I. Introduction

Over the years I have heard a number of complaints about the impenetrable literature on measure-valued branching processes or Dawson-Watanabe superprocesses. These concerns have in part been addressed by some recent publications including Don Dawson’s St. Flour notes (Dawson (1993)), Eugene Dynkin’s monograph (Dynkin (1994)) and Jean-Francois Le Gall’s ETH Lecture Notes (Le Gall (1999)). Nonetheless, one still hears that several topics are only accessible to experts. However, each time I asked a colleague what topics they would like to see treated in these notes, I got a different suggestion. Although there are some other less flattering explanations, I would like to think the lack of a clear consensus is a reflection of the large number of different entry points to the subject. The Fleming-Viot processes, used to model genotype frequencies in population genetics, arise by conditioning the total mass of a superprocess to be one (Etheridge and March (1991)). When densities exist (as for super-Brownian motion in one spatial dimension) they typically are solutions of parabolic stochastic pde’s driven by a white noise and methods developed for their study often have application to large classes of stochastic pde’s (e.g. Mueller and Perkins (1992), Krylov (1997b), Mytnik (1998) and Section III.4). Dawson-Watanabe superprocesses arise as scaling limits of interacting particle systems (Cox, Durrett and Perkins (1999, 2000)) and of oriented percolation at criticality (recent work of van der Hofstad and Slade (2000)). Rescaled lattice trees above eight dimensions converge to the integral of the super-Brownian cluster conditioned to have mass one (Derbez and Slade (1998)). There are close connections with class of nonlinear pde’s and the interaction between these fields has led to results for both (Dynkin and Kuznetsov (1996,1998), Le Gall (1999) and Section III.5). They provide a rich source of exotic path properties and an interesting collection of random fractals which are amenable to detailed study (Perkins (1988), Perkins and Taylor (1998), and Chapter III).

Those looking for an overview of all of these developments will not find them here. If you are looking for “the big picture” you should consult Dawson (1993) or Etheridge (2000). My goal in these notes is two-fold. The first is to give a largely self-contained graduate level course on what John Walsh would call “the worm’s-eye view of superprocesses”. The second is to present some of the topics and methods used in the study of interactive measure-valued models.

Chapters II and III grew out of a set of notes I used in a one-semester graduate course on Superprocesses. A version of these notes, recorded by John Walsh in a legible and accurate hand, has found its way to parts of the community and in fact been referenced in a number of papers. Although I have updated parts of these notes I have not tried to introduce a good deal of the more modern machinary, notably Le Gall’s snake and Donnelly and Kurtz’s particle representation. In part this is pedagogical. I felt a direct manipulation of branching particle systems (as in II.3,II.4) allows one to quickly gain a good intuition for superprocesses, historical processes, their martingale problems and canonical measures. All of these topics are described in Chapter II. In the case of Le Gall’s snake, Le Gall (1999) gives an excellent and authoritative treatment. Chapter III takes a look at the qualitative properties of Dawson-Watanabe superprocesses. Aside from answering a number of natural questions, this allows us to demonstrate the effectiveness of the various tools used to study branching diffusions including the related nonlinear parabolic pde,
historical processes, cluster representations and the martingale problem. Although many of the results presented here are definitive, a number of open problems and conjectures are stated. Most of the Exercises in these Chapters play a crucial role in the presentation and are highly recommended.

My objective in Chapters II and III is to present the basic theory in a middling degree of generality. The researcher looking for a good reference may be disappointed that we are only considering finite variance branching mechanisms, finite initial conditions and Markov spatial motions with a semigroup acting on the space of bounded continuous functions on a Polish space \( E \). The graduate student learning the subject or the instructor teaching a course, may be thankful for the same restrictions. I have included such appendages as location dependent branching and drifts as they motivate some of the interactions studied in Chapters IV and V. Aside from the survey in Section III.7, every effort has been made to provide complete proofs in Chapters II and III. The reader is assumed to have a good understanding of continuous parameter Markov processes and stochastic calculus—for example, the first five Chapters of Ethier and Kurtz (1986) provide ample background. Some of the general tightness results for stochastic processes are stated with references (notably Lemma II.4.5 and (II.4.10), (II.4.11)) but these are topics best dealt with in another course. Finally, although the Hausdorff measure and polar set results in Sections III.3 and III.5 are first stated in their most general forms, complete proofs are then given for slightly weaker versions. This means that at times when these results are used, the proofs may not be self-contained in the critical dimensions (e.g. in Theorem III.6.3 when \( d = 4 \)).

A topic which was included in the original notes but not here is the Fleming-Viot process (but see Exercise IV.1.2). The interplay between these two runs throughout Don Dawson’s St. Flour notes. The reader should really consult the article by Ethier and Kurtz (1993) to complete the course.

The fact that we are able to give such a complete theory and description of Dawson-Watanabe superprocesses stems from the strong independence assumptions underlying the model which in turn produces a rather large tool kit for their study. Chapters IV and V study measure-valued processes which may have state-dependent drifts, spatial motions and branching rates (the latter is discussed only briefly). All of the techniques used to study ordinary superprocesses become invalid or must be substantially altered if such interactions are introduced into the model. This is an ongoing area of active research and the emphasis here is on introducing some approaches which are currently being used. In Chapter IV, a competing species model is used to motivate careful presentations of Dawson’s Girsanov theorem for interactive drifts and of the construction of collision local time for a class of measure-valued processes. In Chapter V, a strong equation driven by a historical Brownian motion is used to model state dependent spatial motions. Section IV.4 gives a discussion of the competing species models in higher dimensions and Section V.5 describes what is known about the martingale problems for these spatial interactions. The other sections in these chapters are again self-contained with complete arguments.

There are no new results contained in these notes. Some of the results although stated and well-known are perhaps not actually proved in the literature (e.g. the disconnectedness results in III.6) and some of the proofs presented here are, I hope, cleaner and shorter. I noticed that some of the theorems were originally derived
using nonstandard analysis and I have standardized the arguments (often using the historical process) to make them more accessible. This saddens me a bit as I feel the nonstandard view, clumsy as it is at times, is pedagogically superior and allows one to come up with novel insights.

As one can see from the outline of the actual lectures, at St. Flour some time was spent on rescaled limits of the voter model and the contact process, but these topics have not made it into these notes. A copy of some notes prepared with Ted Cox and Rick Durrett on this subject was distributed at St. Flour and is available from me (or them) upon request. We were trying to unify and extend these results. As new applications are still emerging, I decided it would be better to wait until they find a more definitive form than rush and include them here. Those who have seen earlier versions of these notes will know that I also had planned to include a detailed presentation of the particle representations of Donnelly and Kurtz (1999). In this case I have no real excuse for not including them aside from running out of time and a desire to keep the total number of pages under control. They certainly are one of the most important techniques available for treating interactive measure-valued models and hence should have been included in the second part of these notes.

There a number of people to thank. First the organizers and audience of the 1999 St. Flour Summer School in Probability for an enjoyable and at times exhausting $2\frac{1}{2}$ weeks. A number of suggestions and corrections from the participants has improved these notes. The Fields Institute invited me to present a shortened and dry run of these lectures in February and March, and the audience tolerated some experiments which were not entirely successful. Thanks especially to Siva Athreya, Eric Derbez, Min Kang, George Skoulakis, Dean Slonowsky, Vrontos Spyros, Hanno Treial and Xiaowen Zhou. Most of my own contributions to the subject have been joint and a sincere thanks goes to my co-authors who have contributed to the results presented at St. Flour and who have made the subject so enjoyable for me: Martin Barlow, Ted Cox, Don Dawson, Rick Durrett, Steve Evans, Jean-Francois Le Gall and Carl Mueller. Finally a special thanks to Don Dawson and John Walsh who introduced me to the subject and have provided ideas which can be seen throughout these notes.
Superprocesses at Saint-Flour
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ISBN: 978-3-642-25431-4