Chapter 5
The Motivic Logarithm for Curves

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Abstract The paper explains how in Kim's approach to diophantine equations étale cohomology can be replaced by motivic cohomology. For this Beilinson's construction of the motivic logarithm suffices, and it is not necessary to construct a category of mixed motives as it is done by Deligne–Goncharov for rational curves.

5.1 Introduction

The purpose of this note is to exhibit the definition of a motivic logarithm for smooth curves, following Beilinson. I personally prefer the name polylogarithm but I have learned that there is some opposition to this because the name polylogarithm is already in use for different (although in my opinion related) objects. Many finer properties of motivic categories are only shown over fields, but our definition makes sense over any base. We show that its different realisations give the logarithm in étale and crystalline cohomology. For rational curves this has been done in Deligne–Goncharov. They in fact achieve much more by also defining a category of mixed Tate-motives which contains the motivic logarithm. Here we are more modest and only construct the logarithm itself, without exhibiting it as an object in a category of mixed motives. It is something like the free tensor-algebra in the reduced motive of the curve, and the motivic fundamental group has as its Lie-algebra the free Lie-algebra. In fact we work with $\mathbb{Q}$-coefficients where nilpotent groups and nilpotent Lie-algebras correspond via the Hausdorff series, see [Bou75, Chap. 2, 6]. At the end we try to define the notion of a motivic torsor. Unfortunately the desired properties need some additional vanishing assumptions. For the moment these are known only for Tate-motives over a field, where however we already can cite Deligne–Goncharov [DG05].

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Our main motivation for this work was the new method of Kim [Kim05] in diophantine geometry, and its improvement by Hadian-Jazi [HJ10] which involves motivic cohomology. However so far the results are not really useful in that context because of our general inability to compute motivic cohomology groups. The few known cases concern Tate-motives mainly of number fields and use the connection with algebraic K-theory. In this context it might be useful to define some regulator with values in K-theory. For example this might work over the integers while the most essential properties of motivic cohomology are only shown over a field.

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We assume that S is an arbitrary base-scheme and

$$\mathrm{X} \to \mathrm{S}$$

a relative smooth curve with geometrically irreducible fibres. Furthermore we assume that X admits an embedding

$$\mathrm{X} \subseteq \bar{\mathrm{X}}$$

into a smooth projective curve $\bar{\mathrm{X}}$ such that the complement

$$\mathrm{D} = \bar{\mathrm{X}} - \mathrm{X}$$

is a divisor which is finite étale over S. We denote by $\mathcal{M}$ the Karoubian hull of the category of smooth S-schemes, with maps given on connected components by $\mathbb{Z}$-linear combinations of actual maps of schemes. That is an object of $\mathcal{M}$ is defined by a smooth S-scheme T together with a projector in the locally constant combinations of elements of $\mathrm{End}_{\mathrm{S}}(\mathrm{T})$. $\mathcal{M}$ is an additive category (sums are disjoint unions) with tensor product given by the fibered product of smooth S-schemes. The same is true for the category $\mathrm{K}(\mathcal{M})$ of finite complexes with entries in $\mathcal{M}$, where the tensor product is the usual tensorproduct of complexes which is symmetric with the usual sign-rule. Objects of $\mathrm{K}(\mathcal{M})$ have well defined cohomology for any reasonable cohomological functor on the category of smooth S-schemes, and $\mathcal{M}$ is sufficiently big to allow our basic construction. For a smooth S-scheme X we denote by M(X) its image in $\mathcal{M}$.

Later we shall pass to coefficients $\mathbb{Q}$ but keep the same notations.

5.2 Enveloping Algebras and Symmetric Groups

This section collects some general remarks for later use. All algebras are algebras over a base field k of characteristic zero. Its purpose is to construct certain operators (linear combinations of permutations) which operate on the free Lie algebra but can also be applied to the motivic logarithm. Assume first that $\mathfrak{g}$ is a Lie-algebra over k. Define its central series $\mathrm{Z}^n(\mathfrak{g})$ by the rule that Z^n is the subspace generated by n-fold commutators, so

$$\mathrm{Z}^1(\mathfrak{g}) = \mathfrak{g}\,.$$

Furthermore $\mathrm{U}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{g})$ denote the enveloping algebra and the symmetric algebra. The algebra $\mathrm{U}(\mathfrak{g})$ admits a cocommutative and coassociative coproduct

$$c\,:\;\mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$$

which is a homomorphism of rings and satisfies

$$c(x) = x \otimes 1 + 1 \otimes x$$

for $x \in \mathfrak{g}$. If we compose with the multiplication on $\mathrm{U}(\mathfrak{g})$ we obtain a linear operator λ on $\mathrm{U}(\mathfrak{g})$. It has the following alternate description:

There exists, see [Bou75, Ch. 1, 2, 7], a $\mathfrak{g}$-linear isomorphism between $\mathrm{S}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g})$ which sends a monomial

$$x_1 \cdot \dots \cdot x_n \in \mathrm{S}(\mathfrak{g})$$

to the average over the symmetric group S_n of all permuted products in $\mathrm{U}(\mathfrak{g})$. One easily checks that λ operates on the image of $\mathrm{S}^n(\mathfrak{g})$ by multiplication by 2^n. Thus λ is a diagonalisable automorphism of $\mathrm{U}(\mathfrak{g})$ with eigenvalues 2^n. If

$$\mathrm{I} \subset \mathrm{U}(\mathfrak{g})$$

denotes the augmentation-ideal one easily sees that I^n is the image of the subspace of weight $\geq n$ in $\mathrm{S}(\mathfrak{g})$. Here the weight of an element $x \in \mathfrak{g}$ is the maximal n such that $x \in \mathrm{Z}^n(\mathfrak{g})$, or ∞, and the weight of a monomial $x_1 \dots x_n$ is the sum of the weights of its components. It follows that λ respects I^n and that on the quotient

$$\mathrm{U}(\mathfrak{g})/\mathrm{I}^n$$

the $\lambda = 2$-eigenspace is isomorphic to

$$\mathfrak{g}/\mathrm{Z}^n(\mathfrak{g})\,.$$

As a generalisation, modify c to ($x \in \mathrm{U}(\mathfrak{g})$)

$$\tilde{c}(x) = c(x) - x \otimes 1 - 1 \otimes x$$

and consider the complex

$$k \to \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \to \dots$$

where the differential is the alternating sum

$$d(x_1 \otimes x_2 \otimes \dots) = \tilde{c}(x_1) \otimes x_2 \otimes \dots - x_1 \otimes \tilde{c}(x_2) \otimes \dots$$

The square of the differential vanishes because of the coassociativity of c. Via the symmetrisation map this complex is isomorphic to the analogeous complex for the symmetric algebra $\mathrm{S}(\mathfrak{g})$.

Lemma 1. *The cohomology of the $\tilde{c}$-complex is isomorphic to the exterior algebra $\wedge(\mathfrak{g})$.*

Proof. This is an assertion about k-vector spaces. For a finite dimensional vector space W let the dual W^t operate via the diagonal on the standard simplicial model for EW^t, that is the simplicial scheme with entries $\mathrm{W}^{t,n+1}$ in degree n. The complex of regular algebraic functions on EW^t gives a resolution of k by injective S(W)-modules. Our complex is the complex of W^t-invariants via the diagonal action whose cohomology is

$$\mathrm{Ext}^{\bullet}_{\mathrm{S}(\mathrm{W}^t)}(k,k) = \wedge(\mathrm{W}) . \qquad \square$$

Remark 2. It is easy to see that the differential, a graded derivation, vanishes on the subspaces

$$\mathfrak{g}^{\otimes n} \subset \mathrm{U}(\mathfrak{g})^{\otimes n} ,$$

and the subspaces $\wedge^n(\mathfrak{g})$ of antisymmetric elements represent the cohomology.

Cocycles representing these classes are for example cup-products of linear functions on W^t, or their antisymmetrisations. Translated to Lie-algebras we obtain that our original complex has cohomology $\wedge(\mathfrak{g})$ and all classes are represented by antisymmetric elements. That is let the groups S_n operate on $\mathrm{U}(\mathfrak{g})^{\otimes n}$ and denote by ϵ_n the projector onto antisymmetric elements. Although the different ϵ_n's do not commute with the differentials they still annihilate their image because of the symmetry of $\tilde{c}$. As they operate as the identity on suitable representatives of the cohomology classes they induce an injection of the n-th cohomology into $\mathrm{U}(\mathfrak{g})^{\otimes n}$. So a cocycle is a coboundary if it is annihilated by ϵ_n.

Finally assume that $\mathfrak{g}$ is the free Lie-algebra on a k-vectorspace V. Then

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathrm{V})$$

is the free tensoralgebra in V, with the coproduct

$$c : \mathrm{T}(\mathrm{V}) \to \mathrm{T}(\mathrm{V}) \otimes \mathrm{T}(\mathrm{V})$$

induced by the diagonal on V, and thus given by the shuffle formula

$$c(v_1 \otimes v_2 \cdots \otimes v_n) = \sum v_{\mathrm{A}} \otimes v_{\mathrm{B}} ,$$

where the sum is over all disjoint decompositions

$$\{1, \dots, n\} = \mathrm{A} \sqcup \mathrm{B} ,$$

and v_A, v_B denote the products (in natural order) of the v_i with $i \in A, B$. It follows that the operator λ respects the direct sum $V^{\otimes n}$ and acts on it via a certain element of the group-ring $\mathbb{Z}[S_n]$ which is independant of V. As the operation of $\mathbb{Q}[S_n]$ on $V^{\otimes n}$ is faithful if V has dimension at least n it follows that this element is semisimple with eigenvalues $\{1, 2, \ldots, 2^n\}$, that is its action by left-multiplication on $\mathbb{Q}[S_n]$ has this property. The projection onto the $\lambda = 2$-eigenspace is then defined by a certain universal element

$$e_n \in \mathbb{Q}[S_n]$$

which is a polynomial in λ. Its image consists of the image of the free Lie-algebra, that is of all Lie-polynomials of length n in its generators. For example for $n = 2$ we obtain all commutators, so

$$e_2 = (1 - \sigma)/2$$

with σ the transposition in S_2. The e_n have the property that for any k-vectorspace V the direct sum

$$\sum_n e_n(V^{\otimes n})$$

is a Lie-algebra, that is we have in $\mathbb{Q}[S_{m+n}]$ the identity (in hopefully suggestive notation)

$$(1 - e_{m+n})(e_m \otimes e_n - e_n \otimes e_m) = 0 \,.$$

A general formula for e_n can be found in [Bou75, Ch. II §3.2].

We give another argument which generalises to other cases. The actions of the algebraic group GL(V) and the finite group S_n on $V^{\otimes n}$ commute, and it is known, see [Wey46, Thm. 4.4.E], that the commutator of GL(V) in $\mathrm{End}(V^{\otimes n})$ is the image of the group ring $k[S_n]$, equal to it if $\dim(V) \geq n$. Also we know that the action of GL(V) is reductive. Thus any GL(V)-invariant subspace is the image of a projector e in $k[S_n]$, and the right ideal generated by e in $k[S_n]$ is canonical. So for the Lie-polynomials of degree n we obtain such a projector in $\mathbb{Q}[S_n]$ which can be chosen independantly of V.

We apply this as follows to our complex with entries $T(V)^{\otimes n}$ and differentials d the alternating sums of $\tilde{c}$'s: There exist universal matrices u, v, w with entries elements of $\mathbb{Q}[S_l]$ (l suitable) such that on $T(V)^{\otimes n}$ we have the identity

$$\mathrm{id} = d \circ u + v \circ \epsilon_n + w \circ d \,.$$

For applications we also need a superversion which amounts to the same with signs added at suitable locations: now $\mathfrak{g}$ is graded into even and odd parts, and for two odd elements the commutator has to be replaced by the supercommutator. By $S(\mathfrak{g})$ we denote the supersymmetric algebra, that is the tensorproduct of the symmetric algebra on even elements and the alternating algebra on odd elements. Again

$$S(\mathfrak{g}) \cong U(\mathfrak{g}) \,.$$

The enveloping algebra $U(\mathfrak{g})$ is also $\mathbb{Z}/(2)$-graded and the coproduct

$$c : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$$

is a ring homomorphism induced by the diagonal on $\mathfrak{g}$. However the multiplication rule on the tensorproduct is "super", that is two odd elements in the factors anticommute. The definition of λ is the same as before, and it operates on the image of $\mathrm{S}^n(\mathfrak{g})$ as 2^n. Furthermore λ respects the powers I^n of the augmentation ideal $\mathrm{I} \subset \mathrm{U}(\mathfrak{g})$ and the $\lambda = 2$-eigenspace on $\mathrm{U}(\mathfrak{g})/\mathrm{I}^n$ is $\mathfrak{g}/\mathrm{Z}^n(\mathfrak{g})$.

Finally, if V is an odd k-vectorspace we can apply this to the free Lie-algebra in V. The coproduct on

$$\mathrm{U}(\mathfrak{g}) = \mathrm{T}(\mathrm{V})$$

is given by the shuffle formula

$$c(v_1 \otimes \ldots \otimes v_n) = \sum_{\mathrm{A,B}} \pm x_{\mathrm{A}} \otimes x_{\mathrm{B}}$$

where the sum is over all disjoint decompositions of $\{1, \ldots, n\}$ and the sign is that of the shuffle, i.e., that of the permutation which defines it. It follows that the $\lambda = 2$-eigenspace in $\mathrm{V}^{\otimes n}$ is again defined by a universal projector

$$f_n \in \mathbb{Q}[\mathrm{S}_n]$$

which differs from the previous e_n by an application of the sign-character. For example

$$f_2 = (1 + \sigma)/2\,.$$

Also as before

$$(1 - f_{m+n})(f_m \otimes f_n - (-1)^{mn} f_n \otimes f_m) = 0\,.$$

Finally, also the results about $\tilde{c}$ carry over.

5.3 The Definition of the Motivic Logarithm

Assume we are given an S-point $\mathrm{X} \in \mathrm{X}(\mathrm{S})$. The composition of projection and the inclusion

$$\mathrm{X} \to \mathrm{S} \to \mathrm{X}$$

is an idempotent e_x acting on M(X), and $\mathrm{M}(\mathrm{X})^\circ$ denotes the image of $1 - e_x$, the **reduced homology**. Similarly a product like

$$\mathrm{M}(\mathrm{X})^\circ \otimes \mathrm{M}(\mathrm{X})^\circ \otimes \mathrm{M}(\mathrm{X})$$

denotes the direct summand of $\mathrm{M}(\mathrm{X} \times \mathrm{X} \times \mathrm{X})$ where we apply the above idempotent to the first two factors.

Now define a projective system of complexes in $\mathrm{K}(\mathcal{M})$ by chosing for $\underline{\mathrm{P}}_n$ the complex

$$\mathrm{M(S)} \to \mathrm{M(X)}^{\circ} \to \mathrm{M(X)}^{\circ\otimes 2} \to \cdots \to \mathrm{M(X)}^{\circ,\otimes n}$$

where the maps are induced by the alternating sums of the adjacent diagonals $\delta_{i,i+1}$ which double the argument in position i, for $0 \le i \le n$. In particular, the first map is 0. It is obviously related to the simplicial object $\mathrm{C}osk_0(\mathrm{X})$ but should not be confused with the usual chain complex whose differentials have the opposite direction. By the equality

$$\delta \cdot e_x = (e_x \otimes e_x) \cdot \delta$$

for the diagonal

$$\delta : \mathrm{M(X)} \to \mathrm{M(X \times X)} = \mathrm{M(X)} \otimes \mathrm{M(X)}$$

the diagonals induce maps on the reduced quotient of $\mathrm{M(X)}^{\otimes m}$. There are compatible associative products

$$\underline{\mathrm{P}}_n \otimes \underline{\mathrm{P}}_n \to \underline{\mathrm{P}}_n$$

induced from the juxtaposition

$$\mathrm{M(X)}^{\otimes a} \otimes \mathrm{M(X)}^{\otimes b} \to \mathrm{M(X)}^{\otimes a+b}$$

which sends $(x_1, \ldots, x_a) \otimes (x_{a+1}, \ldots, x_{a+b})$ to $(x_1, \ldots, x_{a+b})$. Also we have graded cocommutative and coassociative shuffle coproducts

$$\underline{\mathrm{P}}_{m+n} \to \underline{\mathrm{P}}_m \otimes \underline{\mathrm{P}}_n$$

induced from

$$(x_1, \ldots, x_{a+b}) \mapsto \sum_{\sigma} \mathrm{sign}(\sigma)(x_{\sigma(1)}, \ldots, x_{\sigma(a)}) \otimes (x_{\sigma(a+1)}, \ldots, x_{\sigma(a+b)}),$$

where the sum is over all permutions of $\{1, \ldots, a+b\}$ which are monotone on $\{1, \ldots, a\}$ and on $\{a+1, \ldots, a+b\}$. The sum of these makes the right unbounded complex

$$\underline{\mathrm{P}} = \varprojlim \underline{\mathrm{P}}_n$$

a cocommutative Hopf-algebra.

If we pass to $\mathbb{Q}$-coefficients we obtain as a direct summand a super Lie algebra as follows.

Proposition 3. *The direct summands $f_n(\mathrm{M(X)}^{\circ\otimes n})$ are preserved by the differentials (the f_n are polynomials in the composition of multiplication and comultiplication) and they form a super Lie algebra $\underline{\mathrm{L}}$ with truncations $\underline{\mathrm{L}}_n$.*

Proof. This follows from the rule

$$(1 - f_{m+n})(f_m \otimes f_n - (-1)^{mn} f_n \otimes f_m) = 0. \qquad \square$$

If we are given another S-point $y \in \mathrm{X(S)}$ define a new complex $\underline{\mathrm{P}}(y)$ by the rule that its terms are the same as for $\underline{\mathrm{P}}$ but that we add to the differential

$$\underline{\mathrm{X}}_n \to \underline{\mathrm{X}}_{n+1}$$

the map induced by

$$(x_1,\ldots,x_n) \to -(y,x_1,\ldots,x_n)\,.$$

The same formulas as before define a cocommutative and coassociative coproduct

$$\underline{\mathrm{P}}(y) \to \underline{\mathrm{P}}(y) \otimes \underline{\mathrm{P}}(y)$$

as well as an associative product

$$\underline{\mathrm{P}}(y) \otimes \underline{\mathrm{P}} \to \underline{\mathrm{P}}(y)\,.$$

The terms of the complex for $\underline{\mathrm{P}}(y)$ are the same as those for $\underline{\mathrm{P}}$ but the differential differs by left-multiplication with the element

$$a = x - y$$

in degree zero. The element a satisfies the Maurer-Cartan equation

$$d(a) + a \otimes a = 0\,.$$

If we pass to $\mathbb{Q}$-coefficients the element a lies in the super Lie algebra $\underline{\mathrm{L}}$ and the sum of the differential d and the superbracket with a has square zero, therefore defines a new complex. If we vary y we can replace the base S by X and get a universal $\underline{\mathrm{P}}(y)$ over X.

5.4 The Étale Realisation

Assume ℓ is a prime invertible in S. The étale log is the universal unipotent $\mathbb{Z}_\ell$-sheaf on X trivialised at x. One epigonal (to [Del89]) reference is [Fal07]. We claim that unless X is projective of relative genus zero it can be realised as the étale homology relative S, namely the dual of higher direct images of $\underline{\mathrm{P}}(y)$ for the universal section y over X given by the diagonal in $\mathrm{X} \times \mathrm{X}$. This can be seen as follows.

Obviously there exists a spectral sequence starting with the homology of powers $\mathrm{M(X)}^{\circ,\otimes m}$ for $0 \le m \le n$, shifted by degree $-m$, and converging to the homology of $\underline{\mathrm{P}}_n$. If X is affine all nonzero terms have homological degree 0, so the spectral sequence degenerates and the homology is a repeated extension of the powers of the Tate-module $\mathrm{T}_\ell(\mathrm{X})^{\otimes m}$. If X is projective the homology of $\mathrm{M(X)}^{\circ,\otimes m}$ can be computed by the Künneth-formula, which gives a direct sum of tensor products where each factor is either T_ℓ (in degree 1) or $\mathbb{Z}_\ell(1)$ (in degree 2). If we shift by $-m$ this is the direct sum of powers $\mathrm{T}_\ell(\mathrm{X})^{\otimes m-i}(i)$ placed in degree i, for $0 \le i \le m$. The first differential in the spectral sequence multiplies these by the homology class of the diagonal in $\mathrm{X} \times \mathrm{X}$ (the class of y vanishes in reduced étale homology). This class is a sum

$$c_\Delta = \sum \alpha_j \otimes \beta_j\,,$$

with α_j and β_j forming a dual basis for the dual of the cupproduct on $\mathrm{T}_\ell(\mathrm{X})$. We claim that after the first differential only terms associated to $\mathrm{M}(\mathrm{X})^{\circ,\otimes m}$ survive. This comes down to the following.

Lemma 4. *Suppose R is a commutative ring,* T *a free* R*-module of rank* $r \geq 2$*,* α_j *and* β_j *two sets of basis for* R*,* $c = \sum_j \alpha_j \otimes \beta_j \in \mathrm{T}^{\otimes 2}$*. For each m consider the complex* $\mathrm{M}_*^{(m)}$ *given by*

$$\mathrm{M}_m \to \mathrm{M}_{m-1} \to \cdots \to \mathrm{M}_0$$

where $\mathrm{M}_l \subseteq \mathrm{T}^{\otimes m}$ *is the sum over all* $l+1$*-tuples* $(a_0, \ldots, a_l)$ *with* $\sum_i a_i = m - 2l$*, of*

$$\mathrm{T}^{\otimes a_0} \otimes c \otimes \mathrm{T}^{\otimes a_1} \otimes c \ldots c \otimes \mathrm{T}^{\otimes a_l} .$$

The differentials in the complex are the sums of the inclusions $(0 \leq i < l)$ *with signs*

$$\mathrm{T}^{\otimes a_i} \otimes c \otimes \mathrm{T}^{\otimes a_{i+1}} \subset \mathrm{T}^{\otimes a_i + 2 + a_{i+1}} .$$

The sign for the i-th inclusion is

$$(-1)^{a_0 + \ldots a_i + i} .$$

Then this complex is exact in degrees $\neq 0$.

Proof. We denote by $\mathbb{T}_m$ the quotient of M_0 under the image of M_1. The direct sum

$$\mathbb{T} = \bigoplus_m \mathbb{T}_m$$

is the quotient of the tensoralgebra of T under the two sided ideal generated by c. We claim first that multiplication by any basiselement α is injective on $\mathbb{T}$.

We may assume that $\alpha = \alpha_1$, as c can be written using any basis. The assertion clearly holds for multiplication on $\mathbb{T}_0$. If the assertion holds on $\mathbb{T}_l$ suppose $z \in \mathbb{T}_{l+1}$ is annihilated by α_1. As $\mathbb{T}_{l+2}$ is the quotient of $\mathrm{T} \otimes \mathbb{T}_{l+1}$ under $c \otimes \mathbb{T}_l$ we have a relation

$$\alpha_1 \otimes z = c \otimes y = \sum \alpha_j \otimes \beta_j y .$$

Thus for $j \neq 1$, which is possible as $r \geq 2$, we have

$$\beta_j y = 0$$

and by induction, since β_j is also part of a basis,

$$y = 0, \; z = 0 .$$

Now for the assertion of the lemma use induction over m. The case $m = 1$, or even $m = 2$, is trivial. In general the subcomplex consisting of direct summands with $a_0 > 0$ is the tensorproduct of T and the complex for $m - 1$, thus exact in positive degrees. The quotient is the tensorproduct of c and the complex for $m - 2$ and has non-trivial homology $\mathbb{T}_{m-2}$ only in degree 0. The connecting map

$$\mathbb{T}_{m-2} \to \mathrm{T} \otimes \mathbb{T}_{m-1}$$

is given by multiplication by c and injective by the proceeding arguments: decompose the T on the right according to the basis α_i and consider components. □

We derive from this the following proposition.

Proposition 5. *The spectral sequence for the étale homology of* $\underline{\mathrm{P}}_n$ *degenerates after the first differential. The étale homology is free over* $\mathbb{Z}_\ell$ *and concentrated in degrees between* 0 *and* n*. The homology in degree zero is equal to the truncated étale logarithm. The map*

$$\underline{\mathrm{P}}_{n+1} \to \underline{\mathrm{P}}_n$$

induces zero on homology in strictly positive degrees. The pro-object of étale realisations of $\underline{\mathrm{P}}_n$ *is isomorphic in the derived category to the pro-object of étale logarithms.*

Proof. The first step in the spectral sequence is the sum of the complexes $\mathrm{M}_*^{(m)}$ as above, truncated at level n. We derive that after the first differential of the spectral sequence the surviving terms are locally free and either correspond to homology in degree 0, or to higher homology and then are subspaces of the homology of $\mathrm{M(X)}^{\circ,\otimes n+1}$. Furthermore we use a weight argument to show that all higher differentials vanish, so that the homology is locally free (it is the last homology of a truncation of the complex in the lemma) and mixed of certain weights, and this also applies to the homology of $\underline{\mathrm{P}}_n$.

To introduce weights we may assume by base change that S is of finite type over $\mathrm{Spec}(\mathbb{Z})$, and consider the eigenvalues of Frobenius at closed points of S. Then all terms in the complex $\mathrm{M}_*^{(m)}$ are pure of weight $-m$. Thus for degree 0-homology the weights lie between $-n$ and 0, while for higher homology in degree $i > 0$ the weight is $-(n+i)$. As the higher differentials in the spectral sequence respect weights they must vanish. Also because of weights under the projection $\underline{\mathrm{P}}_{m+1} \to \underline{\mathrm{P}}_m$ the induced maps in strictly positive homological degree vanish. It follows that the projection $\underline{\mathrm{P}}_{2n} \to \underline{\mathrm{P}}_n$ induces in homology a map of complexes which factors canonically over the projection to H_0.

This H_0 has a filtration with subquotients the same as for the étale log, see for example [Fal07], the discussion on page 178. Furthermore for $n = 1$ the extension

$$0 \to \mathrm{T}_\ell \to \mathrm{H}_0 \to \mathbb{Z}_\ell \to 0$$

is induced from the diagonal and thus coincides with the first step of the étale logarithm. Thus we get a homomorphism from the universal étale logarithm to our H_0 which is compatible with the multiplication by the fibre at x, and induces an isomorphism on the first two graded subquotients of the filtration. It follows easily that it induces isomorphisms on all graded subquotients, and that the étale H_0 coincides with the étale log. □

As usual the products and coproducts on the $\underline{\mathrm{P}}(y)$ make the homology of $\underline{\mathrm{P}}$ the affine algebra of a prounipotent group-scheme G_{et} over $\mathbb{Z}_\ell$, and the homology of $\underline{\mathrm{P}}(y)$ that of a torsor over G_{et}.

The pro-ℓ fundamental groups of the fibers of X/S form a *local system*

$$G_{et}$$

on S, or more precisely there exists a profinite group which is an extension of the fundamental group of S by this pro-ℓ group. This extension has a splitting defined by x and thus becomes a semidirect product. The projective system of ℓ-adic homologies of $\underline{P}_n$ is identified with the local system given by the completed group-algebra $\mathcal{A}$, that is it is as a pro-object isomorphic to the projective system of quotients $\mathcal{A}/\mathcal{I}^n$ where $\mathcal{I} \subset \mathcal{A}$ is the augmentation ideal. More precisely the 0-th homology of $\underline{P}_n$ corresponds to the universal unipotent sheaf of length n, that is to $\mathcal{A}/\mathcal{I}^{n+1}$.

From the structure of the fundamental group we know that it is a free pro-ℓ group, divided by one relation if X/S is projective. This relation is the commutator of generators. The completed group-algebra of the free group is a completed free tensoralgebra. The one relation corresponds modulo the cube of the augmentation-ideal to the element c from above, and becomes equal to it after applying a suitable automorphism. It follows that in any case $\mathcal{A}$ is non canonically isomorphic respecting augmentations to the completed tensor-algebra

$$\mathbb{T} = \prod \mathbb{T}_n .$$

If we pass to $\mathbb{Q}_\ell$-coefficients the Lie algebra of G_{et}/Z^{n+1} is obtained from $\mathcal{A}/\mathcal{I}^{n+1}$ by applying the operators f_m, for $m \leq n$, to $\underline{P}_n$ or the operators e_n to the subquotients of its homology. The group G_{et}/Z^{n+1} is isomorphic to its Lie algebra via the exponential map, and the multiplication on it is defined by the Hausdorff-series.

5.5 The de Rham and Crystalline Realisation

The arguments are essentially the same as in the previous section. The relative de Rham homology of $\underline{P}_n$ is defined by dualising the double complex derived from the de Rham complexes on $\bar{X}^n$ with logarithmic poles along D. It admits a Hodge filtration. Also, if $S_0 \subset S$ is a closed subscheme defined by an ideal with divided power-structure, the de Rham cohomology (without the Hodge filtration) depends only on the restriction of $(\bar{X}, D)$ to S_0. In fact it is defined for a relative curve over X_0. The remaining arguments carry over verbatim, for example, the weight argument uses that locally the pair $(\bar{X}, D)$ comes by pullback from a smooth $\mathbb{Z}$-scheme. We obtain unipotent flat group-schemes G_{cr} and G_{DR}. The latter admits a Hodge filtration by flat closed subschemes $F^i(G_{DR})$, and the G_{DR}-torsors defined by $\underline{P}(y)$ reduce to $F^0(G_{DR})$-torsors. A possible reference is again [Fal07]. However we should note that while our category $\mathcal{M}$ admits crystalline realisations there is no obvious extension to the motivic category as the action of correspondences on crystalline cohomology still poses some problems.

5.6 The Motivic Realisation

Voevodsky defines a derived category of motives as a quotient of the derived category of Nisnevich sheaves with transfers. The $\underline{\mathrm{P}}_n$ obviously correspond to complexes of such sheaves and we obtain objects in this derived category. Again passing to $\mathbb{Q}$-coefficients we may apply the operators f_n to get (super) Lie algebras. All this can be done before taking the quotient under Nisnevich coverings and $\mathbb{A}^1$-homotopy, but passing to it we get objects in the motivic category. Also the $\underline{\mathrm{P}}(y)$ become objects in the derived category which obviously should be torsors. However the definition of a *torsor* in this context is not so obvious.

5.7 Torsors in Triangulated Categories

We have to study mixed extensions in derived categories. One of the fundamental difficulties in the theory is that mapping cones are only defined up to non canonical isomorphism. To get around this we have to make vanishing assumptions under which all objects become sufficiently welldefined. Recall the definition of extensions in a triangulated category. An extension of B by A is an exact triangle

$$\mathrm{A} \to \mathrm{E} \to \mathrm{B} \to \mathrm{A}[1]\,.$$

An isomorphism of extensions is an isomorphism of E's commuting with the maps from A and those into B. Isomorphism classes are classified by the map $\mathrm{B} \to \mathrm{A}[1]$ as follows. If E and E′ induce the same map there is a map of triangles, by axiom TR3, which is necessarily an isomorphism.

The difference of two such maps is induced by a map of E's which induces 0 on A and B. It is induced by a

$$\alpha : \mathrm{E} \to \mathrm{A}$$

such that its composition with $\mathrm{A} \to \mathrm{E}$ (from both sides, that is applied twice) vanishes. If

$$\mathrm{Hom}(\mathrm{A}, \mathrm{B}[-1]) = (0)$$

it induces a trivial endomorphism of A and thus is induced from a map $\mathrm{B} \to \mathrm{A}$. On the other hand such maps operate obviously as automorphisms of any extension. If in addition

$$\mathrm{Hom}(\mathrm{A}, \mathrm{A}[-1]) = \mathrm{Hom}(\mathrm{B}, \mathrm{B}[-1]) = (0)$$

this operation is free.

A similar problem is the classification of mixed extensions. Given A, B, C and extensions, i.e., exact triangles,

$$\mathrm{A} \to \mathrm{D} \to \mathrm{B} \to \mathrm{A}[1]$$

and

$$B \to E \to C \to B[1]\,,$$

we consider F's which lie in triangles

$$D \to F \to C \to D[1]$$

as well as

$$A \to F \to E \to A[1]$$

such that the three possible compositions (from A to F, from F to C, and from D to E) coincide. We are interested in automorphisms, isomorphism classes, and existence conditions. The last is easy: the composition of the two classifying maps

$$C \to B[1] \to A[2]$$

must vanish. Indeed, this is necessary as the first map goes to zero if composed with

$$B[1] \to E[1]$$

and the second extends to E[1]. Conversely if the composition vanishes we get a lift

$$C \to D[1]\,,$$

thus an extension F of C by D which induces the extension E.

For the classification of isomorphism classes we show that under suitable vanishing conditions they form a principal homogeneous space under

$$\mathrm{Hom}(C, A[1])\,.$$

Assume right away that

$$\mathrm{Hom}(A, B[-1]) = \mathrm{Hom}(A, C[-1]) = \mathrm{Hom}(B, B[-1]) = \mathrm{Hom}(B, C[-1]) = (0)\,.$$

Then we obtain an operation of $\mathrm{Hom}(C, A[1])$ on the isomorphism classes of mixed extensions. Namely assume we have a mixed extension F and an

$$A \to G \to C \to A[1]\,.$$

The direct sum $F \oplus G$ maps to $C \oplus C$, and we denote by H the preimage of the diagonal, that is H lies in an exact triangle

$$H \to F \oplus G \to C \to H[1]$$

and is unique up to non-unique isomorphism. The inclusions of A and D into the direct summands factor over H, uniquely up to maps into $C[-1]$, so they are unique. Especially the antidiagonal in $A \oplus A$ comes from a unique $A \to H$. Finally define F' is a mapping cokernel of this map, that is it lies in an exact triangle

$$A \to H \to F' \to A[1]\,.$$

The map from H to C factors uniquely over F', and so does the projection to E. The maps define exact triangles

$$D \to F' \to C \to D[1]$$

and

$$A \to F' \to E \to A[1] .$$

Furthermore the composition $D \to E$ coincides with the original composition, that is the factorisation over B, so that F' is a mixed extension.

Conversely given mixed extensions F and F' modify $F \oplus F'$ by taking a preimage of the diagonal $E \subset E \oplus E$ and dividing by the diagonal $D \subset D \oplus D$, which amounts to forming mapping cones. Call the result G. For the second we need to lift the inclusion of the diagonal D which is unique up to an element of $\mathrm{Hom}(D, E[-1])$ which by our vanishing conditions is zero. The inclusions of A into F, F' as well as the projections to C extend/factor uniquely and define an exact triangle

$$A \to G \to C \to A[1] ,$$

and one easily sees that this gives an inverse.

Finally an automorphism of F is given by the sum of the identity and compatible endomorphisms of

$$A \to F \to E \to A[1]$$

and

$$D \to F \to C \to D[1]$$

which vanish on the extremes, that is by compatible morphisms

$$E \to A, C \to D .$$

The obstruction that the first factors over C lies in $\mathrm{Hom}(B, A)$ and induces zero in $\mathrm{Hom}(B, D)$, thus comes from an element in

$$\mathrm{Hom}(B, B[-1]) = (0) .$$

Thus the factorisation exists and we get a map

$$C \to A .$$

By a dual argument also the map from C to D factors over A. The difference of the two maps from C to A vanishes if we compose to get a map

$$E \to D .$$

The obstruction for it to vanish is first get an element in $\mathrm{Hom}(E, B[-1])$ and then one in $\mathrm{Hom}(B, A[-1])$. Both groups vanish, so finally the automorphisms of F are the sum of the identity and a unique element of $\mathrm{Hom}(C, A)$.

We subsume the above in the following lemma.

Lemma 6. *Assume the vanishing of*

$$\mathrm{Hom}(\mathrm{A},\mathrm{B}[-1]) = \mathrm{Hom}(\mathrm{A},\mathrm{C}[-1]) = \mathrm{Hom}(\mathrm{B},\mathrm{B}[-1]) = \mathrm{Hom}(\mathrm{B},\mathrm{C}[-1]) = (0)\ .$$

Then the obstruction to extend given extensions of B *by* A *and* C *by* B *to a mixed extension lies in* $\mathrm{Hom}(\mathrm{C},\mathrm{A}[2])$. *If it vanishes the isomorphism classes of solutions form a torsor under* $\mathrm{Hom}(\mathrm{C},\mathrm{A}[1])$, *and the automorphisms of any solutions are* $\mathrm{Hom}(\mathrm{C},\mathrm{A})$.

We apply these generalities to the motivic log, to define **motivic $\underline{\mathrm{L}}_n$-torsors**. We work in the derived category of finitely filtered Nisnevich sheaves with transfer, that is Nisnevich sheaves $\mathcal{F}$ with a finite decreasing filtration of length n, for a fixed n,

$$\mathcal{F} = \mathrm{G}^0(\mathcal{F}) \supset \mathrm{G}^1(\mathcal{F}) \supset \cdots \supset \mathrm{G}^{n+1}(\mathcal{F}) = (0)\ .$$

The transfer should respect filtrations. For example $\underline{\mathrm{P}}_n$ with its stupid filtration defines a complex in this category, with graded

$$gr^i_{\mathrm{G}}(\underline{\mathrm{P}}_n) = \mathrm{M}(\mathrm{X})^{\circ,\otimes i}[-i]\ .$$

In the derived category we invert as usual filtered quasi-isomorphisms.

We also assume that our complexes are bounded above, and for the derived product

$$\mathrm{A} \otimes^{\mathbb{L}} \mathrm{B}$$

we take the usual derived product divided by elements of filtration degree $> n+1$. We know that $\underline{\mathrm{P}}_n$ is an algebra and coalgebra in this category. Any such complex can be resolved by complexes whose associated gradeds are injective. If we further assume that the associated gradeds lie in the triangulated category $\mathrm{DM}^{\mathrm{eff}}_{gm}$, see [Voe00], generated by motives of smooth schemes we get by truncation right bounded resolutions which are sufficiently Ext-acyclic as to compute cohomology (defined as maps in the derived category but usually difficult to compute unless one has injective resolutions).

In the following we consider filtered right $\underline{\mathrm{P}}_n$-modules, that is right bounded filtered complexes $\mathrm{K}^\bullet$ together with a map of complexes

$$\mathrm{K}^\bullet \otimes \underline{\mathrm{P}}_n \to \mathrm{K}^\bullet$$

which is strictly associative (no homotopies involved). The coproduct on $\underline{\mathrm{P}}_n$ defines a $\underline{\mathrm{P}}_n$-module structure on the derived tensorproduct of two $\underline{\mathrm{P}}_n$-modules. The exact forgetful functor to filtered complexes has a right-adjoint which maps $\mathrm{K}^\bullet$ to the internal Hom of filtration preserving maps

$$\mathrm{Hom}(\underline{\mathrm{P}}_n, \mathrm{K}^\bullet)\ ,$$

defined in the obvious way from the rule

$$\mathrm{Hom}(\mathrm{M}(\mathrm{X}),\mathcal{F})(\mathrm{T}) = \mathcal{F}(\mathrm{X}\times\mathrm{T})\,.$$

Applied to filtered injective resolutions without $\underline{\mathrm{P}}_n$-module structure we obtain acyclic resolutions of $\underline{\mathrm{P}}_n$-modules which allow us to compute cohomology. It then coincides with the usual Nisnevich cohomology.

Also as usual we can pass to $\mathbb{A}^1$-homotopy invariant objects by applying the functor C_* with

$$\mathrm{C}_*(\mathcal{F})(\mathrm{T}) = \mathcal{F}(\mathrm{T}\times\Delta^\bullet)\,,$$

and define motivic cohomology by

$$\mathrm{H}^i_{\mathcal{M}}(\mathrm{S},\mathcal{F}) = \mathrm{H}^i(\mathrm{S},\mathrm{C}_*(\mathcal{F}))\,.$$

So from now on we compute in the triangulated category of filtered (with filtration degrees between 0 and n) bounded above complexes of $\mathbb{Q}$-Nisnevich sheaves with transfer which are $\underline{\mathrm{P}}_n$-modules, modulo filtered quasi-isomorphisms and modulo $\mathbb{A}^1$-homotopy. All our objects will lie in the category generated by geometric effective motives, so have finite Ext-dimension. Finally, to apply our previous theory of mixed extensions we make the general assumption that for $i \leq j$ the cohomology

$$\mathrm{H}^{-1}_{\mathcal{M}}(\mathrm{S},\mathrm{Hom}(gr^i_{\mathrm{G}}(\underline{\mathrm{P}}_n),gr^j_{\mathrm{G}}(\underline{\mathrm{P}}_n)) = \mathrm{H}^{j-i-1}_{\mathcal{M}}(\mathrm{S},\mathrm{Hom}(\mathrm{M}(\mathrm{X})^{\circ,\otimes i},\mathrm{M}(\mathrm{X})^{\circ,\otimes j})) = (0)$$

vanishes.

We now define a $\underline{\mathrm{L}}_n$-torsor as a filtered $\underline{\mathrm{P}}_n$-module $\underline{\mathrm{Q}}_n$ together with cocommutative and coassociative coproduct

$$c:\ \underline{\mathrm{Q}}_n \to \underline{\mathrm{Q}}_n \otimes^{\mathbb{L}} \underline{\mathrm{Q}}_n\,.$$

Also we assume that we have isomorphisms compatible with the multiplication

$$gr^i_{\mathrm{G}}(\underline{\mathrm{Q}}_n) \cong \mathrm{C}_*(gr^i_{\mathrm{G}}(\underline{\mathrm{P}}_n))\,.$$

Examples of such objects are the previous $\underline{\mathrm{P}}_n(y)$.

Obviously, for $n = 0$, there is only one isomorphism class of such objects. Furthermore, a $\underline{\mathrm{Q}}_n$ induces naturally a $\underline{\mathrm{Q}}_{n-1}$. Conversely given a $\underline{\mathrm{Q}}_{n-1}$ let us analyse the possible lifts to $\underline{\mathrm{Q}}_n$. Necessarily we have

$$\mathrm{G}^1(\underline{\mathrm{Q}}_n) = \underline{\mathrm{Q}}_{n-1} \otimes^{\mathbb{L}}_{\underline{\mathrm{P}}_{n-1}} \mathrm{G}^1(\underline{\mathrm{P}}_{n-1})\,.$$

Here the tensor product is defined as follows. The complex $\mathrm{G}^1(\underline{\mathrm{P}}_n)$ has the same terms as

$$\underline{\mathrm{P}}_{n-1} \otimes \mathrm{M}(\mathrm{X})^\circ[-1]\,,$$

but with the differential modified by adding multiplication by the diagonal in $\underline{\mathrm{P}}^1_n(\mathrm{X})$. This diagonal satisfies the Maurer-Cartan equation, so we may define the tensor-product as

$$\underline{\mathrm{Q}}_{n-1} \otimes \mathrm{M}(\mathrm{X})^\circ[-1]$$

with multiplication by the same element added to the differential.

Thus $\underline{Q}_n$ becomes a mixed extension with the following subquotients $C_*(\mathbb{Z})$, $\underline{Q}_{n-2} \otimes^{\mathbb{L}}_{\underline{P}_{n-2}} G^1(\underline{P}_{n-1})$, and $C_*(gr^n_G(\underline{P}_n)$. The obstruction z to get such an extensions lies in

$$H^2_{\mathcal{M}}(S, gr^n_G(\underline{P}_n) = H^{2-n}_{\mathcal{M}}(S, L(M(X)^{\circ,\otimes n})) .$$

By the existence of the coproduct on $\underline{Q}_{n-1}$ it satisfies the equation

$$c(z) = z \otimes 1 + 1 \otimes z ,$$

thus lies in the cohomology of the direct summand which is the n-th graded of the Lie algebra.

If z vanishes two mixed extensions differ up to isomorphisms by a class in $H^1_{\mathcal{M}}(S, gr^n_G(\underline{P}_n))$. The obstruction w to extend the coproduct lies in

$$H^1_{\mathcal{M}}(S, gr^n_G(\underline{P}_n \otimes \underline{P}_n))$$

(we have a mixed extension for the tensorproduct), is annihilated by the counits in each factor, and satisfies

$$(\mathrm{id} \otimes c)(w) + 1 \otimes w = (c \otimes \mathrm{id})(w) + w \otimes 1 .$$

If we modify c by the rule

$$\tilde{c}(z) = c(z) - z \otimes 1 - 1 \otimes z$$

this can be written as

$$(\mathrm{id} \otimes \tilde{c})(w) = (\tilde{c} \otimes \mathrm{id})(w) .$$

The obstruction is symmetric and by the general results on Lie algebras, see Lemma 1 and the discussion following it, w lies in the image of $\tilde{c}$:

We know that a certain complex formed from free Lie algebras is acyclic and thus null homotopic. The homotopies are given by elements of group rings $\mathbb{Q}[S_n]$, namely the elements u, v, w from the end of Sect. 5.2. Applying the same elements to the $\underline{P}_n$ gives null-homotopies for the motivic cohomology.

Thus changing the mixed extension $\underline{Q}_n$ by a class in $H^1_{\mathcal{M}}(S, gr^n_G(\underline{P}_n))$ we can make it zero, so the comultiplication extends.

For coassociativity we similarly obtain an obstruction in $H^0_{\mathcal{M}}(S, gr^n_G(\underline{P}^{\otimes 3}))$ which satisfies the cocycle condition and whose (super-)antisymmetric projection vanishes. So it is a coboundary of an element in $H^0_{\mathcal{M}}(S, gr^n_G(\underline{P}^{\otimes 2}))$ which can be chosen symmetric. Thus finally our torsor extends if the obstruction vanishes.

By similar but simpler arguments the isomorphism classes of extensions form a torsor under the first motivic cohomology of the grade n-part of the Lie algebra, and the automorphisms are $H^0_{\mathcal{M}}$.

Remark 7. For complete curves the vanishing assumptions refer to the reduced subspace of the motivic cohomology (where $i \leq j$)

$$\mathrm{H}^{2i-1}_{\mathcal{M}}(\mathrm{X}^{i+j}, \mathbb{Q}(j)) .$$

For an affine curve we need partially compact support. For rational curves (where we deal with Tate-motives) we need the vanishing for $l \geq 0$ of

$$\mathrm{H}^{-1}_{\mathcal{M}}(\mathrm{S}, \mathbb{Q}(l)) .$$

5.8 Representability

Spaces of torsors with an obstruction theory as in the last chapter admit a versal representative as follows. We assume that

$$\mathrm{H}^{1}_{\mathcal{M}}(\mathrm{S}, gr^{n}_{\mathrm{G}}(\underline{\mathrm{L}}))$$

is a finitely generated $\mathbb{Q}$-vectorspace. For any $\mathbb{Q}$-algebra R we can by the same procedure as before define torsors with coefficients in R, such that the obstruction to liftings are classified by H^2, the liftings by H^1 and the automorphisms of liftings by H^0, all with coefficients in R. Then there exist a finitely generated commutative $\mathbb{Q}$-algebra R_n and a versal $\underline{\mathrm{L}}_n$-torsor over R_n, such that any other such torsor over an R is obtained from it via pushout $\mathrm{R}_n \to \mathrm{R}$.

Namely the assertion holds for $n = 0$. Assume we have constructed R_{n-1} the obstruction to lift the versal torsor to $\underline{\mathrm{L}}_n$ lies in

$$\mathrm{H}^{2}_{\mathcal{M}}(\mathrm{S}, gr^{n}_{\mathrm{G}}(\underline{\mathrm{L}}_n)) \otimes_{\mathbb{Q}} \mathrm{R}_{n-1} .$$

Writing it as an R_{n-1}-linear combination of basis elements of $\mathrm{H}^{2}_{\mathcal{M}}(\mathrm{S}, gr^{n}_{\mathrm{G}}(\underline{\mathrm{L}}_n))$ the coefficients generate an ideal

$$\mathrm{I}_{n-1} \subset \mathrm{R}_{n-1} .$$

The torsor then lifts over $\mathrm{R}_{n-1}/\mathrm{I}_{n-1}$ and we chose one such lift. If $\mathrm{H}^{1}_{\mathcal{M}}(\mathrm{S}, gr^{n}_{\mathrm{G}}(\underline{\mathrm{L}}_n))$ has dimension l we then obtain a versal torsor over

$$\mathrm{R}_n = \mathrm{R}_{n-1}/\mathrm{I}_{n-1}[\mathrm{T}_1, \ldots, \mathrm{T}_l]$$

by modifying the lift with the element of H^1 corresponding to the linear combination of basis elements with coefficients T_i.

If in addition the $\mathrm{H}^{0}_{\mathcal{M}}(\mathrm{S}, gr^{j}_{\mathrm{G}}(\underline{\mathrm{L}}_n))$ vanish this torsor is universal and R_n represents the motivic torsors. In any case its dimension is bounded by the sum of the dimensions of $\mathrm{H}^{1}_{\mathcal{M}}(\mathrm{S}, gr^{j}_{\mathrm{G}}(\underline{\mathrm{L}}_n))$.

Finally by comparison to étale cohomology, for example [SV96, Th. 7.6], the homologies of the $\underline{\mathrm{Q}}_n$ define torsors under the étale unipotent fundamental group and also algebraic maps of the representation spaces. The torsors given by points $y \in \mathrm{X}(\mathrm{S})$ correspond.

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