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## Preface

This book is an attempt to present a coherent account of Oka theory, from the classical Oka-Grauert theory originating in the works of Kiyoshi Oka and Hans Grauert to the contemporary developments initiated by Mikhael Gromov.

At the core of Oka theory lies the heuristic *Oka principle*, a term coined by Jean-Pierre Serre in 1951: *Analytic problems on Stein manifolds admit analytic solutions if there are no topological obstructions*. The Cartan-Serre Theorems A and B are primary examples. The main exponent of the classical Oka-Grauert theory is the equivalence between topological and holomorphic classification of principal fiber bundles over Stein spaces. On the interface with affine algebraic geometry the Oka principle holds only rarely, while in projective geometry we have Serre's GAGA principle, the equivalence of analytic and algebraic coherent sheaves on compact projective algebraic varieties. In smooth geometry there is the analogous *homotopy principle* originating in the Smale-Hirsch homotopy classification of smooth immersions.

Modern Oka theory focuses on those properties of a complex manifold  $Y$  which insure that any continuous map  $X \rightarrow Y$  from a Stein source space  $X$  can be deformed to a holomorphic map; the same property is considered for sections of a holomorphic submersion  $Y \rightarrow X$ . By including the Runge approximation and the Cartan extension condition one obtains several ostensibly different Oka properties. Gromov's main result is that a geometric condition called ellipticity – the existence of a dominating holomorphic spray on  $Y$  – implies all forms of the Oka principle for maps or sections  $X \rightarrow Y$ . Subsequent research culminated in the result that all Oka properties of a complex manifold  $Y$  are equivalent to the following Runge approximation property:

*A complex manifold  $Y$  is said to be an Oka manifold if every holomorphic map  $f: K \rightarrow Y$  from a neighborhood of a compact convex set  $K \subset \mathbb{C}^n$  to  $Y$  can be approximated uniformly on  $K$  by entire maps  $\mathbb{C}^n \rightarrow Y$ .*

The related concept of an *Oka map* pertains to the Oka principle for lifting holomorphic maps from Stein sources. The class of Oka manifolds is dual to

the class of Stein manifolds in a sense that can be made precise by means of abstract homotopy theory. Finnur Lárusson constructed a model category containing all complex manifolds in which Stein manifolds are cofibrant, Oka manifolds are fibrant, and Oka maps are fibrations. This means that

*Stein manifolds are the natural sources of holomorphic maps, while Oka manifolds are the natural targets.*

Oka manifolds seem to be few and special; in particular, no compact complex manifold of Kodaira general type is Oka. However, special and highly symmetric objects are often more interesting than average generic ones.

A few words about the content. Chapter 1 contains some preparatory material, and Chapter 2 is a brief survey of Stein space theory. In Chapter 3 we construct open Stein neighborhoods of certain types of sets in complex spaces that are used in Oka theory. Chapter 4 contains an exposition of the theory of holomorphic automorphisms of Euclidean spaces and of the density property, a subject closely intertwined with our main theme. In Chapter 5 we develop Oka theory for stratified fiber bundles with Oka fibers (this includes the classical Oka-Grauert theory), and in Chapter 6 we treat Oka-Gromov theory for stratified subelliptic submersions over Stein spaces. Chapters 7 and 8 contain applications ranging from classical to the recent ones. In Chapter 8 we present results on regular holomorphic maps of Stein manifolds; highlights include the optimal embedding theorems for Stein manifolds and Stein spaces, proper holomorphic embeddings of some bordered Riemann surfaces into  $\mathbb{C}^2$ , and the construction of noncritical holomorphic functions, submersions and foliations on Stein manifolds. In Chapter 9 we explore implications of Seiberg-Witten theory to the geometry of Stein surfaces, and we present the Eliashberg-Gompf construction of Stein structures on manifolds with suitable handlebody decomposition. A part of this story is the *Soft Oka principle*.

This book would not have existed without my collaboration with Jasna Prezelj who explained parts of Gromov's work on the Oka principle in her dissertation (University of Ljubljana, 2000). Josip Globevnik suggested that we look into this subject, while many years earlier Edgar Lee Stout proposed that I study the Oka-Grauert principle. My very special thanks go to the colleagues who read parts of the text and offered suggestions for improvements: Barbara Drinovec-Drnovšek, Frank Kutzschebauch, Finnur Lárusson, Takeo Ohsawa, Marko Slapar, and Erlend Fornæss Wold. I am grateful to Reinhold Remmert for his invitation to write a volume for the *Ergebnisse* series, and to the staff of Springer-Verlag for their professional help.

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