Preface

During the more than 100 years of its existence, the notion of the fundamental group has undergone a considerable evolution. It started by Henri Poincaré when topology as a subject was still in its infancy. The fundamental group in this setup measures the complexity of a pointed topological space by means of an algebraic invariant, a discrete group, composed of deformation classes of based closed loops *within* the space. In this way, for example, the monodromy of a holomorphic function on a Riemann surface could be captured in a systematic way.

It was through the work of Alexander Grothendieck that, raising into the focus the role played by the fundamental group in governing covering spaces, so spaces *over* the given space, a unification of the topological fundamental group with Galois theory of algebra and arithmetic could be achieved. In some sense the roles have been reversed in this *discrete Tannakian* approach of abstract Galois categories: first, we describe a suitable class of objects that captures *monodromy*, and then, by abstract properties of this class alone and moreover uniquely determined by it, we find a pro-finite group that describes this category completely as the category of discrete objects continuously acted upon by that group.

But the different incarnations of a fundamental group do not stop here. The concept of describing a fundamental group through its category of objects upon which the group naturally acts finds its pro-algebraic realisation in the theory of Tannakian categories that, when applied to vector bundles with flat connections, or to smooth ℓ-adic étale sheaves, or to iso-crystals or ..., gives rise to the corresponding fundamental group, each within its natural category as a habitat.

In more recent years, the influence of the fundamental group on the geometry of Kähler manifolds or algebraic varieties has become apparent. Moreover, the program of anabelian geometry as initiated by Alexander Grothendieck realised some spectacular achievements through the work of the Japanese school of Hiroaki Nakamura, Akio Tamagawa and Shinichi Mochizuki culminating in the proof that hyperbolic curves over p-adic fields are determined by the outer Galois action of the absolute Galois group of the base field on the étale fundamental group of the curve.

A natural next target for pieces of arithmetic captured by the fundamental group are rational points, the genuine object of study of Diophantine geometry. Here there are two related strands: Grothendieck's section conjecture in the realm of the étale arithmetic fundamental group, and second, more recently, Minhyong Kim's idea to use the full strength of the different (motivic) realisations of the fundamental group to obtain a nonabelian unipotent version of the classical Chabauty approach towards rational points. In this approach, one seeks for a nontrivial p-adic Coleman analytic function that finds all global rational points among its zeros, whereby in the one-dimensional case the number of zeros necessarily becomes finite. This has led to a spectacular new proof of Siegel's theorem on the finiteness of S-integral points in some cases and, moreover, raised hope for ultimately (effectively) reproving the Faltings–Mordell theorem. A truely motivic advance of Minhyong Kim's ideas due to Gerd Faltings and Majid Hadian is reported in the present volume.

This volume originates from a special activity at Heidelberg University under the sponsorship of the MAThematics Center Heidelberg (MATCH) that took place in January and February 2010 organised by myself. The aim of the activity was to bring together people working in the different strands and incarnations of the fundamental group all of whose work had a link to arithmetic applications. This was reflected in the working title PIA for our activity, which is the (not quite) acronym for π_1–arithmetic, short for *doing arithmetic with the fundamental group* as your main tool and object of study. PIA survived in the title of the workshop organised during the special activity: *PIA 2010 — The arithmetic of fundamental groups*, which in reversed order gives rise to the title of the present volume.

The workshop took place in Heidelberg, 8–12 February 2010, and the abstracts of all talks are listed at the end of this volume. Many of these accounts are mirrored in the contributions of the present volume. The special activity also comprised expository lecture series by Amnon Besser on Coleman integration, a technique used by the non-abelian Chabauty method, and by Tamás Szamuely on Grothendieck's fundamental group with a view towards anabelian geometry. Lecture notes of these two introductory courses are contained in this volume as a welcome addition to the existing literature of both subjects.

I wish to extend my sincere thanks to the contributors of this volume and to all participants of the special activity in Heidelberg on the arithmetic of fundamental groups, especially to the lecturers giving mini-courses, for the energy and time they have devoted to this event and the preparation of the present collection. Paul Seyfert receives the editor's thanks for sharing his marvelous TEX–expertise and help in typesetting this volume. Furthermore, I would like to take this opportunity to thank Dorothea Heukäufer for her efficient handling of the logistics of the special activity and Laura Croitoru for coding the website. I am very grateful to Sabine Stix for sharing her organisational skills both by providing a backbone for the *to do* list of the whole program and also in caring for our kids Antonia, Jaden and Lucie. Finally, I would like to express my gratitude to Willi Jäger, the former director of MATCH, for his enthusiastic support and for the financial support of MATCH that made PIA 2010 possible and in my opinion a true success.

Heidelberg *Jakob Stix*

http://www.springer.com/978-3-642-23904-5

The Arithmetic of Fundamental Groups
PIA 2010
Stix, J. (Ed.)
2012, XII, 380 p., Hardcover
ISBN: 978-3-642-23904-5