27) the form

$$\nabla \cdot D = \nabla \varepsilon_0 \varepsilon(\omega) E = 0$$

(2.43)

Apartment from the above-mentioned transverse solution with $E \perp k$ there is a new solution which does not exist in vacuum ($\varepsilon_{\text{vac}} \equiv 1$), namely

$$\varepsilon(\omega) = 0.$$  

(2.44a)

This means that we can find longitudinal solutions at the frequencies at which $\varepsilon(\omega)$ vanishes. We call these frequencies correspondingly $\omega_L$ and note that for

$$\varepsilon(\omega_L) = 0; \quad E \parallel k \text{ is possible}$$

(2.44b)
Now let us consider the other fields for this longitudinal wave in matter. From (2.27) we see immediately that we have for the longitudinal modes

\[ D = 0 \quad \text{and} \quad E = -\frac{1}{\varepsilon_0} P. \]  

(2.45)

In matter, the Maxwell’s equation \( \nabla \times E = -\dot{B} \) is still valid. This leads for plane waves in nonmagnetic material to

\[ H_0 = (\omega \mu_0)^{-1} k \times E_0. \]  

(2.46)

For the longitudinal wave it follows from (2.44) that

\[ H = 0 \quad \text{and} \quad B = \mu_0 H = 0 \]  

(2.47)

The longitudinal waves which we found in matter are not electromagnetic waves but pure polarization waves with \( E \) and \( P \) opposed to each other with vanishing \( D, B \) and \( H \).

Until now we were considering the properties of light in the bulk of a medium. The boundary of this medium will need some extra consideration e.g., the interface between vacuum (air) and a semiconductor. This interface is crucial for reflection of light and we examine this problem in Sects. 3.1.1–3.1.4; 5.4.2 and 5.6. Here we only want to state that the boundary conditions allow a surface mode, that is, a wave which propagates along the interface and has field amplitudes which decay exponentially on both sides. These waves are also known as surface polaritons for reasons discussed in more detail in Sect. 5.6.

### 2.5 Photons and Some Aspects of Quantum Mechanics and of Dispersion Relations

Maxwell’s equations are together with the Lorentz force (2.1a–g) the basis of the classical theory of light. They describe problems like light propagation and the diffraction at a slit or a grating e.g., in the frame of Huygen’s principle or of Fourier optics [93S1] or the Ref. [95L1, 01H3, 07M1, 07S1] of Chap. 1.

In the interaction of light with matter, its quantum nature becomes apparent, e.g., in the photoelectric effect which shows that a light field of frequency \( \omega \) can exchange energy with matter only in quanta \( \hbar \omega \). Therefore, the proper description of light is in terms of quantum mechanics or of quantum electrodynamics. However, we shall not go through these theories here in detail nor do we want to address the aspects of quantum statistics of coherent and incoherent light sources, but we present in the following some of their well-known results and refer the reader to the corresponding literature [02L1, 69Q1, 70Q1, 73C1, 73L1, 85G1, 92M1, 94A1, 94B1, 01M1, 01T1, 02B1, 02D1, 02G1, 02Y1, 09A1, 09W1] for a comprehensive discussion and to [00P1, 02L1, 05E1, 16M1] for the beginning of the history of the quantum concept.
The electromagnetic fields can be described by their potentials $A$ and $\phi$ by

$$E = -\text{grad} \phi - \dot{A}; \quad B = \nabla \times A.$$  \hfill (2.48)

where $A$ is the so-called vector potential. Since $\nabla \cdot (\nabla \times A) \equiv 0$ the notation of (2.48) fulfills automatically $\nabla \cdot B = 0$ and reduces the six components of $E$ and $B$ to four.

The vector potential $A$ is not exactly defined by (2.48). A gradient of a scalar field can be added. We can choose the so-called Coulomb gauge

$$\nabla \cdot A = 0.$$  \hfill (2.49)

In this case $\phi$ is the usual electrostatic potential obeying the Poisson equation:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0 \varepsilon(\omega)}.$$  \hfill (2.50)

In vacuum we still have $\rho = 0$ and we assume the same for the description of the optical properties of matter.

Now we should carry out the procedure of second quantization, for simplicity again for plane waves. A detailed description of how one begins with Maxwell’s equations and arrives at photons within the framework of second quantization is beyond the scope of this book see [55S1, 71F1, 73H1, 76H1, 80H1, 85G1, 92M1, 94A1, 94B1]. On the other hand we want to avoid that the creation and annihilation operators appear like a “deus ex machina”. Therefore we try at least to outline the procedure.

First we have to write down the classical Hamilton function $H$ which is the total energy of the electromagnetic field using $A$ and $\phi$. Then we must find some new, suitable quantities $p_{k,s}$ and $q_{k,s}$ which are linear in $A$ and which fulfill the canonic equations of motion

$$\frac{\partial H}{\partial q_{k,s}} = -\dot{p}_{k,s}, \quad \frac{\partial H}{\partial p_{k,s}} = -\dot{q}_{k,s}$$  \hfill (2.51)

and are thus canonically conjugate variables. Here $k$ is the wave vector of our plane electromagnetic or $A$-wave and $s$ the two possible transverse polarizations. The Hamilton function reads in these variables:

$$H = \frac{1}{2} \sum_{k,s} (p_{k,s}^2 + \omega_k^2 q_{k,s}^2).$$  \hfill (2.52)

This is the usual form of the harmonic oscillator. The quantization condition

$$p_{k,s} q_{k,s} - q_{k,s} p_{k,s} = \frac{\hbar}{i}$$  \hfill (2.53)
for all $k$ and $s = 1, 2$ gives then the well-known result for the harmonic oscillator: The electromagnetic radiation field has for every $k$ and polarization $s$ energy steps

$$E_k = \left(n_k + \frac{1}{2}\right) \hbar \omega_k \quad \text{with} \quad n_k = 0, 1, 2 \ldots$$ (2.54)

It can exchange energy with other systems only in units of $\hbar \omega$. These energy units or quanta are called photons. The term $\hbar \omega/2$ in (2.54) is the zero-point energy of every mode of the electromagnetic field.

The so-called particle-wave dualism, that is, the fact that light propagates like a wave showing, e.g., diffraction or interference and interacts with matter via particle-like quanta, can be solved by the simple picture that light is an electromagnetic wave, the amplitude of which can have only discrete values so that the energy in the waves just fulfills (2.54).

From the above introduced, or better, postulated quantities $p_{k,s}$ and $q_{k,s}$ we can derive by linear combinations operators $a_{k,s}^\dagger$ and $a_{k,s}$ with the following properties: If $a_{k,s}$ acts on a state which contains $n_{k,s}$ quanta of momentum $k$ and polarization $s$ it produces a new state with $n_{k,s} - 1$ quanta. Correspondingly, $a_{k,s}^\dagger$ increases $n_{k,s}$ by one. We call therefore $a_{k,s}$ and $a_{k,s}^\dagger$ annihilation and creation operators, respectively. Since the operators $a_{k,s}$ and $a_{k,s}^\dagger$ describe bosons (see below), their permutation relation is

$$a_{k,s} a_{k,s}^\dagger - a_{k,s}^\dagger a_{k,s} = 1.$$ (2.55a)

This holds for equal $k$ and $s$. The commutator is zero otherwise.

The operator $a_{k,s}^\dagger a_{k,s}$ acting on a photon state gives the number of photons $n_{k,s}$ times the photon state and is therefore called the number operator. Summing over all possible $k$-values and polarizations $s$ gives finally the Hamilton operator

$$H = \sum_{k,s} \hbar \omega_{k,s} a_{k,s}^\dagger a_{k,s}.$$ (2.55b)

It is clear to the author that the short outline given here is not sufficient to explain the procedure to a reader who is not familiar with it. However, since the intent is not to write a textbook on quantum electrodynamics, we want to stress here only that the electromagnetic radiation field in vacuum can be brought into a mathematical form analogous to that of the harmonic oscillator, and that quantum mechanics gives for every harmonic oscillator the energetically equidistant terms of (2.54).

The harmonic oscillator is one of the fundamental systems, which has been investigated in physics and is understood in great detail. In theoretical physics a problem can be considered as “solved” if it can be rewritten in the form of the harmonic oscillator. Apart from the electromagnetic radiation field in vacuum, we will come across some other systems which are treated in this way. For those readers who are not familiar with the concept of quantization and who wish to study the procedure in a quiet hour by themselves, we recommend the above given references or [08M1].
Here are some more results: A single photon carries a spin component in the direction of propagation $s_z = \pm \hbar$. The two basic polarizations of single quanta of the electromagnetic field, – of the photons – are therefore left and right circular $\sigma^-$ and $\sigma^+$, respectively. A linearly polarized wave can be considered as a coherent superposition of a left and right circularly polarized one with equal frequencies, amplitudes and wave vector $\mathbf{k}$. The term coherent means that two light beams have a fixed-phase relation relative to each other. The component of the angular momentum $s$ in the direction of the quantization axis which is parallel to $\mathbf{k}$ is for photons thus as stated already above

$$s_z = s_\parallel = \pm \hbar. \quad (2.56)$$

This means that photons have integer spin and are bosons. The third possibility $s_\parallel = 0$ expected for spin one particles is forbidden, because longitudinal electromagnetic waves do not exist at least in vacuum.

Photons in thermodynamic equilibrium are described by Bose-statistics. The occupation probability $f_{\text{BE}}$ of a state with frequency $\omega$ is given by

$$f_{\text{BE}} = \left[ \exp (\hbar \omega / k_B T) - 1 \right]^{-1}, \quad (2.57)$$

where $T$ is the absolute temperature and $k_B$ is Boltzmann’s constant.

The chemical potential $\mu$ which could appear in (2.57) is zero in thermal equilibrium, since the number of photons is not conserved.

Approaches to describe non-thermal photon fields e.g. luminescence by a non-vanishing $\mu$ and Kirchhoff’s law in the sense of a generalized Planck’s law are found in [82W1, 92S2, 95D1, 09H1].

The momentum $p$ of a photon with wave vector $\mathbf{k}$ is given, as for all quanta of harmonic waves, by

$$p = \hbar k. \quad (2.58)$$

where $k$ is the real part of the wave vector, which describes as already mentioned the oscillatory, propagating aspect of the plane wave.

To summarize, we can state that photons are bosons with spin $\pm \hbar$, energy $\hbar \omega$ and momentum $\hbar k$ which propagate according to the wave equations.

A very important property of particles in quantum mechanics is their dispersion relation. By this we mean the dependence of energy $E$ or frequency $\omega$ on the wave vector $\mathbf{k}$ i.e., the $E(k)$ or $\omega(k)$ relation. For photons in vacuum we find the classical relation given already in (2.13)

$$E = \hbar \omega = \hbar c k. \quad (2.59)$$

The dispersion relation for photons in vacuum is thus a linear function with slope $\hbar c$ as shown in Fig. 2.2. Correspondingly we find again both for phase and group velocity with (2.15)

$$v_{\text{ph}} = v_g = c. \quad (2.60)$$
We conclude this subsection with an explanation of energy units. In the SI system the energy unit is 1 N m = 1 kg m$^2$/s$^2$ with the following identity relations

\[ 1 \text{ N m} = 1 \text{ kg m}^2/\text{s}^2 = 1 \text{ V A s} = 1 \text{ W s} = 1 \text{ J}. \] (2.61a)

Since the energies of the quanta in optical spectroscopy are much smaller, we frequently use the unit 1 eV. This is the energy that an electron gains if it passes, in vacuum, through a potential difference of one volt, resulting in

\[ 1 \text{ eV} = 1.60217733 \times 10^{-19} \text{ J} \approx 1.6 \times 10^{-19} \text{ J}. \] (2.61b)

In spectroscopy another measure of energy is frequently used the wave number. The definition is as follows. One expresses the energy of a (quasi-) particle by the number of wavelengths per cm of a photon in vacuum with the same energy. So

\[ 1 \text{ eV} \equiv 8065.4 \text{ cm}^{-1} \text{ or } 10^4 \text{ cm}^{-1} \equiv 1.23986 \text{ eV} \] (2.61c)

Another quantity that is sometimes confused with the wave number, which gives the energy and is therefore a scalar quantity, is the wave vector (see Sect. 2.2), since it has also the dimension 1/length.

The amount of the (real part of the) wave vector is given by \( k = 2\pi/\lambda \), where \( \lambda \) is the wavelength of the corresponding quantum or particle (electron, phonon, photon, etc.). The direction of \( k \) is the direction of propagation, i.e., \( k \) is normal to the wavefront, in the case of light in vacuum or in matter normal to \( D \times B \). The quantity \( k \) is very closely related to the (quasi-)momentum of the (quasi-)particle \( p \) through

\[ p = \hbar k. \] (2.58)

For the discussion of the concept of quasi-momentum see, e.g., Sects. 5.2, 5.3 and 5.5 or [98B2]. The dispersion relation of (quasi-) particles is thus given by \( E(k) \).
The wave vector of light is in the visible in vacuum falling in the range of a few times $10^4 \, \text{cm}^{-1}$ while the border of the first Brillouin zone (see Sect. 7.2) is of the order of $10^8 \, \text{cm}^{-1}$.

It is obvious, that a quantity like a wave-number vector (Wellenzahlvektor) is ill-defined and does not exist! To make the difference between wave vector and wave number absolutely clear we give the following example: an optical phonon (see Chaps. 7 and 11) can have for a wave vector zero i.e. $k = 0 \, \text{cm}^{-1}$ an energy of e.g. $500 \, \text{cm}^{-1}$.

### 2.6 Density of States and Occupation Probabilities

A quantity which is crucial in quantum mechanics for the properties of particles is their density of states. It enters, e.g., in Fermi’s golden rule which allows one to calculate transition probabilities and $\sim$-rates. We want to discuss this problem in a general way for systems of different dimensionalities $d = 3, 2$ and 1. We shall need these results later on for low-dimensional semiconductor structures. The discussion of the density of states, especially in various dimensions, is not so commonly treated as the harmonic oscillator, and so we shall spend some time on this problem and dwell more on the details. At the end of this section we shall also state the occupation probability in thermodynamic equilibrium for classical particles, for fermions and bosons.

If we consider a particle which is described by a wave function $\phi(r)$ then the probability $w$ to find it in a small element of space $d\tau = dx dy dz$ around $r$ is

$$w(r)d\tau = \phi^*(r)\phi(r)d\tau$$  \hspace{1cm} (2.62a)

Since the particle has to be somewhere in the system, $w(r)$ has to be “normalized”, that is,

$$\int_{\text{system}} w(r)d\tau = \int_{\text{system}} \phi^*(r)\phi(r)d\tau = 1.$$  \hspace{1cm} (2.62b)

Here, the functions $\phi(r)$ are of the form $\exp(ikr)$. For normalization purposes a factor has to be added

$$\phi(r) = \Omega^{-1/2} \exp(ikr).$$  \hspace{1cm} (2.63)

---

4The letter $\phi$ has been already used for the electrostatic potential e.g., in (2.50). Since there are more different physical quantities than letters of the alphabet, we sometimes use the same letter for different quantities, but from the context it should be clear what is meant.
The normalization condition (2.62b) results in
\[ \Omega^{-1} \int_{\text{system}} \exp(-i\mathbf{k} \cdot \mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r}) d\tau = \Omega^{-1} \int_{\text{system}} d\tau = \Omega^{-1} V_{\text{system}} = 1, \quad (2.64) \]
where \( V_{\text{system}} \) is the volume of our physical system. Consequently \( \Omega \) is just the volume of the system. To avoid a factor of zero in front of the plane-wave term, one assumes that the system is so big that it contains all physically relevant parts, but that it is not infinite. The simplest choice is a box of length \( L \), or, more precisely speaking, a cube in three dimensions, a square in two, and an interval in one. This procedure is known as “normalization in a box”. Consequently we have
\[ V_{\text{system}} = L^d \quad \text{with} \quad d = \text{dimensionality of the system} \quad (2.65) \]
and
\[ \Omega^{1/2} = L^{d/2} \quad \text{for} \quad d = 3, 2, 1. \quad (2.66) \]

The wave vectors which can exist in such a box are limited by the boundary conditions.

If we assume that we have an infinitely high potential barrier around the box, then the wavefunction must have nodes at the walls (Fig. 2.3a). Consequently the components \( k_i \) of \( \mathbf{k} \) must fulfill
\[ k_i = n_i \frac{\pi}{L}; \quad n_i = 1, 2, 3, \ldots; \quad i = 1, \ldots, d, \quad (2.67) \]
where the index \( i \) runs over all dimensions.

Such a wave is a standing wave, i.e., a coherent superposition of two waves with \( \mathbf{k} \) and \(-\mathbf{k}\) and equal amplitudes. In the following we must consider therefore only positive values of \( k \). The various modes are distributed equally spaced over the \( k_i \)-axes with a spacing \( \Delta k_i \),
\[ \Delta k_i = \frac{\pi}{L}. \quad (2.68) \]
In other words, every state (or mode) needs a volume \( V_k \) in \( k \)-space given by
\[ V_k = \left( \frac{\pi}{L} \right)^d. \quad (2.69) \]

Another approach is to impose periodic boundary conditions. Then the plane wave should have equal amplitude and slope on opposite sides of the cube according to Fig. 2.3b. In this case one can fill the infinite space by adding boxes in all dimensions and one finds:
\[ k_i = n'_i \frac{2\pi}{L}; \quad n'_i = \pm 1, \pm 2, \pm 3 \ldots \quad (2.70) \]
Fig. 2.3 Plane waves which have nodes at the boundaries (a) or which obey periodic boundary conditions (b)

This means for $\Delta k_i$

$$\Delta k_i = \frac{2\pi}{L}.$$  \hspace{1cm} (2.71)

In contrast to the case of standing waves, we now have to consider both positive and negative values of $n_i$. This procedure results finally in the same density of states.

As a consequence we find that plane waves have in Cartesian coordinates in $k$-space a constant density on all axes.

This result can also be derived qualitatively from the uncertainty principle. If a particle is confined to a length $L$ in direction $i$ its momentum has an uncertainly
\[ \Delta p_i = \Delta \hbar k_i \geq \hbar / L. \] Consequently two states, which should be distinguishable or “different”, must have \( k_i \) values, which differ by roughly \( 1/L \).

Often one wants to know the number of states in a shell between \( k \) and \( k + dk \) independent of the direction of \( k \). This question can be answered by introducing polar coordinates in \( k \)-space. The differential volume \( dV_k \) of a shell of thickness \( dk \) in a \( d \)-dimensional \( k \)-space is given by

\[
\begin{align*}
dV_k &= 2dk \quad \text{for } d = 1, \\
dV_k &= 2\pi kd k \quad \text{for } d = 2, \\
dV_k &= 4\pi k^2 dk \quad \text{for } d = 3.
\end{align*}
\]

Depending on the boundary condition we have to take into account only positive (2.67), or positive and negative (2.70), values of \( k \) or \( n_i \) with corresponding modifications of the prefactors in (2.72).

The number \( \hat{D}(k) \) of states in \( k \)-space found between \( k \) and \( k + dk \) in polar coordinates is given by dividing \( dV_k \) by the volume for one state and by multiplying by \( g_s \). The quantity \( g_s \) considers degeneracies such as the spin degeneracy. For photons we have \( g_s = 2 \) according to the \( \sigma^+ \) and \( \sigma^- \) polarizations (see above). The results are

\[
\begin{align*}
\hat{D}(k)dk &= g_s \frac{L}{\pi} dk \quad \text{for } d = 1, \\
\hat{D}(k)dk &= g_s \frac{L^2}{2\pi} k dk \quad \text{for } d = 2, \\
\hat{D}(k)dk &= g_s \frac{L^3}{2\pi^2} k^2 dk \quad \text{for } d = 3.
\end{align*}
\]

The derivation of this result is depicted for \( d = 2 \) in Fig. 2.4. If we neglect constant prefactors and divide by \( dk \) we find

\[
\hat{D}(k) \propto g_s L^d k^{d-1}, \quad d = 1, 2, 3 \ldots \quad (2.74)
\]

If we consider not the number of states in the box of volume \( L^d \) but the density of states \( D(k) \) per unit of space (e.g., per cm\(^3\) or m\(^3\)) the term \( L^d \) in (2.74) disappears yielding

\[
D(k) \propto g_s k^{d-1} \quad (2.75)
\]

The concept of periodic boundary conditions yields the same result.

This result has to be expected since the density of states per unit volume must be independent of the size of the box which we have in mind provided the box is sufficiently large.

We want to stress here that we assumed only plane waves but did not make any specific assumptions about which type of particles are represented by these plane waves – photons, electrons etc. Therefore this result is valid for all particles described by plane waves.
The next step is now to calculate the density of states on the energy axis, i.e.,

\[ D(E) \, dE. \]  \hspace{1cm} (2.76)

This quantity gives the number of states in the energy interval from \( E \) to \( E + dE \). To calculate this quantity we need the specific dispersion relation \( E(k) \) and its inverse \( k(E) \) as seen from the identity (2.77):

\[ D(E) \, dE = D[k(E)] \frac{dk}{dE} \cdot dE = D[k(E)] \frac{1}{|\nabla_k E(k)|} \, dE. \]  \hspace{1cm} (2.77)

The term on the right-hand side of (2.77) gives the generalized equation which is also valid for anistropic cases.

In particular for photons in vacuum we have with (2.59)

\[ k = \frac{E}{\hbar c} = \frac{\omega}{c}; \quad \frac{dk}{dE} = \frac{1}{\hbar c} \]  \hspace{1cm} (2.78)

Inserting this result in (2.77), for the case \( d = 3 \) we find

\[ D(E) \, dE = \frac{E^2}{\pi^2(\hbar c)^3} \, dE \]

or

\[ D(\omega) \, d\omega \propto \omega^2 \, d\omega \]  \hspace{1cm} (2.79a)

For massive particles, i.e. particles the dispersion relation of which can be described by an (effective) mass \( m \) according to

\[ E(k) = \frac{\hbar^2 k^2}{2m} \]  \hspace{1cm} (2.79b)
we obtain with (2.77) for a \(d\)-dimensional space

\[
D(E)\,dE \propto g_s E^{d-1} dE.
\] (2.79c)

This formula includes the well-known square root density of states for massive particles in three dimensions.

We repeat again that the density in \(k\)-space is constant on all Cartesian axes in a \(d\)-dimensional space for all particles, which can be described by a plane wave, but the density of states depends on the individual dispersion relation when plotted as a function of energy.

The next quantity, which we need is the occupation probability of the states discussed above. We restrict ourselves in the following to thermodynamic equilibrium. There are three types of statistics which can be considered:

For classical, distinguishable particles, Boltzmann statistics apply:

\[
f_B = \exp[-(E - \mu)/k_B T].
\] (2.80a)

For bosons, i.e., indistinguishable particles with integer spin, photons being an example, one must use the Bose–Einstein statistics

\[
f_{BE} = \{\exp[(E - \mu)/k_B T] - 1\}^{-1}.
\] (2.80b)

Fermions, or indistinguishable particles with half-integer spin e.g., electrons obey the Fermi–Dirac statistics \(f_{FD}\)

\[
f_{FD} = \{\exp[-(E - \mu)/k_B T] + 1\}^{-1}.
\] (2.80c)

The Boltzmann constant is \(k_B\) and the chemical potential is \(\mu\) which gives the average energy necessary to add one more particle to the system. For fermions \(\mu\) is also known as the Fermi energy \(E_F\). The probability to find a particle in the interval from \(E\) to \(E + dE\) is then given by the product of the density of states \(D(E)\) and the occupation probability \(f\)

\[
D(E)\,f(E, T, \mu)\,dE
\] (2.81)

In Fig. 2.5 we plot \(f_B\), \(f_{BE}\) and \(f_{FD}\) as a function of \((E - \mu)/k_B T\).

The Boltzmann statistics shows the well-known exponential dependence. The Fermi–Dirac statistics never exceeds one, realizing thus Pauli’s exclusion principle. The Bose–Einstein statistics has a singularity for \(E = \mu\). This gives rise to Bose–Einstein condensation, or in other words, a macroscopic population of a single state, if \(\mu\) touches a region with a finite density of states. We come back to this aspect in Sect. 20.5. In this case the species with energies \(E = \mu\) and those with \(E > \mu\) must be considered separately. Furthermore it is obvious from Fig. 2.5 that \(f_{BE}\) and \(f_{FD}\) converge to \(f_B\) for \((E - \mu)/kT > 1\).
In Sects. 7.7, 8.2 and 8.9 or 20.5 we discuss the limits, within which classical Boltzmann statistics is a good approximation and when the use of Fermi–Dirac or Bose–Einstein statistics is obligatory.

The chemical potential $\mu$ is zero in thermodynamic equilibrium for quanta the number of which is not conserved, for e.g., photons or phonons. We introduce this topic in Chap. 7.

If the number $N$ or density $n$ of particles in a system is known, as is the case for electrons at non-relativistic energies, then $\mu$ is well defined by (2.82).

$$\int D(E) f(E, \mu, T) dE = n,$$

which says that the density of particles is equal to the integral over the product of the density of states and the probability that a state is occupied.

As an example, we apply now the above statements to photons in a three-dimensional box in thermodynamic equilibrium. With (2.79b) and (2.80b), (2.83) is obtained.

$$N(\omega) d\omega = D(\omega) f_{BE}(\omega, T) d\omega \propto \omega^2 \left[ \exp \left( \frac{\hbar \omega}{k_B T} \right) - 1 \right]^{-1} d\omega.$$  

(2.83)

For the energy content $U(T)$ of the radiation field, the result is

$$U(T) = \hbar \int_0^\infty \omega N(\omega) d\omega \propto \int_0^\infty \omega^3 \left[ \exp \left( \frac{\hbar \omega}{k_B T} \right) - 1 \right]^{-1} d\omega.$$  

(2.84)

Equation 2.83 is nothing other than Planck’s law of black-body radiation. By substituting $x$ for the variable $\hbar \omega / k_B T$ in (2.84) we immediately find the Stefan–Boltzmann $T^4$ law

$$U(T) \propto T^4 \int_0^\infty x^3 (\exp x - 1)^{-1} dx = T^4 A,$$  

(2.85)

where $A$ is a constant.
2.7 Problems

1. The intensity of the sunlight falling on the earth is, for normal incidence and before its passage through the atmosphere, about 1.5 kW m$^{-2}$. Calculate the electric-field strength.

2. Pulsed high power lasers can be easily focussed to a power density $I$ of 10 GW/cm$^2$. Calculate the $E$ and $B$ fields. Compare them with the electric field in an H atom at a distance of one Bohr radius, and the magnetic field on the surface of the earth, respectively.

3. Calculate the number of photon modes in the visible part of the spectrum ($\approx 400$ nm $\lesssim \lambda_v \lesssim 800$ nm) in a box of 1 cm$^3$.

4. Calculate the momentum and energy of a photon with $\lambda_v = 500$ nm. At which acceleration voltage has an electron the same momentum?

5. Show qualitatively the $B$, $H$ and $M$ fields of a homogeneously magnetized, brick-shaped piece of iron and for a hollow sphere with inner radius $R_0$ and outer radius $R_0 + \Delta R$, which is radially magnetized. Use especially for the second case symmetry considerations together with 2.1.

6. Check whether the maximum of $N(\omega)$ in (2.83) shifts in proportion to $T$ (Wien’s law), originally formulated as $\lambda_{\text{max}} \propto T^{-1}$.

7. Compare the contribution of the electric conductivity of a typical semiconductor to that of the polarisation in (2.23) or (2.25). For which frequencies does the second one dominate?

8. Write down the time and space dependence of a spherical wave. Note that the energy flux density varies usually like the amplitude squared. Is it possible to create a spherical vector wave?

9. Inspect (with the help of a textbook or a computer program) the electric field of a static electric dipole and the near and far fields of an oscillating electric dipole. Note that in the near field the electric and magnetic fields are not orthogonal.

10. Show that the definition $v_g = \frac{1}{\hbar} \text{grad}_k E(k)$ leads, for massive and massless particles, directly to the relation (quasi-)momentum $p = \hbar k$.

11. How does the density of states as a function of energy vary for a linear dispersion relation (like photons) in 3, 2 and 1 dimensional systems?

12. Consider or find in a textbook the pattern of the collective motion of the H$_2$O molecules in a surface water wave. Which effects contribute to the restoring force? Are water waves harmonic waves? What happens at a seashore, where the depth of the water decreases gradually? Is there a net transport of matter? Assume that the particles have an electric charge and move relative to a fixed background of opposite charge. Which charge pattern do you expect close to the surface? Compare with Fig. 4.4b.
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