Chapter 2
Standard Form

Given a pseudo-periodic map

$$f : \Sigma_g \to \Sigma_g$$

Nielsen constructed a special homeomorphism which is homotopic to f and plays the role of a "standard form" in the mapping class of f, [53, Sect. 14]. In this Chapter, we will construct a similar standard form, slightly different from Nielsen's, and will show its essential uniqueness. He wanted to avoid fixed points which might appear in annular neighborhoods of cut curves, while we do not care about such fixed points (Compare [22, Theorem 13.3]).

2.1 Definitions and Main Theorem of Chap. 2

Definition 2.1. Let A be an annulus and

$$\phi : [0, 1] \times S^1 \to A$$

a parametrization (i.e. homeomorphism), where $S^1 = \mathbf{R}/\mathbf{Z}$. A homeomorphism

$$f : A \to A$$

which does not interchange the boundary components of A is called a *linear twist with respect to* ϕ, if

$$f\phi(t, x) = \phi(t, x + at + b), (t, x) \in [0, 1] \times S^1,$$

for some $a, b \in \mathbf{Q}$. We say simply that

$$f : A \to A$$

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is a *linear twist* if f is a linear twist with respect to a certain parametrization

$$\phi : [0, 1] \times S^1 \to A.$$

Definition 2.2. Let A and

$$\phi : [0, 1] \times S^1 \to A$$

be as above. A homeomorphism

$$f : A \to A$$

which interchanges the boundary components of A is called a *special (piecewise-linear) twist with respect to* ϕ, if

$$f\phi(t, x) = \begin{cases} \phi\left(1 - t, -x - 3a\left(t - \frac{1}{3}\right)\right), & 0 \le t \le \frac{1}{3} \\ \phi(1 - t, -x), & \frac{1}{3} \le t \le \frac{2}{3} \\ \phi\left(1 - t, -x - 3a\left(t - \frac{2}{3}\right)\right), & \frac{2}{3} \le t \le 1 \end{cases}$$

for some $a \in \mathbf{Q}$. If

$$f : A \to A$$

is a special twist with respect to a certain parametrization

$$\phi : [0, 1] \times S^1 \to A,$$

we simply say that f is a *special twist*.

Remark 2.1. Let

$$\rho : [0, 1] \times S^1 \to [0, 1] \times S^1$$

be defined by

$$\rho(t, x) = \rho(1 - t, -x).$$

If

$$f : A \to A$$

is a special twist with respect to

$$\phi : [0, 1] \times S^1 \to A,$$

then this same f is also a special twist with respect to

$$\phi\rho : [0, 1] \times S^1 \to A.$$

Now we define our standard form.

Definition 2.3. *A pseudo-periodic map*

$$f : \Sigma_g \to \Sigma_g$$

is said to be in *standard form* if the following conditions are satisfied:

(i) There exists a system of disjoint annular neighborhoods $\{A_i\}_{i=1}^r$ of the precise system of cut curves subordinate to f, such that

$$f(\mathscr{A}) = \mathscr{A},$$

where

$$\mathscr{A} = \bigcup_{i=1}^{r} A_i.$$

(ii) The map

$$f \mid \Sigma_g - \mathscr{A} : \Sigma_g - \mathscr{A} \to \Sigma_g - \mathscr{A}$$

is periodic.

(iii) Let k_i be the smallest positive integer such that

$$f^{k_i}(A_i) = A_i, i = 1, 2, \ldots, r.$$

(iii)-a If

$$f^{k_i} \mid A_i : A_i \to A_i$$

does not interchange the boundary components of A_i, then $f^{k_i} \mid A_i$ is a linear twist.

(iii)-b If

$$f^{k_i} \mid A_i : A_i \to A_i$$

interchanges the boundary components of A_i, then $f^{k_i} \mid A_i$ is a special twist.

This whole chapter is devoted to the proof of the following theorem.

Theorem 2.1 (cf. [53, Sect. 15] [22, Theorem 13.3]). *(i) Any pseudo-periodic map*

$$f : \Sigma_g \to \Sigma_g$$

is isotopic to a pseudo-periodic map in standard form.

(ii) Suppose two pseudo-periodic maps in standard form

$$f, f' : \Sigma_g \to \Sigma_g$$

are homotopic, then there is a homeomorphism

$$h : \Sigma_g \to \Sigma_g$$

isotopic to the identity, such that $f = h^{-1} f' h$.

The uniqueness statement (ii) above is stronger than the original one ([53, Sect. 15]).

The proof requires considerations on periodic parts, non-amphidrome annuli, and amphidrome annuli. These preliminary considerations occupy the main body of this chapter.

2.2 Periodic Part

We need the following theorem.

Theorem 2.2. *Let f and f' be periodic maps of a compact surface Σ each component of which has negative Euler characteristic. Suppose*

$$f, f' : (\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma)$$

are homotopic as maps of pairs. Then there exists a homeomorphism $h : \Sigma \to \Sigma$ isotopic to the identity, such that $f = h^{-1} f' h$.

This theorem seems folklore among specialists. A. Edmonds informed, in a letter to the second named author, that C. Frohman had proved a stronger result in his thesis which implied Theorem 2.2. Unfortunately the authors could not find any reference giving an explicit proof. A proof will be given in Appendix A.

2.3 Non-amphidrome Annuli

Lemma 2.1 (Linearization). *Let A be an annulus,*

$$f : A \to A$$

a homeomorphism which does not interchange the boundary components. Suppose

$$f \mid \partial A : \partial A \to \partial A$$

is periodic. Then there exists an isotopy

$$f_\tau : A \to A, \quad 0 \le \tau \le 1,$$

such that

$$f_0 = f, f_\tau \mid \partial A = f \mid \partial A, \textit{and}$$

*$f_1 : A \to A$ is a linear twist. (*An isotopy as this will be referred to hereafter as *"rel.∂").*

Proof. Let $\partial_0 A$ and $\partial_1 A$ denote the two boundary components. Since

$$f \mid \partial A : \partial A \to \partial A$$

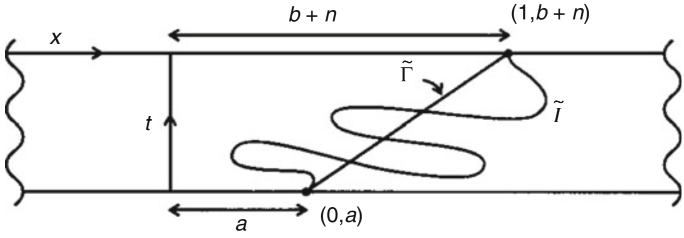

Fig. 2.1 Linearization

is periodic, there are homeomorphisms

$$\varphi : S^1 \to \partial_0 A$$

and

$$\psi : S^1 \to \partial_1 A$$

such that

$$f\varphi(x) = \varphi(x + a), f\psi(x) = \psi(x + b),$$

for $a, b \in \mathbf{Q}$.

Note that φ and ψ are homotopic as maps from S^1 into A. Thus there is a homeomorphism

$$\phi : [0, 1] \times S^1 \to A$$

such that

$$\phi(0, x) = \varphi(x), \phi(1, x) = \psi(x)$$

([21, Lemma 2.4]). Now we identify A with $[0, 1] \times S^1$ through ϕ. We lift the curve $f([0, 1] \times \{0\})$ to the universal covering $[0, 1] \times \mathbf{R}$. Let $\tilde{I}$ be the lift starting at $(0, a)$. Then it ends at $(1, b + n)$, for some $n \in \mathbf{Z}$. (See Fig. 2.1)

Let $\tilde{\Gamma}$ be a line segment in $[0, 1] \times \mathbf{R}$ joining $(0, a)$ and $(1, b+n)$. Then $\tilde{\Gamma}$ projects to a "linear" arc Γ in $[0, 1] \times S^1$ connecting $(0, a)$ to $(1, b)$. By an innermost arc argument, f is rel.∂ isotopic to a homeomorphism

$$f' : [0, 1] \times S^1 \to [0, 1] \times S^1$$

satisfying

$$f'([0, 1] \times \{0\}) = \Gamma.$$

Also by composing f' with an isotopy which keeps ∂A pointwise fixed and moves along Γ, f' becomes isotopic to a homeomorphism $f^{(2)}$ such that

$$f^{(2)}(t, 0) = (t, *) \in \Gamma,$$

i.e., $f^{(2)}$ preserves the t-level when restricted to a map from $[0, 1] \times \{0\}$ onto Γ.

Define
$$f^{(3)} : [0, 1] \times S^1 \to [0, 1] \times S^1$$
by
$$f^{(3)}(t, x) = (t, x + a(1 - t) + (b + n)t).$$
$f^{(3)}$ is a linear twist which coincides with $f^{(2)}$ on $\partial A \cup \Gamma$. Thus by the Alexander trick, $f^{(2)}$ is isotopic to $f^{(3)}$ keeping the points of $\partial A \cup \Gamma$ fixed. □

It will be convenient to extend the notion of screw number of a curve to an annulus. Let
$$\mathscr{A} = \bigcup_{i=1}^{r} A_i$$
be a disjoint union of annuli $f : \mathscr{A} \to \mathscr{A}$ a homeomorphism. Suppose the restriction of f to the boundary
$$\partial\mathscr{A} = \bigcup_{i=1}^{r} \partial A_i$$
is periodic. Let A_i be an annulus in $\mathscr{A}$. Let α be the smallest positive integer such that (i) $f^{\alpha}(A_i) = A_i$; and (ii) f^{α} does not interchange the boundary components. Let l be a non-zero integer such that $f^l \mid \partial A_i =$ the identity. Then l is a multiple of α, and $f^l : A_i \to A_i$ is the result of e full Dehn-twists, e being an integer.

Definition 2.4. The rational number $e\alpha/l$ is called the *screw number* of f in A_i and is denoted by $s(A_i)$. It measures the amount of Dehn twist performed by f^{α} in A_i.

The number $s(A_i)$ is independent of the choice of l. Of course if $\{A_i\}_{i=1}^r$ is an invariant system of annular neighborhoods of a precise cut system $\{C_i\}_{i=1}^r$ subordinate to a pseudo-periodic map
$$f : \Sigma_g \to \Sigma_{g'}$$
then
$$s(A_i) = s(C_i), \quad i = 1, 2, \ldots, r.$$

An annulus A_i is said to be *amphidrome* if there is an integer γ such that $f^{\gamma}(A_i) = A_i$ and f^{γ} interchanges the boundary components.

The *valency* (m, λ, σ) of a boundary curve $\overrightarrow{C}$ of $\mathscr{A}$ oriented by the orientation induced from $\mathscr{A}$ is defined as the valency of $-\overrightarrow{C}$ with respect to the periodic map
$$\partial\mathscr{A} \to \partial\mathscr{A}.$$

As a corollary to (the proof of) Lemma 2.1, we have the following:

Corollary 2.1. *Let* $\mathscr{A} = \bigcup_{i=1}^{r} A_i$ *be a disjoint union of annuli,*

$$f : \mathscr{A} \to \mathscr{A}$$

a homeomorphism such

$$f \mid \partial\mathscr{A} \to \partial\mathscr{A}$$

is periodic. Let A_i *be a non-amphidrome annulus of* $\mathscr{A}$ *with*

$$\partial A_i = \partial_0 A_i \cup \partial_1 A_1.$$

Let

$$(m_i^0, \lambda_i^0, \sigma_i^0)$$

and

$$(m_i^1, \lambda_i^1, \sigma_i^1)$$

be the valencies of $\partial_0 A_i$*, and* $\partial_1 A_i$*, respectively,* $s(A_i)$ *the screw number. Then*

(i) the equality

$$m_i^0 = m_i^1$$

holds; and

(ii) the number

$$s(A_i) + \delta_i^0/\lambda_i^0 + \delta_i^1/\lambda_i^1$$

is an integer, where the integer δ_i^ν *is determined by*

$$\sigma_i^\nu \delta_i^\nu \equiv 1 \pmod{\lambda_i^\nu}$$

and

$$0 \le \delta_i^\nu < \lambda_i^\nu, \quad \nu = 0, 1.$$

Proof. The oriented A_i will be denoted by $\overrightarrow{A_i}$; the induced orientation in ∂A_i by $\partial \overrightarrow{A}_i$. Remember that the valency of ∂A_i is defined as the valency of $-\partial \overrightarrow{A}_i$.

(i) m_i^ν was defined to be the smallest positive integer such that

$$f^{m_i^\nu}(\partial_\nu \overrightarrow{A}_i) = \partial_\nu \overrightarrow{A}_i, \quad \nu = 0, 1.$$

Since A_i is not amphidrome,

$$f^m(\partial_\nu \overrightarrow{A}_i) = \partial_\nu \overrightarrow{A}_i$$

if and only if

$$f^m(A_i) = A_i, \quad \nu = 0, 1.$$

This proves $m_i^0 = m_i^1$.

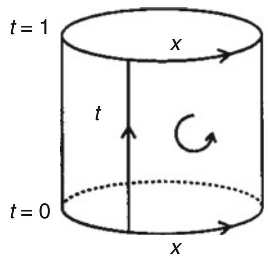

Fig. 2.2 The orientation of the annulus used in Corollary 2.1

(ii) Let us give an orientation to $[0, 1] \times S^1$ as Fig. 2.2 indicates. Then the orientation of $\{1\} \times S^1$ given by the x-direction is

$$-\partial_1(\overrightarrow{[0,1] \times S^1})$$

and the orientation of $\{0\} \times S^1$ given by the x-direction is

$$\partial_0(\overrightarrow{[0,1] \times S^1}).$$

Let m denote the common number $m_i^0 = m_i^1$, and consider

$$f^m : A_i \to A_i$$

as the homeomorphism (not yet normalized) $f : A \to A$ of Lemma 2.1.

Take a parametrization

$$\phi : [0, 1] \times S^1 \to A_i$$

for which

$$f^m \phi(0, x) = \phi(0, x + a)$$

and

$$f^m \phi(1, x) = \phi(1, x + b)$$

for some $a, b \in \mathbf{Q}$, and identify A_i with $[0, 1] \times S^1$ through ϕ.

Then by the geometric meaning of $\delta_i^\nu / \lambda_i^\nu$ $(\nu = 0, 1)$, we have

$$\frac{\delta_i^1}{\lambda_i^1} \equiv b \pmod 1,$$

$$\frac{\delta_i^0}{\lambda_i^0} \equiv -a \pmod 1.$$

Recall that if, as in Lemma 2.1,

$$f^m([0, 1] \times \{0\})$$

is lifted to a curve in $[0, 1] \times \mathbf{R}$ joining $(0, a)$ to $(1, b + n)$, then

$$s(A_i) = -(b + n - a).$$

(Remember the Convention (†) on the sign of a Dehn twist.)

Therefore,

$$s(A_i) + \frac{\delta_i^0}{\lambda_i^0} + \frac{\delta_i^1}{\lambda_i^1} \equiv -(b + n - a) - a + b \equiv 0 \pmod 1$$

□

Corollary 2.2. *Let*

$$f : A \to A$$

be a linear twist with respect to a parametrization

$$\phi : [0, 1] \times S^1 \to A.$$

Then the equation giving the linear twist is determined, up to the ambiguity of an integer l, by the screw number $s(A)$ and the valency

$$(m^0, \lambda^0, \sigma^0)$$

of

$$\partial_0 A(\{0\} \times S^1)$$

as follows:

$$f\phi(t, x) = \phi\left(t, x - s(A)t - \frac{\delta^0}{\lambda^0} + l\right), \quad l \in \mathbf{Z},$$

where δ^0 is determined by

$$\delta^0 \sigma^0 \equiv 1 \pmod{\lambda^0}, \quad 0 \le \delta^0 < \lambda^0.$$

The proof is immediate from the argument of Corollary 2.1.

Lemma 2.2 (Uniqueness of linearization). *Let*

$$f, f' : A \to A$$

be linear twists of an annulus A. Suppose that

$$f \mid \partial A = f' \mid \partial A,$$

and that the screw number of f in A is equal to the screw number of f' in A. Then there is an isotopy

$$h_\tau : A \to A, \quad 0 \le \tau \le 1,$$

such that

(i) $h_0 = id_A$,
(ii) $h_\tau^{-1}(f' \mid \partial A)\, h_\tau = f' \mid \partial A (= f \mid \partial A)$ *on* ∂A, *and*
(iii) $f = h_1^{-1} f' h_1$.

Remark 2.2. This shows that the quality of being linear is independent of the parametrization ϕ up to a sort of isotopy given by conditions (i), (ii) and (iii) above. Essentially this isotopy is boundary equivariant.

Proof (of Lemma 2.2). Let ϕ and $\phi' : [0,1] \times S^1 \to A$ be the parametrizations for which f and f' are linear twists respectively. By considering $\phi'\rho$ instead of ϕ', if necessary, we may assume

$$\phi(\{0\} \times S^1) = \phi'(\{0\} \times S^1)$$

and

$$\phi(\{1\} \times S^1) = \phi'(\{1\} \times S^1),$$

where

$$\rho : [0,1] \times S^1 \to [0,1] \times S^1$$

is defined by

$$\rho(t, x) = (1 - t, -x).$$

For a while we confine ourselves to the boundary component

$$\partial_0 A = \phi(\{0\} \times S^1),$$

and denote the restrictions

$$\phi \mid \{0\} \times S^1 : \{0\} \times S^1 \to \partial_0 A$$

and

$$\phi' \mid \{0\} \times S^1 : \{0\} \times S^1 \to \partial_0 A$$

simply by

$$\phi : S^1 \to \partial_0 A$$

and

$$\phi' : S^1 \to \partial_0 A,$$

respectively. By the definition of a (**Q**-) linear twist, the action of

$$f \mid \partial_0 A = f' \mid \partial_0 A$$

on $\partial_0 A$ is topologically equivalent to a rotation of finite order, say, $\lambda > 0$.

Since

$$\phi : S^1 \to \partial_0 A$$

and

$$\phi' : S^1 \to \partial_0 A$$

are "linear" parametrizations for this same action we have

$$(\phi')^{-1}\phi\left(x + \frac{1}{\lambda}\right) = (\phi')^{-1}\phi(x) + \frac{1}{\lambda}$$

(Recall $S^1 = \mathbf{R}/\mathbf{Z}$ here). However,

$$(\phi')^{-1}\phi : S^1 \to S^1$$

need not be a "linear" rotation. This requires additional technicality in the first half of the proof below.

Working in the universal covering $\mathbf{R}$, we define an isotopy

$$l_\tau : \mathbf{R} \to \mathbf{R}, \quad 0 \le \tau \le 1,$$

by

$$l_\tau(x) = (1-\tau)(\phi')^{-1}\phi(x) + \tau x, \quad x \in \mathbf{R}$$

Obviously we have

$$l_0 = (\phi')^{-1}\phi, l_1 = id_{\mathbf{R}},$$

and

$$l_\tau(x + 1/\lambda) = l_\tau(x) + 1/\lambda.$$

The last property assures that l_τ projects to an isotopy of $S^1 = \mathbf{R}/\mathbf{Z}$ which we denote by the same notation $l_\tau : S^1 \to S^1$. It satisfies

$$l_0 = (\phi')^{-1}\phi, l_1 = id_{S^1},$$

and

$$l_\tau(x + 1/\lambda) = l_\tau(x) + 1/\lambda.$$

Define

$$g_\tau : \partial_0 A \to \partial_0 A, (0 \le \tau \le 1)$$

by

$$g_\tau = \phi' l_\tau \phi^{-1}.$$

Then

$$g_0 = id$$

and

$$g_1 = \phi' \phi^{-1}.$$

Also g_τ satisfies the condition

$$g_\tau^{-1}(f' \mid \partial_0 A) g_\tau = f' \mid \partial_0 A \text{ on } \partial_0 A.$$

To see this, recall that

$$(\phi')^{-1}(f' \mid \partial_0 A)\phi' : S^1 \to S^1$$

and

$$\phi^{-1}(f' \mid \partial_0 A)\phi = \phi^{-1}(f \mid \partial_0 A)\phi : S^1 \to S^1$$

are the same rotation of order λ;

$$\phi^{-1}(f' \mid \partial_0 A)\phi(x) = (\phi')^{-1}(f' \mid \partial_0 A)\phi'(x) = x + \frac{\delta}{\lambda}, \tag{2.1}$$

where δ is an integer such that $\gcd(\delta, \lambda) = 1$. On the other hand,

$$\begin{aligned} \phi^{-1} g_\tau^{-1}(f' \mid \partial_o A) g_\tau \phi(x) &= l_\tau^{-1}(\phi')^{-1}(f' \mid \partial_0 A)\phi' l_\tau(x) \\ &= l_\tau^{-1}\left(l_\tau(x) + \frac{\delta}{\lambda}\right) \quad \text{by} \quad (2.1) \\ &= x + \frac{\delta}{\lambda}. \end{aligned}$$

Thus

$$\phi^{-1} g_\tau^{-1}(f' \mid \partial_0 A) g_\tau \phi = \phi^{-1}(f' \mid \partial_0 A)\phi,$$

so

$$g_\tau^{-1}(f' \mid \partial_0 A) g_\tau = f' \mid \partial_0 A$$

as asserted.

We extend $g_\tau : \partial_0 A \to \partial_0 A$ to an isotopy

$$\overline{g}_\tau : A \to A, \quad 0 \le \tau \le 1,$$

in such a way that

$$\overline{g}_0 = id_A, \quad \overline{g}_\tau \mid \partial_0 A = g_\tau, \quad \text{and} \quad \overline{g}_\tau \mid \partial_1 A = \text{identity}.$$

Then the isotopy $\overline{g}_\tau$ satisfies conditions (i) and (ii) of Lemma 2.2, and

$$(\overline{g}_1)^{-1} f' \overline{g}_1 : A \to A$$

is a linear twist with respect to the parametrization

$$(\overline{g}_1)^{-1}\phi' : [0, 1] \times S^1 \to A.$$

Moreover, this parametrization $(\overline{g}_1)^{-1}\phi'$ satisfies

$$(\overline{g}_1)^{-1}\phi' \mid \{0\} \times S^1 = \phi \mid \{0\} \times S^1$$

Therefore, taking

$$(\overline{g}_1)^{-1} f' \overline{g}_1$$

instead of f' if necessary, we may assume the parametrization

$$\phi' : [0, 1] \times S^1 \to A$$

for f' satisfies

$$\phi' \mid \{0\} \times S^1 = \phi \mid \{0\} \times S^1.$$

Applying the same argument to $\{1\} \times S^1$, we may assume

$$\phi' \mid \{0, 1\} \times S^1 = \phi \mid \{0, 1\} \times S^1.$$

This completes the preparatory argument.

Now by the technique of the proof of Lemma 2.1, there exists an isotopy

$$\Phi_\tau : [0, 1] \times S^1 \to [0, 1] \times S^1, \quad 0 \le \tau \le 1,$$

such that

1. $\Phi_0 = (\phi')^{-1}\phi$,

2. Φ_1 is linear, that is
$$\Phi_1(t, x) = (t, x + at + b)$$
for some $a, b \in \mathbf{Q}$, and

3. the restriction
$$\Phi_\tau \mid \{0, 1\} \times S^1$$
equals $(\phi')^{-1}\phi \mid \{0, 1\} \times S^1$ (= the identity of $\{0, 1\} \times S^1$).

Define

$$h_\tau : A \to A, \quad 0 \le \tau \le 1,$$

by

$$h_\tau = \phi' \Phi_\tau \phi^{-1}.$$

Then h_τ satisfies (i) $h_0 = id_A$, (ii) $h_\tau \mid \partial A = id_{\partial A}$, in particular

$$h_\tau^{-1}(f' \mid \partial A)h_\tau = f' \mid \partial A.$$

Moreover, one can verify

$$(h_1^{-1} f' h_1)\phi(t, x) = \phi(t, x + ct + d) \text{ for some } c, d \in \mathbf{Q}.$$

Thus the homeomorphisms f and $h_1^{-1} f' h_1 : A \to A$ are both linear with respect to the same ϕ; they coincide on the boundary ∂A, and have the same screw number by the assumption. Therefore, (iii) $f = h_1^{-1} f' h_1$. □

Putting together Lemmmas 2.1 and 2.2, and generalizing them to a disjoint union of annuli, we obtain the following theorem.

Theorem 2.3. (i) (LINEARIZATION) *Let*

$$\mathscr{A} = \bigcup_{i=1}^{r} A_i$$

be a disjoint union of annuli,

$$f : \mathscr{A} \to \mathscr{A}$$

a homeomorphism such that

$$f(A_i) = A_{i+1}, \quad i = 1, 2, \ldots, r-1,$$

and $f(A_r) = A_1$. *Suppose that*

$$f \mid \partial\mathscr{A} : \partial\mathscr{A} \to \partial\mathscr{A}$$

is periodic and that no annulus in $\mathscr{A}$ *is amphidrome with respect to* f. *Then* f *is rel.*∂ *isotopic to a homeomorphism* $f' : \mathscr{A} \to \mathscr{A}$ *such that*

$$(f')^r \mid A_i : A_i \to A_i$$

is a linear twist for each $i = 1, 2, \ldots, r$.
(ii) (UNIQUENESS OF LINEARIZATION) *Let*

$$f, f' : \mathscr{A} \to \mathscr{A}$$

be homeomorphisms such that

$$f(A_i) = f'(A_i) = A_{i+1}, \quad i = 1, 2, \ldots, r-1,$$

and

$$f(A_r) = f'(A_r) = A_1.$$

Suppose that $(f)^r \mid A_i$ *and*

$$(f')^r \mid A_i : A_i \to A_i$$

are linear twists for each $i = 1, 2, \ldots, r$, *and that* $f \mid \partial\mathscr{A} = f' \mid \partial\mathscr{A}$, *and that*

$$f, f' : \mathscr{A} \to \mathscr{A}$$

are mutually isotopic by a rel.∂ isotopy. Then there is an isotopy

$$h_\tau : \mathscr{A} \to \mathscr{A}, \quad 0 \le \tau \le 1,$$

such that

1. $h_0 = id_{\mathscr{A}}$
2. $h_\tau^{-1}(f' \,|\, \partial\mathscr{A})\, h_\tau = f' \,|\, \partial\mathscr{A} (= f \,|\, \partial\mathscr{A})$ *on* $\partial\mathscr{A}$, *and*
3. $f = h_1^{-1} f' h_1$.

Proof (of (i) LINEARIZATION*).* Let

$$\phi : [0,1] \times S^1 \to A_1$$

be a parametrization for which

$$f^r : A_1 \to A_1$$

is "linear" on ∂A_1. We adopt

$$f^{i-1}\phi : [0,1] \times S^1 \to A_i$$

as a parametrization of A_i, $i = 1, 2, \ldots, r$. By essentially the same argument as in Lemma 2.1,

$$f \,|\, A_r : A_r \to A_1$$

is *rel.*∂ isotopic to a map

$$f' \,|\, A_r : A_r \to A_1$$

which is "linear" with respect to the parametrizations

$$f^{r-1}\phi : [0,1] \times S^1 \to A_r$$

and

$$\phi : [0,1] \times S^1 \to A_1.$$

We define

$$f' : \mathscr{A} \to \mathscr{A}$$

by setting

$$f' \,|\, A_i = f \,|\, A_i : A_i \to A_{i+1},$$

for $i = 1, 2, \ldots, r-1$, and by taking the above $f' \,|\, A_r : A_r \to A_1$, for $i = r$.

Then

$$(f')^r \,|\, A_1 : A_1 \to A_1$$

satisfies

$$\begin{aligned}(f')^r\phi(t,x) &= (f' \mid A_r)(f' \mid A_{r-1})\cdots(f' \mid A_1)\phi(t,x)\\ &= (f' \mid A_r)f^{r-1}\phi(t,x)\\ &= \phi(t, x+at+b),\ \text{for some } a, b \in \mathbf{Q}\end{aligned}$$

Thus

$$(f')^r \mid A_1 : A_1 \to A_1$$

is a linear twist with respect to $\phi : [0,1] \times S^1 \to A_1$.

Similarly, if $r \geq 2$,

$$(f')^r \mid A_2 : A_2 \to A_2$$

satisfies

$$\begin{aligned}(f')^r f\phi(t,x) &= (f' \mid A_1)(f' \mid A_r)\cdots(f' \mid A_2)(f' \mid A_1)\phi(t,x)\\ &= (f \mid A_1)(f' \mid A_r)f^{r-1}\phi(t,x)\\ &= (f \mid A_1)\phi(t, x+at+b)\\ &= f\phi(t, x+at+b).\end{aligned}$$

Thus

$$(f')^r \mid A_2 : A_2 \to A_2$$

is a linear twist with respect to

$$f\phi : [0,1] \times S^1 \to A_2,$$

and so on. This proves (i). □

Proof (of (ii) UNIQUENESS OF LINEARIZATION*).* Applying Lemma 2.2 to

$$f^r \mid A_1 : A_1 \to A_1$$

and

$$(f')^r \mid A_1 : A_1 \to A_1,$$

we find an isotopy

$$g_\tau^{(1)} : A_1 \to A_1, \quad 0 \leq \tau \leq 1,$$

such that

$$g_0^{(1)} = id_{A_1}, (g_\tau^{(1)})^{-1}((f')^r \mid \partial A_1)g_\tau^{(1)} = (f')^r \mid \partial A_1,$$

and

$$f^r \mid A_1 = (g_1^{(1)})^{-1}((f')^r \mid A_1)g_\tau^{(1)}.$$

Define isotopies

$$g_\tau^{(i)} : A_i \to A_i, \quad 0 \leq \tau \leq 1, \quad i = 2, \ldots, r$$

by the formula

$$g_\tau^{(i)} = (f \mid A_{i-1}) \cdots (f \mid A_1) g_\tau^{(1)} (f \mid A_1)^{-1} \cdots (f \mid A_{i-1})^{-1}$$

(or equivalently, by an inductive formula

$$g_\tau^{(i)} = (f \mid A_{i-1}) g_\tau^{(i-1)} (f \mid A_{i-1})^{-1},$$

$i = 2, \ldots, r$) and define an isotopy

$$g_\tau : \mathscr{A} \to \mathscr{A}, \quad 0 \leq \tau \leq 1,$$

by setting

$$g_\tau \mid A_i = g_\tau^{(i)}, \quad i = 1, 2, \ldots, r$$

Then g_τ satisfies $g_o = id_{\mathscr{A}}$,

$$g_\tau^{-1}(f' \mid \partial\mathscr{A}) g_\tau = f' \mid \partial\mathscr{A},$$

and

$$(g_1^{-1} f' g_1)^r \mid A_1 = f^r \mid A_1.$$

Therefore, replacing f' by $g_1^{-1} f' g_1$ if necessary, we may assume

$$(f')^r \mid A_1 = f^r \mid A_1.$$

By the assumption,

$$f : (\mathscr{A}, \partial\mathscr{A}) \to (\mathscr{A}, \partial\mathscr{A})$$

is rel.∂ isotopic to

$$f' : (\mathscr{A}, \partial\mathscr{A}) \to (\mathscr{A}, \partial\mathscr{A}).$$

In particular, there is an isotopy

$$h_\tau^{(2)} : A_2 \to A_2, \quad 0 \leq \tau \leq 1,$$

such that $h_0^{(2)} = id_{A_2}$, $h_\tau^{(2)} \mid \partial A_2 = id_{\partial A_2}$, and $h_1^{(2)}(f \mid A_1) = f' \mid A_1$.

Then

$$(f \mid A_2)(h_1^{(2)})^{-1} : A_2 \to A_3$$

is rel.∂ isotopic to

$$f' \mid A_2 : A_2 \to A_3,$$

so there is an isotopy

$$h_\tau^{(3)} : A_3 \to A_3, \quad 0 \leq \tau \leq 1,$$

such that $h_0^{(3)} = id_{A_3}$, $h_\tau^{(3)} \mid \partial A_3 = id_{\partial A_3}$, and $h_1^{(3)}(f \mid A_2)(h_1^{(2)})^{-1} = f' \mid A_2$.

Proceeding in this way, we can construct isotopies

$$h_\tau^{(i)} : A_i \to A_i, \quad 0 \le \tau \le 1, i = 2, \dots, r,$$

such that $h_0^{(i)} = id_{A_i}, h_\tau^{(i)} \mid \partial A_i = id_{\partial A_i}$, and

$$h_1^{(i)}(f \mid A_{i-1})(h_1^{(i-1)})^{-1} = f' \mid A_{i-1}.$$

We will examine

$$(f \mid A_r)(h_1^{(r)})^{-1} : A_r \to A_1.$$

For this purpose, set

$$P = h_1^{(r)}(f \mid A_{r-1})(f \mid A_{r-2}) \cdots (f \mid A_1) : A_1 \to A_r.$$

Then

$$\begin{aligned} P &= h_1^{(r)}(f \mid A_{r-1})(h_1^{(r-1)})^{-1} h_1^{(r-1)}(f \mid A_{r-2})(h_1^{(r-2)})^{-1} \cdots \\ &\quad h_1^{(3)}(f \mid A_2)(h_1^{(2)})^{-1} h_1^{(2)}(f \mid A_1) \\ &= (f')^{r-1} \mid A_1. \end{aligned}$$

We have

$$(f \mid A_r)(h_1^{(r)})^{-1} P = f^r \mid A_1 = (f')^r \mid A_1 = (f' \mid A_r) P$$

implying

$$(f \mid A_r)(h_1^{(r)})^{-1} = f' \mid A_r : A_r \to A_1$$

Finally define an isotopy

$$h_\tau : \mathscr{A} \to \mathscr{A}, \quad 0 \le \tau \le 1,$$

by setting

$$\begin{aligned} h_\tau \mid A_1 &= id, \\ h_\tau \mid A_i &= h_\tau^{(i)}, \quad i = 2, \dots, r. \end{aligned}$$

Then,

$$h_0 = id_{\mathscr{A}}, h_\tau \mid \partial\mathscr{A} = id_{\partial\mathscr{A}}, \text{ and } f' = h_1 f h_1^{-1} \text{ (i.e., } f = h_1^{-1} f' h_1)$$

as asserted. □

2.4 Amphidrome Annuli

Lemma 2.3 (Specialization). *Let A be an annulus,*

$$f : A \to A$$

an (orientation-preserving) homeomorphism which interchanges the boundary components. Suppose

$$f \mid \partial A : \partial A \to \partial A$$

is periodic. Then there exists a rel.∂ isotopy

$$f_\tau : A \to A, \quad 0 \le \tau \le 1,$$

such that $f_0 = f$ and

$$f_1 : A \to A$$

is a special twist.

We need a sublemma.

Sublemma 1 Let *A and*

$$f : A \to A$$

be as in Lemma 2.3. Then there is a parametrization

$$\phi : [0,1] \times S^1 \to A$$

such that

$$f\phi(0,x) = \phi(1, -x + a)$$
$$f\phi(1,x) = \phi(0, -x - a)$$

for some $a \in Q$.

Proof. Let $\partial A = \partial_0 A \cup \partial_1 A$. Since

$$f^2 \mid \partial_0 A : \partial_0 A \to \partial_0 A$$

is a rotation of finite order, there is a parametrization

$$\varphi : S^1 \to \partial_0 A$$

such that

$$f^2 \varphi(x) = \varphi(x - 2b)$$

for some $b \in \mathbf{Q}$. Take a parametrization

$$\psi : S^1 \to \partial_1 A$$

for which

$$f\varphi(x) = \psi(-x + a)$$

holds, where $a \in \mathbf{Q}$ is arbitrary at this point.

Now,

$$\begin{aligned} f\psi(x) &= (f^2\varphi)(-x + a) \\ &= \varphi(-x + a - 2b) \end{aligned}$$

We have already chosen the number a satisfying

$$f\varphi(x) = \psi(-x + a)$$

and we want a to satisfy also

$$f\psi(x) = \varphi(-x - a).$$

To attain this $a \in \mathbf{Q}$ must be chosen so that

$$a - 2b \equiv -a \pmod 1,$$

that is

$$a \equiv b \text{ or } b + \frac{1}{2} \pmod 1.$$

This ambiguity cannot be settled now.

Since

$$\varphi : S^1 \to \partial_0 A$$

and

$$\psi : S^1 \to \partial_1 A$$

are homotopic as maps from S^1 into A, there is a homeomorphism

$$\phi : [0, 1] \times S^1 \to A$$

such that $\phi(0, x) = \varphi(x)$ and $\phi(1, x) = \psi(x)$.

Then

$$\begin{aligned} f\phi(0,\ x) &= \phi(1,\ -x + a), \\ f\phi(1,\ x) &= \phi(0,\ -x - a), \end{aligned}$$

as required. But remember that there are two possible values for $a \pmod 1$. ☐

This sublemma has the following corollary.

Corollary 2.3. *Let*

$$\mathscr{A} = \bigcup_{i=1}^{r} A_i$$

be a disjoint union of annuli,

$$f : \mathscr{A} \to \mathscr{A}$$

a homeomorphism such that

$$f \mid \partial\mathscr{A} : \partial\mathscr{A} \to \partial\mathscr{A}$$

is periodic. Suppose A_i be an amphidrome annulus in $\mathscr{A}$ with respect to f. Let

$$(m_i^0, \lambda_i^0, \sigma_i^0)$$

and

$$(m_i^1, \lambda_i^1, \sigma_i^1)$$

be the valencies of $\partial_0 A$ and $\partial_1 A$, respectively, and $s(A_i)$ the screw number of f in A_i. Then

(i) $m_i^0 = m_i^1 =$ *an even number,*
(ii) $(\lambda_i^0, \sigma_i^0) = (\lambda_i^1, \sigma_i^1)$
(iii) $(1/2)s(A_i) + \delta_i/\lambda_i$ *is an integer, where δ_i is an integer determined by $\sigma_i\delta_i \equiv 1 \pmod{\lambda_i}$ and $0 \le \delta_i < \lambda_i$. (Here λ_i denotes $\lambda_i^0 = \lambda_i^1$, and σ_i denotes $\sigma_i^0 = \sigma_i^1$.)*
(iv) (λ_i, σ_i)*is uniquely determined by $s(A_i)$.*

Proof. Let k be the smallest positive integer such that $f^k(A_i) = A_i$. Since A_i is amphidrome, f^k interchanges the boundary components of A_i. Thus $2k$ is the smallest positive integer such that $f^{2k}(A_i) = A_i$ does not interchange the boundary components. This implies (i) $m_i^0 = m_i^1 = 2k$.

Obviously,

$$f^k \mid \partial_0 A : \partial_0 A \to \partial_1 A$$

is equivariant with respect to the actions of $f^{2k} \mid \partial_0 A$ on $\partial_0 A$ and $f^{2k} \mid \partial_1 A$ on $\partial_1 A$. This proves (ii) $(\lambda_i^0, \sigma_i^0) = (\lambda_i^1, \sigma_i^1)$.

To prove (iii), consider

$$f^k : A_i \to A_i$$

as f in sublemma 1 and take a parametrization

$$\phi : [0, 1] \times S^1 \to A_i$$

there. We identify A_i with $[0, 1] \times S^1$ through ϕ. We give to $[0, 1] \times S^1$ the same orientation as in the proof of Corollary 2.1 (Fig. 2.2).

Note that

$$f^k(0, x) = (1, -x + a), f^k(1, x) = (0, -x - a).$$

Let us pass to the universal covering $[0, 1] \times \mathbf{R}$.

Let $\tilde{I}$ be the lift of the arc

$$f^k([0, 1] \times \{0\})$$

which starts at

$$(0, -a) \in [0, 1] \times \mathbf{R}.$$

Then it ends at $(1, a + n)$ for some $n \in \mathbf{Z}$. Thus there is a lift

$$\tilde{f}^k : [0, 1] \times \mathbf{R} \to [0, 1] \times \mathbf{R}$$

of

$$f^k : [0, 1] \times S^1 \to [0, 1] \times S^1$$

satisfying

$$\tilde{f}^k(0, x) = (1, -x + a + n),$$
$$\tilde{f}^k(1, x) = (0, -x - a).$$

We have

$$(\tilde{f}^k)^2(0, x) = (0, x - 2a - n),$$
$$(\tilde{f}^k)^2(1, x) = (1, x + 2a + n).$$

The curve $f^{2k}([0, 1] \times \{0\})$ is lifted to a curve joining $(0, -2a - n)$ to $(1, 2a + n)$. This implies

$$s(A_i) = -4a - 2n. \tag{2.2}$$

(Recall the convention of the sign of a Dehn twist in Convention (†).)

On the other hand, by the geometric meaning of δ_i/λ_i, we have

$$\frac{\delta_i}{\lambda_i} \equiv 2a \pmod 1.$$

Thus

$$\frac{1}{2}s(A_i) + \frac{\delta_i}{\lambda_i} \equiv -2a - n + 2a \equiv 0 \pmod 1,$$

which proves (iii).

Since $0 \leq \sigma_i < 1$ and $\sigma_i\delta_i \equiv 1 \pmod{\lambda_i}$, assertion (iv) follows from (iii). □

Now we can fix the ambiguity of the number a unsettled in Sublemma 1. We observed in the above proof that

$$\frac{1}{2}s(A_i) = -2a - n.$$

Hence by shifting a by $1/2$ if necessary, we can make n an even number. In other words, we can (and will) take a so that

$$a \equiv -\frac{1}{4}s(A_i) \pmod 1.$$

Let us restate Sublemma 1 as Corollary 2.4, taking this choice into account.

Corollary 2.4. *Let A and*

$$f : A \to A$$

be as in Lemma 2.3. *Let $s(A)$ be the screw number of f in A. Then there is a parametrization*

$$\phi : [0, 1] \times S^1 \to A$$

such that

$$f\phi(0, x) = \phi\left(1, -x - \frac{1}{4}s(A)\right),$$

$$f\phi(1, x) = \phi\left(0, -x + \frac{1}{4}s(A)\right).$$

Remark 2.3. In this way f^2 rotates both sides of A by half the screw number in *opposite* directions.

Proof (of Lemma 2.3). Let

$$\phi : [0, 1] \times S^1 \to A$$

be the parametrization of Corollary 2.4, and identify A with $[0, 1] \times S^1$ through this ϕ. Let

$$\tilde{f} : [0, 1] \times \mathbf{R} \to [0, 1] \times \mathbf{R}$$

be the lift of

$$f : [0, 1] \times S^1 \to [0, 1] \times S^1$$

such that

$$\tilde{f}(1, x) = \left(0, -x + \frac{1}{4}s(A)\right).$$

Then

$$\tilde{f}(0, x) = (1, -x - \frac{1}{4}s(A) + n)$$

for some $n \in \mathbf{Z}$, but this n must be 0 because, as we observed in the proof of Corollary 2.3, if the curve

$$\tilde{f}([0, 1] \times \{0\})$$

connects $(0, 1/4s(A))$ and $(1, -1/4s(A) + n)$, then

$$s(A) = -4(-\frac{1}{4}s(A)) - 2n.$$

(See (2.2) in the proof.) This implies $n = 0$, and we get

$$\tilde{f}(0, x) = \left(1, -x - \frac{1}{4}s(A)\right).$$

Define a piecewise-linear homeomorphism

$$\tilde{f}' : [0, 1] \times \mathbf{R} \to [0, 1] \times \mathbf{R}$$

by setting

$$\tilde{f}'(t, x) = \begin{cases} \left(1 - t, -x + \frac{3}{4}s(A)\left(t - \frac{1}{3}\right)\right), & 0 \le t \le \frac{1}{3}, \\ (1 - t, -x), & \frac{1}{3} \le t \le \frac{2}{3}, \\ \left(1 - t, -x + \frac{3}{4}s(A)\left(t - \frac{2}{3}\right)\right), & \frac{2}{3} \le t \le 1. \end{cases}$$

Note that

$$\tilde{f}' \mid \{0, 1\} \times \mathbf{R} = \tilde{f} \mid \{0, 1\} \times \mathbf{R}.$$

This homeomorphism $\tilde{f}'$ projects to a special twist

$$f' : [0, 1] \times S^1 \to [0, 1] \times S^1.$$

By essentially the same argument in the proof of Lemma 2.1,

$$f : [0, 1] \times S^1 \to [0, 1] \times S^1$$

is *rel.*∂ isotopic to f'. This completes the proof of Lemma 2.3. □

The following corollary will be obvious from the above argument.

Corollary 2.5. *Let $f : A \to A$ be a special twist with respect to a parametrization*

$$\phi : [0, 1] \times S^1 \to A.$$

Then, in contradistinction with the case of linear twists (Corollary 2.2), the equation defining a special twist is uniquely determined by the screw number $s(A)$ as follows:

$$f\phi(t,x) = \begin{cases} \phi\big(1-t, -x + \frac{3}{4}s(A)\left(t-\frac{1}{3}\right)\big), & 0 \le t \le \frac{1}{3}, \\ \phi(1-t, -x), & \frac{1}{3} \le t \le \frac{2}{3}, \\ \phi\big(1-t, -x + \frac{3}{4}s(A)\left(t-\frac{2}{3}\right)\big), & \frac{2}{3} \le t \le 1. \end{cases}$$

Lemma 2.4 (UNIQUENESS OF SPECIALIZATION). *Let*

$$f, f' : A \to A$$

be special twists of an annulus A. Suppose that

$$f \mid \partial A = f' \mid \partial A$$

and that the screw number of f in A is equal to the screw number of f' in A. Then there is an isotopy

$$h_\tau : A \to A, \quad 0 \le \tau \le 1,$$

such that

(i) $h_0 = id_A$,
(ii) $h_\tau^{-1}(f' \mid \partial A)h_\tau = f' \mid \partial A \ (= f \mid \partial A) on \partial A$, *and*
(iii) $f = h_1^{-1} f' h_1$.

Proof (of Lemma 2.4). The idea is the same as in Lemma 2.2. Let ϕ and

$$\phi' : [0.1] \times S^1 \to A$$

be the parametrizations with respect to which f and f' are special twists. After a preliminary isotopy, we may assume

$$\phi \mid \{0,1\} \times S^1 = \phi' \mid \{0,1\} \times S^1.$$

(Cf. the proof of Lemma 2.2.). Then, the next Claim shows that ϕ and ϕ' differ by an *even* number of full twists.

Claim (A). There exist lifts

$$\tilde{\phi}, \tilde{\phi}' : [0,1] \times \mathbf{R} \to \tilde{A}$$

of

$$\phi, \phi' : [0,1] \times S^1 \to A$$

such that

$$(\tilde{\phi}')^{-1}\tilde{\phi}(0,x) = (0, x+m),$$
$$(\tilde{\phi}')^{-1}\tilde{\phi}(1,x) = (1, x-m),$$

for some $m \in \mathbf{Z}$.

Proof (of Claim (A)). For simplicity, we identify A with $[0, 1] \times S^1$ through ϕ'. By Corollary 2.5,

$$f' : [0, 1] \times S^1 \to [0, 1] \times S^1$$

is lifted to

$$\tilde{f}' : [0, 1] \times \mathbf{R} \to [0, 1] \times \mathbf{R}$$

such that the image of the interval $[0, 1] \times \{0\}$ under $\widetilde{f'}$ is a piecewise-linear arc connecting $(0, 1/4s)$ and $(1, -1/4s)$, where $s = s(A)$. Since $f \mid \partial A = f' \mid \partial A$, and the screw number $s(A)$ is common to f and f',

$$f : [0, 1] \times S^1 \to [0, 1] \times S^1$$

can also be lifted to

$$\tilde{f} : [0, 1] \times \mathbf{R} \to [0, 1] \times \mathbf{R}$$

such that $f([0, 1] \times \{0\})$ is a (not necessarily piecewise-linear) arc connecting $(0, 1/4s)$ and $(1, -1/4s)$. In particular,

$$\tilde{f}(0, x) = \left(1, -x - \frac{1}{4}s\right), \quad \tilde{f}(1, x) = \left(0, -x + \frac{1}{4}s\right).$$

By Corollary 2.5, there is a lift

$$\tilde{\phi} : [0, 1] \times \mathbf{R} \to \tilde{A} (= [0, 1] \times \mathbf{R} \text{ via } \tilde{\phi}')$$

such that

$$(\tilde{\phi})^{-1} \tilde{f} \tilde{\phi}([0, 1] \times \{0\})$$

is a piecewise-linear arc connecting $(0, 1/4s)$ and $(1, -1/4s)$. Since

$$\phi \mid \{0, 1\} \times S^1 = \phi' \mid \{0, 1\} \times S^1,$$

$\tilde{\phi}$ satisfies

$$\tilde{\phi}(0, x) = (0, x + m),$$
$$\tilde{\phi}(1, x) = (1, x + n),$$

for some $m, n \in \mathbf{Z}$. Then

$$\tilde{\phi}^{-1} \tilde{f} \tilde{\phi}(0, x) = \left(1, -x - m - n - \frac{1}{4}s\right),$$

$$\tilde{\phi}^{-1} \tilde{f} \tilde{\phi}(1, x) = \left(0, -x - m - n + \frac{1}{4}s\right).$$

By our choice of $\tilde{\phi}$, $m + n = 0$. Thus

$$(\tilde{\phi}')^{-1}\tilde{\phi} : [0, 1] \times \mathbf{R} \to [0, 1] \times \mathbf{R}$$

satisfies the formula stated in Claim (A). □

Proof (of Lemma 2.4: continued). Just as in the proof of Lemma 2.2, there is an isotopy

$$\Phi_\tau : [0, 1] \times S^1 \to [0, 1] \times S^1$$

such that

1. $\Phi_0 = (\phi')^{-1}\phi$,
2. Φ_1 is linear;

$$\Phi_1(t, x) = (t, x - 2mt + m),$$

   m being the integer of Claim (A).
3. The restriction

$$\Phi_\tau \,|\, \{0, 1\} \times S^1$$

   equals

$$(\phi')^{-1}\phi \,|\, \{0, 1\} \times S^1$$

   which is the identity on $\{0, 1\} \times S^1$.

Claim (B). If $f' : A \to A$ is a special twist with respect to a parametrization

$$\phi' : [0, 1] \times S^1 \to A,$$

then f' is also a special twist with respect to

$$\phi'\Phi_1 : [0, 1] \times S^1 \to A,$$

where

$$\Phi_1 : [0, 1] \times S^1 \to [0, 1] \times S^1$$

is given by

$$\Phi_1(t, x) = (t, x - 2mt + m)$$

for some $m \in \mathbf{Z}$.

Proof (of Claim (B)). Compute

$$\Phi_1^{-1}(\phi')^{-1} f' \phi' \Phi_1(t, x).$$

But beware that it is here where the full force of Claim (A) is used. □

Define an isotopy

$$h_\tau : A \to A, \quad 0 \le \tau \le 1,$$

by

$$h_\tau = \phi' \Phi_\tau \phi^{-1}.$$

Then h_τ satisfies (i) $h_0 = id_A$, (ii) $h_\tau \mid \partial A = id_{\partial A}$. Moreover, using Claim (B), one can verify that

$$h_1^{-1} f' h_1 (= \phi \Phi_1^{-1} (\phi')^{-1} f' \phi' \Phi_1 \phi^{-1})$$

is a special twist with respect to $\phi : [0,1] \times S^1 \to A$. Both $h_1^{-1} f' h_1$ and f are special twists with respect to the same parametrization

$$\phi : [0,1] \times S^1 \to A.$$

Since they coincide on ∂A and have the same screw number in A then (iii) $f = h_1^{-1} f' h_1$. □

Let us generalize Lemmas 2.3 and 2.4 to the case of a disjoint union of annuli.

Theorem 2.4. (i) *(*SPECIALIZATION*) Let*

$$\mathscr{A} = \bigcup_{i=1}^{r} A_i$$

be a disjoint union of annuli,

$$f : \mathscr{A} \to \mathscr{A}$$

a homeomorphism such that

$$f(A_i) = A_{i+1}, \quad i = 1, 2, \ldots, r-1,$$

and $f(A_r) = A_1$. Suppose that

$$f \mid \partial\mathscr{A} : \partial\mathscr{A} \to \partial\mathscr{A}$$

is periodic and that each A_i is amphidrome with respect to f. Then f is rel.∂ isotopic to a homeomorphism

$$f' : \mathscr{A} \to \mathscr{A}$$

such that

$$(f')^r \mid A_i : A_i \to A_i$$

is a special twist for each $i = 1, 2, \ldots, r$.
(ii) *(*UNIQUENESS OF SPECIALIZATION*) Let*

$$f, f' : \mathscr{A} \to \mathscr{A}$$

be homeomorphisms such that

$$f(A_i) = f'(A_i) = A_{i+1}, \quad i = 1, 2, \ldots, r-1,$$

and

$$f(A_r) = f'(A_r) = A_1.$$

Suppose that $f^r \mid A_i$ *and*

$$(f')^r \mid A_i : A_i \to A_i$$

are special twists for each $i = 1, 2, \ldots, r$, *and that*

$$f \mid \partial\mathscr{A} = f' \mid \partial\mathscr{A},$$

and that

$$f, f' : \mathscr{A} \to \mathscr{A}$$

are mutually isotopic by a rel.∂ isotopy. Then there is an isotopy

$$h_\tau : A \to A, \quad 0 \le \tau \le 1,$$

such that

(i) $h_0 = id_{\mathscr{A}}$,
(ii) $h_\tau^{-1}(f' \mid \partial A)h_\tau = f' \mid \partial A (= f \mid \partial A)$ *on* ∂A, *and*
(iii) $f = h_1^{-1} f' h_1$.

Proof (of (i) SPECIALIZATION*).* Since A_1 is amphidrome,

$$f^r : A_1 \to A_1$$

interchanges the boundary components. Take a parametrization

$$\phi : [0, 1] \times S^1 \to A_1$$

for which

$$f^r : A_1 \to A_1$$

satisfies

$$f^r \phi(0, x) = \phi\left(1, -x - \frac{1}{4}s(A_1)\right),$$

$$f^r \phi(1, x) = \phi\left(0, -x + \frac{1}{4}s(A_1)\right).$$

(See Corollary 2.4).

As in the proof of Theorem 2.3 (i), we adopt

$$f^{i-1}\phi : [0, 1] \times S^1 \to A_i$$

as a parametrization of A_i, $i = 1, 2, \ldots, r$. By our choice of

$$\phi : [0, 1] \times S^1 \to A_1,$$

the map

$$f \mid A_r : A_r \to A_1$$

trivially satisfies

$$(f \mid A_r) f^{r-1} \phi(0, x) = \phi\left(1, -x - \frac{1}{4} s(A_1)\right),$$

$$(f \mid A_r) f^{r-1} \phi(1, x) = \phi\left(0, -x + \frac{1}{4} s(A_1)\right).$$

Then

$$f \mid A_r : A_r \to A_1$$

is *rel.*∂ isotopic to a homeomorphism

$$f' \mid A_r : A_r \to A_1$$

defined by

$$(f' \mid A_r) f^{r-1} \phi(t, x) = \begin{cases} \phi\left(1 - t, -x + \frac{3}{4} s(A_1)\left(t - \frac{1}{3}\right)\right), & 0 \le t \le \frac{1}{3}, \\ \phi(1 - t, -x), & \frac{1}{3} \le t \le \frac{2}{3}, \\ \phi\left(1 - t, -x + \frac{3}{4} s(A_1)\left(t - \frac{2}{3}\right)\right), & \frac{2}{3} \le t \le 1. \end{cases}$$

(See the proof of Lemma 2.3). □

We define

$$f' : \mathscr{A} \to \mathscr{A}$$

by setting

$$f' \mid A_i = f \mid A_i : A_i \to A_{i+1},$$

for $i = 1, 2, \ldots, r - 1$, and by taking the above

$$f' \mid A_r : A_r \to A_1$$

for $i = r$. Then by the same argument as in the proof of Theorem 2.3(i), we can show that

$$(f')^r \mid A_i : A_i \to A_i$$

is a special twist with respect to the parametrization

$$f^{i-1} \phi : [0, 1] \times S^1 \to A_i,$$

for $i = 1, 2, \dots, r$. Clearly f' is rel.∂ isotopic to f. We are done.

Proof (of (ii) UNIQUENESS OF SPECIALIZATION*).* Applying Lemma 2.4 to

$$f^r \mid A_1 : A_1 \to A_1,$$

and

$$(f')^r \mid A_1 : A_1 \to A_1$$

we find an isotopy

$$g_\tau^{(1)} : A_1 \to A_1, \quad 0 \le \tau \le 1,$$

such that $g_0^{(1)} = id_{A_1}$,

$$(g_\tau^{(1)})^{-1}((f')^r \mid \partial A_1) g_\tau^{(1)} = (f')^r \mid \partial A_1$$

and

$$f^r \mid A_1 = (g_1^{(1)})^{-1}((f')^r \mid A_1) g_1^{(1)}.$$

Then the rest of the proof is exactly the same as the proof of Theorem 2.3 (ii). This completes the proof. □

Now we are in a position to prove the main theorem of this Chap. 2.

2.5 Proof of Theorem 2.1

Proof (of (i)). We must show that a given pseudo-periodic map

$$f : \Sigma_g \to \Sigma_g$$

is isotopic to a pseudo-periodic map in standard form. Let $\{C_i\}_{i=1}^r$ be the precise system of cut curves subordinate to f (Chap. 1). We may assume

$$f(\mathscr{C}) = \mathscr{C},$$

$\mathscr{C}$ being $\bigcup_{i=1}^r C_i$. We choose a system of annular neighborhoods $\{A_i\}_{i=1}^r$ of $\{C_i\}_{i=1}^r$. We may assume

$$f(\mathscr{A}) = \mathscr{A},$$

where

$$\mathscr{A} = \bigcup_{i=1}^r A_i.$$

By the definition of a pseudo-periodic map,

$$f \mid (\Sigma_g - \mathscr{C}) : \Sigma_g - \mathscr{C} \to \Sigma_g - \mathscr{C}$$

is isotopic to a periodic map. Then

$$f \mid \mathscr{B} : \mathscr{B} \to \mathscr{B}$$

is also isotopic to a periodic map (Cf. [52]), where

$$\mathscr{B} = \Sigma_g - Int(\mathscr{A}),$$

so we may assume that $f \mid \mathscr{B}$ is already periodic.

Decompose the finite set $\{A_i\}_{i=1}^{r}$ into cyclic orbits under the permutation caused by f, and decompose $\mathscr{A}$ into

$$\mathscr{A}^{(1)} \cup \mathscr{A}^{(2)} \cup \cdots \cup \mathscr{A}^{(s)}$$

accordingly. Since f cyclically permutes the connected components of $\mathscr{A}^{(\nu)}$, all the annuli contained in $\mathscr{A}^{(\nu)}$ are simultaneously non-amphidrome or amphidrome for each $\nu = 1, 2, \ldots, s$. Apply Theorem 2.3 (i) or Theorem 2.4 (i) as the case may be, then

$$f \mid \mathscr{A}^{(\nu)} : \mathscr{A}^{(\nu)} \to \mathscr{A}^{(\nu)}$$

is rel.∂ isotopic to a homeomorphism

$$f' \mid \mathscr{A}^{(\nu)} : \mathscr{A}^{(\nu)} \to \mathscr{A}^{(\nu)}$$

such that, for each annulus $A_j^{(\nu)}$ in $A^{(\nu)}$,

$$(f' \mid \mathscr{A}^{(\nu)})^{r_\nu} : A_j^{(\nu)} \to A_j^{(\nu)}$$

is a linear twist or a special twist, where r_ν denotes the number of the annuli in $\mathscr{A}^{(\nu)}$.

Applying this isotopy for each $\nu = 1, 2, \ldots, s$, we get a pseudo-periodic map

$$f' : \Sigma_g \to \Sigma_g$$

in standard form, which is isotopic to the original

$$f : \Sigma_g \to \Sigma_g.$$

This completes the proof of (i). □

Proof (of (ii)). Suppose we are given two pseudo-periodic maps

$$f, f' : \Sigma_g \to \Sigma_g$$

in standard form. Suppose they are homotopic. We will show the existence of a homeomorphism

$$h : \Sigma_g \to \Sigma_g$$

isotopic to the identity and such that $f = h^{-1} f' h$.

By Theorem 2.2 (i), there exists a homeomorphism

$$g : \Sigma_g \to \Sigma_g$$

which is isotopic to the identity and sends the precise system of cut curves for f to that for f'. Replacing f' by $g^{-1}f'g$, we may assume that the precise system of cut curves is common to f and f'. Let us denote it by $\{C_i\}_{i=1}^r$. Let $\{A_i\}_{i=1}^r$ and $\{A_i'\}_{i=1}^r$ be the systems of annular neighborhoods of $\{C_i\}_{i=1}^r$ which are invariant under the action of f and of f', respectively. Again there exists a homeomorphism

$$g : \Sigma_g \to \Sigma_g$$

which is isotopic to the identity and sends A_i to A_i'. Replacing f' by $g^{-1}f'g$, we may assume that $\{A_i\}_{i=1}^r$ is common to f and f'. Let us denote $\bigcup_{i=1}^r A_i$ by $\mathscr{A}$ and $\Sigma_g - Int(\mathscr{A})$ by $\mathscr{B}$ as before. □

Claim (C). $f \mid \mathscr{B}$ and $f' \mid \mathscr{B}$ are homotopic as maps of pairs:

$$(\mathscr{B}, \partial\mathscr{B}) \to (\mathscr{B}, \partial\mathscr{B}).$$

Proof. We may assume that $\{C_i\}_{i=1}^r$ are closed geodesic with respect to a certain metric Σ_g. Since

$$f, f' : \Sigma_g \to \Sigma_g$$

are homotopic, they are isotopic, [10, 21]. Let

$$f_\tau : \Sigma_g \to \Sigma_g, \quad 0 \le \tau \le 1,$$

be the isotopy with $f_0 = f$ and $f_1 = f'$. This isotopy gives a homeomorphism

$$F : \Sigma_g \times [0,1] \to \Sigma_g \times [0,1]$$

defined by

$$F(p,\tau) = (f_\tau(p), \tau)$$

for

$$(p,\tau) \in \Sigma_p \times [0,1].$$

Since $\Sigma_g \times [0,1]$ is an irreducible 3-manifold (cf. [68]), we can apply an innermost disk argument to achieve

$$F(\mathscr{C} \times [0,1]) = \mathscr{C} \times [0,1].$$

Also by moving F by a rel.∂ isotopy, we may assume

$$F(\mathscr{A} \times [0,1]) = \mathscr{A} \times [0,1].$$

Then

$$F \mid \mathscr{B} \times [0,1] : \mathscr{B} \times [0,1] \to \mathscr{B} \times [0,1]$$

gives a homotopy between $f \mid \mathscr{B}$ and $f' \mid \mathscr{B}$. □

Now we can apply Theorem 2.2, and obtain a homeomorphism

$$h \mid \mathscr{B} : \mathscr{B} \to \mathscr{B}$$

isotopic to the identity $id_{\mathscr{B}}$ and such that

$$f \mid \mathscr{B} = (h \mid \mathscr{B})^{-1}(f' \mid \mathscr{B})(h\mathscr{B}).$$

$h \mid \mathscr{B}$ extends to a homeomorphism

$$h : \Sigma_g \to \Sigma_g$$

which is isotopic to the identity. Replacing f' by $h^{-1} f' h$, we may assume

$$f \mid \mathscr{B} = f' \mid \mathscr{B}.$$

Claim (D). $f \mid \mathscr{A}$ and

$$f' \mid \mathscr{A} : (\mathscr{A}, \partial\mathscr{A}) \to (\mathscr{A}, \partial\mathscr{A})$$

are isotopic by a rel.∂ isotopy.

Proof. Let A_i be an annulus of $\mathscr{A}$. f and f' cause the same permutation on the set of annuli $\{A_i\}_{i=1}^r$ because

$$f \mid \mathscr{B} = f' \mid \mathscr{B}.$$

We have

$$f(A_1) = f'(A_1),$$

which we denote A_2, for simplicity. Let L be a "straight" line in A_1 connecting a point $p_0 \in \partial_0 A_1$ and another $p_1 \in \partial_1 A_1$. Then the images $f(L)$ and $f'(L)$ are arcs in A_2 connecting

$$q_0 := f(p_0) = f'(p_0)$$

and

$$q_1 := f(p_1) = f'(p_1).$$

The arcs $f(L)$ and $f'(L)$ are homotopic in A_2 fixing the end points $\{q_0, q_1\}$. (*Proof.* Note that

$$(f')^{-1} f : \Sigma_g \to \Sigma_g$$

is a pseudo-periodic map, because

$$(f')^{-1} f \mid \mathscr{B} = id_{\mathscr{B}}.$$

If the assertion above is not correct then $(f')^{-1}f$ would have non-zero screw number about C_1, the center line of A_1. But this is impossible by Theorem 2.2, because

$$(f')^{-1}f : \Sigma_g \to \Sigma_g$$

is homotopic to id_{Σ_g} by the assumption of Theorem 2.1(ii)). Then by the ordinary innermost arc argument, together with the Alexander trick,

$$f \mid A_1 : A_1 \to A_2$$

is rel.∂ isotopic to

$$f' \mid A_1 : A_1 \to A_2.$$

Doing the same argument for each $A_i, i = 1, 2, \ldots, r$, we get Claim (B). As we did in the proof of Theorem 2.1 (i), decompose $\mathscr{A}$ into cyclic orbits

$$\mathscr{A}^{(1)} \cup \mathscr{A}^{(2)} \cup \cdots \cup \mathscr{A}^{(s)}$$

under the action of f. Consider an orbit $\mathscr{A}^{(\nu)}$. By the assumption of Theorem 2.1 (ii), $f^{r_\nu} \mid A_j^{(\nu)}$ and

$$(f')^{r_\nu} \mid A_j^{(\nu)} : A_j^{(\nu)} \to A_j^{(\nu)}$$

are simultaneously linear twists or special twists, where $A_j^{(\nu)}$ is an annulus of $\mathscr{A}^{(\nu)}$ and

$$r_\nu = \#(\mathscr{A}^{(\nu)}).$$

Also by Claim (B), $f \mid \mathscr{A}^{(\nu)}$ and

$$f' \mid \mathscr{A}^{(\nu)} : (\mathscr{A}^{(\nu)}, \partial\mathscr{A}^{(\nu)}) \to (\mathscr{A}^{(\nu)}, \partial\mathscr{A}^{(\nu)})$$

are isotopic by a rel.∂ isotopy. Then we can apply Theorem 2.3 (ii) or Theorem 2.4 (ii) as the case may be, and obtain an isotopy

$$h_\tau^{(\nu)} : \mathscr{A}^{(\nu)} \to \mathscr{A}^{(\nu)}, \quad 0 \le \tau \le 1,$$

such that

1. $h_0^{(\nu)} = id$.
2. $(h_\tau^{(\nu)})^{-1}(f' \mid \partial A^{(\nu)})h_\tau^{(\nu)} = f' \mid \partial\mathscr{A}^{(\nu)} (= f \mid \partial\mathscr{A}^{(\nu)})$ on $\partial\mathscr{A}^{(\nu)}$, and
3. $f \mid \mathscr{A}^{(\nu)} = (h_1^{(\nu)})^{-1}(f' \mid \mathscr{A}^{(\nu)})h_1^{(\nu)}$.

The condition (2) above says that

$$h_\tau^{(\nu)} : \mathscr{A}^{(\nu)} \to \mathscr{A}^{(\nu)}$$

is equivariant on $\partial\mathscr{A}^{(\nu)}$ with respect to the action of

$$f' \mid \partial\mathscr{A}^{(\nu)} \ (= f \mid \partial\mathscr{A}^{(\nu)}).$$

Therefore, we can extend the isotopy

$$h_\tau^{(\nu)}, \quad 0 \le \tau \le 1,$$

equivariantly into collar neighborhoods $(\partial\mathscr{B}) \times [0,\varepsilon)$ of $\mathscr{B}$ so that, beyond the collar, the extension gives the identity on

$$\mathscr{B} - (\partial\mathscr{B}) \times [0,\varepsilon).$$

Doing the same construction and the extension for each $\nu = 1, 2, \ldots, s$, we get an isotopy

$$h_\tau : \Sigma_g \to \Sigma_g, \quad 0 \le \tau \le 1,$$

such that

(a) $h_\tau(\mathscr{A}) = \mathscr{A}$ and $h_\tau(\mathscr{B}) = \mathscr{B}$,
(b) $h_0 = id$,
(c) $h_\tau \mid \mathscr{B} \to \mathscr{B}$ is equivariant with respect to $f' \mid \mathscr{B} = f \mid \mathscr{B}$, i.e.

$$(h_\tau \mid \mathscr{B})^{-1}(f' \mid \mathscr{B})(h_\tau \mid \mathscr{B}) = f' \mid \mathscr{B}(= f \mid \mathscr{B}), \text{ and}$$

(d) $h_\tau \mid \mathscr{A}^{(\nu)} = h_\tau^{(\nu)}, \nu = 1, 2, \ldots, s$.

Now, since

$$f \mid \mathscr{B} = (h_1 \mid \mathscr{B})^{-1}(f' \mid \mathscr{B})(h_1 \mid \mathscr{B})$$

and

$$f \mid \mathscr{A}^{(\nu)} = (h_1 \mid \mathscr{A}^{(\nu)})^{-1}(f' \mid \mathscr{A}^{(\nu)})(h_1 \mid \mathscr{A}^{(\nu)}), \nu = 1, 2, \ldots, s,$$

it is evident that

$$f = h_1^{-1} f' h_1.$$

This completes the proof of Theorem 2.1 (ii). □

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