2 An Overview of General Equilibrium Theory

2.1 Agents, Behavior and Markets

As economists we are usually interested in how production is organized and in how whatever is produced is eventually distributed among consumers. All these activities take place within specific institutions we know as markets. What condition markets' outcomes, i.e. prices of goods and services and quantities traded, are agents' behavioral characteristics and the market mechanisms that emanate from them, namely, the so-called law of supply and demand. It is common to distinguish two large and distinct groups of agents—households and firms. Each of these groups plays a different role in the marketplace and in the whole economic system as well, depending on the particular type of commodity being traded.

The role of firms is to organize and accomplish the production process of commodities or services, which are then supplied and sold in markets. For production to take place, firms become demanders and buyers of primary (or non-produced, like labor or capital) and non-primary (or produced, like iron or energy) goods and services that will be subjected to the transformation process we call production. The goods that enter into the production process are referred to as inputs while the goods that result are called outputs. In playing this common role, firms are assumed to share a common grand objective: of all possible production plans, they will choose the one that yields the greater return to the firm. For return we mean the difference between the income associated to selling the production plan and the expenditure in inputs needed to make that particular plan possible. This behavioral assumption is basic in economics and is called profit maximization. Nonetheless, the specific way these production activities are undertaken may differ among firms, the reason being that not all firms share or can effectively use the same technology. By technology we mean all the available ways, or recipes, for some good or service to be produced. Electricity producing firms, for instance, aim at generating their output in the most profitable way for them but it is clear that different technological options exist. While some firms will produce electricity using plants of nuclear power, other firms may use coal, water, solar or wind power.

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The role of households is to demand commodities so that they can satisfy their consumption needs. At the same time, households own labor and capital assets that they offer to the firms that need them to carry out the production process. In deciding how much to demand of each good and how much to offer of their assets, consumers value the return they would obtain from the possible consumptions of goods that can be made possible by the sale of their assets. This judgment takes place in terms of an internal value system we call preferences or utility. The assumption that households select the consumption schedule that best fits their value system, under whatever their available income makes possible, is called preference or utility maximization. Utility is a classical concept in economics and even though it is unnecessary in modern microeconomics, it is retained because it happens to be equivalent, and much more convenient to use, than preferences. As is the case with firms, we can find within the households' group different ways of achieving the same objectives, mainly defined by individual preferences, or more simply utility levels. As we well know, there are different ways of feeling content. Some of us prefer riding a bike while enjoying the scenery. Instead, others take pleasure in relaxing at home watching a science-fiction movie, reading a best-selling novel, or listening to classical music. Similarly, even when sharing the same basic preferences, some consumers might be wealthier than others and this will most likely lead to consuming a larger amount of each commodity.

What happens when these two groups of agents interact in the marketplace? Despite the abovementioned heterogeneity in preferences, income levels and technology, we can aggregate – for each possible set of prices – all agents' decisions in two market blocks or sides, i.e. the demand block and the supply block. Households and firms might be either behind the demand or the supply side. In the market for primary inputs, for instance, firms demand labor and capital services while households supply them. In the market for non-primary inputs, or intermediate commodities, some firms are demanders and other firms are suppliers. Lastly, in the markets for goods and services oriented to final uses households demand these commodities for consumption while firms supply them. Households and firms are both demanders and suppliers. Once the two sides of the market are well-defined, the next question is whether these two blocks can reach a "trade agreement" based on a mutually compatible price. In other words, a price at which the specific amount being demanded coincides with the specific amount being produced. This type of coincidental situations, with the matching of demand with supply, is referred to as market equilibrium. In fact, market equilibrium is all about finding prices that are capable of yielding this type of agreement between the two market blocks.

There are other types of economic situations that condition the way prices are set and thus how markets agreements are achieved. Besides agents' behavior, for instance, the number of market participants and the way information is shared among them constitute key determinants for markets' mechanisms and, consequently, for the equilibrium outcome in the markets. Depending on the rules shaping how equilibrium prices are set, markets can be perfectly competitive or imperfectly competitive. In the first case, when markets are perfectly competitive, prices are publicly known by all participants and they have no way to individually exert any influence over those prices. A condition for this is to have large markets. When the number of participants in a market is large, then each one of them is negligible given the size of the market and no one can have an individual influence on prices. We will say that all agents participating in these markets are price takers. It is also worth noting that when a market is considered to be perfectly competitive, there is an implicit assumption that has to do with the production technology and, more specifically, with the link between the unit costs and the scale of production. It is postulated that in perfectly competitive economies this link is constant, in other words, production exhibits Constant Returns to Scale. Decreasing returns to scale are ruled out since it is assumed that replication of any production plan is always possible, provided there are no indivisibilities. Finally, the presence of increasing returns to scale, as a possible alternative in technology, is incompatible with perfectly competitive markets since price-taking behavior would imply that firms incur in losses. In this case, the setting of non-competitive prices by firms opens some room for strategic behavior leading to a rich constellation of possible market organizations. In this introductory book, we will focus our attention exclusively on competitive markets.

Perfectly competitive markets are very appealing since they turn out to have nice properties in terms of welfare. The result that prices equal marginal rates of transformation is a necessary condition and, under convex preferences and convex choice sets, a sufficient one for optimality. Therefore, the way equilibrium prices are set in perfectly competitive markets relies on an efficient mechanism that could not be present with non-convexities in the production set (see Villar 1996). The equilibrium outcome of perfectly competitive markets is known as Walrasian equilibrium in honor of the French mathematical economist Léon Walras (1834–1910), a leading figure of the "marginalist" school at that moment.

The equilibrium outcome of a particular competitive market can be analyzed essentially from two perspectives: from a partial equilibrium perspective or from a general equilibrium perspective. Partial equilibrium implies analyzing one market in isolation from all other markets. This is in fact the very definition of partial equilibrium analysis where only direct effects are taken into account while omitting possible indirect and induced or feedback impacts that occur simultaneously in other interrelated markets. In doing so, as we indicated in the introduction of this book, we make use of the *caeteris paribus* assumption. The general equilibrium approach, however, considers the economy as a closed and interdependent system of markets where equilibrium prices and quantities are the result of all kind of economy-wide interactions, that is, in an equilibrium there is a reflection of the role played by all direct, indirect and induced effects.

Once we have commented on the basic flavor of how market institutions work and, more specifically, have described the essentials of the Walrasian approach to economics, we devote the following two sections of this chapter to present the normative and positive aspects of Walrasian equilibrium theory. We will use the simple case of a pure exchange economy because of its transparency and also because almost everything which is conceptually relevant to the analysis can be discussed using this setup. Technical details will be, for the most part, omitted since we want to focus in the issues rather than in the mathematics.

2.2 Positive Analysis of the Walrasian Equilibrium

2.2.1 Walras' Law and the Walrasian Equilibrium: Definitions

Economists are very much interested in ascertaining how the market allocation problem is solved, i.e. which are the laws and the conditions that regulate the way markets work. In this section we describe these mechanisms in the context of Walrasian markets. In doing so, we will confine our attention to the simplest case of a pure exchange economy, or Edgeworth box economy, where no production possibilities are considered. In this economy the role of agents is simply to trade among themselves the existing stock of commodities. In this pure exchange economy there is a finite number of N different commodities (listed as $i = 1, 2,\ldots, N$) and H different consumers or households or, more generally, agents (listed as $h = (1,\ 2, ..., H)$). A vector such as $x = (x_1, ..., x_i, ...x_N) \in \mathbb{R}^N$ will represent a listing of quantities of those N goods. Each commodity is valued with a non negative unit of market value that we call price and represent by $p = (p_1, ..., p_i, ..., p_N) \in \mathbb{R}^N_+$. Since production possibilities are not considered in this economy, there are initial commodity endowments. Each agent h initially owns a part of these endowments, i.e. $e_h \in \mathbb{R}^N_+$, that along with prices define for all agents their initial wealth and thus their budget constraint.

Agents want to consume goods and this is only possible from the existing level of endowments. These consumptions are a reflection of the preference system of agents once feasibility via prices is established. We will assume that for each agent h preferences in consumption for commodity i are reflected through a non-negative demand function $\chi_{ih}(p, e_h)$ that is continuous and homogenous of degree zero in prices. Each of these individual commodity demands are the result of a well defined restricted optimization problem, but we bypass the details here. The most common way is to assume that preferences are utility representable and thus demand functions are derived from solving the utility maximization problem of every agent, subject to the restriction that the only possible consumptions are those within the agent's budget constraint.

Since the analysis of this pure exchange economy is done from a general equilibrium perspective, we need to impose some "laws" to make this economy become a closed and interdependent system. Therefore, we first notice that the following restriction is satisfied:

$$\sum_{i=1}^{N} p_i \cdot \chi_{ih}(p, e_h) = \sum_{i=1}^{N} p_i \cdot e_{ih} \quad \forall h = 1, ..., H \tag{2.1}$$

Restriction (2.1) says that for each agent the value of the total consumption of available resources should be equal to the value of the owned wealth, or endowment. Consequently, this restriction will be also verified, by summation for all agents, at the level of the whole economy. It is known as Walras' law:

$$\sum_{h=1}^{H}\sum_{i=1}^{N} p_i \cdot \chi_{ih}(p, e_h) = \sum_{h=1}^{H}\sum_{i=1}^{N} p_i \cdot e_{ih} \tag{2.2a}$$

and with some algebraic rearrangement:

$$\begin{aligned}\sum_{i=1}^{N} p_i \sum_{h=1}^{H} (\chi_{ih}(p, e_h) - e_{ih}) &= \sum_{i=1}^{N} p_i \left(\sum_{h=1}^{H} \chi_{ih}(p, e_h) - \sum_{a=1}^{H} e_{ih}\right) \\ &= \sum_{i=1}^{N} p_i \cdot (\chi_i(p) - e_i) = 0 \end{aligned} \tag{2.2b}$$

We have added up all individual demands and endowments and introduced market demand $\chi_i(p)$ and market supply e_i, or total endowment, for each good i. We now introduce the market excess demand function as their difference, that is, $\zeta_i(p) = \chi_i(p) - e_i$. The final and most compact expression for Walras' law will read:

$$\sum_{i=1}^{N} p_i \cdot \zeta_i(p) = 0 \tag{2.2c}$$

This equality is telling us something quite interesting: the value of market excess demands equals zero at all prices, whether or not they are equilibrium prices. Thus, Walras' law is a necessary condition for markets to be in equilibrium but it is not sufficient.

For a set of prices to become an equilibrium in this closed and interdependent economic system the sufficient condition, using market demand and supply, reads as:

$$\zeta_i(p^*) = \chi_i(p^*) - e_i = 0 \tag{2.3}$$

for all markets $i = 1, 2, \ldots, N$ and $p^* > 0$

A Walrasian equilibrium is defined as an allocation of the available level of goods and a set of prices such that condition (2.3) is fulfilled. The allocation is given by the demand functions at the equilibrium prices. Notice that (2.3) is a system of N equations (one for each of the N goods) with N unknowns (one for each of the N prices). This condition of exact equality between demand and supply is in fact a bit stronger than needed. We use it here because it helps in simplifying the presentation of the issues without delving unnecessarily into some technicalities. All that is required, in fact, is that demand is no greater than supply at the equilibrium prices, and hence it is possible in principle that some of the goods are free goods, but then their price should be zero. We restrict ourselves to exact equality and positive prices for all goods as expressed in (2.3), which implicitly requires some type of monotonicity property on consumers' preferences. This desirability assumption, by the way, justifies as well that the budget constraints (2.1) hold as equalities and so by implication does Walras' law.

A key corollary that stems from Walras' law is that *"if all markets but one are cleared, then the remaining market must also be cleared"*. So, to determine equilibrium prices in this economy, we just need to pick up N-1 of the equations from (2.3) and find a solution to that reduced size system. But notice that now we have more variables to determine (N) than available independent equations (N-1). An alternative way of presenting this corollary is by noticing that if there is excess demand in a certain market i ($\zeta_i(p)>0$) then there must be an excess supply in some other market j ($\zeta_j(p)<0$), or the other way around.

For a better understanding of the implications of Walras' law and the definition of Walrasian equilibrium, we move now to the details of a more specific example. We will consider an Edgeworth box economy with two agents, i.e. $h = (1,\ 2)$, and two commodities, i.e. $i = (1,\ 2)$. Agents' preferences in this economy will follow the pattern of Cobb–Douglas utility functions $u_1(x_1, x_2) = x_1^{\alpha^1} \cdot x_2^{1-\alpha^1}$ for agent 1 and $u_2(x_1, x_2) = x_1^{\alpha^2} \cdot x_2^{1-\alpha^2}$ for agent 2. Their consumption possibilities will be limited by the aggregate stocks form their initial property of endowments, so for agent 1 we write $e_1 = (e_{11}, e_{21})$ while for 2 we have $e_2 = (e_{12}, e_{22})$.

The commodity demand functions consistent with solving the utility maximization problem for agent 1 are:

$$\chi_{11}(p, e_1) = \frac{\alpha^1 \cdot (p_1 \cdot e_{11} + p_2 \cdot e_{21})}{p_1} \qquad \chi_{21}(p, e_1) = \frac{(1-\alpha^1) \cdot (p_1 \cdot e_{11} + p_2 \cdot e_{21})}{p_2} \tag{2.4}$$

Similarly for the case of agent *2*:

$$\chi_{12}(p, e_2) = \frac{\alpha^2 \cdot (p_1 \cdot e_{12} + p_2 \cdot e_{22})}{p_1} \qquad \chi_{22}(p, e_2) = \frac{(1-\alpha^2) \cdot (p_1 \cdot e_{12} + p_2 \cdot e_{22})}{p_2} \tag{2.5}$$

We now write, for this simple Cobb-Douglas economy, the two equilibrium equations corresponding to expression (2.3) above:

$$\frac{\alpha^1 \cdot (p_1^* \cdot e_{11} + p_2^* \cdot e_{21})}{p_1^*} + \frac{\alpha^2 \cdot (p_1^* \cdot e_{12} + p_2^* \cdot e_{22})}{p_1^*} = e_{11} + e_{12}$$

$$\frac{(1-\alpha^1) \cdot (p_1^* \cdot e_{11} + p_2^* \cdot e_{21})}{p_2^*} + \frac{(1-\alpha^2) \cdot (p_1^* \cdot e_{12} + p_2^* \cdot e_{22})}{p_2^*} = e_{21} + e_{22} \tag{2.6}$$

Remember though that the corollary of Walras' law implies one of these two equations is redundant. We can focus in solving either one of them, and discard the

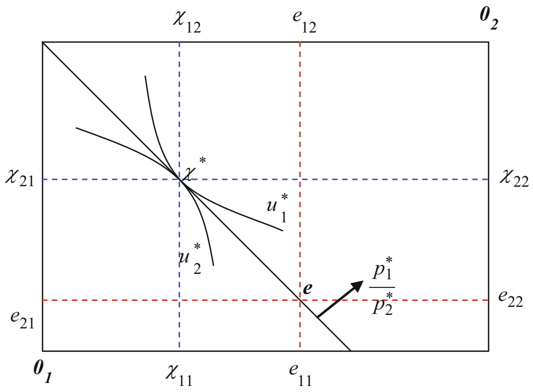


Fig. 2.1 Walrasian equilibrium in a pure exchange economy

other one. Let's check the solution from using the first equation. With a little bit of algebra we would find:

$$\frac{p_2^*}{p_1^*} = \frac{(1-\alpha^1)\cdot e_{11} + (1-\alpha^2)\cdot e_{12}}{\alpha^1 \cdot e_{21} + \alpha^2 \cdot e_{22}} \tag{2.7}$$

Notice that if $p*$ solves the equation, so does $c \cdot p^*$ where c refers to any positive number. Any set of prices that fulfils condition (2.7) will clear both markets. The reader can try and solve the equilibrium set of prices using instead the second equation in expression (2.6). No surprise here, the same ratio of prices will be found. This statement can be checked graphically in the Edgeworth box economy of Fig. 2.1 above where the slope of the budget sets for both agents is given by the price ratio p_1^*/p_2^*. Each agent maximizes his or her utility at the relative price p_1^*/p_2^* and their individual demands are at the same time exactly compatible with the available supply of goods determined by the initial endowment point e.

Notice that only relative equilibrium prices can be determined. If we wish to fix all prices at some absolute level, arbitrary of course, we need some "reference" price fixed from outside. This reference price, or unit of value measurement, is usually called the *numéraire*. With an adequate selection of numéraire, all prices are expressed as relative distances to it, both when determining the initial or benchmark equilibrium scenario and when exploring possible changes in this economy in response to variations, for instance, of the initial level of endowments. We will come back to this issue later on along Chap. 3.

2.2.2 Existence and Uniqueness of Walrasian Equilibrium

Léon Walras (1874a, b) was the first to present the equilibrium in a set of markets as the solution of a system of equations, reflecting how commodities are allocated between agents or groups of agents for a specific set of prices. Nevertheless, Walras did not provide any formal proof of the existence of the solution for this system, and in addition he presupposed the solution to be unique.[1] Later on, during the 1930s, the seminal work of Wald (1934, 1935, 1936a, b) and in a more integrated way, the analysis carried out by Arrow and Debreu (1954) contributed to show under which specific but quite general conditions this market system of equations had a solution relevant for both *"descriptive and normative economics"* (Arrow and Debreu 1954, p. 265).

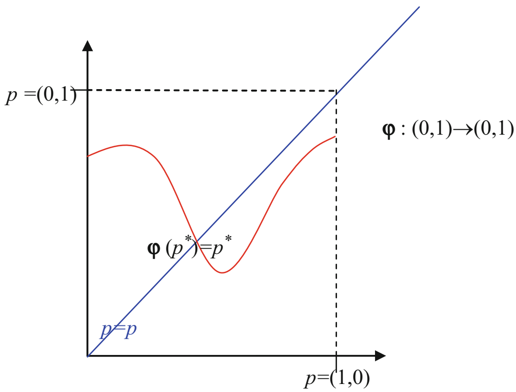
When we think in terms of the pure exchange economy described in the previous section, proving the existence of a general equilibrium is equivalent to prove that there is at least one set of prices, in the non-negative price set, that makes all commodity excess demands equal to zero. By defining general equilibrium prices and allocations in this way, it becomes natural to use theorems from topology to provide a proof of the existence of equilibrium. And within the arsenal of topology's theorems, we find that fixed point theorems have been quite appropriate and powerful mathematical tools for this purpose. The two main theorems that were instrumental to Arrow and Debreu in answering the existence question in Walrasian general equilibrium theory are Brouwer's (1911–1912) and Kakutani's (1941) fixed point theorems, the latter being an extension of the former. Since Brouwer's theorem can be stated with far less mathematical apparel than Kakutani's, we will present it here. First, the unit simplex S is defined as the subset of points p in $\mathbb{R}_+^N$ such that $p_1 + p_2 + \ldots + p_N = 1$ and from the definition we can easily see that S is a closed and convex set. For general equilibrium theory, as in Debreu (1952) and Arrow and Debreu (1954), the proper statement of the theorem takes the following form.

Brouwer's fixed point theorem: Let $\varphi(p)$ be a continuous function $\varphi : S \to S$ with $p \in S$, then there is a $p^* \in S$ such that $\varphi(p^*) = p^*$.

In Fig. 2.2 we illustrate the theorem for the simple situation of a continuous real valued function defined in the unit interval, a case where the theorem is almost graphically self-evident. As we can also see in this Figure, this theorem implies that if the function $\varphi(\cdot)$ is continuous, its graph cannot go from the left edge to the right edge without intersecting the diagonal at least once. That intersection is a fixed point. In the picture there is just one fixed point but the reader can easily visualize that should the graph of the function intersect the diagonal a second time, then it must necessarily do it a third time. In words, there would be in this case a finite and odd number of fixed points. This possibility is in fact a very remarkable result that holds for so-called *regular* economies. In the spirit of Fig. 2.2, an economy is called regular when the mapping $\varphi(\cdot)$ only and always intersects the diagonal. Expressed

[1] Walras (1874a, b) was wrong when posing and facing the question about the uniqueness of equilibrium. In fact, aggregate excess demand function and not only individual excess demand functions must present additional properties for the equilibrium to become unique and stable, since not all the properties of the later are inherited into the former. This is the so-called Sonnenschein-Mantel-Debreu Theorem (1973, 1974, 1974). The concept of uniqueness is also related to the concept of stability (Arrow and Hurwicz 1958, 1959). An excellent exposition of uniqueness and stability of equilibrium for pure exchange economies and economies with production can be found in Elements of General Equilibrium Analysis (1998), Chap. III, written by Timothy J. Kehoe.

Fig. 2.2 Illustration of Brouwer's fixed point theorem in the unit interval



in economics terms it says that in regular economies there will always be when a finite but odd number of equilibrium price configurations. Another outstanding result is that almost all economies are regular. Non-regular economies can nonetheless be devised but in probabilistic terms they would be observed with zero probability. The theory of regular economies is due to Debreu (1970).

The fixed point refers in fact to the equilibrium set of prices. In proving the existence of a fixed point, and thus making the theorem applicable to the economic problem of existence, we need first a continuous function that maps the set of prices into itself. Furthermore, this continuous function should verify the version of Walras' law in expression (2.3). It seems then that one good candidate function should refer to a continuous transformation of the excess demand functions, since a continuous transformation of continuous functions is also continuous. Remember also that excess demand functions are homogenous of degree zero in prices, and thanks to this prices can be transformed into points belonging to the unit simplex S using the normalization $\sum_{i=1}^{N} p_i$.

How can the continuous function $\varphi(\cdot)$ be defined?. The literature has provided alternative ways of defining this mapping (Gale 1955; Nikaido 1956; Debreu 1956), but a function $\varphi(\cdot)$ with a possible economic interpretation should always be preferred. To this effect, the mapping $\varphi(\cdot)$ could be viewed as a "price adjustment function" mimicking the way the so-called Walrasian auctioneer would adjust initial prices following the law of demand and supply until equilibrium prices are achieved (Gale 1955), that is, the fixed point is determined. The Walrasian auctioneer is just a symbolic way to represent the way markets are supposed to adjust.

Consider this function:

$$\varphi_i(p) = \frac{p_i + \max[0, \zeta_i(p)]}{1 + \sum_{j=1}^{N} \max[0, \zeta_j(p)]} \tag{2.8}$$

The function in (2.8) is indeed a price adjustment function and it is known as the Gale-Nikaido mapping. To see how this price adjustment function works, take an initial set of prices p. If such p were an equilibrium, then for all i $\zeta_i(p) = 0$ and by the definition in (2.8) we would have $\varphi_i(p) = p_i$, since all the max operators would be zero. In other words, if p is an equilibrium then p is also a fixed point of the mapping in (2.8). Now, we will not be usually so lucky and when picking up a price vector p it will most likely not be an equilibrium price. This means that in some market i its excess demand function in (2.3) will not be zero. If $\zeta_i(p)>0$ then the numerator of (2.8) indicates that the price of good i should be increased precisely by the value of the positive excess demand. The denominator ensures that the new adjusted price remains in the simplex. Notice also that by summation for all i in (2.8) we would find that $\sum_{i=1}^{N} \varphi_i(p) = 1$ and so price adjustments take always place in the simplex S. If instead $\zeta_i(p)<0$ then by the corollary of Walras' law there has to be another market with positive excess demand, say market k where $\zeta_k(p)>0$. The adjustment rule now acts modifying the price of good k, and so on. The Walrasian auctioneer will raise prices wherever there is a positive excess demand in order to make the consumption of those goods less attractive to households. Since prices are relative when raising the price p_k for the good with a positive excess demand, the auctioneer is implicitly reducing p_i in the market with a negative excess demand.

The idea behind the adjustment function, or Walrasian auctioneer, is that markets work to smooth out the differences between demand and supply. These price adjustments will cease when all excess demand functions are zero for some positive price vector p^*:

$$\zeta_i(p^*) = 0 \qquad \forall i = 1, 2, \ldots, N \tag{2.9}$$

which is the definition of equilibrium prices in our pure exchange economy.

The next step is to show that the mapping $\varphi(p)$ in expression (2.8) has a fixed point and that the fixed point is indeed an equilibrium. The first part is just an application of Brouwer's theorem since the mapping $\varphi(p)$ satisfies all of its requirements. It is continuous, it is defined on the simplex S and its images are all in the simplex S. As a consequence there is a vector p^* such that $\varphi(p^*) = p^*$. The second part involves showing that for this vector p^* we indeed have $\zeta_i(p^*) = 0$ for all i. Since $\varphi_i(p^*) = p_i^*$ we can replace it in expression (2.8):

$$p_i^* = \frac{p_i^* + \max[0, \zeta_i(p^*)]}{1 + \sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right]} \tag{2.10}$$

Rearranging terms and simplifying we would find:

$$p_i^* \cdot \sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right] = \max[0, \zeta_i(p^*)] \tag{2.11}$$

Multiply now by the excess demand function for good i and add all of them up to obtain:

$$\left(\sum_{i=1}^{N} p_i^* \cdot \zeta_i(p^*)\right) \cdot \left(\sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right]\right) = \sum_{i=1}^{N} \max\left[0, \zeta_i^2(p^*)\right] \tag{2.12}$$

By Walras' law the left-hand side of (2.12) is necessarily zero and so must therefore be the right-hand side. From here it must follow that all $\zeta_i(p^*) = 0$. Should one of them not be zero, say $\zeta_k(p^*) \neq 0$, then $\zeta_k^2(p^*)>0$ and the right hand side could not be zero, a contradiction.

We have seen that an equilibrium price vector p^* is a fixed point for the mapping $\varphi(p)$ and a fixed point p^* for this mapping is an equilibrium for the economy. This is a result often overlooked or forgotten but very relevant since it says that the existence theorem and the fixed point theorem are in fact equivalent statements.

We now review the question of the number of equilibria. We commented before, while reviewing Fig. 2.2, that such a number will always be finite and odd for the vast majority of economies. This issue is very relevant in fact for the appropriateness of the method of comparative statistics, which is routinely used in applied general equilibrium models for the evaluation of specific policies (Kehoe 1985, 1991). The exploration of the economy-wide impacts of policies is undertaken comparing the initial equilibrium (i.e. the benchmark) with the equilibrium that would ensue after the policy change is enacted and absorbed (i.e. the counterfactual). If the equilibrium is unique for each set of structural and policy parameters defining the economy and it behaves continuously, then comparative statics make sense. We can compare the two equilibria and from their comparison we can extract valuable information on how the policy affects the economy. If multiple equilibria are possible, however, we can get into methodological trouble. The reason is that a change in parameters may move the economy to one of the different equilibria with no a priori information on which one will actually be, nor if other policy changes will yield a transition to a different equilibrium. In short, in the presence of non-uniqueness it would be difficult to really know what we are comparing. Notice also that when an economy has a unique equilibrium point for each possible parameter configuration, then it is both globally and locally unique and it does not matter for comparative statics if we perform an experiment where a parameter change is large or small. Anything goes and it goes nicely. When an economy has multiple equilibria, the next question is whether those equilibrium points are locally unique. If they are, comparative statics would still make sense, provided changes are small enough to be contained in a close neighborhood of the initial equilibrium configuration. With large parameter changes, however, comparative statics is methodologically risky since we do not really know where the economy may be jumping to.

The theoretical conditions that guarantee the uniqueness of equilibrium have been widely explored by many economists (Wald 1936b; Arrow and Hahn 1971;

Kehoe 1980, 1985, 1991; Kehoe and Mas-Colell 1984; Kehoe and Whalley 1985 and Mas-Colell 1991). For a pure exchange economy, uniqueness of equilibrium can be assured if the aggregate excess demand functions satisfy the so-called gross substitutability property. This property rules out all type of complementarities in demand. Translating it into our notation it means that if we have two price vectors, p and p', such that for some good j we have $p_j' > p_j$ but $p_i' = p_i$ for all $i \neq j$ then $\zeta_i(p') > \zeta_i(p)$ for $i \neq j$. This would imply that if the price of one commodity j increases then the excess demand of the other commodities $i \neq j$ must increase too; using derivatives we would write $\partial\zeta_i(p)/\partial p_j > 0$, and vice-versa. This condition of gross substitutability is sufficient but not necessary for the globally uniqueness of equilibrium. The flavor of the proof can be seen straightforwardly by considering two different equilibrium prices, call them p' and p, that are not proportional (otherwise they would just be different normalizations of each other and we would be done). If they are equilibrium prices then it will be the case that for all goods i $\zeta_i(p') = \zeta_i(p) = 0$. Without loss of generality (via price normalization if need be) we can look for the good k such that that $p_k' = p_k$ while $p_j' \geq p_j$ with strict inequality for some of the $j \neq k$. In this case, say $\bar{j}$, raise the price from $p_{\bar{j}}'$ to $p_{\bar{j}}$ and by the gross substitutability property the excess demand of good k will increase, that is, $\zeta_k(p') > \zeta_k(p)$, which is a violation of the equilibrium condition. Thus, prices cannot be different.

In our pure exchange economy, a sufficient condition for gross substitutability would be that the elasticity of substitution in consumption for each agent is larger or equal than one (Kehoe 1992). Which additional assumptions do we need if the gross substitutability assumption is not fulfilled, for example when the elasticity of substitution is lower than one? One such property on the excess demand functions is the extension to them of the weak axiom of revealed preference, a concept originally applied to individual choice. We will omit the details here and refer the reader to Kehoe (1992, 1998).

Uniqueness, global or at least local when there are multiple equilibria, is fundamental from the perspective of the type of applied models we will be building and dealing with in the next chapters. Since these models are commonly used to evaluate actual policy changes, we must make sure that the comparison of equilibria makes methodological sense. The theoretical conditions in the uniqueness literature may be too strong for applied models and some testing of these models is called for. In this direction, Kehoe and Whalley (1985) have performed extensive numerical calculations searching for multiple equilibria in applied general equilibrium models of the *USA* and Mexico using one and two-dimensional search grids, respectively. They report that no such multiplicity has been found. Even though their conclusion is not based on theoretical arguments, the fact that the examined applied models are quite standard provides researchers with reassurance that uniqueness is the common situation in empirical analysis. This reinforces the reliability of the routine comparative statics exercises, at least from an empirical perspective.

2.3 Normative Properties of Walrasian Equilibrium

After reviewing some of the basic results concerning the positive aspects of Walrasian general equilibrium theory, we now turn to the description of its normative aspects. The nice welfare properties of the competitive equilibrium are what have made this concept so appealing to generations of economists. These welfare properties can be summarized with the help of the two well-known fundamental theorems of welfare economics (Arrow 1951). The first of these theorems states that any Walrasian allocation is a Pareto efficient allocation as well. The second theorem says that any Pareto efficient allocation can be implemented as a Walrasian allocation with the use of appropriate lump-sum transfers, i.e. transfers based on a reshuffling of the initial endowments, thus not wasting any of the economy's initial resources.

An allocation is said to be Pareto efficient if there is no other feasible allocation where at least one agent is strictly better off while nobody else is worse off. In the context of our Edgeworth box economy, the first theorem implies that if the economy is in a Walrasian equilibrium, then there is no alternative feasible allocation at which every agent in this economy is at least as well off and some agent is strictly better off. In other words, there is no way for the agents of this economy to collectively agree to move to a different feasible allocation. If they moved from the market equilibrium, somebody would certainly be worse off.

In our pure exchange economy, agents are rational in the sense that they will participate in the exchange process and trade using their initial endowments as long as that makes them at least as well as not trading. Among all the Pareto efficient allocations, the one marked as *a* in Fig. 2.3 makes agent 1 indifferent between trading and not trading, with allocation *b* playing the same role for agent 2. The curve that connects point *a* with *b* is known as the contract curve and represents all the feasible allocations at which all agents do at least as well as in their initial endowments. All the allocations in the contract curve represent a subset of the Pareto efficient allocations, i.e. the line that connects point O_1 and O_2 in Fig. 2.3.

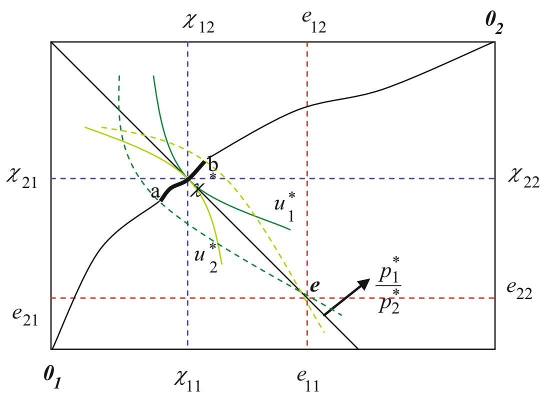


Fig. 2.3 First welfare theorem

The Walrasian auctioneer will adjust prices till reaching equilibrium prices, i.e. p_1^*/p_2^* where excess demands are zero. Note that since the equilibrium allocation χ^* lies on the contract curve, this allocation is Pareto efficient.

We now provide a quick proof of the first welfare theorem. Let us consider that there is another feasible allocation $\hat{\chi}$ that Pareto dominates the Walrasian allocation χ^*. This would imply that at the equilibrium prices p^*:

$$\sum_{i=1}^{N} p_i^* \hat{\chi}_{ih} \geq \sum_{i=1}^{N} p_i^* \chi_{ih}^* \text{ for all } h \in H \tag{2.13}$$

and

$$\sum_{i=1}^{N} p_i^* \hat{\chi}_{ih'} > \sum_{i=1}^{N} p_i^* \chi_{ih'}^* \text{ for at least one agent } h' \in H \tag{2.14}$$

The reason is that if $\hat{\chi}$ is at least as good as χ^* then it must be the case that $\hat{\chi}$ is beyond the budget sets of all agents, otherwise being at least as good it would have been selected. Adding up (2.13) and (2.14) for the set of agents we obtain:

$$\sum_{h=1}^{H} \sum_{i=1}^{N} p_i^* \hat{\chi}_{ih} > \sum_{h=1}^{H} \sum_{i=1}^{N} p_i^* \chi_{ih}^* = \sum_{h=1}^{H} \sum_{i=1}^{N} p_i^* e_{ih} \tag{2.15}$$

Expression (2.15) is in fact a contradiction since it implies that $\hat{\chi}$ could not be a feasible allocation.

We should point out a major distinction between the concepts of Pareto efficiency and Walrasian equilibrium. While Pareto efficiency specifies a subset of allocations which depend on total endowments and individual preferences alone, a Walrasian equilibrium relates to two set of variables, allocations and prices, and depends on individual endowments and individual preferences. In this more stringent sense, it is clear that not all Pareto efficient allocations will be a Walrasian equilibrium. Nonetheless, with appropriately designed transfers of endowments, we can transform any Pareto efficient allocation into a Walrasian equilibrium, provided some basic technical assumptions (such as convexity in preferences, and in technology when production possibilities are present) are satisfied. This is what is stated in the second welfare theorem.

We reproduce in Fig. 2.4 above two Pareto efficient allocations, χ^* and χ'^*. They can also be seen as Walrasian allocations conditional to two different distributions of the total endowment in the economy, i.e. $e \neq e'$. Using Fig. 2.4 we can visualize the essence of what lies behind the second welfare theorem. Suppose an hypothetical central planner, or some social welfare function, deems allocation χ'^* to be socially more desirable than the current market allocation χ^*. Suppose also the planning authority wants to reach allocation χ'^* through the workings of the market. What should this authority do to reach this goal? The actual details do not matter but

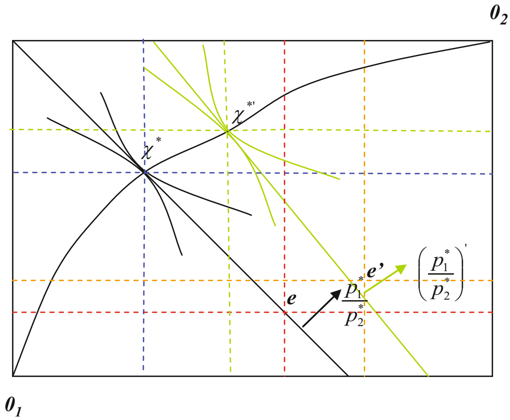


Fig. 2.4 Second welfare theorem

the second welfare theorem would suggest the redistribution of initial endowments from *e* to *e*', for instance, and then let the market work out the new equilibrium for the new endowment distribution. Any such redistribution of the physical assets of the economy is called a lump-sum transfer, i.e. $t = e' - e$ in our example of Fig. 2.4, since total endowments in this economy remained unchanged. Notice too that *e*' in Fig. 2.4 is just one of the very many possible redistribution schemes. The message here is that there is quite a bit of room in the design of redistributive policy actions and that efficiency and equity considerations can be separated.

2.4 Summary

General equilibrium theory has provided economics with a very sound set of powerful tools and has instilled discipline of though and analytical rigor to countless generations of economists. The influence of general equilibrium has been pervasive even beyond its original and traditional microeconomics setting. Modern macroeconomics, for instance, is general equilibrium in an aggregate, dynamic time set-up. Industrial organization has also evolved from its partial equilibrium beginnings and has adopted some of the general equilibrium flavor. International trade theory is nothing but general equilibrium between countries or regions.

An actual economy, after all, is composed of numerous agents, markets and institutions, all of them interacting in the social medium and general equilibrium theory gives us insights as to what can we expect, in economic terms, from such a complex interaction. Theoretical models are of course ideal constructs and they are developed using assumptions that sometimes are hard to fathom in practical terms. The transition from these ideal models to more mundane applied models is justified, on the one hand, by the well established body of theoretical results and, on the other hand, by the need to have operational tools which are able to explore the intricacies of complex issues; in other words, by the need to evaluate the actual economic policies being implemented in the real world by all levels of government. As we will see in the next chapters, applied general equilibrium is one very powerful tool to this effect.

In this chapter we have attempted to present, in an admittedly brief way, the main properties of the Walrasian equilibrium concept and provide some insights on their theoretical relevance, from both a positive and normative perspective. Even though the Walrasian theoretical model is just that, a theoretical model, it is nonetheless the point of departure and provides the necessary blueprint for carrying out sound and well founded applied economic analysis.

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