Stochastic PDAE-Model and Associated Monte-Carlo Simulations for Elastic Threads in Turbulent Flows

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Abstract Considering the motion of a long slender elastic thread in turbulent flows, a stochastic aerodynamic drag force concept was derived for a one-way coupling on top of a k-ϵ turbulence description in Marheineke and Wegener (SIAM J. Appl. Math. 66:1703–1726, 2006). In this paper we present a generalization of this concept that allows the simulation of practically relevant fluid-solid interactions and yields very convincing results in comparison to experiments. Thereby, it reduces the complex problem to two surrogate models: a universally valid drag model for all Reynolds number regimes and incident flow directions and a turbulence correlation model.

1 Stochastic Elastic Generalized String Model

Consider a single elastic thread of slenderness ratio $\delta = \mathrm{d}/\mathrm{l} \ll 1$ with length l and circular cross-sections of typical diameter d that is immersed in a subsonic highly turbulent air flow with small pressure gradients and Mach number $\mathrm{Ma} < 1/3$. Its dynamics is mainly due to the acting aerodynamic force. The determination of this force requires in principle a two-way coupling of solid structure and fluid flow with no-slip interface conditions. In case of slender threads and turbulent flows, the needed high resolution and adaptive grid refinement make the direct numerical simulation of the coupled fluid-solid-problem not only extremely costly

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and complex, but also still impossible for practically relevant applications. Since the thread's influence on the turbulent flow is negligibly small due to the slender geometry, it makes sense to associate to the force a stochastic drag that characterizes the turbulent flow effects on the thread and allows a one-way coupling, [5, 6].

We represent the thread as arc-length parameterized time-dependent curve $\mathbf{r} : [0, l] \times \mathbb{R}_0^+ \to \mathbb{R}^3$ with line weight $(\rho \mathrm{A})$. Then, its dynamics can be asymptotically modeled by a system of stochastic partial differential equations with algebraic constraint of inextensibility, i.e.,

$$\|\partial_s \mathbf{r}\|_2 = 1 \tag{1a}$$

$$\begin{aligned}(\rho \mathrm{A})\, \partial_{tt}\mathbf{r}\, ds\, dt = &\{\, \partial_s(\mathrm{T}\, \partial_s \mathbf{r} - \partial_s(\mathrm{EI}\, \partial_{ss}\mathbf{r})) + (\rho \mathrm{A})\, \mathbf{g} + \mathbf{a}(\mathbf{r}, \partial_t \mathbf{r}, \partial_s \mathbf{r}, s, t) \,\}\, ds\, dt \\ &+ \mathbf{A}(\mathbf{r}, \partial_t \mathbf{r}, \partial_s \mathbf{r}, s, t) \cdot d\mathbf{w}_{s,t}\end{aligned} \tag{1b}$$

supplemented with appropriate initial and boundary conditions, where

$$\mathbf{a}(\mathbf{x}, \mathbf{w}, \boldsymbol{\tau}, s, t) = \mathbf{m}(\boldsymbol{\tau}, \bar{\mathbf{u}}(\mathbf{x}, t) - \mathbf{w}, k(\mathbf{x}, t), \nu(\mathbf{x}, t), \rho(\mathbf{x}, t), d(s)), \tag{1c}$$

$$\begin{aligned}\mathbf{A}(\mathbf{x}, \mathbf{w}, \boldsymbol{\tau}, s, t) = &\ \mathbf{L}(\boldsymbol{\tau}, \bar{\mathbf{u}}(\mathbf{x}, t) - \mathbf{w}, k(\mathbf{x}, t), \nu(\mathbf{x}, t), \rho(\mathbf{x}, t), d(s)) \\ &\cdot \mathbf{D}(\boldsymbol{\tau}, \bar{\mathbf{u}}(\mathbf{x}, t) - \mathbf{w}, k(\mathbf{x}, t), \epsilon(\mathbf{x}, t), \nu(\mathbf{x}, t))\end{aligned} \tag{1d}$$

and $\|.\|_2$ is the Euclidean norm. This stochastic elastic generalized string model is deduced from the dynamical Kirchhoff-Love equations [1] for a Cosserat rod being capable of large, geometrically nonlinear deformations, neglecting torsion, [4, 5]. In (1b) the change of the momentum is balanced by the acting internal and external forces. The internal line forces stem from bending stiffness indicated by Young's modulus and the moment of inertia (EI) as well as from traction. The tractive force $\mathrm{T} : [0, l] \times \mathbb{R}_0^+ \to \mathbb{R}$ can be viewed as Lagrangian multiplier to (1a). The external line forces come from gravity $\mathbf{g}$ and aerodynamics $\mathbf{a}$, $\mathbf{A}$.

The aerodynamic force is derived on basis of a stochastic k-ϵ turbulence model. Expressing the instantaneous flow velocity as sum of a mean and a fluctuating part, the Reynolds-averaged Navier-Stokes equations (RANS) yield a deterministic description for the mean velocity $\bar{\mathbf{u}} : \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R}^3$, whereas two further transport equations for the kinetic turbulent energy $k : \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R}^+$ and dissipation rate $\epsilon : \mathbb{R}^3 \times \mathbb{R}_0^+ \to \mathbb{R}^+$ characterize the random fluctuations $\mathbf{u}'$ according to $k = \mathbb{E}[\mathbf{u}' \cdot \mathbf{u}']/2$ and $\epsilon = \nu\, \mathbb{E}[\nabla \mathbf{u}' : \nabla \mathbf{u}']$ with kinematic viscosity ν, density ρ and expectation $\mathbb{E}[.]$. Analogously, the aerodynamic force is split into a mean and a fluctuating part. Acting as additive Gaussian noise in (1b), it depends on the flow quantities $\bar{\mathbf{u}}$, k, ϵ, and ν, ρ, cf. (1c), (1d). Thereby, the deterministic mean force $\mathbf{m} : S^2 \times \mathbb{R}^3 \times (\mathbb{R}^+)^4 \to \mathbb{R}^3$ as well as the associated splitting operator $\mathbf{L} : S^2 \times \mathbb{R}^3 \times (\mathbb{R}^+)^4 \to \mathbb{R}^{3\times3}$ are determined by the chosen air drag model $\mathbf{f}$ which is a function of the mean relative velocity between fluid and thread, $\bar{\mathbf{u}}(\mathbf{r}, t) - \partial_t \mathbf{r}$, and the thread tangent $\partial_s \mathbf{r}$. The correlated fluctuations are asymptotically approximated by Gaussian white noise with turbulence-dependent amplitude, where $(\mathbf{w}_{s,t}, (s,t) \in [0,1] \times \mathbb{R}_0^+)$ denotes a $\mathbb{R}^3$-valued Wiener process (Brownian motion). The amplitude $\mathbf{D} : S^2 \times \mathbb{R}^3 \times (\mathbb{R}^+)^3 \to \mathbb{R}^{3\times 3}$ represents the integral effects of the localized centered Gaussian velocity fluctuations on the relevant thread scales by containing the necessary information of the spatial and temporal correlations of the double-velocity fluctuations $\boldsymbol{\gamma} = \mathbb{E}[\mathbf{u}' \otimes \mathbf{u}']$.

Consequently, the performance of the aerodynamic force mainly relies on two models, i.e. the air drag model $\mathbf{f}$ (inducing $\mathbf{m}$ and $\mathbf{L}$) and the turbulence correlation approximation $\boldsymbol{\gamma}$ (inducing $\mathbf{D}$). Applying the Global-from-Local Concept of [5] we present here local models that we globalize by superposition, for details see [6]. So, we handle the delicate interaction problem by help of two surrogate models: a drag model for an incompressible flow around an inclined infinitely long circular cylinder and a correlation model for incompressible homogeneous isotropic turbulence.

1.0.1 Air Drag Model

In an incompressible flow, the force $\mathbf{f}$ acting on a fixed, infinitely long circular cylinder is exclusively caused by friction and inertia, [7–9]. It depends on the material and geometrical properties (fluid density ρ, kinematic viscosity ν, cylinder diameter d) and the specific inflow situation (inflow velocity $\mathbf{v}$, cylinder orientation $\boldsymbol{\tau}$, $\|\boldsymbol{\tau}\|_2 = 1$). Non-dimensionalizing the line force $\mathbf{f}$ and flow velocity $\mathbf{v}$ with the typical mass $\rho \mathrm{d}^3$, length d and time d^2/ν yields a reduction of the dependencies,

$$\mathbf{f}(\boldsymbol{\tau}, \mathbf{v}, \nu, \rho, \mathrm{d}) = \frac{\rho \nu^2}{\mathrm{d}} f\left(\boldsymbol{\tau}, \frac{\mathrm{d}}{\nu}\mathbf{v}\right), \qquad \mathbf{v} = \frac{\nu}{\mathrm{d}} v.$$

We focus on the dimensionless quantity $f(\boldsymbol{\tau}, v)$. Assuming $v \not\parallel \boldsymbol{\tau}$ at first, we introduce a $(\boldsymbol{\tau}, v)$-induced orthonormal basis $(n, b, \boldsymbol{\tau})$ by

$$n = \frac{v - v_\tau \boldsymbol{\tau}}{v_n}, \quad b = \boldsymbol{\tau} \times n, \qquad v_\tau = v \cdot \boldsymbol{\tau}, \quad v_n = \sqrt{v^2 - v_\tau^2}.$$

Because of the rotational invariance of the force, its components depend only on the scalar products $v \cdot \boldsymbol{\tau}$ and v^2. The binormal force component vanishes in case of a circular cylinder due to symmetry reasons such that

$$f(\boldsymbol{\tau}, v) = f_n(v_n, v_\tau) n + f_\tau(v_n, v_\tau) \boldsymbol{\tau}$$

holds. For the dependencies of normal and tangential component, Hoerner [3] postulated an independence principle which is strictly proved for a stationary flow in [6].

Theorem 1 (Independence Principle)

- *The normal force f_n is independent of the tangential velocity v_τ.*
- *The tangential force f_τ depends linearly on the tangential velocity v_τ.*

The force can be expressed in terms of normal and tangential drag c_n, c_τ or resistance r_n, r_τ functions

$$f_n(v_n, v_\tau) = v_n^2\, c_n(v_n) = v_n\, r_n(v_n), \qquad f_\tau(v_n, v_\tau) = v_\tau\, v_n\, c_\tau(v_n) = v_\tau\, r_\tau(v_n).$$

Model 2 (Universal Drag) *[6] The continuously differentiable drag functions c_n, c_τ are composed of Oseen theory, Taylor heuristic and numerical simulations. They are experimentally validated and hold true for all Reynolds number regimes, $v \not\parallel \boldsymbol{\tau}$*

$$c_n(v_n) = \begin{cases} 4\pi/(S v_n)\left[1 - v_n^2(S^2 - S/2 + 5/16)/(32S)\right] & v_n < v_1 \\ \exp\left(\sum_{j=0}^{3} p_{n,j} \ln^j v_n\right) & v_1 \le v_n \le v_2 \\ 2/\sqrt{v_n} + 0.5 & v_2 < v_n \end{cases}$$

$$c_\tau(v_n) = \begin{cases} 4\pi/((2S-1)v_n)\left[1 - v_n^2(2S^2 - 2S + 1)/(16(2S-1))\right] & v_n < v_1 \\ \exp\left(\sum_{j=0}^{3} p_{\tau,j} \ln^j v_n\right) & v_1 \le v_n \le v_2 \\ \gamma/\sqrt{v_n} & v_2 < v_n \end{cases}$$

with $S(v_n) = 2.0022 - \ln v_n$, transition points $v_1 = 0.1$, $v_2 = 100$, amplitude $\gamma = 2$. The $\mathscr{C}^1$-regularity involves the parameters $p_{n,0} = 1.6911$, $p_{n,1} = -6.7222 \cdot 10^{-1}$, $p_{n,2} = 3.3287 \cdot 10^{-2}$, $p_{n,3} = 3.5015 \cdot 10^{-3}$ and $p_{\tau,0} = 1.1552$, $p_{\tau,1} = -6.8479 \cdot 10^{-1}$, $p_{\tau,2} = 1.4884 \cdot 10^{-2}$, $p_{\tau,3} = 7.4966 \cdot 10^{-4}$.

To be also applicable in the special case of a transversal incident flow $v \parallel \boldsymbol{\tau}$ and to allow for a realistic smooth force f, the drag need to be adapted for $v_n \to 0$. Taking into account Stokes theory for finitely long cylinders a δ-based regularization of the associated resistance functions r_n, r_τ is proposed in [6]. It matches Stokes resistance coefficients of higher order for $v_n \ll 1$ to those of Model 2, assuming $\delta < 3.5 \cdot 10^{-2}$.

Coming back to the turbulent flow around a moving long flexible thread, we generalize the drag by glueing together the locally valid results for the cylinder. The force acting on the thread at a certain position is then given by

$$f(\partial_s r, u(r,t) - \partial_t r) = f(\partial_s r, (\bar{u}(r,t) - \partial_t r) + u'(r,t)),$$

where r and $u = \bar{u} + u'$ describe the dimensionless thread curve and flow velocity. Analogously to the RANS-averaging ansatz, the force is approximated by an appropriately chosen linear Gaussian process that is split into a mean part m and a fluctuation part. The drag fluctuations inherit the stochastic properties of the turbulence by being modeled linearly in the locally isotropic, centered Gaussian velocity fluctuations with the matrix-valued linearization operator L. In particular, we have

$$m(\boldsymbol{\tau}, v, k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f\left(\boldsymbol{\tau}, v + \sqrt{\frac{2k}{3}}\boldsymbol{\xi}\right) \exp\left(\frac{-\boldsymbol{\xi}^2}{2}\right) \mathrm{d}\boldsymbol{\xi}$$

$$L(\boldsymbol{\tau}, v, k) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{3}{2k}} \int_{\mathbb{R}^3} f\left(\boldsymbol{\tau}, v + \sqrt{\frac{2k}{3}}\boldsymbol{\xi}\right) \otimes \boldsymbol{\xi} \exp\left(\frac{-\boldsymbol{\xi}^2}{2}\right) \mathrm{d}\boldsymbol{\xi},$$

depending on thread tangent $\boldsymbol{\tau}$, mean relative velocity v between mean flow and thread and turbulent kinetic energy k. From the dimensionless quantities we return to the associated dimensional ones in (1) by using

$$\mathbf{m}(\boldsymbol{\tau}, \mathbf{v}, \mathrm{k}, v, \rho, \mathrm{d}) = \frac{\rho \nu^2}{\mathrm{d}} m\left(\boldsymbol{\tau}, \frac{\mathrm{d}}{\nu}\mathbf{v}, \frac{\mathrm{d}^2}{\nu^2}\mathrm{k}\right), \quad \mathbf{L}(\boldsymbol{\tau}, \mathbf{v}, \mathrm{k}, v, \rho, \mathrm{d}) = \rho \nu L\left(\boldsymbol{\tau}, \frac{\mathrm{d}}{\nu}\mathbf{v}, \frac{\mathrm{d}^2}{\nu^2}\mathrm{k}\right).$$

1.0.2 Turbulence Correlation Model

The amplitude $\mathbf{D}$ represents the integral effects of the spatial and temporal correlations of the double-velocity fluctuations on the relevant thread scales. In an incompressible, homogeneous and isotropic turbulent flow, $\mathbf{D}$ depends on the turbulence properties (turbulent kinetic energy k, dissipation rate ϵ, kinematic viscosity ν) and the specific thread-flow relation (mean relative velocity $\mathbf{v}$, thread tangent $\boldsymbol{\tau}$, $\|\boldsymbol{\tau}\|_2 = 1$). Non-dimensionalizing the correlation representant $\mathbf{D}$, mean velocity $\mathbf{v}$ and viscosity ν with the typical turbulent length $\mathrm{k}^{3/2}/\epsilon$ and time k/ϵ yields a reduction of the dependencies,

$$\mathbf{D}(\boldsymbol{\tau}, \mathbf{v}, \mathrm{k}, \epsilon, \nu) = \frac{\mathrm{k}^{7/4}}{\epsilon} D\left(\boldsymbol{\tau}, \frac{1}{\sqrt{\mathrm{k}}}\mathbf{v}, \frac{\epsilon}{\mathrm{k}^2}\nu\right), \qquad \mathbf{v} = \sqrt{\mathrm{k}}v, \quad \nu = \frac{\mathrm{k}^2}{\epsilon}\zeta.$$

We proceed with the dimensionless quantity $D(\boldsymbol{\tau}, v, \zeta)$. Considering an advection-driven flow, the correlations $\boldsymbol{\gamma}$ of the velocity fluctuations can be modeled by help of an initial correlation tensor $\boldsymbol{\gamma}_0$ and a temporal decay function φ, i.e.

$$\boldsymbol{\gamma}(x + \hat{x}, t + \hat{t}, \hat{x}, \hat{t}) = \mathbb{E}[u'(x + \hat{x}, t + \hat{t}) \otimes u'(\hat{x}, \hat{t})] = \boldsymbol{\gamma}_0(x - t\bar{u})\, \varphi(t).$$

The Fourier transform $\mathscr{F}_{\boldsymbol{\gamma}_0}$ of the initial correlations is the spectral density which is exclusively determined by the scalar-valued energy spectrum E in case of incompressible isotropic turbulence. Gathering the existing knowledge [2] about E we use

Model 3 (Energy Spectrum) *[5] The continuously differentiable energy spectrum*

$$E(\kappa, \zeta) = C_\mathrm{K} \begin{cases} \kappa_1^{-5/3} \sum_{j=4}^{6} a_j \left(\frac{\kappa}{\kappa_1}\right)^j & \kappa < \kappa_1 \\ \kappa^{-5/3} & \kappa_1 \le \kappa \le \kappa_2 \,, \\ \kappa_2^{-5/3} \sum_{j=7}^{9} b_j \left(\frac{\kappa}{\kappa_2}\right)^{-j} & \kappa_2 < \kappa \end{cases}$$

where the ζ-dependent transition wave numbers κ_1 and κ_2 are implicitly given by

$$\int_0^\infty E(\kappa,\zeta)\,\mathrm{d}\kappa = 1, \qquad \int_0^\infty E(\kappa,\zeta)\,\kappa^2\,\mathrm{d}\kappa = \frac{1}{2\zeta},$$

induces a velocity fluctuation field that agrees with Kolmogorov's 5/3-Law and the k-ϵ *turbulence model. The Kolmogorov constant is* $C_\mathrm{K} = 1/2$. *and the regularity parameters are* $a_4 = 230/9$, $a_5 = -391/9$, $a_6 = 170/9$, $b_7 = 209/9$, $b_8 = -352/9$, $b_9 = 152/9$.

The restriction $0 < \zeta < \zeta_{crit} \approx 3.86$ coming from the condition $0 < \kappa_1 < \kappa_2 < \infty$ is practically irrelevant, since the turbulence theory presupposes $\zeta \ll 1$. The temporal decay is assumed to be $\varphi(t) = \exp(t^2/2)$ with Fourier transform $\mathscr{F}_\varphi$. Then, the tensor-valued amplitude D formulated in the $(\boldsymbol{\tau}, v)$-induced orthonormal basis is

$$\begin{aligned}
D(\boldsymbol{\tau}, v, \zeta) &= d_n(v_n,\zeta)\, n \otimes n \;+\; d_b(v_n,\zeta)\, b \otimes b + d_\tau(v_n,\zeta)\, \boldsymbol{\tau} \otimes \boldsymbol{\tau} \\
d^2_{n,b,\tau}(v_n,\zeta) &= 4\pi \int_0^\infty \frac{E(\kappa,\zeta)}{\kappa}\, l_{n,b,\tau}(v_n\kappa)\,\mathrm{d}\kappa, \\
l_{n,b,\tau}(\kappa) &= \int_0^{\pi/2} \{\sin^2\beta, \cos^2\beta, 1\}\, \mathscr{F}_\varphi(\kappa\cos\beta)\,\mathrm{d}\beta
\end{aligned}$$

Since $d_n^2 + d_b^2 = d_\tau^2$ holds, the effort for the computation of D reduces to the evaluation of two scalar-valued functions d_τ, d_b depending on two parameters.

2 Results and Discussion

The system (1) of stochastic partial differential equation that models the thread dynamics in turbulent flows is implemented in the software tool FIDYST,[1] where it is solved by a method of lines. The use of a spatial finite difference method of higher order ensures the appropriate approximation of the algebraic constraint. The Box-Muller method generates the Gaussian deviates for the stochastic force. Incorporating the force amplitude explicitely, the time integration is realized by a semi-implicit Euler method with step size control.

So far, our proposed stochastic force model is successfully applied to the simulation of thread-turbulence interactions in technical textile manufacturing. In [6] for example, we show its performance in a specific industrial melt-spinning process of nonwoven materials where hundreds of threads are computed in parallel by Monte-Carlo simulations. The numerical results turn out to coincide very well

[1]FIDYST: Fiber Dynamics Simulation Tool developed at Fraunhofer ITWM, Kaiserslautern, for details see [4].

with experimental data. For further applications, numerical simulations and figures we refer to the contribution by Olawsky et al. in this book.

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