Preface

This monograph originates from lectures given at the General Relativity Trimester at the Institut Henri Poincaré in Paris [1]; at the VII Mexican School on Gravitation and Mathematical Physics in Playa del Carmen (Mexico) [2]; and at the 2008 International Summer School on Computational Methods in Gravitation and Astrophysics held in Pohang (Korea) [3]. It is devoted to the 3+1 formalism of general relativity, which constitutes among other things, the theoretical foundations for numerical relativity. Numerical techniques are not covered here. For a pedagogical introduction to them, we recommend instead the lectures by Choptuik [4] (finite differences) and the review article by Grandclément and Novak [5] (spectral methods), as well as the numerical relativity textbooks by Alcubierre [6], Bona, Palenzuela-Luque and Bona-Casas [7] and Baumgarte and Shapiro [8].

The prerequisites are those of a general relativity course, at the undergraduate or graduate level, like the textbooks by Hartle [9] or Carroll [10], or part I of Wald's book [11], as well as track 1 of the book by Misner, Thorne and Wheeler [12]. The fact that the present text consists of lecture notes implies two things:

- the calculations are rather detailed (the experienced reader might say *too* detailed), with an attempt to made them self-consistent and complete, trying to use as infrequently as possible the famous phrases "as shown in paper XXX" or "see paper XXX for details";
- the bibliographical references do not constitute an extensive survey of the literature on the subject: articles have been cited in so far as they have a direct connection with the main text.

The book starts with a chapter setting the mathematical background, which is differential geometry, at a basic level (Chap. 2). This is followed by two purely geometrical chapters devoted to the study of a single hypersurface embedded in spacetime (Chap. 3) and to the foliation (or slicing) of spacetime by a family of spacelike hypersurfaces (Chap. 4). The presentation is divided in two chapters to distinguish between concepts which are meaningful for a single hypersurface and those that rely on a foliation. The decomposition of the Einstein equation relative to the foliation is given in Chap. 5, giving rise to the Cauchy problem with constraints, which constitutes the core of the 3+1 formalism. The ADM Hamiltonian formulation of general relativity is also introduced in this chapter. Chapter 6 is devoted to the decomposition of the matter and electromagnetic field equations, focusing on the astrophysically relevant cases of a perfect fluid and a perfect conductor (ideal MHD). An important technical chapter occurs then: Chap. 7 introduces some conformal transformation of the 3-metric on each hypersurface and the corresponding rewriting of the 3+1 Einstein equations. As a by-product, we also discuss the Isenberg-Wilson-Mathews (or conformally flat) approximation to general relativity. Chapter 8 details the various global quantities associated with asymptotic flatness (ADM mass, ADM linear momentum and angular momentum) or with some symmetries (Komar mass and Komar angular momentum). In Chap. 9, we study the initial data problem, presenting with some examples two classical methods: the conformal transversetraceless method and the conformal thin-sandwich one. Both methods rely on the conformal decomposition that has been introduced in Chap. 7. The choice of spacetime coordinates within the 3+1 framework is discussed in Chap. 10, starting from the choice of foliation before discussing the choice of the three coordinates in each leaf of the foliation. The major coordinate families used in modern numerical relativity are reviewed. Finally Chap. 11 presents various schemes for the time integration of the 3+1 Einstein equations, putting some emphasis on the most successful scheme to date, the BSSN one. Appendix A is devoted to basic tools of the 3+1 formalism: the conformal Killing operator and the related vector Laplacian, whereas Appendix B provides some computer algebra codes based on the Sage system.

A web page is dedicated to the book, at the URL

http://relativite.obspm.fr/3p1

This page contains the errata, the clickable list of references, the computer algebra codes described in Appendix B and various supplementary material. Readers are invited to use this page to report any error that they may find in the text.

I am deeply indebted to Michał Bejger, Philippe Grandclément, Alexandre Le Tiec, Yuichiro Sekiguchi and Nicolas Vasset for the careful reading of a preliminary version of these notes. I am very grateful to my friends and colleagues Thomas Baumgarte, Michał Bejger, Luc Blanchet, Silvano Bonazzola, Brandon Carter, Isabel Cordero-Carrión, Thibault Damour, Nathalie Deruelle, Guillaume Faye, John Friedman, Philippe Grandclément, José Maria Ibáñez, José Luis Jaramillo, Jean-Pierre Lasota, Jérôme Novak, Jean-Philippe Nicolas, Motoyuki Saijo, Masaru Shibata, Keisuke Taniguchi, Koji Uryu, Nicolas Vasset and Loïc Villain, for the numerous and fruitful discussions that we had about general relativity and the 3+1 formalism. I also warmly thank Robert Beig and Christian Caron for their invitation to publish this text in the Lecture Notes in Physics series.

Meudon, September 2011 Éric Gourgoulhon

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http://www.springer.com/978-3-642-24524-4

3+1 Formalism in General Relativity
Bases of Numerical Relativity
Gourgoulhon, É.
2012, XVII, 294 p. 29 illus., Softcover
ISBN: 978-3-642-24524-4