Chapter 2

Small Random Perturbations on a Finite Time Interval

1 Zeroth Order Approximation

In the space R^r we consider the following system of ordinary differential equations:

$$X_t^\varepsilon = b(X_t^\varepsilon, \varepsilon\xi_t), \qquad X_0^\varepsilon = x. \tag{1.1}$$

Here $\xi_t(\omega)$, $t \geq 0$ is a random process on a probability space $\{\Omega, \mathscr{F}, \mathsf{P}\}$ with values in R^l and ε is a small numerical parameter. We assume that the trajectories of $\xi_t(\omega)$ are right continuous, bounded and have at most a finite number of points of discontinuity on every interval $[0,T]$, $T < \infty$. At the points of discontinuity of ξ_t, where as a rule, (1.1) cannot be satisfied, we impose the requirement of continuity of X_t^ε. The vector field $b(x,y) = (b^1(x,y), \ldots, b^r(x,y))$, $x \in R^r$, $y \in R^l$ is assumed to be jointly continuous in its variables. Under these conditions the solution of problem (1.1) exists for almost all $\omega \in \Omega$ on a sufficiently small interval $[0,T]$, $T = T(\omega)$.

Let $b(x,0) = b(x)$. We consider the random process X_t^ε as a result of small perturbations of the system

$$\dot{x}_t = b(x_t), \qquad x_0 = x. \tag{1.2}$$

Theorem 1.1. *Assume that the vector field $b(x,y)$, $x \in R^r$, $y \in R^l$ is continuous and that (1.2) has a unique solution on the interval $[0,T]$. Then, for sufficiently small ε, the solution of (1.1) is defined for $t \in [0,T]$ and*

$$\mathsf{P}\Big\{\lim_{\varepsilon\to 0} \max_{0\leq t\leq T} |X_t^\varepsilon - x_t| = 0\Big\} = 1.$$

Strictly speaking, this result does not have a probabilistic character and belongs to the theory of ordinary differential equations. We are not going to give a detailed proof but only note that the existence of the solution X_t^ε on the whole interval $[0,T]$ follows from the proof of Peano's theorem on the existence of the solution of an ordinary differential equation (cf., for example, Coddington and Levinson [1]) and the convergence follows from Arzela's theorem on compactness of sets in $\mathbf{C}_{0T}$ if we take into account that the solution of (1.2) is unique.

Now in R^r we consider the stochastic differential equation with a small parameter

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$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon\sigma(X_t^\varepsilon)\dot{w}_t, \qquad X_t^\varepsilon = x. \tag{1.3}$$

This equation might be considered as a special case of (1.1) with $b(x, y) = b(x) + \sigma(x)y$. Nevertheless, here for y we have substituted a white noise process, whose trajectories are not only discontinuous functions but distributions in the general case. Therefore the convergence of the solution of (1.3) to the solution of (1.2), which is obtained for $\varepsilon = 0$, has to be considered separately.

Theorem 1.2. *Assume that the coefficients of* (1.3) *satisfy a Lipschitz condition and increase no faster than linearly*:

$$\sum_i [b^i(x) - b^i(y)]^2 + \sum_{i,j} [\sigma_j^i(x) - \sigma_j^i(y)]^2 \le K^2|x - y|^2,$$
$$\sum_i [b^i(x)]^2 + \sum_{i,j} [\sigma_j^i(x)]^2 \le K^2(1 + |x|^2).$$

Then for all $t > 0$ *and* $\delta > 0$ *we have*

$$\mathsf{M}|X_t^\varepsilon - x_t|^2 \le \varepsilon^2 a(t), \qquad \lim_{\varepsilon\to 0} \mathsf{P}\left\{\max_{0\le s\le t} |X_s^\varepsilon - x_s| > \delta\right\} = 0,$$

where $a(t)$ *is a monotone increasing function, which is expressed in terms of* $|x|$ *and* K.

For the proof, we need the following lemma, which we shall use several times in what follows.

Lemma 1.1. *Let* $m(t)$, $t \in [0, T]$, *be a nonnegative function satisfying the relation*

$$m(t) \le C + \alpha \int_0^t m(s)\, ds, \quad t \in [0, T], \tag{1.4}$$

with C, $\alpha > 0$. *Then*

$$m(t) \le Ce^{\alpha t}$$

for $t \in [0, T]$.

Proof. From inequality (1.4) we obtain

$$m(t)\left(C + \alpha \int_0^t m(s)\, ds\right)^{-1} \le 1.$$

Integrating both sides from 0 to *t*, we obtain

$$\ln\left(C + \alpha \int_0^t m(s)\, ds\right) - \ln C \le \alpha t,$$

which implies that

$$C + \alpha \int_0^t m(s)\, ds \le Ce^{\alpha t}.$$

The last inequality and (1.4) imply the assertion of the lemma.

Now we begin the proof of the theorem. We prove that $\mathsf{M}|X_t^\varepsilon|^2$ is bounded uniformly in $\varepsilon \in [0,1]$. For this, we apply Itô's formula (cf. Sect. 3, Chap. 1) to the function $1 + |X_t^\varepsilon|^2$. Taking into account that the mathematical expectation of the stochastic integral in this formula vanishes, we obtain

$$1 + \mathsf{M}|X_t^\varepsilon|^2 = 1 + |x|^2 + 2\int_0^t \mathsf{M}(X_s^\varepsilon, b(X_s^\varepsilon))\,ds + \varepsilon^2 \int_0^t \mathsf{M}\sum_{i,j}[\sigma_j^i(X_s^\varepsilon)]^2\,ds.$$

Since the coefficients of (1.3) increase no faster than linearly, the last relation implies the estimate

$$\begin{aligned} 1 + \mathsf{M}|X_t^\varepsilon|^2 &\le 1 + |x|^2 + 2\int_0^t \mathsf{M}\sqrt{|X_s^\varepsilon|^2 K^2(1 + |X_s^\varepsilon|^2)}\,ds \\ &\quad + \varepsilon^2 K^2 \int_0^t (1 + M|X_s^\varepsilon|^2)\,ds \\ &\le 1 + |x|^2 + (2K + \varepsilon^2 K^2)\int_0^t (1 + \mathsf{M}|X_s^\varepsilon|^2)\,ds. \end{aligned}$$

Using Lemma 1.1, we conclude that

$$1 + \mathsf{M}|X_t^\varepsilon|^2 < (1 + |x|^2)\exp[(2K + \varepsilon^2 K^2)t]. \tag{1.5}$$

Now we apply Itô's formula to the function $|X_t^\varepsilon - x_t|^2$ and take the mathematical expectation on both sides of the equality:

$$\mathsf{M}|X_t^\varepsilon - x_t|^2 = 2\int_0^t \mathsf{M}(X_s^\varepsilon - x_s, b(X_s^\varepsilon) - b(x_s))\,ds + \varepsilon^2 \int_0^t \mathsf{M}\sum_{i,j}[\sigma_j^i(X_s^\varepsilon)]^2\,ds.$$

It follows from this relation that

$$\mathsf{M}|X_t^\varepsilon - x_t|^2 \le 2K\int_0^t \mathsf{M}|X_s^\varepsilon - x_s|^2\,ds + \varepsilon^2 K^2 \int_0^t (1 + \mathsf{M}|X_s^\varepsilon|^2)\,ds,$$

and using Lemma 1.1, we obtain

$$\mathsf{M}|X_t^\varepsilon - x_t|^2 \le e^{2Kt}\cdot \varepsilon^2 K^2 \int_0^t (1 + \mathsf{M}|X_s^\varepsilon|^2)\,ds.$$

Combining the last inequality and (1.5), we obtain the first assertion of the theorem:

$$\mathsf{M}|X_t^\varepsilon - x_t|^2 \le \varepsilon^2 K^2 e^{2Kt}(1+|x|^2)\int_0^t \exp[(2K+\varepsilon^2K^2)s]\,ds \le \varepsilon^2 a(t).$$

Now we prove the second assertion of Theorem 1.2. It follows from the definition of X_t^ε and x_t that

$$\max_{0\le s\le t}|X_s^\varepsilon - x_s| \le \int_0^t |b(X_s^\varepsilon) - b(x_s)|\,ds + \varepsilon \max_{0\le s\le t}\left|\int_0^s \sigma(X_v^\varepsilon)\,dw_v\right|. \quad (1.6)$$

From Chebyshev's inequality and the first assertion of the theorem we obtain an estimate of the first term on the right side of (1.6):

$$\begin{aligned}\mathsf{P}\left\{\int_0^t |b(X_s^\varepsilon)-b(x_s)|\,ds > \frac{\delta}{2}\right\} &\le 4\delta^{-2}\mathsf{M}\left[\int_0^t |b(X_s^\varepsilon)-b(x_s)|\,ds\right]^2\\ &\le 4tK^2\delta^{-2}\int_0^t \mathsf{M}|X_s^\varepsilon - x_s|^2\,ds\\ &\le 4tK^2\delta^{-2}\varepsilon^2\int_0^t a(s)\,ds\\ &= \varepsilon^2\delta^{-2}a_1(t). \end{aligned} \quad (1.7)$$

The estimation of the second term in (1.6) can be accomplished with use of the generalized Kolmogorov inequality for stochastic integrals:

$$\begin{aligned}\mathsf{P}\left\{\varepsilon\max_{0\le s\le t}\left|\int_0^s \sigma(X_v^\varepsilon)\,dw_v\right| > \frac{\delta}{2}\right\} &\le 4\delta^{-2}\varepsilon^2\int_0^t \sum_{i,j}\mathsf{M}[\sigma_j^i(X_s^\varepsilon)]^2\,ds\\ &= \varepsilon^2\delta^{-2}a_2(t). \end{aligned} \quad (1.8)$$

Estimates (1.6)–(1.8) imply the last assertion of the theorem. □

In some respect we make more stringent assumptions in Theorem 1.2 than in Theorem 1.1. We assumed that the coefficients satisfied a Lipschitz condition instead of continuity. However, we obtained a stronger result in that not only did we prove that X_t^ε converges to x_t, but we also obtain estimates of the rate of convergence. If we make even more stringent assumptions concerning the smoothness of the coefficients, then the difference $X_t^\varepsilon - x_t$ can be estimated more accurately. We shall return to this question in the next section. Now we will obtain a result on the zeroth approximation for a differential equation with a right side of a sufficiently general form.

We consider the differential equation

$$\dot{X}_t^\varepsilon = b(\varepsilon, t, X_t^\varepsilon, \omega), \qquad X_0^\varepsilon = x$$

in R^n. Here $b(\varepsilon,t,x,\omega) = (b^1(\varepsilon,t,x,\omega),\dots,b^r(\varepsilon,t,x,\omega))$ is an r-dimensional vector defined for $x\in R^r$, $t \geqq 0$, $\varepsilon > 0$ and $\omega\in\Omega$.

We assume that the field $b(\varepsilon,t,x,\omega)$ is continuous in t and x for almost all ω for any $\varepsilon > 0$,

$$\sup_{t\geq\Delta,\varepsilon\in(0,1]} \mathsf{M}|b(\varepsilon,t,x,\omega)|^2 < \infty,$$

and for some $K > 0$ we have

$$\sup_{t\geq 0,\varepsilon\in(0,1]} |b(\varepsilon,t,x,\omega) - b(\varepsilon,t,y,\omega)| \leq K|x-y|$$

almost surely for any x, $y \in R^r$, $t \geq 0$, $\varepsilon > 0$. We note that continuity in ε for fixed t, x, ω is not assumed.

Theorem 1.3. *We assume that there exists a continuous function* $\bar{b}(t,x)$, $t > 0$, $x \in R^r$ *such that for any* $\delta > 0, T > 0, x \in R^r$ *we have*

$$\lim_{\varepsilon\to 0} \mathsf{P}\left\{\left|\int_{t_0}^{t_0+T} b(\varepsilon,t,x,\omega)\,dt - \int_{t_0}^{t_0+T} \bar{b}(t,x)\,dt\right| > \delta\right\} = 0 \tag{1.9}$$

uniformly in $t_0 \geq 0$. *Then the equation*

$$\dot{\bar{x}}_t = \bar{b}(t,\bar{x}_t), \qquad \bar{x}_0 = x \tag{1.10}$$

has a unique solution and

$$\lim_{\varepsilon\to 0} \mathsf{P}\left\{\max_{0\leq t\leq T} |X_t^\varepsilon - \bar{x}_t| > \delta\right\} = 0$$

for every $T > 0$ *and* $\delta > 0$.

Proof. First we note that the function $\bar{b}(t,x)$ satisfies a Lipschitz condition in x with the same constant as the function $b(\varepsilon,t,x,\omega)$. Indeed, since the function $\bar{b}(t,x)$ is continuous, by the mean value theorem we have

$$\int_t^{t+\Delta} \bar{b}(s,x)\,ds = \bar{b}(t,x)\Delta + o(\Delta), \quad \Delta \to 0.$$

Taking account of (1.9), we obtain that

$$\begin{aligned} |\bar{b}(t,x) - \bar{b}(t,y)| &= \frac{1}{\Delta}\left|\int_t^{t+\Delta} \bar{b}(s,x)\,ds - \int_t^{t+\Delta} \bar{b}(s,y)\,ds\right| + \frac{o(\Delta)}{\Delta} \\ &\leq \frac{1}{\Delta}\left|\int_t^{t+\Delta} b(\varepsilon,s,x,\omega)\,ds - \int_t^{t+\Delta} b(\varepsilon,s,y,\omega)\,ds\right| \\ &\quad + \frac{o(\Delta)}{\Delta} + \delta_\varepsilon \\ &\leq K|x-y| + \frac{o(\Delta)}{\Delta} + \delta_\varepsilon, \end{aligned}$$

where $\delta_\varepsilon = \delta_\varepsilon(t,\omega) \to 0$ in probability as $\varepsilon \to 0$.

Since this inequality holds for arbitrary small ε and Δ we have

$$|\bar{b}(t,x) - \bar{b}(t,y)| \le K|x-y|. \tag{1.11}$$

It follows from (1.11) that (1.10) has a unique solution.

By the definition of X_t^ε and $\bar{x}_t$ we have

$$\begin{aligned} X_t^\varepsilon - \bar{x}_t &= \int_0^t [b(\varepsilon, s, X_s^\varepsilon, \omega) - \bar{b}(s, \bar{x}_s)]\, ds \\ &= \int_0^t [b(\varepsilon, s, X_s^\varepsilon, \omega) - b(\varepsilon, s, \bar{x}_s, \omega)]\, ds \\ &\quad + \int_0^t [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds. \end{aligned}$$

Define $m(t) = m^\varepsilon(t) = \max_{0 \le s \le t} |X_s^\varepsilon - \bar{x}_s|$. Using the preceding formula, we obtain the inequality

$$m(t) \le K \cdot \int_0^t m(s)\, ds + \max_{0 \le t_1 \le t} \left| \int_0^{t_1} [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds \right|.$$

Then we obtain by Lemma 1.1 that

$$m(T) \le e^{KT} \max_{0 \le t \le T} \left| \int_0^t [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds \right|, \tag{1.12}$$

where T is an arbitrary positive number.

We now show that the maximum on the right side of (1.12) converges to zero in probability as $\varepsilon \to 0$. Let n be a large integer, which we will choose later. Using the Lipschitz condition we have for $t \in [0, T]$,

$$\begin{aligned} &\int_0^t [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds \\ &\quad = \sum_{k=0}^{n-1} \int_{kt/n}^{(k+1)t/n} [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds \\ &\quad = \sum_{k=0}^{n-1} \int_{kt/n}^{(k+1)t/n} [b(\varepsilon, s, \bar{x}_{kt/n}, \omega) - \bar{b}(s, \bar{x}_{kt/n})]\, ds \\ &\qquad + \sum_{k=0}^{n-1} \int_{kt/n}^{(k+1)t/n} [b(\varepsilon, s, \bar{x}_s, \omega) - b(\varepsilon, s, \bar{x}_{kt/n}, \omega)]\, ds \\ &\qquad + \sum_{k=0}^{n-1} \int_{kt/n}^{(k+1)t/n} [\bar{b}(s, \bar{x}_{kt/n}) - \bar{b}(s, \bar{x}_s)]\, ds \\ &\quad = \sum_{k=0}^{n-1} \int_{kt/n}^{(k+1)t/n} [b(\varepsilon, s, \bar{x}_{kt/n}, \omega) - \bar{b}(s, \bar{x}_{kt/n})]\, ds + \rho_{n,t}^\varepsilon, \end{aligned} \tag{1.13}$$

where $|\rho^\varepsilon_{n,t}| < C/n$ and C is a constant depending on the Lipschitz constant K and T.

By condition (1.9), the sum on the last side of the formula converges to zero in probability for given n. Consequently, (1.13) implies that

$$\lim_{\varepsilon\to 0} \mathsf{P}\left\{\max_{0\le k\le n}\left|\int_0^{kT/n} [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds\right| > \frac{\delta}{2}\right\} = 0 \tag{1.14}$$

for $n > 4C/\delta$.

Moreover, we note that

$$\begin{aligned}
&\mathsf{P}\left\{\max_{0\le t\le T}\left|\int_0^{t} [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\, ds\right| > \delta\right\}\\
&\quad\le \mathsf{P}\left\{\max_{0\le k\le n}\left|\int_0^{kT/n} [b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)]\right| ds > \frac{\delta}{2}\right\}\\
&\qquad + \mathsf{P}\left\{\max_{0\le k\le n}\int_{kT/n}^{(k+1)T/n} |b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)|\, ds > \frac{\delta}{2}\right\}.
\end{aligned} \tag{1.15}$$

We estimate the last term by means of Chebyshev's inequality:

$$\begin{aligned}
&\mathsf{P}\left\{\max_{k}\int_{kT/n}^{(k+1)T/n} |b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)|\, ds > \frac{\delta}{2}\right\}\\
&\quad\le n\cdot\max_{k} \mathsf{P}\left\{\int_{kT/n}^{(k+1)T/n} |b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)|\, ds > \frac{\delta}{2}\right\}\\
&\quad\le n\frac{4}{\delta^2}\frac{T^2}{n^2}\sup_{\varepsilon\ge 0, \varepsilon\in(0,1]} \mathsf{M}|b(\varepsilon, s, \bar{x}_s, \omega) - \bar{b}(s, \bar{x}_s)|^2\\
&\quad\le \frac{4T^2}{n\delta^2}\sup_{s\ge 0, \varepsilon\in(0,1]} \mathsf{M}[|\bar{b}(s, \bar{x}_s)| + |b(\varepsilon, s, 0, \omega)| + K|\bar{x}_s|]^2\\
&\quad\le \frac{C_1 T}{n\delta^2},
\end{aligned} \tag{1.16}$$

where C_1 is a constant. Here we have used the fact that

$$\sup_{s\ge 0, \varepsilon\in(0,1]} \mathsf{M}|b(\varepsilon, s, 0, \omega)|^2 < \infty.$$

It follows from (1.14)–(1.16) that the right side of (1.12) converges to zero in probability as $\varepsilon \to 0$. This completes the proof of Theorem 1.3. □

The random process X^ε_t considered in Theorem 1.3 can be viewed as a result of random perturbations of system (1.10). We shall return to the study of similar perturbations in Chap. 7.

2 Expansion in Powers of a Small Parameter

We return to the study of (1.1) and (1.3). In this section we obtain an expansion of X_t^ε in powers of the small parameter ε provided that the functions $b(x, y)$ are sufficiently smooth.

We follow the usual approach of perturbation theory to obtain an expansion

$$X_t^\varepsilon = X_t^{(0)} + \varepsilon X_t^{(1)} + \cdots + \varepsilon^k X_t^{(k)} + \cdots \tag{2.1}$$

of X_t^ε in powers of ε. We substitute this expansion with unknown coefficients $X_t^{(0)}, \ldots, X_t^{(k)}, \ldots$ into (1.1) and expand the right sides in powers of ε. Equating the coefficients of the same powers on the left and right, we obtain differential equations for the successive calculation of the coefficients $X_t^{(0)}, X_t^{(1)}, \ldots$ in (2.1).

We discuss how the right side of (1.1) is expanded in powers of ε. Let $X(\varepsilon)$ be any power series with coefficients from R^r:

$$X(\varepsilon) = c_0 + c_1\varepsilon + \cdots + c_k\varepsilon^k + \cdots .$$

We write

$$\Phi_k = \Phi_k(c_0, c_1, \ldots, c_k, y) = \frac{1}{k!} \frac{d^k b(X(\varepsilon), \varepsilon y)}{d\varepsilon^k}\bigg|_{\varepsilon=0}.$$

It is easy to see that Φ_k depends linearly on c_k for $k \geq 1$ and Φ_k is a polynomial of degree k in the variable y. In particular,

$$\begin{aligned}
\Phi_0 &= b(c_0, 0), \\
\Phi_1 &= B_1(c_0, 0)c_1 + B_2(c_0, 0)y,
\end{aligned}$$

where $B_1(x, y) = (\partial b^i(x, y)/\partial x^k)$ is a square matrix of order r and $B_2(x, y) = (\partial b^i(x, y)/\partial y^k)$ is a matrix having r rows and l columns. It is clear from the definition of Φ_k that the difference $\Phi_k - B_1(c_0, 0)c_k = \Psi_k(c_0, c_1, \ldots, c_{k-1}, y)$ is independent of c_k.

Carrying out the above program, we expand both sides of (1.1) in powers of ε:

$$\begin{aligned}
&\dot{X}_t^{(0)} + \varepsilon \dot{X}_t^{(t)} + \cdots + \varepsilon^k \dot{X}_t^{(k)} + \cdots \\
&\quad = \Phi_0(X_t^{(0)}, \xi_t) + \varepsilon \Phi_1(X_t^{(0)}, X_t^{(1)}, \xi_t) + \cdots \\
&\qquad + \varepsilon^k \Phi_k(X_t^{(0)}, \ldots, X_t^{(k)}, \xi_t) + \cdots .
\end{aligned}$$

Hence we obtain the differential equations

$$
\begin{aligned}
\dot{X}_t^{(0)} &= \Phi_0(X_t^{(0)}, \xi_t) = b(X_t^{(0)}, 0), \\
\dot{X}_t^{(1)} &= \Phi_1(X_t^{(0)}, X_t^{(1)}, \xi_t) = B_1(X_t^{(0)}, 0)X_t^{(1)} + B_2(X_t^{(0)}, 0)\xi_t, \\
&\ \vdots \\
\dot{X}_t^{(k)} &= \Phi_k(X_t^{(0)}, \ldots, X_t^{(k)}, \xi_t) \\
&= B_1(X_t^{(0)}, 0)X_t^{(k)} + \Psi_k(X_t^{(0)}, \ldots, X_t^{(k-1)}, \xi_t). \\
&\ \vdots
\end{aligned}
\tag{2.2}
$$

To these differential equations we add the initial conditions $X_0^{(0)} = x, X_0^{(1)} = 0, \ldots, X_0^{(k)} = 0, \ldots$. If $b(x, y)$ is sufficiently smooth, then (2.2), together with the initial conditions, determine the functions $X_t^{(0)}, X_t^{(1)}, \ldots, X_t^{(k)}, \ldots$ uniquely. The zeroth approximation is determined from the first equation of system (2.2), which coincides with (1.2). If $X_t^{(0)}$ is known, then the second equation in (2.2) is a linear equation in $X_t^{(1)}$. In general, if the functions $X_t^{(0)}, \ldots, X_t^{(k-1)}$ are known, then the equation for $X_t^{(k)}$ will be a nonhomogeneous linear equation with coefficients depending on time.

Theorem 2.1. *Suppose the trajectories of a process $\xi_t(\omega)$ are continuous with probability 1 and that the function $b(x, y)$, $x \in R^r$, $y \in R^l$, has $k+1$ bounded continuous partial derivatives with respect to x and y. Then*

$$X_t^{\varepsilon} = X_t^{(0)} + \varepsilon X_t^{(1)} + \cdots + \varepsilon^k X_t^{(k)} + R_{k+1}^{\varepsilon}(t),$$

where the functions $X_t^{(i)}$, $i = 0, 1, \ldots, k$, are determined from system (2.2) and

$$\sup_{0 \le t \le T} |R_{k+1}^{\varepsilon}(t)| < C(\omega)\varepsilon^{k+1}, \qquad \mathsf{P}\{C(\omega) < \infty\} = 1.$$

Proof. From the definition of X_t^{ε}, $X_t^{(i)}$, $i = 0, 1, \ldots, k$, it follows that the function $R_{k+1}^{\varepsilon}(t) = X_t^{\varepsilon} - \sum_{i=0}^{k} \varepsilon^i X_t^{(i)}$ satisfies the relation

$$
\begin{aligned}
\dot{R}_{k+1}^{\varepsilon}(t) &= b(X_t^{\varepsilon}, \varepsilon\xi_t) - \sum_{i=0}^{k} \varepsilon^i \Phi_i(X_t^{(0)}, \ldots, X_t^{(i)}, \xi_t) \\
&= \left[b(X_t^{\varepsilon}, \varepsilon\xi_t) - b\left(\sum_{i=0}^{k} \varepsilon^i X_t^{(i)}, \varepsilon\xi_t \right) \right] \\
&\quad + \left[b\left(\sum_{i=0}^{k} \varepsilon^i X_t^{(i)}, \varepsilon\xi_t \right) - \sum_{i=0}^{k} \varepsilon^i \Phi_i(X_t^{(0)}, \ldots, X_t^{(i)}, \xi_t) \right].
\end{aligned}
\tag{2.3}
$$

Since the first derivatives of $b(x, y)$ are bounded then the first term on the right side of (2.3) can be estimated in the following way:

$$\left| b(X_t^\varepsilon, \varepsilon\xi_t) - b\left(\sum_{i=0}^{k} \varepsilon^i X_t^{(i)}, \varepsilon\xi_t\right) \right| \le K_1 |R_{k+1}^\varepsilon(t)|, \tag{2.4}$$

where K_1 is a constant.

In the Taylor series of $b(\sum_{i=0}^{k} \varepsilon^i X_t^{(i)}, \varepsilon\xi_t)$ about $(X_t^{(0)}, 0)$, the coefficients of ε^i are equal to Φ_i up to $i = k$. It follows that

$$\left| b\left(\sum_{i=0}^{k} \varepsilon^i X_t^{(i)}, \varepsilon\xi_t\right) - \sum_{i=0}^{k} \varepsilon^i \Phi_i(X_t^{(0)}, \dots, X_t^{(i)}, \xi_t) \right| \\ \le \sum_{\substack{0 \le j \le k-1 \\ 1 \le i_1 \le k; \dots; 1 \le i_j \le k}} K_{i_1,\dots,i_j} \varepsilon^{i_1 + \cdots + i_j + k + 1 - j} |X_t^{(i_1)}| \cdots |X_t^{(i_j)}| |\xi_t|^{k+1-j}. \tag{2.5}$$

Here $K_{i_1,\dots,i_j}$ are constants depending on the maximum absolute value of the $(k+1)$st derivatives of $b(x, y)$, on $i_1, \dots, i_j$ and on the dimension.

The following lemma provides an estimate for $|X_t^{(i)}|$.

Lemma 2.1. *There exist constants $C_i < \infty$ such that*

$$|X_t^{(i)}| \le C_i \cdot \left(\max_{0 \le s \le t} |\xi_s|\right)^i$$

for all $t \le T$.

The lemma can be proved by induction (using (2.2), of course).

If we integrate (2.3) from 0 to t and take into account that $R_{k+1}^\varepsilon(0) = 0$ and inequalities (2.4), (2.5) and Lemma 2.1, as well, we obtain

$$|R_{k+1}^\varepsilon(t)| \le K_1 \int_0^t |R_{k+1}^\varepsilon(s)|\, ds + K_2 t \sum_{i=k+1}^{k(k+1)} \left[\varepsilon \cdot \max_{0 \le s \le t} |\xi_s|\right]^i,$$

where K_2 is a constant. For $\varepsilon \le (2 \max_{0 \le s \le T} |\xi_s|)^{-1}$ the sum on the right side does not exceed $2\varepsilon^{k+1}(\max_{0 \le s \le t} |\xi_s|)^{k+1}$. Using Lemma 1.1, we obtain

$$|R_{k+1}^\varepsilon(t)| \le \varepsilon^{k+1} 2K_2 e^{K_1 t} \left(\max_{0 \le s \le t} |\xi_s|\right)^{k+1}.$$

This completes the proof of the theorem. □

Consequently, if $b(x, y)$ is sufficiently smooth, then X_t^ε can be calculated to any accuracy. For this we have to integrate the equations for $X_t^{(i)}$. All these equations are linear and have approximately the same structure. The zeroth approximation $X_t^{(0)}$ is a nonrandom function while all approximations beginning with the first one are random processes. We remark that $X_t^{(1)}$ is determined from the equation

$$\dot{X}_t^{(1)} = B_1(X_t^{(0)}, 0)X_t^{(1)} + B_2(X_t^{(0)}, 0)\xi_t; \qquad X_0^{(1)} = 0.$$

It is clear from this that $X_t^{(1)}$ can be obtained from ξ_t by means of a linear (nonrandom) transformation. In particular, if ξ_t is a Gaussian process, then $X_t^{(1)}$ is also Gaussian, and consequently, the approximation $X_t^{(0)}+\varepsilon X_t^{(1)}$ of X_t^ε to within values of order ε^2 is a Gaussian process.

We discuss the one-dimensional case in more detail: X_t^ε is a process in R^1 and ξ_t is a one-dimensional process. Then the equation for $X_t^{(1)}$ can be solved by quadratures:

$$X_t^{(1)} = \int_0^t b_2'(X_s^{(0)}, 0)\xi_s \exp\left\{\int_s^t b_1'(X_u^{(0)}, 0)\, du\right\} ds.$$

We write out the equation for $X_t^{(2)}$:

$$\begin{aligned} \dot{X}_t^{(2)} &= b_1'(X_t^{(0)}, 0)X_t^{(2)} + \frac{1}{2}[b_{11}''(X_t^{(0)}, 0)(X_t^{(1)})^2 \\ &\quad + 2b_{12}''(X_t^{(0)}, 0)X_t^{(1)}\xi_t + b_{22}''(X_t^{(0)}, 0)\xi_t^2], \\ X_0^{(2)} &= 0. \end{aligned}$$

Here b_1' and b_2' are the derivatives of $b(x, y)$ with respect to x and y and the b_{ij}'' are the second derivatives of the same function. The equation for $X_t^{(2)}$ can also be solved by quadratures. $X_t^{(2)}$ is a quadratic functional of the process ξ_t. The equations for $X_t^{(i)}$ look similar for $i = 3, 4, \ldots$ and can be integrated successively by quadratures.

These equations become especially simple if $X_0 = x$ is an equilibrium position of the unperturbed system. In this case the functions $X_t^{(i)}$ can be found as solutions of nonhomogeneous linear equations with constant coefficients.

Theorem 2.1 can be used to calculate the expansions, in powers of a small parameter, of smooth functions of X_t^ε and their mathematical expectations. For example, if the first and second derivatives of a function $f(x)$ are bounded, then

$$\begin{aligned} \mathsf{M}f(X_t^\varepsilon) &= \mathsf{M}[f(X_t^{(0)}) + (\nabla f(X_t^{(0)}), X_t^{(1)})\varepsilon + O(\varepsilon^2)] \\ &= f(X_t^{(0)}) + \varepsilon(\nabla f(X_t^{(0)}), \mathsf{M}X_t^{(1)}) + O(\varepsilon^2), \end{aligned}$$

where the function $m(t) = \mathsf{M}X_t^{(1)}$ is a solution of the differential equation

$$\dot{m}(t) = B_1(X_t^{(0)}, 0)m(t) + B_2(X_t^{(0)}, 0)\mathsf{M}\xi_t.$$

We now consider (1.3). Formally we can consider (1.3) as a special case of (1.1) with $b(x, y) = b(x) + \sigma(x)y$. Assuming $b(x)$ and $\sigma(x)$ are sufficiently smooth, we can write down (2.2) for this case:

$$\begin{aligned}
\dot{X}_t^{(0)} &= b(X_t^{(0)}), \qquad X_0^{(0)} = X_0^\varepsilon = x, \\
\dot{X}_t^{(1)} &= B(X_t^{(0)})X_t^{(1)} + \sigma(X_t^{(0)})\dot{w}_t, \qquad X_0^{(1)} = 0, \\
&\vdots \\
\dot{X}_t^{(k)} &= \Phi_k(X_t^{(0)}, \dots, X_t^{(k)}, \dot{w}_t), \qquad X_0^{(k)} = 0. \\
&\vdots
\end{aligned} \tag{2.6}$$

Here $B(x) = (\partial b^i(x)/\partial x^j)$. These equations are all linear in $\dot{w}_t$. The set of the first k equations of system (2.6) may be considered as a stochastic differential equation for the process $X_t^{k+1} = (X_t^{(0)}, X_t^{(1)}, \dots, X_t^{(k)})$. If $b(x)$ and $\sigma(x)$ have bounded derivatives up to order $k+1$, then this stochastic differential equation has a unique solution and determines a $(k+1)$-dimensional Markov process. As in the case of (1.1), the zeroth approximation is a deterministic motion along the trajectories of the unperturbed dynamical system (1.2). The process $X_t^{(1)}$ can be determined from a stochastic differential equation whose drift vector depends linearly on $X_t^{(1)}$ and the diffusion coefficients depend only on t. It can be verified easily that $X_t^{(1)}$ is a Gaussian process. (It follows, for example, from the fact that the solution of a stochastic differential equation can be constructed by the method of successive approximations.) Therefore, the solution of (1.3) to within values of order ε^2 is a Gaussian Markov process $X_t^{(0)} + \varepsilon X_t^{(1)}$ which is nonhomogeneous in time.

As an example, we consider the one-dimensional stochastic differential equation

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon \dot{w}_t, \qquad X_0^\varepsilon = x.$$

The zeroth approximation $X_t^{(0)}$ is the solution of the equation $\dot{X}_t^{(0)} = b(X_t^{(0)})$, $X_0^{(0)} = x$. For $X_t^{(1)}$ we obtain the equation

$$\dot{X}_t^{(1)} = b'(X_t^{(0)})X_t^{(1)} + \dot{w}_t.$$

If we consider $X_t^{(0)}$ known, then the solution of this equation can be written in the form

$$X_t^{(1)} = \int_0^t \exp\left\{\int_s^t b'(X_u^{(0)})\,du\right\} dw_s.$$

Therefore

$$X_t^\varepsilon = X_t^{(0)} + \varepsilon \int_0^t \exp\left\{\int_s^t b'(X_u^0)\,du\right\} dw_s + o(\varepsilon).$$

We formulate the following theorem concerning the expansion of the solution of the stochastic differential equation in powers of a small parameter ε.

Theorem 2.2. *Suppose the coefficients $b^i(x)$ and $\sigma^i_j(x)$ have bounded partial derivatives up to order $k+1$ inclusive.*

Then for the solution X^ε_t of (1.3) we have the expansion

$$X^\varepsilon_t = X^{(0)}_t + \varepsilon X^{(1)}_t + \cdots + \varepsilon^k X^{(k)}_t + R^\varepsilon_{k+1}(t), \tag{2.7}$$

where $X^{(0)}_t, X^{(1)}_t, \ldots, X^{(k)}_t$ are determined from (2.6). The random process $X^{k+1}_t = (X^{(0)}_t, \ldots, X^{(k)}_t)$ is determined by the first $k+1$ equations of system (2.6). The process X^ε_t is approximated to within values of order ε^2 by the Gaussian process $X^{(0)}_t + \varepsilon X^{(1)}_t$. The remainder in (2.7) satisfies the inequality

$$\sup_{0\le t\le T} (\mathsf{M}|R^\varepsilon_{k+1}(t)|^2)^{1/2} \le C\varepsilon^{k+1}, \quad C < \infty.$$

The proof of this theorem differs from that of Theorem 2.1 in some technical details only. However, these technical details require tedious calculations connected with the proof of the differentiability of a solution of a stochastic differential equation with respect to a parameter. These questions are outside the scope of our main theme, and therefore, we do not give the proof of Theorem 2.2 here. The proof can be found in Blagoveshchenskii and Freidlin [1] and Blagoveshchenskii [1].

From the expansion

$$X^\varepsilon_t = X^{(0)}_t + \varepsilon X^{(1)}_t + \cdots + \varepsilon^k X^{(k)}_t + o(\varepsilon^k) \tag{2.8}$$

of the realizations of X^ε_t it is easy to obtain expansions in powers of ε for smooth functionals of realizations. Let the functional F be Fréchet differentiable at the point $X^{(0)}$. The derivative of F at this point is a linear functional $F'(X^{(0)}; h)$. In this case we have

$$F(X^\varepsilon) = F(X^{(0)}) + \varepsilon F'(X^{(0)}; X^{(1)}) + o(\varepsilon) \tag{2.9}$$

as $\varepsilon \to 0$, where $o(\varepsilon)$ is understood in the same way as in (2.8) (uniformly in $t \in [0, T]$ for almost all ω or in the sense of convergence in probability uniformly in t).

In the case where $X^{(1)}_t$ is a Gaussian random process, from the expansion (2.9) we obtain that the value of the functional $F(X^\varepsilon)$ is asymptotically normal with standard deviation proportional to ε. The coefficient of proportionality in this asymptotic standard deviation can be expressed in terms of the derivative $F'(X^{(0)}; h)$ and the correlation function of $X^{(1)}_t$ (the asymptotic mean is equal to $F(X^{(0)})$ provided that $\mathsf{M}X^{(1)}_t = 0$).

Now consider the case where F is twice differentiable, the second derivative being a bilinear functional $F''(X^{(0)}; h_1, h_2)$. We obtain the following expansion to within values of order $o(\varepsilon^2)$ as $\varepsilon \to 0$:

$$
\begin{aligned}
F(X^\varepsilon) = F(X^{(0)}) &+ \varepsilon F'(X^{(0)}; X^{(1)}) \\
&+ \varepsilon^2 \left[\frac{1}{2} F''(X^{(0)}; X^{(1)}, X^{(1)}) + F'(X^{(0)}; X^{(2)}) \right] \\
&+ o(\varepsilon^2); \qquad (2.10)
\end{aligned}
$$

and so on.

For example, if the functional F has the form

$$F(\varphi) = \int_0^T g(\varphi_t)\, dt, \qquad (2.11)$$

then formulas (2.9) and (2.10) take the forms (for the sake of simple notation, we consider the one-dimensional case):

$$\int_0^T g(X_t^\varepsilon)\, dt = \int_0^T g(X_t^{(0)})\, dt + \varepsilon \int_0^T g'(X_t^{(0)}) X_t^{(1)}\, dt + o(\varepsilon); \qquad (2.12)$$

$$
\begin{aligned}
\int_0^T g(X_t^\varepsilon)\, dt = \int_0^T g(X_t^{(0)})\, dt &+ \varepsilon \int_0^T g'(X_t^{(0)}) X_t^{(1)}\, dt \\
&+ \varepsilon^2 \left[\frac{1}{2} \int_0^T g''(X_t^{(0)}) (X_t^{(1)})^2\, dt \right. \\
&\left. + \int_0^T g'(X_t^{(0)}) X_t^{(2)}\, dt \right] + o(\varepsilon^2). \qquad (2.13)
\end{aligned}
$$

In problems connected with random processes, we often have to consider functionals defined in terms of the first exit time of a domain D. In the case of a domain with a smooth boundary, the functional $\tau(\varphi) = \min\{t : \varphi_t \notin D\}$ will not be Fréchet differentiable or even continuous at all points φ of the space of continuous functions. Nevertheless, it will be differentiable at all points φ for which φ_t has a derivative for $t = \tau(\varphi)$ whose direction is not tangent to the boundary. We shall not prove this in the language of derivatives of functionals but rather formulate it directly in the language of expansions in powers of ε.

Theorem 2.3. *Suppose* (2.8) *holds with* $k = 1$. *Let* t_0 *be the first time of exit of* $X_t^{(0)}$ *from a domain* D *and let* τ^ε *be the first time of exit of* X_t^ε *from* D. *Let the boundary* ∂D *of* D *be once differentiable at the point* $X_{t_0}^{(0)}$ *and let* n *be the exterior normal at this point. Suppose that* $(\dot{X}_{t_0}^{(0)}, n) > 0$. *Then we have*

$$\tau^\varepsilon = t_0 - \varepsilon \frac{(X_{t_0}^{(1)}, n)}{(\dot{X}_{t_0}^{(0)}, n)} + o(\varepsilon), \qquad (2.14)$$

$$X_{\tau^\varepsilon}^\varepsilon = X_{t_0}^{(0)} + \varepsilon \left[X_{t_0}^{(1)} - \dot{X}_{t_0}^{(0)} \frac{(X_{t_0}^{(1)}, n)}{(\dot{X}_{t_0}^{(0)}, n)} \right] + o(\varepsilon) \qquad (2.15)$$

as $\varepsilon \to 0$ (*here* $o(\varepsilon)$ *is understood in the sense of convergence with probability* 1 *or convergence in probability depending on how* $o(\varepsilon)$ *is interpreted in the expansion* (2.8)).

Proof. We use the expansion (2.8) on the interval $[0, T]$, where $T > t_0$. First we obtain that $\tau^\varepsilon \to t_0$ as $\varepsilon \to 0$. From this we obtain

$$\begin{aligned} X^\varepsilon_{\tau^\varepsilon} &= X^{(0)}_{\tau^\varepsilon} + \varepsilon X^{(1)}_{\tau^\varepsilon} + o(\varepsilon) \\ &= X^{(0)}_{t_0} + (\tau^\varepsilon - t_0)\dot{X}^{(0)}_{t_0} + o(\tau^\varepsilon - t_0) + \varepsilon X^{(1)}_{t_0} + o(\varepsilon). \end{aligned} \tag{2.16}$$

Taking the scalar product of (2.16) and n, we obtain

$$(X^\varepsilon_{\tau^\varepsilon} - X^{(0)}_{t_0}, n) = (\tau^\varepsilon - t_0)(\dot{X}^{(0)}_{t_0}, n) + o(\tau^\varepsilon - t_0) + \varepsilon(X^{(1)}_{t_0}, n) + o(\varepsilon). \tag{2.17}$$

On the other hand, because of the smoothness of ∂D at the point $X^{(0)}_{t_0}$, the scalar product on the left side of (2.17) will be infinitesimal compared to $X^\varepsilon_{\tau^\varepsilon} - X^{(0)}_{t_0}$. It follows from this and from (2.16) that

$$(X^\varepsilon_{\tau^\varepsilon} - X^{(0)}_{t_0}, n) = o(\tau^\varepsilon - t_0) + o(\varepsilon). \tag{2.18}$$

From (2.17) and (2.18) we obtain the expansion (2.14) for τ^ε. Substituting the expansion in (2.16) again, we obtain (2.15).

The coefficient of ε in the expansion (2.15) can be obtained by projecting $X^{(1)}_{t_0}$ parallel to $X^{(0)}_{t_0}$ onto the tangent hyperplane at $X^{(0)}_{t_0}$.

If the expansion (2.8) holds with $k = 2$, the function $X^{(0)}_t$ is twice differentiable and the random function $X^{(1)}_t$ is once differentiable, then we can obtain an expansion of τ^ε and $X^\varepsilon_{\tau^\varepsilon}$ to within $o(\varepsilon^2)$ (although the corresponding functional is twice differentiable only on some subspace). On the other hand, if $X^{(1)}_t$ is not differentiable (this happens in the case of diffusion processes with small diffusion, considered in Theorem 2.2) then we do not obtain an expansion for τ^ε to within $o(\varepsilon^2)$. We explain why this is so.

The fact is that in the proof of Theorem 2.3 we did not use the circumstance that τ^ε is exactly the first time of reaching the boundary but only that it is a time when X^ε_t is on the boundary, converging to t_0. If we consider a process X^ε_t of a simple form: $X^\varepsilon_t = x_0 + t + \varepsilon w_t$, then the first time τ^ε of reaching a point $x_1 > x_0$ and the last time σ^ε of being at x_1 differ by a quantity of order ε^2. Indeed, by virtue of the strong Markov property with respect to the Markov time τ^ε, we obtain that the distribution of $\sigma^\varepsilon - \tau^\varepsilon$ is the same as that of the random variable $\zeta^\varepsilon = \max\{t : t + \varepsilon w_t = 0\}$. Then, we use the fact that $\varepsilon^{-2}(t\varepsilon^2 + \varepsilon w_{t\varepsilon^2}) = t + \varepsilon^{-1} w_{t\varepsilon^2} = t + \tilde{w}_t$, where $\tilde{w}_t$ is again a Wiener process issued from zero and $\zeta^\varepsilon = \varepsilon^2 \tilde{\zeta}$, where $\tilde{\zeta} = \max\{t : t + \tilde{w}_t = 0\}$.

3 Elliptic and Parabolic Differential Equations with a Small Parameter at the Derivatives of Highest Order

In the theory of differential equations of elliptic or parabolic type, much attention is devoted to the study of the behavior, as $\varepsilon \to 0$, of solutions of boundary value problems for equations of the form $L^\varepsilon u^\varepsilon + c(x)u^\varepsilon = f(x)$ or $\partial v^\varepsilon/\partial t = L^\varepsilon v^\varepsilon + c(x)v^\varepsilon + g(x)$, where L^ε is an elliptic differential operator with a small parameter at the derivatives of highest order:

$$L^\varepsilon = \frac{\varepsilon^2}{2}\sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial^2}{\partial x^i\,\partial x^j} + \sum_{i=1}^{r} b^i(x)\frac{\partial}{\partial x^i}.$$

As was said in Chap. 1, with every such operator L^ε (whose coefficients are assumed to be sufficiently regular) there is associated a diffusion process $X_t^{\varepsilon,x}$. This diffusion process can be given by means of the stochastic equation

$$\dot{X}_t^{\varepsilon,x} = b(X_t^{\varepsilon,x}) + \varepsilon\sigma(X_t^{\varepsilon,x})\dot{w}_t, \qquad X_0^{\varepsilon,x} = x, \tag{3.1}$$

where $\sigma(x)\sigma^*(x) = (a^{ij}(x))$, $b(x) = (b^1(x), \dots, b^r(x))$. For this process we shall sometimes use the notation $X_t^{\varepsilon,x}$, sometimes $X_t^\varepsilon(x)$ (in the framework of the notion of a Markov family), sometimes X_t^ε and in which case we shall write the index x in the probability and consider the Markov process $(X_t^\varepsilon, \mathsf{P}_x)$.

In the preceding two sections of this chapter we obtained several results concerning the behavior of solutions $X_t^{\varepsilon,x}(\omega)$ of (3.1) as $\varepsilon \to 0$. Since the solutions of the boundary value problems for L^ε can be written as mean values of some functionals of the trajectories of the family $(X_t^{\varepsilon,x}, \mathsf{P})$ results concerning the behavior of solutions of boundary value problems as $\varepsilon \to 0$ can be obtained from the behavior of $X_t^{\varepsilon,x}(\omega)$ as $\varepsilon \to 0$. The present section is devoted to these questions.

We consider the Cauchy problem

$$\begin{aligned}
&\frac{\partial v^\varepsilon(t,x)}{\partial t} = L^\varepsilon v^\varepsilon(t,x) + c(x)v^\varepsilon(t,x) + g(x); \quad t > 0,\ x \in R^r, \\
&v^\varepsilon(0,x) = f(x)
\end{aligned} \tag{3.2}$$

for $\varepsilon > 0$ and together with it the problem for the first-order operator which is obtained for $\varepsilon = 0$:

$$\frac{\partial v^0(t,x)}{\partial t} = L^0 v^0 + c(x)v^0 + g(x); \quad t > 0,\ x \in R^r,\ v^0(0,x) = f(x). \tag{3.3}$$

We assume that the following conditions are satisfied.

(1) the function $c(x)$ is uniformly continuous and bounded for $x \in R^r$;
(2) the coefficients of L^1 satisfy a Lipschitz condition;

(3) $k^{-2}\sum\lambda_t^2 \le \sum_{i,j=1}^r a^{ij}(x)\lambda_i\lambda_j \le k^2\sum\lambda_i^2$ for any real $\lambda_1, \lambda_2, \dots, \lambda_r$ and $x \in R^r$, where k^2 is a positive constant.

Under these conditions, the solutions of problems (3.2) and (3.3) exists and are unique.

All results of this paragraph remain valid in the case where the form $\sum a^{ij}(x)\lambda_i\lambda_j$ is only nonnegative definite. However, in the case of degeneracies the formulation of boundary value problems has to be adjusted and the notion of a generalized solution has to be introduced. We shall make the adjustments necessary in the case of degeneracies after an analysis of the nondegenerate case.

Theorem 3.1. *If conditions* (1)–(3) *are satisfied, then the limit* $\lim_{\varepsilon\to 0} v^\varepsilon(t,x) = v^0(t,x)$ *exists for every bounded continuous initial function* $f(x)$, $x \in R^r$. *The function* $v^0(t,x)$ *is a solution of problem* (3.3).

For the proof we note first of all that if condition (3) is satisfied, then there exists a matrix $\sigma(x)$ with entries satisfying a Lipschitz condition for which $\sigma(x)\sigma^*(x) = (a^{ij}(x))$ (cf. Sect. 5, Chap. 1).

The solution of (3.2) can be represented in the following way:

$$v^\varepsilon(t,x) = \mathsf{M} f(X_t^{\varepsilon,x}) \exp\left[\int_0^t c(X_s^{\varepsilon,x})\,ds\right] + \mathsf{M}\int_0^t g(X_s^{\varepsilon,x}) \exp\left[\int_0^s c(X_u^{\varepsilon,x})\,du\right] ds, \tag{3.4}$$

where $X_t^{\varepsilon,x}$ is the Markov family constructed by means of (3.1). It follows from Theorem 1.2 that the processes $X_s^{\varepsilon,x}(\omega)$ converge to $X_s^{0,x}$ (the solution of (1.2) with initial condition $X_0^{0,x} = x$) in probability uniformly on the interval $[0,t]$ as $\varepsilon \to 0$. Taking into account that there is a bounded continuous functional of $X_s^{\varepsilon,x}(\omega)$ under the sign of mathematical expectation in (3.4), by the Lebesgue dominated convergence theorem we conclude that

$$\lim_{\varepsilon\downarrow 0} v^\varepsilon(t,x) = f(X_t^{0,x}) \exp\left[\int_0^t c(X_s^{0,x})\,ds\right] + \int_0^t g(X_s^{0,x}) \exp\left[\int_0^s c(X_u^{0,x})\,du\right] ds.$$

An easy substitution shows that the function on the right side of the equality is a solution of problem (3.3). Theorem 3.1 is proved. □

If we assume that the coefficients of L^ε have bounded derivatives up to order $k+1$ inclusive, then the matrix $\sigma(x)$ can be chosen so that its entries also have $k+1$ bounded derivatives. In this case, by virtue of Theorem 2.2 we can write down an expansion for $X_t^{\varepsilon,x}$ in powers of ε up to order k. If the functions $f(x)$, $c(x)$, and $g(x)$ have $k+1$ bounded derivatives, then, as follows from (2.7), we have an expansion in powers of ε up to order k with remainder of order ε^{k+1}.

Hence, for example, if $g(x) \equiv c(x) \equiv 0$ and $r = 1$, then the solution of problem (3.2) can be written in the form

$$
\begin{aligned}
v^\varepsilon(t,x) &= \mathsf{M}_x f(X_t^\varepsilon) \\
&= \mathsf{M}_x f(X_t^{(0)} + \varepsilon X_t^{(1)} + \cdots + \varepsilon^k X_t^{(k)} + R_{k+1}^\varepsilon(t)) \\
&= \sum_{i=0}^{k} \varepsilon^i \mathsf{M}_x G_i + O(\varepsilon^{k+1}),
\end{aligned} \tag{3.5}
$$

where $X_t^{(0)}, X_t^{(1)}, \ldots, X_t^{(k)}$ are the coefficients mentioned in Theorem 2.2 of the expansion of X_t in powers of the small parameter;

$$G_i = G_i(X_t^{(0)}, \ldots, X_t^{(i)}) = \frac{1}{i!}\frac{d^i}{d\varepsilon^i} f(X_t^{(0)} + \varepsilon X_t^{(1)} + \cdots + \varepsilon^k X_t^{(k)})|_{\varepsilon=0}.$$

We can derive from formula (3.5) and the equations defining the processes $X_t^{(i)}$ that the coefficients of the odd powers of ε vanish. The coefficients of ε^{2m} are the solutions of some first-order partial differential equations; they can, of course, be found by solving systems of ordinary differential equations.

We illustrate the method of finding the coefficients of the expansion of $v^\varepsilon(t,x)$ in the simplest case, i.e., for dimension 1 and up to terms of order ε^2. For the coefficients of the expansion in powers of e of the solution of the stochastic differential equation

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon\sigma(X_t^\varepsilon)\dot{w}_t, \qquad X_0^\varepsilon = x, \tag{3.6}$$

we write out the first three equations in (2.6):

$$\dot{X}_t^{(0)} = b(X_t^{(0)}), \qquad X_0^{(0)} = x; \tag{3.7}$$

$$\dot{X}_t^{(1)} = b'(X_t^{(0)})X_t^{(1)} + \sigma(X_t^{(0)})\dot{w}_t, \qquad X_0^{(1)} = 0, \tag{3.8}$$

$$
\begin{gathered}
\dot{X}_t^{(2)} = b'(X_t^{(0)})X_t^{(2)} + \frac{1}{2}b''(X_t^{(0)})(X_t^{(1)})^2 + \sigma'(X_t^{(0)})X_t^{(1)}\dot{w}_t, \\
X_0^{(2)} = 0.
\end{gathered} \tag{3.9}
$$

The function $X_t^{(0)}$ is nonrandom and another notation for it is $x_t(x)$.

If f is a twice continuously differentiable function, then we have the expansion

$$
\begin{aligned}
f(X_t^\varepsilon) &= f(X_t^{(0)}) + \varepsilon f'(X_t^{(0)})X_t^{(1)} \\
&\quad + \varepsilon^2\left[f'(X_t^{(0)})X_t^{(2)} + \frac{1}{2}f''(X_t^{(0)})(X_t^{(1)})^2\right] + o(\varepsilon^2).
\end{aligned} \tag{3.10}
$$

We take mathematical expectation on both sides:

$$v^\varepsilon(t,x) = \mathsf{M}_x f(x_t^\varepsilon) = f(x_t,(x)) + \varepsilon f'(x_t(x))\mathsf{M}_x X_t^{(1)} \\ + \varepsilon^2[f'(x_t(x))\mathsf{M}_x X_t^{(2)} + \frac{1}{2} f''(x_t(x))\mathsf{M}_x (X_t^{(1)})^2] + o(\varepsilon^2). \quad (3.11)$$

Since the process $X_t^{(1)}$ is Gaussian with zero mean, the coefficient of ε vanishes. To obtain $\mathsf{M}_x(X_t^{(1)})^2$, we apply formula (3.8) and Itô's formula:

$$\frac{d}{dt}(X_t^{(1)})^2 = 2b'(X_t^{(0)})(X_t^{(1)})^2 + 2\sigma(X_t^{(0)})b'(X_t^{(0)})\dot{w}_t + \sigma(X_t^{(0)})^2. \quad (3.12)$$

Taking mathematical expectation on both sides, we obtain the following nonhomogeneous linear differential equation for $\mathsf{M}_x(X_t^{(1)})^2$ with initial condition $\mathsf{M}_x(X_0^{(1)})^2 = 0$:

$$\frac{d}{dt}\mathsf{M}(X_t^{(1)})^2 = 2b'(x_t(x))\mathsf{M}_x(X_t^{(1)})^2 + a(x_t(x)). \quad (3.13)$$

Solving this equation, we also find the solution of the equation for $\mathsf{M}_x X_t^{(2)}$, which can be obtained by taking the mathematical expectation of (3.9):

$$\frac{d}{dt}\mathsf{M}_x X_t^{(2)} = b'(x_t(x))\mathsf{M}_x X_t^{(2)} + \frac{1}{2}b''(x_t(x))\mathsf{M}_x(X_t^{(1)})^2, \qquad \mathsf{M}_x X_0^{(2)} = 0. \quad (3.14)$$

Hence for the determination of the coefficients of the expansion of $v^\varepsilon(t,x)$ in powers of ε to within order 2, it is sufficient to solve the nonlinear equation $\dot{x}_t(x) = b(x_t(x))$ and the two linear equations (3.13) and (3.14).

The same result can be obtained in a simpler way by using standard methods of the theory of differential equations. Nevertheless, methods of probability theory can also be applied to less standard asymptotic problems. For example, suppose the function f is not smooth at a point $y = x_t(x)$ but has a power-like "corner": $f(z) = f(y) + C|z-y|^\alpha + o(|z-y|^\alpha)$ as $z \to y$, $0 < \alpha \le 1$. We use the expansion

$$X_s^\varepsilon = x_s(x) + \varepsilon X_s^{(1)} + o(\varepsilon)$$

of X_s^ε. For $s = t$ we obtain

$$X_t^\varepsilon = y + \varepsilon X_t^{(1)} + o(\varepsilon),$$
$$f(X_t^\varepsilon) = f(y) + \varepsilon^\alpha C|X_t^{(1)}|^\alpha + o(\varepsilon^\alpha),$$
$$v_\varepsilon(t,x) = \mathsf{M}_x f(X_t^\varepsilon) = f(y) + \varepsilon^\alpha C\mathsf{M}_x|X_t^{(1)}|^\alpha + o(\varepsilon^\alpha).$$

We obtain the mathematical expectation above by using the fact that $X_t^{(1)}$ is Gaussian; it is equal to

$$\int_{-\infty}^{\infty} |u|^\alpha \frac{1}{\sqrt{2\pi \mathsf{M}_x(X_t^{(1)})^2}} \exp[-u^2/2\mathsf{M}_x(X_t^{(1)})^2]\, du \\ = \frac{(2\mathsf{M}_x(X_t^{(1)})^2)^{\alpha/2}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{\alpha}{2} + \frac{1}{2}\right).$$

If f vanishes in the neighborhood of $X_t^{(0)}$, the position of the unperturbed dynamical system (1.2) at time t, then all terms of (3.5) vanish. It turns out that in this case $v^\varepsilon(t,x)$ is logarithmically equivalent to $\exp\{-C\varepsilon^{-2}\}$, where C is a constant. We return to this case in the following chapter.

Now we consider Dirichlet's problem for the elliptic equation with a small parameter

$$\frac{\varepsilon^2}{2}\sum_{i,j}^{r} a^{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x^i \partial x^j} + \sum_{i=1}^{r} b^i(x)\frac{\partial u^\varepsilon}{\partial x^i} + c(x)u^\varepsilon(x) = L^\varepsilon u^\varepsilon + c(x)u^\varepsilon = g(x),$$
$$u^\varepsilon(x)|_{\partial D} = \psi(x). \tag{3.15}$$

in a bounded domain $D \in R^r$ with boundary ∂D.

We assume that the coefficients satisfy conditions (1)–(3) and $c(x) \le 0$. For the sake of simplicity, the boundary ∂D of D is assumed to be smooth and the function $\psi(x)$, $x \in \partial D$ continuous. Under these conditions, there exists a unique solution of problem (3.15) for every $\varepsilon \neq 0$. This solution can be written in the form (cf. Sect. 5, Chap. 1)

$$\begin{aligned} u^\varepsilon(x) = \mathsf{M}_x \Bigg[&\psi(X^\varepsilon_{\tau^\varepsilon}) \exp\left[\int_0^{\tau^\varepsilon} c(X^\varepsilon_s)\,ds\right] \\ &- \int_0^{\tau^\varepsilon} g(x^\varepsilon_s) \exp\left[\int_0^s c(X^\varepsilon_v)\,dv\right] ds \Bigg], \end{aligned} \tag{3.16}$$

where $(X_t^\varepsilon, \mathsf{P}_x)$ is the Markov process defined by (3.1) and $\tau^\varepsilon = \min\{t : X_t^\varepsilon \notin D\}$. In case we use the notation $X_t^\varepsilon(x)$, we also write $\tau^\varepsilon(x)$.

We shall say that a trajectory $x_t(x)$, $x \in D$ of system (1.2) leaves D in a *regular manner* if $T(x) = \min\{t : x_t(x) \notin D\} < \infty$ and $x_{T(x)+\delta}(x) \notin D \cup \partial D$ for sufficiently small $\delta > 0$.

Theorem 3.2. *Suppose conditions* (1)–(3) *are satisfied and the domain D is bounded and has a smooth boundary. If $c(x) < 0$ for all $x \in D \cup \partial D$ and for a given x, the trajectory $x_t(x)$, $t \ge 0$ does not leave D, then* $\lim_{\varepsilon\to 0} u^\varepsilon(x) = u^0(x)$ *exists and*

$$u^0(x) = -\int_0^\infty g(x_s(x)) \exp\left[\int_0^s c(x_v(x))\,dv\right] ds.$$

If $c(x) \le 0$ for all $x \in D \cup \partial D$ and for a given x, the trajectory $x_t(x)$ leaves D in a regular manner, then

$$\begin{aligned} \lim_{\varepsilon\to 0} u^\varepsilon(x) = u^0(x) = \psi(x_{T(x)}(x)) &\exp\left[\int_0^{T(x)} c(x_s(x))\,ds\right] \\ &- \int_0^{T(x)} g(x_s(x)) \exp\left[\int_0^s c(x_v(x))\,dv\right] ds. \end{aligned}$$

Proof. First let $T(x) = +\infty$. For every $T < \infty$, the distance of the trajectory segment $x_s(x)$, $0 \le s \le T$ from ∂D is positive. We denote this distance by δ_T. For every $\alpha > 0$ and a sufficiently small $\varepsilon_0 > 0$ we have

$$\mathsf{P}\left\{ \max_{0 \le s \le T} |X_s^\varepsilon(x) - x_s(x)| > \frac{\delta_T}{2} \right\} < \alpha \tag{3.17}$$

for $\varepsilon < \varepsilon_0$. This follows from the second assertion of Theorem 1.2. From the definition of δ_T and (3.17) it follows that

$$\mathsf{P}\{\tau^\varepsilon(x) < T\} < \alpha. \tag{3.18}$$

We write

$$c_0 = \min_{x \in D \cup \partial D} |c(x)|, \qquad \psi_0 = \max_{x \in \partial D} |\psi(x)|, \qquad g_0 = \max_{x \in D \cup \partial D} |g(x)|.$$

On the basis of (3.18) we arrive at the following estimation:

$$\begin{aligned} &\left| u^\varepsilon(x) + \int_0^\infty g(x_s(x)) \exp\left[\int_0^s c(x_v(x))\, dv \right] ds \right| \\ &\le \psi_0 e^{-c_0 T} + \int_T^\infty g_0 e^{-c_0 s}\, ds + \alpha(\psi_0 + g_0 c_0^{-1}) \\ &\quad + \mathsf{M} \int_0^T \left| g(X_s^\varepsilon(x)) \exp\left[\int_0^s c(X_v^\varepsilon(x))\, dv \right] \right. \\ &\quad \left. - g(x_s(x)) \exp\left[\int_0^s c(x_v(x))\, dv \right] \right| ds. \end{aligned}$$

Since α and $e^{-c_0 T}$ can be chosen arbitrarily small for ε sufficiently small and $\sup_{0 \le s \le T} |X_s^\varepsilon(x) - x_s(x)| \to 0$ in probability as $\varepsilon \to 0$, the first assertion of the theorem follows from the last inequality.

Now let $x_t(x)$ leave D in a regular manner (Fig. 1). We have $\tau^\varepsilon(x) \to T(x)$ in probability as $\varepsilon \to 0$. Indeed, for every sufficiently small $\delta > 0$ we have

$$x_{T(x)-\delta}(x) \in D, \qquad x_{T(x)+\delta}(x) \notin D \cup \partial D.$$

Let δ_1 be the distance of the trajectory segment $x_s(x)$, $s \in [0, T(x) - \delta]$ from ∂D, let δ_2 be the distance of $x_{T(x)+\delta}(x)$ from ∂D, and let $\bar{\delta} = \min(\delta_1, \delta_2)$. By Theorem 1.2 we have

$$\lim_{\varepsilon \to 0} \mathsf{P}\left\{ \sup_{0 \le s \le T(x)+\delta} |X_s^\varepsilon(x) - x_s(x)| > \bar{\delta} \right\} = 0.$$

This implies that $\tau^\varepsilon(x) \in [T(x) - \delta, T(x) + \delta]$ with probability converging to 1 as $\varepsilon \to 0$. This means that $\tau^\varepsilon(\varepsilon) \to T(x)$ in probability. Using this circumstance and Theorem 1.2, the last assertion of the theorem follows from (3.16).

The passage to the limit under the sign of mathematical expectation is legitimate by virtue of the uniform boundedness of the expression under the sign of mathematical expectation.

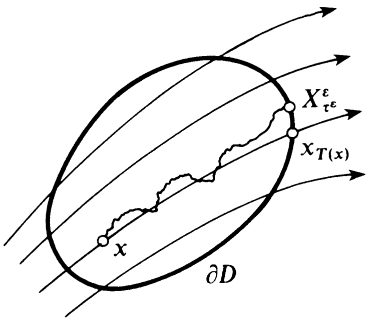

Figure 1.

Now let $c(x)$ be of an arbitrary sign. We only assume that it is continuous. In this case problem (3.15) may go out to the spectrum in general: its solution may not exist for every right side and may not be unique. As was discussed in Sect. 5, Chap. 1, in order that this does not occur it is sufficient that $c(x) \le c_0$ for $x \in D$ and $\mathsf{M}_x e^{c_0 \tau} < \infty$.

Lemma 3.1. *Suppose that for every $x \in D$, the trajectory $x_t(x)$ leaves D in a regular manner and $T(x) \le T_0 < \infty$ for $x \in D$. For some $\delta > 0$, let*

$$\max_{T(x) \le t \le T(x)+\delta} \rho(x_t(x), D \cup \partial D) \ge c > 0$$

for all $x \in D$. Then for any λ there exist $A(\lambda)$ and $\varepsilon(\lambda) > 0$ such that

$$\sup_{x \in D} \mathsf{M}_x e^{\lambda \tau^\varepsilon} \le A(\lambda) < \infty$$

for $\varepsilon \le \varepsilon(\lambda)$.

Proof. As follows from the analysis carried out in the proof of Theorem 3.2, if $x_t(x)$ leaves D in a regular manner, then $\tau^\varepsilon(x) \to T(x)$ in probability as $\varepsilon \to 0$. The conditions $T(x) \le T_0$ and $\max \rho(x_t(x), D \cup \partial D) \ge c$ imply that for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ we have

$$\mathsf{P}\{|\tau^\varepsilon(x) - T(x)| > \delta\} < \delta$$

for all $x \in D$. This implies that

$$\sup_{x \in D} \mathsf{P}_x\{\tau^\varepsilon > 2T_0\} < \delta. \tag{3.19}$$

Moreover, using (3.19) and the Markov property of $(X_t^\varepsilon, \mathsf{P}_x)$, we obtain

$$\begin{aligned} \sup_{x \in D} \mathsf{P}_x\{\tau^\varepsilon > n \cdot 2T_0\} &= \sup_{x \in D} \mathsf{M}_x\{\tau^\varepsilon > (n-1) \cdot 2T_0; \mathsf{P}_{X^\varepsilon_{(n-1)2T_0}}\{\tau^\varepsilon > 2T_0\}\} \\ &\le \delta \cdot \sup_{x \in D} \mathsf{P}_x\{\tau^\varepsilon > (n-1)2T_0\}. \end{aligned}$$

It follows from this inequality that

$$\mathsf{P}_x\{\tau^\varepsilon > n \cdot 2T_0\} < \delta^n$$

for every integer n and $x \in D$.

Since δ can be chosen arbitrarily small for ε sufficiently small, from the last inequality we obtain the assertion of the lemma:

$$\mathsf{M}_x e^{\lambda\tau^\varepsilon} \le \sum_{n=0}^{\infty} e^{\lambda\cdot 2T_0(n+1)} \mathsf{P}_x\{\tau^\varepsilon > 2T_0 n\}$$
$$\le e^{\lambda\cdot 2T_0} \cdot \sum_{n=0}^{\infty} (e^{\lambda\cdot 2T_0}\delta)^n = A(\lambda) < \infty. \quad \square$$

Corollary. *For every $k > 0$ there exists a constant $B = B(k)$ such that $t^k \le Be^t$. By Lemma 3.1 this implies that*

$$\mathsf{M}_x(\tau^\varepsilon)^k \le B(k) M_x e^{\tau^\varepsilon} \le B(k)A(1) = \tilde{A} < \infty.$$

Theorem 3.3. *Suppose that conditions* (1)–(3) *are satisfied, the domain D is bounded and has a smooth boundary ∂D and the function $\psi(x)$ is continuous on ∂D. Suppose furthermore that for all $x \in D$, the trajectories $x_t(x)$ leave D in a regular manner,*

$$\sup_{x\in D} T(x) \le T_0 < \infty,$$

and

$$\max_{T(x)\le t\le T(x)+\delta} \rho(x_t(x), D \cup \partial D) \ge c > 0.$$

Then for every continuous function $c(x)$, $x \in D \cup \partial D$, the problem (3.15) *has a unique solution for sufficiently small ε and*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = u^0(x) = \psi(x_{T(x)}(x)) \exp\left[\int_0^{T(x)} c(x_s(x))\, ds\right]$$
$$- \int_0^{T(x)} g(x_s(x)) \exp\left[\int_0^s c(x_v(x))\, dv\right] ds.$$

Proof. As was indicated in Sect. 5 in Chap. 1, the existence of a unique solution of problem (3.15) and the validity of (3.16) are guaranteed if

$$\mathsf{M}_x \exp\Big\{\tau^\varepsilon \cdot \max_{x\in D\cup\partial D} c(x)\Big\} \le A < \infty.$$

Therefore, the first assertion of the theorem follows from Lemma 3.1.

The second assertion follows from (3.16) if we note that for any $t > 0$ and $x \in D$ we have

$$\tau^\varepsilon(x) \to T(x), \qquad \sup_{0 \le s \le t} |X_s^\varepsilon(x) - x_s(x)| \to 0$$

in probability as $\varepsilon \to 0$ and note that the mathematical expectation of the square of the random variable in (3.16) under the sign of mathematical expectation is bounded uniformly in $\varepsilon < \varepsilon_0$ provided that ε_0 is sufficiently small.

Remark 3.1. If $T(x) < \infty$ but the trajectory $x_t(x)$ does not leave D in a regular manner, then, as follows from simple examples, the limit function may have discontinuities on this trajectory.

Remark 3.2. It is easy to verify that the limit function $u^0(x)$ in Theorem 3.3 satisfies the following first-order equation obtained for $\varepsilon = 0$:

$$L^0 u^0(x) + c(x)u^0(x) = \sum_{i=1}^{r} b^i(x) \frac{\partial u^0}{\partial x^i} + c(x)u^0(x) = g(x).$$

The function $u^0(x)$ is chosen from the solutions of this equation by the condition that it coincides with $\psi(x)$ at those points of the boundary of D through which the trajectories $x_t(x)$ leave D.

Remark 3.3. Now let us allow the matrix $(a^{ij}(x))$ to have degeneracies. In this case problem (3.15) must be modified. First, we cannot prescribe boundary conditions at all points of the boundary. This is easily seen from the example of first-order equations; boundary conditions will not be assumed at some points of the boundary. Second, a classical solution may not exist even in the case of infinitely differentiable coefficients, and it is necessary to introduce the notion of a generalized solution. Third and finally, a generalized solution may not be unique without additional assumptions. To construct a theory of such equations with a nonnegative characteristic form, we can use methods of probability theory. The first results in this area were actually obtained in this way (Freidlin [1], [4], [6]). Some of these results were subsequently obtained by traditional methods of the theory of differential equations. If the entries of $(a^{ij}(x))$ have bounded second derivatives, then there exists a factorization $(a^{ij}(x)) = \sigma(x)\sigma^*(x)$, where the entries of $\sigma(x)$ satisfy a Lipschitz condition. In this case the process $(X^\varepsilon, \mathsf{P}_x)$ corresponding to the operator L^ε is constructed by means of (3.1). In Freidlin's publications [1] and [4], this process is used to make precise the formulation of boundary value problems for L^ε, to introduce the notion of a generalized solution, to prove existence and uniqueness theorems, and to study the smoothness of a generalized solution.

In particular, if the functions $a^{ij}(x)$ have bounded second derivatives and satisfy the hypotheses of Theorem 3.2 or Theorem 3.3, respectively (with the exception of nondegeneracy), then for every sufficiently small ε, the generalized solution exists, is unique, and satisfies (3.16). In this case the assertion of Theorem 3.2 (Theorem 3.3) also holds if by $u^\varepsilon(x)$ we understand the generalized solution.

After similar adjustments, Theorem 3.1 also remains valid.

Theorems 3.2 and 3.3 used results concerning the limit behavior of X_t^ε which are of the type of law of large numbers. From finer results (expansions in powers of ε) we can obtain finer consequences concerning the asymptotics of the solution of Dirichlet's problem. Concerning the expansion of the solution in powers of a small parameter (in the case of smooth boundary conditions), the best results are not obtained by methods of pure probability theory but rather by purely analytical or combined (cf. Holland [1]) methods. We consider an example with nonsmooth boundary conditions.

Let the characteristic $x_t(x)$, $t \geq 0$ issued from an interior point x of a domain D with a smooth boundary leave the domain, intersecting its boundary for the value t_0 of the parameter; at the point $y = x_{t_0}(x)$ the vector $b(y)$ is directed strictly outside the domain. Let u^ε be a solution of the Dirichlet problem $L^\varepsilon u^\varepsilon = 0$, $u^\varepsilon \to 1$ as we approach some subdomain Γ_1 of the boundary and $u^\varepsilon \to 0$ as we approach the interior points of $\partial D \backslash \Gamma_1$ (and u^ε is assumed to be bounded everywhere). Suppose that the surface area of the boundary of Γ_1 is equal to zero. Then the solution $u^\varepsilon(x)$ is unique and can be represented in the form

$$u^\varepsilon(x) = \mathsf{M}_x \chi_{\Gamma_1}(X^\varepsilon_{\tau^\varepsilon}).$$

If y is an interior point of Γ_1 or $\partial D \backslash \Gamma_1$, then the value of u^ε at the point x converges to 1 or 0, respectively, as $\varepsilon \to 0$ (results concerning the rate of convergence must rely on results of the type of large deviations; cf. Chap. 6, Theorems 2.1 and 2.2). On the other hand, if y belongs to the boundary of the domain Γ_1, then the expansion (2.15) reduces the problem of asymptotics of $u^\varepsilon(x)$ to the problem of asymptotics of the probability that the Gaussian random vector $X_{t_0}^{(0)} - \dot{X}_{t_0}^{(0)}[(X_{t_0}^{(1)}, n)/(\dot{X}_{t_0}^{(0)}, n)]$ hits the ε^{-1} times magnified projection of Γ_1 onto the tangent plane (tangent line in the two-dimensional case). In particular, in the two-dimensional case if Γ_1 is a segment of an arc with y as one endpoint, then $\lim_{\varepsilon \to 0} u^\varepsilon(x) = \frac{1}{2}$. The same is true in the higher dimensional case provided that the boundary of Γ_1 is smooth at y. If this boundary has a "corner" at y, then the problem reduces to the problem of the probability that a normal random vector with mean zero falls into an angle (solid angle, cone) with vertex at zero. Using an affine transformation, one can calculate the angle (solid angle).

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