Chapter 2
Basic Differential Geometry

Abstract This first chapter recapitulates the basic concepts of differential geometry that are used throughout the book. This encompasses differentiable manifolds, tensor fields, affine connections, metric tensors, pseudo-Riemannian manifolds, Levi–Civita connections, curvature tensors and Lie derivatives. The dimension of the manifold and the signature of the metric are kept general so that the results can be subsequently applied either to the whole spacetime or to some submanifold of it.

2.1 Introduction

The mathematical language of general relativity is mostly differential geometry. We recall in this chapter basic definitions and results in this field, which we will use throughout the book. The reader who has some knowledge of general relativity should be familiar with most of them. We recall them here to make the text fairly self-contained and also to provide definitions with sufficient generality, not limited to the dimension 4—the standard spacetime dimension. Indeed we will manipulate manifolds whose dimension differs from 4, such as hypersurfaces (the building blocks of the 3+1 formalism !) or 2-dimensional surfaces. In the same spirit, we do not stick to Lorentzian metrics (such as the spacetime one) but discuss pseudo-Riemannian metrics, which encompass both Lorentzian metrics and Riemannian ones. Accordingly, in this chapter, $\mathscr{M}$ denotes a generic manifold of any dimension and $\boldsymbol{g}$ a pseudo-Riemannian metric on $\mathscr{M}$. In the subsequent chapters, the symbol $\mathscr{M}$ will be restricted to the spacetime manifold and the symbol $\boldsymbol{g}$ to a Lorentzian metric on $\mathscr{M}$.

This chapter is not intended to a be a lecture on differential geometry, but a collection of basic definitions and useful results. In particular, contrary to the other chapters, we state many results without proofs, referring the reader to classical textbooks on the topic [1–6].

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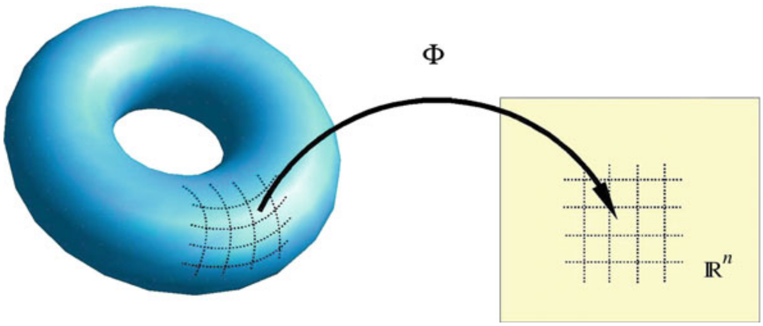

Fig. 2.1 Locally a manifold resembles $\mathbb{R}^n$ ($n = 2$ on the figure), but this is not necessarily the case at the global level

2.2 Differentiable Manifolds

2.2.1 Notion of Manifold

Given an integer $n \geq 1$, a ***manifold of dimension*** n is a topological space $\mathscr{M}$ obeying the following properties:

1. $\mathscr{M}$ is a ***separated space*** (also called ***Hausdorff space***): any two distinct points of $\mathscr{M}$ admit disjoint open neighbourhoods.
2. $\mathscr{M}$ has a ***countable base***[1]: there exists a countable family $(\mathscr{U}_k)_{k\in\mathbb{N}}$ of open sets of $\mathscr{M}$ such that any open set of $\mathscr{M}$ can be written as the union (possibly infinite) of some members of the above family.
3. Around each point of $\mathscr{M}$, there exists a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

Property 1 excludes manifolds with "forks" and is very reasonable from a physical point of view: it allows to distinguish between two points even after a small perturbation. Property 2 excludes "too large" manifolds; in particular it permits setting up the theory of integration on manifolds. It also allows for a differentiable manifold of dimension n to be embedded smoothly into the Euclidean space $\mathbb{R}^{2n}$ (Whitney theorem). Property 3 expresses the essence of a manifold: it means that, locally, one can label the points of $\mathscr{M}$ in a continuous way by n real numbers $(x^\alpha)_{\alpha\in\{0,\ldots,n-1\}}$, which are called ***coordinates*** (cf. Fig. 2.1). More precisely, given an open subset $\mathscr{U} \subset \mathscr{M}$, a ***coordinate system*** or ***chart*** on $\mathscr{U}$ is a homeomorphism[2]

$$\begin{aligned} \Phi : \mathscr{U} \subset \mathscr{M} &\longrightarrow \Phi(\mathscr{U}) \subset \mathbb{R}^n \\ p &\longmapsto (x^0, \ldots, x^{n-1}). \end{aligned} \tag{2.1}$$

[1] In the language of topology, one says that $\mathscr{M}$ is a *second-countable space*.

[2] Let us recall that a ***homeomorphism*** between two topological spaces (here $\mathscr{U}$ and $\Phi(\mathscr{U})$) is a one-to-one map Φ such that both Φ and Φ^{-1} are continuous.

Remark 2.1 In relativity, it is customary to label the n coordinates by an index ranging from 0 to $n-1$. Actually, this convention is mostly used when $\mathscr{M}$ is the spacetime manifold ($n = 4$ in standard general relativity). The computer-oriented reader will have noticed the similarity with the index ranging of arrays in the C/C++ or Python programming languages.

Remark 2.2 Strictly speaking the definition given above is that of a ***topological manifold***. We are saying *manifold* for short.

Usually, one needs more than one coordinate system to cover $\mathscr{M}$. An ***atlas*** on $\mathscr{M}$ is a finite set of couples $(\mathscr{U}_k, \Phi_k)_{1\le k\le K}$, where K is a non-zero integer, $\mathscr{U}_k$ an open set of $\mathscr{M}$ and Φ_k a chart on $\mathscr{U}_k$, such that the union of all $\mathscr{U}_k$ covers $\mathscr{M}$:

$$\bigcup_{k=1}^{K} \mathscr{U}_k = \mathscr{M}. \tag{2.2}$$

The above definition of a manifold lies at the *topological* level (Remark 2.2), meaning that one has the notion of continuity, but not of differentiability. To provide the latter, one should rely on the differentiable structure of $\mathbb{R}^n$, via the atlases: a ***differentiable manifold*** is a manifold $\mathscr{M}$ equipped with an atlas $(\mathscr{U}_k, \Phi_k)_{1\le k\le K}$ such that for any non-empty intersection $\mathscr{U}_i \cap \mathscr{U}_j$, the mapping

$$\Phi_i \circ \Phi_j^{-1} : \Phi_j(\mathscr{U}_i \cap \mathscr{U}_j) \subset \mathbb{R}^n \longrightarrow \Phi_i(\mathscr{U}_i \cap \mathscr{U}_j) \subset \mathbb{R}^n \tag{2.3}$$

is differentiable (i.e. C^∞). Note that the above mapping is from an open set of $\mathbb{R}^n$ to an open set of $\mathbb{R}^n$, so that the invoked differentiability is nothing but that of $\mathbb{R}^n$. The atlas $(\mathscr{U}_k, \Phi_k)_{1\le k\le K}$ is called a ***differentiable atlas***. In the following, we consider only differentiable manifolds.

Remark 2.3 We are using the word *differentiable* for C^∞, i.e. *smooth*.

Given two differentiable manifolds, $\mathscr{M}$ and $\mathscr{M}'$, of respective dimensions n and n', we say that a map $\phi : \mathscr{M} \to \mathscr{M}'$ is ***differentiable*** iff in some (and hence all) coordinate systems of $\mathscr{M}$ and $\mathscr{M}'$ (belonging to the differentiable atlases of $\mathscr{M}$ and $\mathscr{M}'$), the coordinates of the image $\phi(p)$ are differentiable functions $\mathbb{R}^n \to \mathbb{R}^{n'}$ of the coordinates of p. The map ϕ is said to be a ***diffeomorphism*** iff it is one-to-one and both ϕ and ϕ^{-1} are differentiable. This implies $n = n'$.

Remark 2.4 Strictly speaking a differentiable manifold is a couple $(\mathscr{M}, \mathscr{A})$ where $\mathscr{A}$ is a (maximal) differentiable atlas on $\mathscr{M}$. Indeed a given (topological) manifold $\mathscr{M}$ can have non-equivalent differentiable structures, as shown by Milnor (1956) [7] in the specific case of the unit sphere of dimension 7, $\mathbb{S}^7$: there exist differentiable manifolds, the so-called *exotic spheres*, that are homeomorphic to $\mathbb{S}^7$ but not diffeomorphic to $\mathbb{S}^7$. On the other side, for $n \le 6$, there is a unique differentiable structure for the sphere $\mathbb{S}^n$. Moreover, any manifold of dimension $n \le 3$ admits a unique differentiable structure. Amazingly, in the case of $\mathbb{R}^n$, there exists a unique differentiable structure (the standard one) for any $n \neq 4$, but for $n = 4$ (the space-time case !) there exist uncountably many non-equivalent differentiable structures, the so-called *exotic* $\mathbb{R}^4$ [8].

2.2.2 Vectors on a Manifold

On a manifold, vectors are defined as tangent vectors to a curve. A ***curve*** is a subset $\mathscr{C} \subset \mathscr{M}$ which is the image of a differentiable function $\mathbb{R} \to \mathscr{M}$:

$$\begin{aligned} P : \mathbb{R} &\longrightarrow \mathscr{M} \\ \lambda &\longmapsto p = P(\lambda) \in \mathscr{C}. \end{aligned} \tag{2.4}$$

Hence $\mathscr{C} = \{P(\lambda) | \lambda \in \mathbb{R}\}$. The function P is called a ***parametrization*** of $\mathscr{C}$ and the variable λ is called a ***parameter along*** $\mathscr{C}$. Given a coordinate system (x^α) in a neighbourhood of a point $p \in \mathscr{C}$, the parametrization P is defined by n functions $X^\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$x^\alpha(P(\lambda)) = X^\alpha(\lambda). \tag{2.5}$$

A ***scalar field*** on $\mathscr{M}$ is a function $f : \mathscr{M} \to \mathbb{R}$. In practice, we will always consider differentiable scalar fields. At a point $p = P(\lambda) \in \mathscr{C}$, the ***vector tangent to*** $\mathscr{C}$ associated with the parametrization P is the operator $\boldsymbol{v}$ which maps every scalar field f to the real number

$$\boldsymbol{v}(f) = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_{\mathscr{C}} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(P(\lambda + \varepsilon)) - f(P(\lambda)) \right]. \tag{2.6}$$

Given a coordinate system (x^α) around some point $p \in \mathscr{M}$, there are n curves $\mathscr{C}_\alpha$ through p associated with (x^α) and called the ***coordinate lines***: for each $\alpha \in \{0, \ldots, n-1\}$, $\mathscr{C}_\alpha$ is defined as the curve through p parametrized by $\lambda = x^\alpha$ and having constant coordinates x^β for all $\beta \neq \alpha$. The vector tangent to $\mathscr{C}_\alpha$ parametrized by x^α is denoted $\boldsymbol{\partial}_\alpha$. Its action on a scalar field f is by definition

$$\boldsymbol{\partial}_\alpha(f) = \left.\frac{\mathrm{d}f}{\mathrm{d}x^\alpha}\right|_{\mathscr{C}_\alpha} = \left.\frac{\mathrm{d}f}{\mathrm{d}x^\alpha}\right|_{\substack{x^\beta = \mathrm{const} \\ \beta \neq \alpha}}.$$

Considering f as a function of the coordinates $(x^0, \ldots, x^{n-1})$ (whereas strictly speaking it is a function of the points on $\mathscr{M}$) we recognize in the last term the partial derivative of f with respect to x^α. Hence

$$\boxed{\boldsymbol{\partial}_\alpha(f) = \frac{\partial f}{\partial x^\alpha}}. \tag{2.7}$$

Similarly, we may rewrite (2.6) as

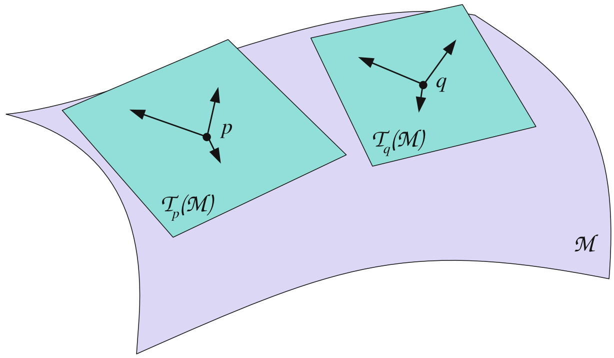

Fig. 2.2 The vectors at two points p and q on the manifold $\mathscr{M}$ belong to two different vector spaces: the tangent spaces $\mathscr{T}_p(\mathscr{M})$ and $\mathscr{T}_q(\mathscr{M})$

$$\begin{aligned}\boldsymbol{v}(f) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[f(X^0(\lambda+\varepsilon),\ldots,X^{n-1}(\lambda+\varepsilon)) - f(X^0(\lambda),\ldots,X^{n-1}(\lambda))\right]\\ &= \frac{\partial f}{\partial x^\alpha}\frac{\mathrm{d}X^\alpha}{\mathrm{d}\lambda} = \boldsymbol{\partial}_\alpha(f)\frac{\mathrm{d}X^\alpha}{\mathrm{d}\lambda}.\end{aligned}$$

In the above equation and throughout the all book, we are using ***Einstein summation convention***: a repeated index implies a summation over all the possible values of this index (here from $\alpha = 0$ to $\alpha = n-1$). The above identity being valid for any scalar field f, we conclude that

$$\boxed{\boldsymbol{v} = v^\alpha\,\boldsymbol{\partial}_\alpha}, \tag{2.8}$$

with the n real numbers

$$v^\alpha := \frac{\mathrm{d}X^\alpha}{\mathrm{d}\lambda}, \quad 0 \le \alpha \le n-1. \tag{2.9}$$

Since every vector tangent to a curve at p is expressible as (2.8), we conclude that the set of all vectors tangent to a curve at p is a vector space of dimension n and that $(\boldsymbol{\partial}_\alpha)$ constitutes a basis of it. This vector space is called the ***tangent vector space to*** $\mathscr{M}$ ***at*** p and is denoted $\mathscr{T}_p(\mathscr{M})$. The elements of $\mathscr{T}_p(\mathscr{M})$ are simply called ***vectors*** at p. The basis $(\boldsymbol{\partial}_\alpha)$ is called the ***natural basis*** associated with the coordinates (x^α) and the coefficients v^α in (2.8) are called the ***components of the vector v with respect to the coordinates***(x^α). The tangent vector space is represented at two different points in Fig. 2.2.

Contrary to what happens for an affine space, one cannot, in general, define a vector connecting two points p and q on a manifold, except if p and q are infinitesimally close to each other. Indeed, in the latter case, we may define the ***infinitesimal displacement vector from*** p ***to*** q as the vector $\mathrm{d}\boldsymbol{\ell} \in \mathscr{T}_p(\mathscr{M})$ whose action on a scalar field f is

$$\mathrm{d}\boldsymbol{\ell}(f) = \mathrm{d}f|_{p\to q} = f(q) - f(p). \tag{2.10}$$

Since p and q are infinitesimally close, there is a unique (piece of) curve $\mathscr{C}$ going from p to q and one has

$$\mathrm{d}\boldsymbol{\ell} = \boldsymbol{v}\mathrm{d}\lambda, \tag{2.11}$$

where λ is a parameter along $\mathscr{C}$, $\boldsymbol{v}$ the associated tangent vector at p and $\mathrm{d}\lambda$ the parameter increment from p to q: $p = P(\lambda)$ and $q = P(\lambda + \mathrm{d}\lambda)$. The relation (2.11) follows immediately from the definition (2.6) of $\boldsymbol{v}$. Given a coordinate system, let (x^α) be the coordinates of p and $(x^\alpha + \mathrm{d}x^\alpha)$ those of q. Then from Eq. (2.10),

$$\mathrm{d}\boldsymbol{\ell}(f) = \mathrm{d}f = \frac{\partial f}{\partial x^\alpha}\mathrm{d}x^\alpha = \mathrm{d}x^\alpha \boldsymbol{\partial}_\alpha(f).$$

The scalar field f being arbitrary, we conclude that

$$\boxed{\mathrm{d}\boldsymbol{\ell} = \mathrm{d}x^\alpha \boldsymbol{\partial}_\alpha}. \tag{2.12}$$

In other words, the components of the infinitesimal displacement vector with respect to the coordinates (x^α) are nothing but the infinitesimal coordinate increments $\mathrm{d}x^\alpha$.

2.2.3 Linear Forms

A fundamental operation on vectors consists in mapping them to real numbers, and this in a linear way. More precisely, at each point $p \in \mathscr{M}$, one defines a ***linear form*** as a mapping[3]

$$\begin{aligned} \boldsymbol{\omega} : \mathscr{T}_p(\mathscr{M}) &\longrightarrow \mathbb{R} \\ \boldsymbol{v} &\longmapsto \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle \end{aligned} \tag{2.13}$$

that is linear: $\langle \boldsymbol{\omega}, \lambda\boldsymbol{v} + \boldsymbol{u} \rangle = \lambda\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle + \langle \boldsymbol{\omega}, \boldsymbol{u} \rangle$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathscr{T}_p(\mathscr{M})$ and $\lambda \in \mathbb{R}$. The set of all linear forms at p constitutes a n-dimensional vector space, which is called the ***dual space of*** $\boldsymbol{\mathscr{T}_p(\mathscr{M})}$ and denoted by $\mathscr{T}_p^*(\mathscr{M})$. Given the natural basis $(\boldsymbol{\partial}_\alpha)$ of $\mathscr{T}_p(\mathscr{M})$ associated with some coordinates (x^α), there is a unique basis of $\mathscr{T}_p^*(\mathscr{M})$, denoted by $(\mathbf{d}x^\alpha)$, such that

$$\boxed{\langle \mathbf{d}x^\alpha, \boldsymbol{\partial}_\beta \rangle = \delta^\alpha{}_\beta}, \tag{2.14}$$

where $\delta^\alpha{}_\beta$ is the ***Kronecker symbol***: $\delta^\alpha{}_\beta = 1$ if $\alpha = \beta$ and 0 otherwise. The basis $(\mathbf{d}x^\alpha)$ is called the ***dual basis*** of the basis $(\boldsymbol{\partial}_\alpha)$. The notation $(\mathbf{d}x^\alpha)$ stems from the

[3] We are using the same bra-ket notation as in quantum mechanics to denote the action of a linear form on a vector.

fact that if we apply the linear form $\mathbf{d}x^\alpha$ to the infinitesimal displacement vector (2.12), we get nothing but the number $\mathrm{d}x^\alpha$:

$$\langle \mathbf{d}x^\alpha, \mathrm{d}\boldsymbol{\ell}\rangle = \langle \mathbf{d}x^\alpha, \mathrm{d}x^\beta \boldsymbol{\partial}_\beta\rangle = \mathrm{d}x^\beta \underbrace{\langle \mathbf{d}x^\alpha, \boldsymbol{\partial}_\beta\rangle}_{\delta^\alpha{}_\beta} = \mathrm{d}x^\alpha. \tag{2.15}$$

Remark 2.5 Do not confuse the linear form $\mathbf{d}x^\alpha$ with the infinitesimal increment $\mathrm{d}x^\alpha$ of the coordinate x^α.

The dual basis can be used to expand any linear form $\boldsymbol{\omega}$, thereby defining its ***components*** ω_α ***with respect to the coordinates*** (x^α):

$$\boldsymbol{\omega} = \omega_\alpha \mathbf{d}x^\alpha. \tag{2.16}$$

In terms of components, the action of a linear form on a vector takes then a very simple form:

$$\boxed{\langle \boldsymbol{\omega}, \boldsymbol{v}\rangle = \omega_\alpha v^\alpha}. \tag{2.17}$$

This follows immediately from (2.16), (2.8) and (2.14).

A field of linear forms, i.e. a (smooth) map which associates to each point $p \in \mathscr{M}$ a member of $\mathscr{T}_p(\mathscr{M})$ is called a ***1-form***. Given a smooth scalar field f on $\mathscr{M}$, there exists a 1-form canonically associated with it, called the ***gradient of*** f and denoted $\boldsymbol{\nabla} f$. At each point $p \in \mathscr{M}$, $\boldsymbol{\nabla} f$ is the unique linear form which, once applied to the infinitesimal displacement vector $\mathrm{d}\boldsymbol{\ell}$ from p to a nearby point q, gives the change in f between points p and q:

$$\mathrm{d}f := f(q) - f(p) = \langle \boldsymbol{\nabla} f, \mathrm{d}\boldsymbol{\ell}\rangle. \tag{2.18}$$

Since $\mathrm{d}f = \partial f/\partial x^\alpha \mathrm{d}x^\alpha$, Eq. (2.15) implies that the components of the gradient with respect to the dual basis are nothing but the partial derivatives of f with respect to the coordinates (x^α):

$$\boxed{\boldsymbol{\nabla} f = \frac{\partial f}{\partial x^\alpha}\mathbf{d}x^\alpha}. \tag{2.19}$$

Remark 2.6 In non-relativistic physics, the gradient is very often considered as a vector field and not as a 1-form. This is so because one associates implicitly a vector $\vec{\boldsymbol{\omega}}$ to any 1-form $\boldsymbol{\omega}$ via the Euclidean scalar product of $\mathbb{R}^3$: $\forall \vec{v} \in \mathbb{R}^3$, $\langle \boldsymbol{\omega}, \vec{v}\rangle = \vec{\boldsymbol{\omega}} \cdot \vec{v}$. Accordingly, formula (2.18) is rewritten as $\mathrm{d}f = \vec{\boldsymbol{\nabla}} f \cdot \mathrm{d}\boldsymbol{\ell}$. But we should keep in mind that, fundamentally, the gradient is a linear form and not a vector.

Remark 2.7 For a fixed value of α, the coordinate x^α can be considered as a scalar field on $\mathscr{M}$. If we apply (2.19) to $f = x^\alpha$, we then get

$$\boldsymbol{\nabla} x^\alpha = \mathbf{d}x^\alpha. \tag{2.20}$$

Hence the dual basis to the natural basis $(\boldsymbol{\partial}_\alpha)$ is formed by the gradients of the coordinates.

Of course the natural bases are not the only possible bases in the vector space $\mathscr{T}_p(\mathscr{M})$. One may use a basis $(\boldsymbol{e}_\alpha)$ that is not related to a coordinate system on $\mathscr{M}$, for instance an orthonormal basis with respect to some metric. There exists then a unique basis $(\boldsymbol{e}^\alpha)$ of the dual space $\mathscr{T}_p^*(\mathscr{M})$ such that[4]

$$\boxed{\langle \boldsymbol{e}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha{}_\beta}. \tag{2.21}$$

$(\boldsymbol{e}^\alpha)$ is called the ***dual basis*** to $(\boldsymbol{e}_\alpha)$. The relation (2.14) is a special case of (2.21), for which $\boldsymbol{e}_\alpha = \boldsymbol{\partial}_\alpha$ and $\boldsymbol{e}^\alpha = \mathbf{d}x^\alpha$.

2.2.4 Tensors

Tensors are generalizations of both vectors and linear forms. At a point $p \in \mathscr{M}$, a ***tensor of type*** (k, ℓ) with $(k, \ell) \in \mathbb{N}^2$, also called ***tensor*** k ***times contravariant and*** ℓ ***times covariant***, is a mapping

$$\begin{aligned} \boldsymbol{T} : \underbrace{\mathscr{T}_p^*(\mathscr{M}) \times \cdots \times \mathscr{T}_p^*(\mathscr{M})}_{k \text{ times}} \times \underbrace{\mathscr{T}_p(\mathscr{M}) \times \cdots \times \mathscr{T}_p(\mathscr{M})}_{\ell \text{ times}} &\longrightarrow \mathbb{R} \\ (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell) &\longmapsto \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell) \end{aligned} \tag{2.22}$$

that is linear with respect to each of its arguments. The integer $k+\ell$ is called the tensor ***valence***, or sometimes the tensor ***rank*** or ***order***. Let us recall the canonical duality $\mathscr{T}_p^{**}(\mathscr{M}) = \mathscr{T}_p(\mathscr{M})$, which means that every vector $\boldsymbol{v}$ can be considered as a linear form on the space $\mathscr{T}_p^*(\mathscr{M})$, via $\mathscr{T}_p^*(\mathscr{M}) \to \mathbb{R}$, $\boldsymbol{\omega} \mapsto \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle$. Accordingly a vector is a tensor of type $(1,0)$. A linear form is a tensor of type $(0,1)$. A tensor of type $(0,2)$ is called a ***bilinear form***. It maps couples of vectors to real numbers, in a linear way for each vector.

Given a basis $(\boldsymbol{e}_\alpha)$ of $\mathscr{T}_p(\mathscr{M})$ and the corresponding dual basis $(\boldsymbol{e}^\alpha)$ in $\mathscr{T}_p^*(\mathscr{M})$, we can expand any tensor $\boldsymbol{T}$ of type (k, ℓ) as

$$\boxed{\boldsymbol{T} = T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} \, \boldsymbol{e}_{\alpha_1} \otimes \ldots \otimes \boldsymbol{e}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \ldots \otimes \boldsymbol{e}^{\beta_\ell}}, \tag{2.23}$$

where the ***tensor product*** $\boldsymbol{e}_{\alpha_1} \otimes \ldots \otimes \boldsymbol{e}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \ldots \otimes \boldsymbol{e}^{\beta_\ell}$ is the tensor of type (k, ℓ) for which the image of $(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell)$ as in (2.22) is the real number

$$\prod_{i=1}^{k} \langle \boldsymbol{\omega}_i, \boldsymbol{e}_{\alpha_i} \rangle \times \prod_{j=1}^{\ell} \langle \boldsymbol{e}^{\beta_j}, \boldsymbol{v}_j \rangle .$$

[4] Notice that, according to the standard usage, the symbol for the vector $\boldsymbol{e}_\alpha$ and that for the linear form $\boldsymbol{e}^\alpha$ differ only by the position of the index α.

Notice that all the products in the above formula are simply products in $\mathbb{R}$. The $n^{k+\ell}$ scalar coefficients $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$ in (2.23) are called the ***components of the tensor T with respect to the basis*** $(\boldsymbol{e}_\alpha)$. These components are unique and fully characterize the tensor $\boldsymbol{T}$.

Remark 2.8 The notations v^α and ω_α already introduced for the components of a vector $\boldsymbol{v}$ [Eq. (2.8)] or a linear form $\boldsymbol{\omega}$ [Eq. (2.16)] are of course the particular cases $(k,\ell) = (1,0)$ or $(k,\ell) = (0,1)$ of (2.23), with, in addition, $\boldsymbol{e}_\alpha = \boldsymbol{\partial}_\alpha$ and $\boldsymbol{e}^\alpha = \mathbf{d}x^\alpha$.

2.2.5 Fields on a Manifold

A ***tensor field*** of type (k,ℓ) is a map which associates to each point $p \in \mathscr{M}$ a tensor of type (k,ℓ) on $\mathscr{T}_p(\mathscr{M})$. By convention, a scalar field is considered as a tensor field of type $(0,0)$. We shall consider only smooth fields.

Given an integer p, a ***p-form*** is a tensor field of type $(0,p)$, i.e. a field of p-linear forms, that is fully antisymmetric whenever $p \geq 2$. This definition generalizes that of a 1-form given in Sect. 2.2.3.

A ***frame field*** or ***moving frame*** is a n-uplet of vector fields $(\boldsymbol{e}_\alpha)$ such that at each point $p \in \mathscr{M}$, $(\boldsymbol{e}_\alpha(p))$ is a basis of the tangent space $\mathscr{T}_p(\mathscr{M})$. If $n = 4$, a frame field is also called a ***tetrad*** and if $n = 3$, it is called a ***triad***.

Given a vector field $\boldsymbol{v}$ and a scalar field f, the function $\mathscr{M} \to \mathbb{R}$, $p \mapsto \boldsymbol{v}|_p(f)$ clearly defines a scalar field on $\mathscr{M}$, which we denote naturally by $\boldsymbol{v}(f)$. We may then define the ***commutator of two vector fields u*** and $\boldsymbol{v}$ as the vector field $[\boldsymbol{u},\boldsymbol{v}]$ whose action on a scalar field f is

$$[\boldsymbol{u},\boldsymbol{v}](f) := \boldsymbol{u}(\boldsymbol{v}(f)) - \boldsymbol{v}(\boldsymbol{u}(f)). \tag{2.24}$$

With respect to a coordinate system (x^α), it is not difficult, via (2.8), to see that the components of the commutator are

$$\boxed{[\boldsymbol{u},\boldsymbol{v}]^\alpha = u^\mu \frac{\partial v^\alpha}{\partial x^\mu} - v^\mu \frac{\partial u^\alpha}{\partial x^\mu}}. \tag{2.25}$$

2.3 Pseudo-Riemannian Manifolds

2.3.1 Metric Tensor

A ***pseudo-Riemannian manifold*** is a couple $(\mathscr{M}, \boldsymbol{g})$ where $\mathscr{M}$ is a differentiable manifold and $\boldsymbol{g}$ is a ***metric tensor*** on $\mathscr{M}$, i.e. a tensor field obeying the following properties:

1. $\boldsymbol{g}$ is a tensor field of type (0, 2): at each point $p \in \mathscr{M}$, $\boldsymbol{g}(p)$ is a bilinear form acting on vectors in the tangent space $\mathscr{T}_p(\mathscr{M})$:

$$\begin{aligned} \boldsymbol{g}(p) : \mathscr{T}_p(\mathscr{M}) \times \mathscr{T}_p(\mathscr{M}) &\longrightarrow \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) \qquad\qquad &\longmapsto \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{2.26}$$

2. $\boldsymbol{g}$ is ***symmetric***: $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u})$.
3. $\boldsymbol{g}$ is ***non-degenerate***: at any point $p \in \mathscr{M}$, a vector $\boldsymbol{u}$ such that $\forall \boldsymbol{v} \in \mathscr{T}_p(\mathscr{M})$, $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0$ is necessarily the null vector.

The properties of being symmetric and non-degenerate are typical of a ***scalar product***. Accordingly, one says that two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are ***g-orthogonal*** (or simply ***orthogonal*** if there is no ambiguity) iff $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = 0$. Moreover, when there is no ambiguity on the metric (usually the spacetime metric), we are using a dot to denote the scalar product of two vectors taken with $\boldsymbol{g}$:

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in \mathscr{T}_p(\mathscr{M}) \times \mathscr{T}_p(\mathscr{M}), \quad \boxed{\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})}. \tag{2.27}$$

In a given basis $(\boldsymbol{e}_\alpha)$ of $\mathscr{T}_p(\mathscr{M})$, the components of $\boldsymbol{g}$ is the matrix $(g_{\alpha\beta})$ defined by formula (2.23) with $(k, \ell) = (0, 2)$:

$$\boldsymbol{g} = g_{\alpha\beta} \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta. \tag{2.28}$$

For any couple $(\boldsymbol{u}, \boldsymbol{v})$ of vectors we have then

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\alpha\beta} u^\alpha v^\beta. \tag{2.29}$$

In particular, considering the natural basis associated with some coordinate system (x^α), the scalar square of an infinitesimal displacement vector $\mathrm{d}\boldsymbol{\ell}$ [cf. Eq. (2.10)] is

$$\mathrm{d}s^2 := \boldsymbol{g}(\mathrm{d}\boldsymbol{\ell}, \mathrm{d}\boldsymbol{\ell}) = g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta. \tag{2.30}$$

This formula, which follows from the value (2.12) of the components of $\mathrm{d}\boldsymbol{\ell}$, is called the expression of the ***line element*** on the pseudo-Riemannian manifold $(\mathscr{M}, \boldsymbol{g})$. It is often used to define the metric tensor in general relativity texts. Note that contrary to what the notation may suggest, $\mathrm{d}s^2$ is not necessarily a positive quantity.

2.3.2 Signature and Orthonormal Bases

An important feature of the metric tensor is its ***signature***: in all bases of $\mathscr{T}_p(\mathscr{M})$ where the components $(g_{\alpha\beta})$ form a diagonal matrix, there is necessarily the same number, s say, of negative components and the same number, $n - s$, of positive components. The independence of s from the choice of the basis where $(g_{\alpha\beta})$ is diagonal is a basic result of linear algebra named ***Sylvester's law of inertia***. One writes:

$$\operatorname{sign} \boldsymbol{g} = (\underbrace{-, \ldots, -}_{s \text{ times}}, \underbrace{+, \ldots, +}_{n-s \text{ times}}). \tag{2.31}$$

If $s = 0$, $\boldsymbol{g}$ is called a ***Riemannian metric*** and $(\mathscr{M}, \boldsymbol{g})$ a ***Riemannian manifold***. In this case, $\boldsymbol{g}$ is ***positive-definite***, which means that

$$\forall \boldsymbol{v} \in \mathscr{T}_p(\mathscr{M}), \quad \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) \geq 0 \tag{2.32}$$

and $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$ iff $\boldsymbol{v} = 0$. A standard example of Riemannian metric is of course the scalar product of the Euclidean space $\mathbb{R}^n$.

If $s = 1$, $\boldsymbol{g}$ is called a ***Lorentzian metric*** and $(\mathscr{M}, \boldsymbol{g})$ a ***Lorentzian manifold***. One may then have $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$; vectors for which this occurs are called ***timelike***, whereas vectors for which $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$ are called ***spacelike***, and those for which $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$ are called ***null***. The subset of $\mathscr{T}_p(\mathscr{M})$ formed by all null vectors is termed the ***null cone*** of $\boldsymbol{g}$ at p.

A basis $(\boldsymbol{e}_\alpha)$ of $\mathscr{T}_p(\mathscr{M})$ is called a g-***orthonormal basis*** (or simply ***orthonormal basis*** if there is no ambiguity on the metric) iff [5]

$$\begin{array}{lll} \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha) = -1 & \text{for} & 0 \leq \alpha \leq s - 1 \\ \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\alpha) = 1 & \text{for} & s \leq \alpha \leq n - 1 \\ \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = 0 & \text{for} & \alpha \neq \beta. \end{array} \tag{2.33}$$

2.3.3 Metric Duality

Since the bilinear form $\boldsymbol{g}$ is non-degenerate, its matrix $(g_{\alpha\beta})$ in any basis $(\boldsymbol{e}_\alpha)$ is invertible and the inverse is denoted by $(g^{\alpha\beta})$:

$$\boxed{g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_{\ \beta}}. \tag{2.34}$$

The metric $\boldsymbol{g}$ induces an isomorphism between $\mathscr{T}_p(\mathscr{M})$ (vectors) and $\mathscr{T}_p^*(\mathscr{M})$ (linear forms) which, in index notation, corresponds to the lowering or raising of the index by contraction with $g_{\alpha\beta}$ or $g^{\alpha\beta}$. In the present book, an index-free symbol will always denote a tensor with a fixed covariance type (such as a vector, a 1-form, a bilinear form, etc.). We will therefore use a different symbol to denote its image under the metric isomorphism. In particular, we denote by an underbar the isomorphism $\mathscr{T}_p(\mathscr{M}) \to \mathscr{T}_p^*(\mathscr{M})$ and by an arrow the reverse isomorphism $\mathscr{T}_p^*(\mathscr{M}) \to \mathscr{T}_p(\mathscr{M})$:

1. For any vector $\boldsymbol{u}$ in $\mathscr{T}_p(\mathscr{M})$, $\underline{\boldsymbol{u}}$ stands for the unique linear form such that

$$\forall \boldsymbol{v} \in \mathscr{T}_p(\mathscr{M}), \quad \langle \underline{\boldsymbol{u}}, \boldsymbol{v} \rangle = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}). \tag{2.35}$$

[5] No summation on α is implied.

However, we will omit the underbar on the components of $\underline{\boldsymbol{u}}$, since the position of the index allows us to distinguish between vectors and linear forms, following the standard usage: if the components of $\boldsymbol{u}$ in a given basis $(\boldsymbol{e}_\alpha)$ are denoted by $\boldsymbol{u}^\alpha$, the components of $\underline{\boldsymbol{u}}$ in the dual basis $(\boldsymbol{e}^\alpha)$ are then denoted by u_α and are given by

$$u_\alpha = g_{\alpha\mu} u^\mu. \tag{2.36}$$

2. For any linear form $\boldsymbol{\omega}$ in $\mathscr{T}_p^*(\mathscr{M})$, $\vec{\boldsymbol{\omega}}$ stands for the unique vector of $\mathscr{T}_p(\mathscr{M})$ such that

$$\forall \boldsymbol{v} \in \mathscr{T}_p(\mathscr{M}), \quad \boldsymbol{g}(\vec{\boldsymbol{\omega}}, \boldsymbol{v}) = \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle. \tag{2.37}$$

As for the underbar, we will omit the arrow on the components of $\vec{\boldsymbol{\omega}}$ by denoting them ω^α; they are given by

$$\omega^\alpha = g^{\alpha\mu} \omega_\mu. \tag{2.38}$$

3. We extend the arrow notation to *bilinear* forms on $\mathscr{T}_p(\mathscr{M})$ (type-(0, 2) tensor): for any bilinear form $\boldsymbol{T}$, we denote by $\vec{\boldsymbol{T}}$ the tensor of type (1, 1) such that

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in \mathscr{T}_p(\mathscr{M}) \times \mathscr{T}_p(\mathscr{M}), \quad \boldsymbol{T}(\boldsymbol{u}, \boldsymbol{v}) = \vec{\boldsymbol{T}}(\underline{\boldsymbol{u}}, \boldsymbol{v}) = \boldsymbol{u} \cdot \vec{\boldsymbol{T}}(\boldsymbol{v}), \tag{2.39}$$

and by $\overset{\twoheadrightarrow}{\boldsymbol{T}}$ the tensor of type (2, 0) defined by

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in \mathscr{T}_p(\mathscr{M}) \times \mathscr{T}_p(\mathscr{M}), \quad \boldsymbol{T}(\boldsymbol{u}, \boldsymbol{v}) = \overset{\twoheadrightarrow}{\boldsymbol{T}}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}). \tag{2.40}$$

Note that in the second equality of (2.39), we have considered $\vec{\boldsymbol{T}}$ as an endomorphism $\mathscr{T}_p(\mathscr{M}) \to \mathscr{T}_p(\mathscr{M})$, which is always possible for a tensor of type (1, 1). If $T_{\alpha\beta}$ are the components of $\boldsymbol{T}$ in some basis $(\boldsymbol{e}_\alpha)$, the components of $\vec{\boldsymbol{T}}$ and $\overset{\twoheadrightarrow}{\boldsymbol{T}}$ are respectively

$$(\vec{\boldsymbol{T}})^\alpha{}_\beta = T^\alpha{}_\beta = g^{\alpha\mu} T_{\mu\beta} \tag{2.41}$$

$$(\overset{\twoheadrightarrow}{\boldsymbol{T}})^{\alpha\beta} = T^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} T_{\mu\nu}. \tag{2.42}$$

Remark 2.9 In mathematical literature, the isomorphism induced by $\boldsymbol{g}$ between $\mathscr{T}_p(\mathscr{M})$ and $\mathscr{T}_p^*(\mathscr{M})$ is called the ***musical isomorphism***, because a flat or a sharp symbol is used instead of, respectively, the underbar and the arrow introduced above:

$$\boldsymbol{u}^\flat = \underline{\boldsymbol{u}} \quad \text{and} \quad \boldsymbol{\omega}^\sharp = \vec{\boldsymbol{\omega}}.$$

2.3.4 Levi–Civita Tensor

Let us assume that $\mathscr{M}$ is an ***orientable manifold***, i.e. that there exists a n-form[6] on $\mathscr{M}$ (n being $\mathscr{M}$'s dimension) that is continuous on $\mathscr{M}$ and nowhere vanishing. Then, given a metric $\boldsymbol{g}$ on $\mathscr{M}$, one can show that there exist only two n-forms $\boldsymbol{\varepsilon}$ such that for any $\boldsymbol{g}$-orthonormal basis $(\boldsymbol{e}_\alpha)$,

$$\boldsymbol{\varepsilon}(\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{n-1}) = \pm 1. \tag{2.43}$$

Picking one of these two n-forms amounts to choosing an ***orientation*** for $\mathscr{M}$. The chosen $\boldsymbol{\varepsilon}$ is then called the ***Levi-Civita tensor*** associated with the metric $\boldsymbol{g}$. Bases for which the right-hand side of (2.43) is $+1$ are called ***right-handed***, and those for which it is -1 are called ***left-handed***. More generally, given a (not necessarily orthonormal) basis $(\boldsymbol{e}_\alpha)$ of $\mathscr{T}_p(\mathscr{M})$, one has necessarily $\boldsymbol{\varepsilon}(\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{n-1}) \neq 0$ and one says that the basis is ***right-handed*** or ***left-handed*** iff $\boldsymbol{\varepsilon}(\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{n-1}) > 0$ or < 0, respectively.

If (x^α) is a coordinate system on $\mathscr{M}$ such that the corresponding natural basis $(\boldsymbol{\partial}_\alpha)$ is right-handed, then the components of $\boldsymbol{\varepsilon}$ with respect to (x^α) are given by

$$\boxed{\varepsilon_{\alpha_1 \ldots \alpha_n} = \sqrt{|g|}\, [\alpha_1, \ldots, \alpha_n]}, \tag{2.44}$$

where g stands for the determinant of the matrix of $\boldsymbol{g}$'s components with respect to the coordinates (x^α):

$$\boxed{g := \det(g_{\alpha\beta})} \tag{2.45}$$

and the symbol $[\alpha_1, \ldots, \alpha_n]$ takes the value 0 if any two indices $(\alpha_1, \ldots, \alpha_n)$ are equal and takes the value 1 or -1 if $(\alpha_1, \ldots, \alpha_n)$ is an even or odd permutation, respectively, of $(0, \ldots, n-1)$.

2.4 Covariant Derivative

2.4.1 Affine Connection on a Manifold

Let us denote by $\mathscr{T}(\mathscr{M})$ the space of smooth vector fields on $\mathscr{M}$.[7] Given a vector field $\boldsymbol{v} \in \mathscr{T}(\mathscr{M})$, it is not possible from the manifold structure alone to define its variation between two neighbouring points p and q. Indeed a formula like $\mathrm{d}\boldsymbol{v} := \boldsymbol{v}(q) - \boldsymbol{v}(p)$ is

[6] Cf. Sect. 2.2.5 for the definition of a n-form.

[7] The experienced reader is warned that $\mathscr{T}(\mathscr{M})$ does not stand for the tangent bundle of $\mathscr{M}$; it rather corresponds to the space of smooth cross-sections of that bundle. No confusion should arise because we shall not use the notion of bundle.

meaningless because the vectors $\boldsymbol{v}(q)$ and $\boldsymbol{v}(p)$ belong to two different vector spaces, $\mathscr{T}_q(\mathscr{M})$ and $\mathscr{T}_p(\mathscr{M})$ respectively (cf. Fig. 2.2). Note that for a scalar field, this problem does not arise [cf. Eq. (2.18)]. The solution is to introduce an extra-structure on the manifold, called an *affine connection* because, by defining the variation of a vector field, it allows one to connect the various tangent spaces on the manifold

An ***affine connection*** on $\mathscr{M}$ is a mapping

$$\begin{aligned} \nabla : \mathscr{T}(\mathscr{M}) \times \mathscr{T}(\mathscr{M}) &\longrightarrow \mathscr{T}(\mathscr{M}) \\ (\boldsymbol{u}, \boldsymbol{v}) &\longmapsto \nabla_{\boldsymbol{u}} \boldsymbol{v} \end{aligned} \tag{2.46}$$

which satisfies the following properties:

1. ∇ is bilinear (considering $\mathscr{T}(\mathscr{M})$ as a vector space over $\mathbb{R}$).
2. For any scalar field f and any pair $(\boldsymbol{u}, \boldsymbol{v})$ of vector fields:

$$\nabla_{f\boldsymbol{u}} \boldsymbol{v} = f \nabla_{\boldsymbol{u}} \boldsymbol{v}. \tag{2.47}$$

3. For any scalar field f and any pair $(\boldsymbol{u}, \boldsymbol{v})$ of vector fields, the following Leibniz rule holds:

$$\nabla_{\boldsymbol{u}}(f\boldsymbol{v}) = \langle \nabla f, \boldsymbol{u} \rangle \boldsymbol{v} + f \nabla_{\boldsymbol{u}} \boldsymbol{v}, \tag{2.48}$$

   where ∇f stands for the gradient of f as defined in Sect. 2.2.3.

The vector $\nabla_{\boldsymbol{u}} \boldsymbol{v}$ is called the ***covariant derivative of $\boldsymbol{v}$ along $\boldsymbol{u}$***.

Remark 2.10 Property 2 is not implied by property 1, for f is a scalar field, not a real number. Actually, property 2 ensures that at a given point $p \in \mathscr{M}$, the value of $\nabla_{\boldsymbol{u}} \boldsymbol{v}$ depends only on the vector $\boldsymbol{u}(p) \in \mathscr{T}_p(\mathscr{M})$ and not on the behaviour of $\boldsymbol{u}$ around p; therefore the role of $\boldsymbol{u}$ is only to give the direction of the derivative of $\boldsymbol{v}$.

Given an affine connection, the variation of a vector field $\boldsymbol{v}$ between two neighbouring points p and q, is defined as

$$\mathrm{d}\boldsymbol{v} := \nabla_{\mathrm{d}\boldsymbol{\ell}} \boldsymbol{v}, \tag{2.49}$$

$\mathrm{d}\boldsymbol{\ell}$ being the infinitesimal displacement vector connecting p and q [cf. Eq. (2.10)]. One says that $\boldsymbol{v}$ is ***parallelly transported from p to q with respect to the connection ∇*** iff $\mathrm{d}\boldsymbol{v} = 0$.

Given a frame field $(\boldsymbol{e}_\alpha)$ on $\mathscr{M}$, the ***connection coefficients*** of an affine connection ∇ with respect to $(\boldsymbol{e}_\alpha)$ are the scalar fields $\Gamma^\gamma{}_{\alpha\beta}$ defined by the expansion, at each point $p \in \mathscr{M}$, of the vector $\nabla_{\boldsymbol{e}_\beta} \boldsymbol{e}_\alpha(p)$ onto the basis $(\boldsymbol{e}_\alpha(p))$:

$$\boxed{\nabla_{\boldsymbol{e}_\beta} \boldsymbol{e}_\alpha =: \Gamma^\mu{}_{\alpha\beta} \boldsymbol{e}_\mu}. \tag{2.50}$$

An affine connection is entirely defined by the connection coefficients. In other words, there are as many affine connections on a manifold of dimension n as there are possibilities of choosing n^3 scalar fields $\Gamma^\gamma{}_{\alpha\beta}$.

Given $\boldsymbol{v} \in \mathscr{T}(\mathscr{M})$, one defines a tensor field of type $(1, 1)$, $\boldsymbol{\nabla v}$, called the ***covariant derivative of*** $\boldsymbol{v}$ ***with respect to the affine connection*** $\boldsymbol{\nabla}$, by the following action at each point $p \in \mathscr{M}$:

$$\begin{aligned} \boldsymbol{\nabla v}(p) : \mathscr{T}_p^*(\mathscr{M}) \times \mathscr{T}_p(\mathscr{M}) &\longrightarrow \quad \mathbb{R} \\ (\boldsymbol{\omega}, \boldsymbol{u}) \qquad &\longmapsto \langle \boldsymbol{\omega}, \boldsymbol{\nabla}_{\tilde{\boldsymbol{u}}} \boldsymbol{v}(p) \rangle, \end{aligned} \tag{2.51}$$

where $\tilde{\boldsymbol{u}}$ is any vector field which performs some extension of $\boldsymbol{u}$ around p: $\tilde{\boldsymbol{u}}(p) = \boldsymbol{u}$. As already noted (cf. Remark 2.10), $\boldsymbol{\nabla}_{\tilde{\boldsymbol{u}}} \boldsymbol{v}(p)$ is independent of the choice of $\tilde{\boldsymbol{u}}$, so that the mapping (2.51) is well defined. By comparing with (2.22), we verify that $\boldsymbol{\nabla v}(p)$ is a tensor of type $(1, 1)$.

One can extend the covariant derivative to all tensor fields by (i) demanding that for a scalar field the action of the affine connection is nothing but the gradient (hence the same notation $\boldsymbol{\nabla} f$) and (ii) using the Leibniz rule. As a result, the covariant derivative of a tensor field $\boldsymbol{T}$ of type (k, ℓ) is a tensor field $\boldsymbol{\nabla T}$ of type $(k, \ell + 1)$. Its components with respect a given field frame $(\boldsymbol{e}_\alpha)$ are denoted

$$\nabla_\gamma T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} := (\boldsymbol{\nabla T})^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell \gamma} \tag{2.52}$$

and are given by

$$\begin{aligned} \nabla_\gamma T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} = & \, \boldsymbol{e}_\gamma (T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell}) + \sum_{i=1}^{k} \Gamma^{\alpha_i}{}_{\sigma\gamma} \, T^{\alpha_1 \ldots \overset{i}{\overset{\downarrow}{\sigma}} \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} \\ & - \sum_{i=1}^{\ell} \Gamma^{\sigma}{}_{\beta_i \gamma} \, T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \underset{\underset{i}{\uparrow}}{\sigma} \ldots \beta_\ell}, \end{aligned} \tag{2.53}$$

where $\boldsymbol{e}_\gamma (T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell})$ stands for the action of the vector $\boldsymbol{e}_\gamma$ on the scalar field $T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell}$, resulting from the very definition of a vector (cf. Sect. 2.2.2). In particular, if $(\boldsymbol{e}_\alpha)$ is a natural frame associated with some coordinate system (x^α), then $\boldsymbol{e}_\alpha = \boldsymbol{\partial}_\alpha$ and the above formula becomes [cf. (2.7)]

$$\begin{aligned} \nabla_\gamma T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} = & \frac{\partial}{\partial x^\gamma} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} + \sum_{i=1}^{k} \Gamma^{\alpha_i}{}_{\sigma\gamma} \, T^{\alpha_1 \ldots \overset{i}{\overset{\downarrow}{\sigma}} \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} \\ & - \sum_{i=1}^{\ell} \Gamma^{\sigma}{}_{\beta_i \gamma} \, T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \underset{\underset{i}{\uparrow}}{\sigma} \ldots \beta_\ell}. \end{aligned} \tag{2.54}$$

Remark 2.11 Notice the position of the index γ in Eq. (2.52): it is the last one on the right-hand side. According to (2.23), $\boldsymbol{\nabla T}$ is then expressed as

$$\boldsymbol{\nabla T} = \nabla_\gamma T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} \, \boldsymbol{e}_{\alpha_1} \otimes \ldots \otimes \boldsymbol{e}_{\alpha_k} \otimes \boldsymbol{e}^{\beta_1} \otimes \ldots \otimes \boldsymbol{e}^{\beta_\ell} \otimes \boldsymbol{e}^\gamma. \tag{2.55}$$

Because $\boldsymbol{e}^\gamma$ is the *last* 1-form of the tensorial product on the right-hand side, the notation $T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell; \gamma}$ instead of $\nabla_\gamma T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell}$ would have been more appropriate. The index convention (2.55) agrees with that of MTW [9] [cf. their Eq. (10.17)].

The ***covariant derivative of the tensor field T along a vector v*** is defined by

$$\nabla_{\boldsymbol{v}}\boldsymbol{T} := \nabla\boldsymbol{T}(\underbrace{.,\ldots,.}_{k+\ell\ \text{slots}},\boldsymbol{u}). \tag{2.56}$$

The components of $\nabla_{\boldsymbol{v}}\boldsymbol{T}$ are then $v^\mu\nabla_\mu T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}$. Note that $\nabla_{\boldsymbol{v}}\boldsymbol{T}$ is a tensor field of the same type as $\boldsymbol{T}$. In the particular case of a scalar field f, we will use the notation $\boldsymbol{v}\cdot\nabla f$ for $\nabla_{\boldsymbol{v}} f$:

$$\boldsymbol{v}\cdot\nabla f := \nabla_{\boldsymbol{v}} f = \langle\nabla f,\boldsymbol{v}\rangle = \boldsymbol{v}(f). \tag{2.57}$$

The ***divergence*** with respect to the affine connection ∇ of a tensor field $\boldsymbol{T}$ of type (k,ℓ) with $k\geq 1$ is the tensor field denoted $\nabla\cdot\boldsymbol{T}$ of type $(k-1,\ell)$ and whose components with respect to any frame field are given by

$$(\nabla\cdot\boldsymbol{T})^{\alpha_1\ldots\alpha_{k-1}}{}_{\beta_1\ldots\beta_\ell} = \nabla_\mu \boldsymbol{T}^{\alpha_1\ldots\alpha_{k-1}\mu}{}_{\beta_1\ldots\beta_\ell}. \tag{2.58}$$

Remark 2.12 For the divergence, the contraction is performed on the *last* upper index of $\boldsymbol{T}$.

2.4.2 Levi–Civita Connection

On a pseudo-Riemannian manifold $(\mathscr{M},\boldsymbol{g})$ there is a unique affine connection ∇ such that

1. ∇ is ***torsion-free***, i.e. for any scalar field f, $\nabla\nabla f$ is a field of *symmetric* bilinear forms; in components:

$$\nabla_\alpha\nabla_\beta f = \nabla_\beta\nabla_\alpha f. \tag{2.59}$$

2. The covariant derivative of the metric tensor vanishes identically:

$$\boxed{\nabla\boldsymbol{g} = 0}. \tag{2.60}$$

∇ is called the ***Levi–Civita connection associated with g***. In this book, we shall consider only such connections.

With respect to the Levi–Civita connection, the Levi–Civita tensor $\boldsymbol{\varepsilon}$ introduced in Sect. 2.3.4 shares the same property as $\boldsymbol{g}$:

$$\boxed{\nabla\boldsymbol{\varepsilon} = 0}. \tag{2.61}$$

Given a coordinate system (x^α) on $\mathscr{M}$, the connection coefficients of the Levi–Civita connection with respect to the natural basis $(\boldsymbol{\partial}_\alpha)$ are called the ***Christoffel***

symbols; they can be evaluated from the partial derivatives of the metric components with respect to the coordinates:

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu} \left(\frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} \right). \tag{2.62}$$

Note that the Christoffel symbols are symmetric with respect to the lower two indices.

For the Levi–Civita connection, the expression for the divergence of a vector takes a rather simple form in a natural basis associated with some coordinates (x^{α}). Indeed, combining Eqs. (2.58) and (2.54), we get for $\boldsymbol{v} \in \mathscr{T}(\mathscr{M})$,

$$\nabla \cdot \boldsymbol{v} = \nabla_{\mu} v^{\mu} = \frac{\partial v^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}{}_{\sigma\mu} v^{\sigma}.$$

Now, from (2.62), we have

$$\Gamma^{\mu}{}_{\alpha\mu} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = \frac{1}{2} \frac{\partial}{\partial x^{\alpha}} \ln|g| = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\alpha}} \sqrt{|g|}, \tag{2.63}$$

where $g := \det(g_{\alpha\beta})$ [Eq. (2.45)]. The last but one equality follows from the general law of variation of the determinant of any invertible matrix A:

$$\boxed{\delta(\ln|\det A|) = \mathrm{tr}(A^{-1} \times \delta A)}, \tag{2.64}$$

where δ denotes any variation (derivative) that fulfills the Leibniz rule, tr stands for the trace and $\times$ for the matrix product. We conclude that

$$\boxed{\nabla \cdot \boldsymbol{v} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{|g|}\, v^{\mu} \right)}. \tag{2.65}$$

Similarly, for an antisymmetric tensor field of type (2, 0),

$$\nabla_{\mu} A^{\alpha\mu} = \frac{\partial A^{\alpha\mu}}{\partial x^{\mu}} + \underbrace{\Gamma^{\alpha}{}_{\sigma\mu} A^{\sigma\mu}}_{0} + \Gamma^{\mu}{}_{\sigma\mu} A^{\alpha\sigma} = \frac{\partial A^{\alpha\mu}}{\partial x^{\mu}} + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\sigma}} \sqrt{|g|}\; A^{\alpha\sigma},$$

where we have used the fact that $\Gamma^{\alpha}_{\sigma\mu}$ is symmetric in (σ, μ), whereas $A^{\sigma\mu}$ is antisymmetric. Hence the simple formula for the divergence of an *antisymmetric* tensor field of (2, 0):

$$\boxed{\nabla_{\mu} A^{\alpha\mu} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{|g|} A^{\alpha\mu} \right)}. \tag{2.66}$$

2.4.3 Curvature

2.4.3.1 General Definition

The ***Riemann curvature tensor*** of an affine connection ∇ is defined by

$$\begin{array}{rcl}\mathbf{Riem} : \mathscr{T}^*(\mathscr{M}) \times \mathscr{T}(\mathscr{M})^3 & \longrightarrow & C^\infty(\mathscr{M}, \mathbb{R}) \\ (\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) & \longmapsto & \left\langle \boldsymbol{\omega}, \nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w} - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} \boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]} \boldsymbol{w} \right\rangle,\end{array} \tag{2.67}$$

where $\mathscr{T}^*(\mathscr{M})$ stands for the space of 1-forms on $\mathscr{M}$, $\mathscr{T}(\mathscr{M})$ for the space of vector fields on $\mathscr{M}$ and$C^\infty(\mathscr{M}, \mathbb{R})$ for the space of smooth scalar fields on $\mathscr{M}$. The above formula does define a tensor field on $\mathscr{M}$, i.e. the value of $\mathbf{Riem}(\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})$ at a given point $p \in \mathscr{M}$ depends only upon the values of the fields $\boldsymbol{\omega}$, $\boldsymbol{w}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ at p and not upon their behaviours away from p, as the gradients in Eq. (2.67) might suggest. We denote the components of this tensor in a given basis $(\boldsymbol{e}_\alpha)$, not by $\mathrm{Riem}^\gamma{}_{\delta\alpha\beta}$, but by $R^\gamma{}_{\delta\alpha\beta}$. The definition (2.67) leads then to the following expression, named the ***Ricci identity***:

$$\forall \boldsymbol{w} \in \mathscr{T}(\mathscr{M}), \quad \left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) w^\gamma = R^\gamma_{\ \mu\alpha\beta} w^\mu. \tag{2.68}$$

Remark 2.13 In view of this identity, one may say that the Riemann tensor measures the lack of commutativity of two successive covariant derivatives of a vector field. On the opposite, for a scalar field and a torsion-free connection, two successive covariant derivatives always commute [cf. Eq. (2.59)].

In a coordinate basis, the components of the Riemann tensor are given in terms of the connection coefficients by

$$\boxed{R^\alpha{}_{\beta\mu\nu} = \frac{\partial \Gamma^\alpha{}_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha{}_{\beta\mu}}{\partial x^\nu} + \Gamma^\alpha{}_{\sigma\mu} \Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu} \Gamma^\sigma{}_{\beta\mu}}. \tag{2.69}$$

From the definition (2.67), the Riemann tensor is clearly antisymmetric with respect to its last two arguments $(\boldsymbol{u}, \boldsymbol{v})$:

$$\mathbf{Riem}(., ., \boldsymbol{u}, \boldsymbol{v}) = -\mathbf{Riem}(., ., \boldsymbol{v}, \boldsymbol{u}). \tag{2.70}$$

In addition, it satisfies the cyclic property

$$\mathbf{Riem}(., \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + \mathbf{Riem}(., \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + \mathbf{Riem}(., \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) = 0. \tag{2.71}$$

The covariant derivatives of the Riemann tensor obeys the ***Bianchi identity***

$$\boxed{\nabla_\rho R^\alpha_{\ \beta\mu\nu} + \nabla_\mu R^\alpha_{\ \beta\nu\rho} + \nabla_\nu R^\alpha_{\ \beta\rho\mu} = 0}. \tag{2.72}$$

2.4.3.2 Case of a Pseudo-Riemannian Manifold

The Riemann tensor of the Levi–Civita connection obeys the additional antisymmetry:

$$\mathbf{Riem}(\boldsymbol{\omega}, \boldsymbol{w}, ., .) = -\mathbf{Riem}(\underline{\boldsymbol{w}}, \vec{\boldsymbol{\omega}}, ., .). \tag{2.73}$$

Combined with (2.70) and (2.71), this implies the symmetry property

$$\mathbf{Riem}(\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \mathbf{Riem}(\underline{\boldsymbol{u}}, \boldsymbol{v}, \vec{\boldsymbol{\omega}}, \boldsymbol{w}). \tag{2.74}$$

A pseudo-Riemannian manifold $(\mathscr{M}, \boldsymbol{g})$ with a vanishing Riemann tensor is called a ***flat manifold***; in this case, $\boldsymbol{g}$ is said to be a ***flat metric***. If in addition, it has a Riemannian signature, $\boldsymbol{g}$ is called an ***Euclidean metric***.

2.4.3.3 Ricci Tensor

The ***Ricci tensor*** of the affine connection ∇ is the field of bilinear forms $\boldsymbol{R}$ defined by

$$\begin{array}{rccl} \boldsymbol{R}: & \mathscr{T}(\mathscr{M}) \times \mathscr{T}(\mathscr{M}) & \longrightarrow & C^{\infty}(\mathscr{M}, \mathbb{R}) \\ & (\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & \mathbf{Riem}(\boldsymbol{e}^{\mu}, \boldsymbol{u}, \boldsymbol{e}_{\mu}, \boldsymbol{v}). \end{array} \tag{2.75}$$

This definition is independent of the choice of the basis $(\boldsymbol{e}_\alpha)$ and its dual counterpart $(\boldsymbol{e}^\alpha)$. In terms of components:

$$\boxed{R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta}}. \tag{2.76}$$

Remark 2.14 Following the standard usage, we denote the components of the Riemann and Ricci tensors by the same letter R, the number of indices allowing us to distinguish between the two tensors. On the other hand, we are using different symbols, **Riem** and $\boldsymbol{R}$, when employing the 'intrinsic' notation.

For the Levi–Civita connection associated with the metric $\boldsymbol{g}$, the property (2.74) implies that the Ricci tensor is symmetric:

$$\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{R}(\boldsymbol{v}, \boldsymbol{u}). \tag{2.77}$$

In addition, one defines the ***Ricci scalar*** (also called ***scalar curvature***) as the trace of the Ricci tensor with respect to the metric $\boldsymbol{g}$:

$$R := g^{\mu\nu} R_{\mu\nu}. \tag{2.78}$$

The Bianchi identity (2.72) implies the divergence-free property

$$\boxed{\nabla \cdot \vec{\boldsymbol{G}} = 0}, \tag{2.79}$$

where $\vec{G}$ in the type-(1, 1) tensor associated by metric duality [cf. (2.39)] to the ***Einstein tensor***:

$$\boxed{\boldsymbol{G} := \boldsymbol{R} - \frac{1}{2} R \boldsymbol{g}}. \tag{2.80}$$

Equation (2.79) is called the ***contracted Bianchi identity***.

2.4.4 Weyl Tensor

Let $(\mathscr{M}, \boldsymbol{g})$ be a pseudo-Riemannian manifold of dimension n.

For $n = 1$, the Riemann tensor vanishes identically, i.e. $(\mathscr{M}, \boldsymbol{g})$ is necessarily flat. The reader who has in mind a curved line in the Euclidean plane $\mathbb{R}^2$ might be surprised by the above statement. This is because the Riemann tensor represents the *intrinsic* curvature of a manifold. For a line, the curvature that is not vanishing is the *extrinsic* curvature, i.e. the curvature resulting from the embedding of the line in $\mathbb{R}^2$. We shall discuss in more details the concepts of intrinsic and extrinsic curvatures in Chap. 3.

For $n = 2$, the Riemann tensor is entirely determined by the knowledge of the Ricci scalar R, according to the formula:

$$R^{\gamma}{}_{\delta\alpha\beta} = R\left(\delta^{\gamma}{}_{\alpha} g_{\delta\beta} - \delta^{\gamma}{}_{\beta} g_{\delta\alpha}\right) \qquad (n = 2). \tag{2.81}$$

For $n = 3$, the Riemann tensor is entirely determined by the knowledge of the Ricci tensor, according to

$$\begin{aligned} R^{\gamma}{}_{\delta\alpha\beta} = & R^{\gamma}{}_{\alpha} g_{\delta\beta} - R^{\gamma}{}_{\beta} g_{\delta\alpha} + \delta^{\gamma}{}_{\alpha} R_{\delta\beta} - \delta^{\gamma}{}_{\beta} R_{\delta\alpha} \\ & + \frac{R}{2}\left(\delta^{\gamma}{}_{\beta} g_{\delta\alpha} - \delta^{\gamma}{}_{\alpha} g_{\delta\beta}\right) \qquad (n = 3). \end{aligned} \tag{2.82}$$

For $n \geq 4$, the Riemann tensor can be split into (i) a "trace-trace" part, represented by the Ricci scalar R [Eq. (2.78)], (ii) a "trace" part, represented by the Ricci tensor $\boldsymbol{R}$ [Eq. (2.76)], and (iii) a "traceless" part, which is constituted by the ***Weyl conformal curvature tensor***, $\boldsymbol{C}$:

$$\begin{aligned} R^{\gamma}_{\ \delta\alpha\beta} = & C^{\gamma}_{\ \delta\alpha\beta} + \frac{1}{n-2}\left(R^{\gamma}_{\alpha} g_{\delta\beta} - R^{\gamma}_{\beta} g_{\delta\alpha} + \delta^{\gamma}_{\alpha} R_{\delta\beta} - \delta^{\gamma}_{\beta} R_{\delta\alpha}\right) \\ & + \frac{1}{(n-1)(n-2)} R\left(\delta^{\gamma}_{\beta} g_{\delta\alpha} - \delta^{\gamma}_{\alpha} g_{\delta\beta}\right). \end{aligned} \tag{2.83}$$

The above relation may be taken as the definition of $\boldsymbol{C}$. It implies that $\boldsymbol{C}$ is traceless: $C^{\mu}_{\ \alpha\mu\beta} = 0$. The other possible traces are zero thanks to the symmetry properties of the Riemann tensor.

Remark 2.15 The decomposition (2.83) is also meaningful for $n = 3$, but it then implies that the Weyl tensor vanishes identically [compare with (2.82)].

2.5 Lie Derivative

As discussed in Sect. 2.4.1, the notion of a derivative of a vector field on a manifold $\mathscr{M}$ requires the introduction of some extra-structure on $\mathscr{M}$. In Sect. 2.4.1, this extra-structure was an affine connection and in Sect. 2.4.2 a metric $\boldsymbol{g}$ (which provides naturally an affine connection: the Levi–Civita one). Another possible extra-structure is a “reference” vector field, with respect to which the derivative is to be defined. This is the concept of the *Lie derivative*, which we discuss here.

2.5.1 Lie Derivative of a Vector Field

Consider a vector field $\boldsymbol{u}$ on $\mathscr{M}$, called hereafter the ***flow***. Let $\boldsymbol{v}$ be another vector field on $\mathscr{M}$, the variation of which is to be studied. We can use the flow $\boldsymbol{u}$ to transport the vector $\boldsymbol{v}$ from one point p to a neighbouring one q and then define rigorously the variation of $\boldsymbol{v}$ as the difference between the actual value of $\boldsymbol{v}$ at q and the transported value via $\boldsymbol{u}$. More precisely the definition of the Lie derivative of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$ is as follows (see Fig. 2.3). We first define the image $\Phi_\varepsilon(p)$ of the point p by the transport by an infinitesimal “distance” ε along the field lines of $\boldsymbol{u}$ as $\Phi_\varepsilon(p) = q$, where q is the point close to p such that the infinitesimal displacement vector from p to q is $\overrightarrow{pq} = \varepsilon \boldsymbol{u}(p)$ (cf. Sect. 2.2.2). Besides, if we multiply the vector $\boldsymbol{v}(p)$ by some infinitesimal parameter λ, it becomes an infinitesimal vector at p. Then there exists a unique point p' close to p such that $\lambda \boldsymbol{v}(p) = \overrightarrow{pp'}$. We may transport the point p' to a point q' along the field lines of $\boldsymbol{u}$ by the same “distance” ε as that used to transport p to q: $q' = \Phi_\varepsilon(p')$ (see Fig. 2.3). $\overrightarrow{qq'}$ is then an infinitesimal vector at q and we define the transport by the distance ε of the vector $\boldsymbol{v}(p)$ along the field lines of $\boldsymbol{u}$ according to

$$\Phi_\varepsilon(\boldsymbol{v}(p)) := \frac{1}{\lambda}\overrightarrow{qq'}. \tag{2.84}$$

$\Phi_\varepsilon(\boldsymbol{v}(p))$ is a vector in $\mathscr{T}_q(\mathscr{M})$. We may then subtract it from the actual value of the field $\boldsymbol{v}$ at q and define the ***Lie derivative*** of $\boldsymbol{v}$ along $\boldsymbol{u}$ by

$$\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v} := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\boldsymbol{v}(q) - \Phi_\varepsilon(\boldsymbol{v}(p))\right]. \tag{2.85}$$

Let us consider a coordinate system (x^α) adapted to the field $\boldsymbol{u}$ in the sense that $\boldsymbol{u} = \boldsymbol{\partial}_0$, where $\boldsymbol{\partial}_0$ is the first vector of the natural basis associated with the coordinates (x^α). We have, from the definitions of points q, p' and q',

$$\begin{aligned}
x^\alpha(q) &= x^\alpha(p) + \varepsilon\delta^\alpha{}_0\\
x^\alpha(p') &= x^\alpha(p) + \lambda v^\alpha(p)\\
x^\alpha(q') &= x^\alpha(p') + \varepsilon\delta^\alpha{}_0,
\end{aligned}$$

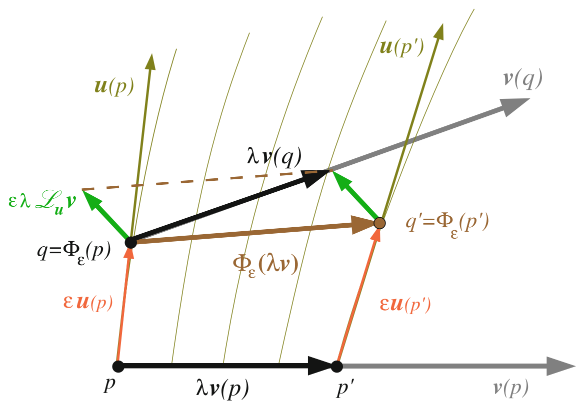

Fig. 2.3 Geometrical construction of the Lie derivative of a vector field: given a small parameter λ, each extremity of the arrow $\lambda\boldsymbol{v}$ is dragged by some small parameter ε along $\boldsymbol{u}$, to form the vector denoted by $\Phi_\varepsilon(\lambda\boldsymbol{v})$. The latter is then compared with the actual value of $\lambda\boldsymbol{v}$ at the point q, the difference (divided by $\lambda\varepsilon$) defining the Lie derivative $\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v}$

so that

$$(qq')^\alpha = x^\alpha(p') - x^\alpha(p) = \lambda v^\alpha(p).$$

Accordingly, (2.84) and (2.85) result in

$$\begin{aligned}(\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v})^\alpha &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[v^\alpha(q) - v^\alpha(p)\right]\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[v^\alpha(x^0+\varepsilon,\ x^1,\ \ldots,\ x^{n-1}) - v^\alpha(x^0,\ x^1,\ \ldots,\ x^{n-1})\right].\end{aligned}$$

Hence, in adapted coordinates, the Lie derivative is simply obtained by taking the partial derivative of the vector components with respect to x^0:

$$\mathscr{L}_{\boldsymbol{u}} v^\alpha = \frac{\partial v^\alpha}{\partial x^0}, \tag{2.86}$$

where we have used the standard notation for the components of a Lie derivative: $\mathscr{L}_{\boldsymbol{u}} v^\alpha := (\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v})^\alpha$. Besides, using the fact that the components of $\boldsymbol{u}$ are $u^\alpha = (1, 0, \ldots, 0)$ in the adapted coordinate system, we notice that the components of the commutator of $\boldsymbol{u}$ and $\boldsymbol{v}$, as given by (2.25), are

$$[\boldsymbol{u}, \boldsymbol{v}]^\alpha = \frac{\partial v^\alpha}{\partial x^0}.$$

This is exactly (2.86): $[\boldsymbol{u}, \boldsymbol{v}]^\alpha = \mathscr{L}_{\boldsymbol{u}} v^\alpha$. We conclude that the Lie derivative of a vector with respect to another one is actually nothing but the commutator of these two vectors:

$$\boxed{\mathscr{L}_{\boldsymbol{u}}\boldsymbol{v} = [\boldsymbol{u}, \boldsymbol{v}]}. \tag{2.87}$$

Thanks to formula (2.25), we may then express the components of the Lie derivative in an arbitrary coordinate system:

$$\boxed{\mathscr{L}_{\boldsymbol{u}} v^\alpha = u^\mu \frac{\partial v^\alpha}{\partial x^\mu} - v^\mu \frac{\partial u^\alpha}{\partial x^\mu}}. \tag{2.88}$$

Thanks to the symmetry property of the Christoffel symbols, the partial derivatives in Eq. (2.88) can be replaced by the Levi–Civita connection $\boldsymbol{\nabla}$ associated with some metric $\boldsymbol{g}$, yielding

$$\mathscr{L}_{\boldsymbol{u}} v^\alpha = u^\mu \nabla_\mu v^\alpha - v^\mu \nabla_\mu u^\alpha. \tag{2.89}$$

2.5.2 Generalization to Any Tensor Field

The Lie derivative is extended to any tensor field by (i) demanding that for a scalar field f, $\mathscr{L}_{\boldsymbol{u}} f = \langle \boldsymbol{\nabla} f, \boldsymbol{u} \rangle$ and (ii) using the Leibniz rule. As a result, the Lie derivative $\mathscr{L}_{\boldsymbol{u}} \boldsymbol{T}$ of a tensor field $\boldsymbol{T}$ of type (k, ℓ) is a tensor field of the same type, the components of which with respect to a given coordinate system (x^α) are

$$\begin{aligned} \mathscr{L}_{\boldsymbol{u}} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} = u^\mu \frac{\partial}{\partial x^\mu} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} - \sum_{i=1}^{k} T^{\alpha_1 \ldots \overset{i}{\overset{\downarrow}{\sigma}} \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} \, \frac{\partial u^{\alpha_i}}{\partial x^\sigma} \\ + \sum_{i=1}^{\ell} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \underset{i}{\underset{\uparrow}{\sigma}} \ldots \beta_\ell} \, \frac{\partial u^\sigma}{\partial x^{\beta_i}}. \end{aligned} \tag{2.90}$$

In particular, for a 1-form,

$$\mathscr{L}_{\boldsymbol{u}} \boldsymbol{\omega}_\alpha = u^\mu \frac{\partial \omega_\alpha}{\partial x^\mu} + \omega_\mu \frac{\partial u^\mu}{\partial x^\alpha}. \tag{2.91}$$

As for the vector case [Eq. (2.88)], the partial derivatives in Eq. (2.90) can be replaced by the covariant derivative $\boldsymbol{\nabla}$ (or any other connection without torsion), yielding

$$\begin{aligned} \mathscr{L}_{\boldsymbol{u}} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} = u^\mu \nabla_\mu T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} - \sum_{i=1}^{k} T^{\alpha_1 \ldots \overset{i}{\overset{\downarrow}{\sigma}} \ldots \alpha_k}{}_{\beta_1 \ldots \beta_\ell} \, \nabla_\sigma u^{\alpha_i} \\ + \sum_{i=1}^{\ell} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \underset{i}{\underset{\uparrow}{\sigma}} \ldots \beta_\ell} \, \nabla_{\beta_i} u^\sigma. \end{aligned} \tag{2.92}$$

Remark 2.16 Both the covariant derivative (affine connection) and the Lie derivative act on any kind of tensor field. For the specific class of tensor fields composed of p-forms (cf. Sect. 2.2.5), there exists a third type of derivative, which does not require any extra-structure on $\mathscr{M}$: the *exterior derivative* $\mathbf{d}$. For a 0-form (scalar field), $\mathbf{d}$ coincides with the gradient, hence the notation $\mathbf{d}x^\alpha$ used to denote the gradient of coordinates [cf. (2.20)]. We shall not use the exterior derivative in this book and so will not discus it further (see the classical textbooks [3, 6, 9] or Ref. [10] for an elementary introduction).

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