
Stein Manifolds

This chapter is a brief survey of the theory of Stein manifolds and Stein spaces, with emphasis on the results that are frequently used in this book. After the initial developments by K. Weierstrass, B. Riemann, F. Hartogs, E. E. Levi, K. Reinhardt, H. Kneser, H. Cartan, P. Thullen and others, the main contributions were made in the period 1942–1965 by Kiyoshi Oka, by the French school around Henri Cartan including Pierre Dolbeault, Alexander Grothendieck and Jean-Pierre Serre, and by the Münster school founded by Heinrich Behnke and including Karl Stein, Hans Grauert, Reinhold Remmert and Friedrich Hirzebruch. In 1942 Oka [385, 388] published the first solution to the Levi problem on two dimensional domains, while the year 1965 marks the publication of Hörmander’s fundamental paper [266] in which the $\bar{\partial}$ -equation was solved by L^2 -methods. (Another contemporary work in this direction is due to Andreotti and Vesentini [22].) Together with the works of Kohn [304, 305] these provide the basis for quantitative methods in complex analysis. Comprehensive accounts of Stein theory are available in [228, 241, 267]. The article of Schumacher [434] provides a solid historical survey. A quick introduction to interesting topics in L^2 -theory can be found in [380].

2.1 Domains of Holomorphy

A basic notion in complex analysis is that of analytic continuation. Karl Weierstrass knew already in 1841 that a holomorphic function in an annulus in the complex plane \mathbb{C} admits a development into what is now called a Laurent series. By estimating the coefficients in this series, Bernhard Riemann showed in his dissertation in 1851 that a function which is analytic in a punctured neighborhood of a point $p \in \mathbb{C}$ and is bounded near p extends to a holomorphic function in a full neighborhood of p . It was known early on that on any open set $D \subset \mathbb{C}$ there exist holomorphic functions that do not extend holomorphically across any boundary point of D . An explicit example on the disc

$\mathbb{D} = \{|z| < 1\}$ is Kronecker's function $f(z) = \sum_{n=1}^{\infty} z^{n^2}$; further examples were given by Weierstrass.

A fundamental discovery was the phenomenon of *simultaneous analytic continuation*. In 1897 Adolph Hurwitz showed in his lecture at the first International Congress of Mathematicians that a holomorphic function of two or more variables does not have any isolated singularities. Much more interesting examples of analytic continuation were found by Friedrich Hartogs in 1906 [247]. The simplest *Hartogs figure* is the domain H in the bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$ defined by

$$H = \left\{ (z, w) \in \mathbb{D}^2 : |z| < \frac{1}{2} \text{ or } |w| > \frac{1}{2} \right\}. \quad (2.1)$$

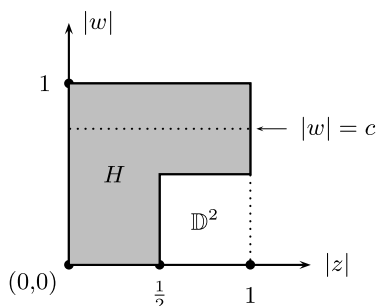


Fig. 2.1. A Hartogs figure in the bidisc

Every function $f \in \mathcal{O}(H)$ extends to a holomorphic function on the bidisc \mathbb{D}^2 . Indeed, pick a number $\frac{1}{2} < c < 1$ and consider the Cauchy integral

$$F(z, w) = \frac{1}{2\pi i} \int_{|\zeta|=c} \frac{f(z, \zeta)}{\zeta - w} d\zeta. \quad |z| < 1, |w| < c,$$

Then F is a holomorphic function on $D = \mathbb{D} \times c\mathbb{D}$ which agrees with f on $H \cap D$. (Since the disc $\{z\} \times c\mathbb{D}$ is contained in H when $|z| < \frac{1}{2}$, we have $f = F$ there by the Cauchy integral formula; the equality elsewhere follows by the identity principle.) This extends f to a holomorphic function on $H \cup D = \mathbb{D}^2$.

Fifteen years later, Karl Reinhardt [410] studied domains of convergence of power series $\sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$ in several variables $z = (z_1, \dots, z_n)$. It is immediate that the domain of convergence is a union of open polydiscs centered at the origin. By introducing the map $\phi: \mathbb{C}^n \rightarrow (\{-\infty\} \cup \mathbb{R})^n$,

$$\phi(z_1, \dots, z_n) = (\log |z_1|, \dots, \log |z_n|),$$

we see that each union of polydiscs is of the form $\Omega = \phi^{-1}(D)$ where D is a domain in $(\{-\infty\} \cup \mathbb{R})^n$ such that $(x_1, \dots, x_n) \in D$ and $y_j \leq x_j$ for $j = 1, \dots, n$ implies that $(y_1, \dots, y_n) \in D$. Reinhardt showed that Ω is the domain of convergence of a power series if and only of the corresponding

domain $D \subset (\{-\infty\} \cup \mathbb{R})^n$ is *convex*. This gives analytic continuation of holomorphic functions from a complete Reinhardt domain $\Omega \subset \mathbb{C}^n$ to the smallest logarithmically convex complete Reinhardt domain $\tilde{\Omega} \subset \mathbb{C}^n$ containing Ω .

In 1932 Hellmuth Kneser reformulated Hartogs's result into a more useful form known as the *Kontinuitätssatz*: Given an embedded family of closed analytic discs $D_t \subset \mathbb{C}^n$ ($t \in [0, 1]$) such that D_0 and all the boundaries bD_t belong to a domain $\Omega \subset \mathbb{C}^n$, every holomorphic function on Ω admits an analytic continuation along this family to a neighborhood of the disc D_1 .

Hartogs' discovery initiated research on 'natural domains' of holomorphic functions. Analytic continuation in general yields a multivalued function. Following an idea of Riemann, multivalued functions are considered as singlevalued functions on *Riemann domains* over \mathbb{C}^n : A complex manifold X together with a locally biholomorphic map $\pi: X \rightarrow \mathbb{C}^n$. The central concept became that of a *domain of holomorphy* – a domain in \mathbb{C}^n , or over \mathbb{C}^n , with a holomorphic function that does not extend holomorphically to any larger domain. Much of the classical theory developed around the problem of characterizing domains of holomorphy, and of constructing the *envelope of holomorphy* $\tilde{\Omega}$ of a given domain $\Omega \subset \mathbb{C}^n$ – the largest domain such that every holomorphic function on Ω extends to a holomorphic function on $\tilde{\Omega}$.

Another important discovery was made by Eugenio E. Levi in 1911 [337]. He investigated domains $D \Subset \mathbb{C}^n$ with \mathcal{C}^2 boundaries. Let $\rho: \mathbb{C}^n \rightarrow \mathbb{R}$ be a \mathcal{C}^2 defining function for D , i.e., $D = \{z \in \mathbb{C}^n: \rho(z) < 0\}$ and $d\rho_z \neq 0$ for every $z \in bD = \{\rho = 0\}$. Levi noticed that, if for some boundary point $p \in bD$ and some vector $v \in T_p^{\mathbb{C}}bD$ that is complex tangential to the boundary (i.e., such that $\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(p)v_j = 0$) the Levi form $\mathcal{L}_{\rho,p}(v) < 0$ is negative, then holomorphic functions on D continue to a neighborhood of p in \mathbb{C}^n . The condition $\mathcal{L}_{\rho,p}(v) < 0$ implies that we can embed a Hartogs pair (H, \mathbb{D}^n) in \mathbb{C}^n such that H is mapped into D but the image of \mathbb{D}^n contains a neighborhood of p . Levi conjectured that any domain $D \subset \mathbb{C}^n$ as in the following definition is a domain of holomorphy; this became known as the *Levi problem*.

Definition 2.1.1. A domain $D = \{\rho < 0\}$ with a \mathcal{C}^2 defining function ρ such that $d\rho \neq 0$ on $bD = \{\rho = 0\}$ is said to be (weakly) Levi pseudoconvex if $\mathcal{L}_{\rho,p}(v) \geq 0$ for every $p \in bD$ and $v \in T_p^{\mathbb{C}}bD$. The domain D is strongly pseudoconvex if $\mathcal{L}_{\rho,p}(v) > 0$ for every $p \in bD$ and $0 \neq v \in T_p^{\mathbb{C}}bD$.

It is easily seen that the definition is independent of the choice of a defining function. A strongly pseudoconvex domain is locally at each boundary point biholomorphic to a piece of a strongly convex domain, and is osculated by a ball in suitable coordinates. This is commonly known as *Narasimhan's lemma*, although it was already known to Kneser in 1936 [298].

An important characterization of domains of holomorphy was obtained by Henri Cartan and Peter Thullen in 1932. To a compact set K in a complex space X we associate its $\mathcal{O}(X)$ -*hull*

$$\widehat{K}_{\mathcal{O}(X)} = \{p \in X: |f(p)| \leq \max_{x \in K} |f(x)|, \forall f \in \mathcal{O}(X)\}. \quad (2.2)$$

If K is a compact set in \mathbb{C}^n then $\widehat{K} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ is the *polynomial hull* of K .

Definition 2.1.2. A compact set K in a complex space X is $\mathcal{O}(X)$ -convex if $K = \widehat{K}_{\mathcal{O}(X)}$; if $X = \mathbb{C}^n$ then such K is said to be *polynomially convex*. A complex space X is *holomorphically convex* if for every compact set $K \subset X$ its $\mathcal{O}(X)$ -hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

Theorem 2.1.3. (Cartan and Thullen [80]) A Riemann domain over \mathbb{C}^n is a domain of holomorphy if and only if it is holomorphically convex.

The hull of any compact set in a domain $\Omega \subset \mathbb{C}^n$ is a bounded closed subset of Ω , but it may fail to be compact as is seen in the Hartogs figure (2.1): Since every $f \in \mathcal{O}(H)$ extends to a function in $\mathcal{O}(\mathbb{D}^2)$, the maximum principle shows that the $\mathcal{O}(H)$ -hull of the circle $\{(z_0, w): |w| = \frac{3}{4}\}$ is the intersection of the disc $\{(z_0, w): |w| \leq \frac{3}{4}\}$ with Ω ; clearly this set is not compact if $\frac{1}{2} < |z_0| < 1$.

Theorem 2.1.3 is not difficult to prove. On the one hand, the derivatives of a holomorphic function $f \in \mathcal{O}(\Omega)$ satisfy the same bounds on $\widehat{K}_{\mathcal{O}(\Omega)}$ as on K , and hence the Taylor series of f centered around a point $p \in \widehat{K}_{\mathcal{O}(\Omega)}$ has the same domain of convergence as for points in K . If Ω is a domain of holomorphy, it follows that for any compact set $K \subset \Omega$ we have

$$\text{dist}(\widehat{K}_{\mathcal{O}(\Omega)}, b\Omega) = \text{dist}(K, b\Omega), \quad (2.3)$$

so $\widehat{K}_{\mathcal{O}(\Omega)}$ is compact. Conversely, using holomorphic convexity one can easily construct holomorphic functions tending to infinity along a given discrete sequence, so Ω is a domain of holomorphy.

A more challenging problem was to find a geometric characterization of domains of holomorphy. It follows from (2.3) that any closed holomorphic disc D in a domain of holomorphy Ω satisfies $\text{dist}(D, b\Omega) = \text{dist}(bD, b\Omega)$. This condition, which can be formulated in terms of *Hartogs pairs* (biholomorphic images of a standard pair $H \subset \mathbb{D}^n$, where H is a Hartogs figure in the polydisc \mathbb{D}^n), is known as *Hartogs pseudoconvexity* of Ω . Essentially it means that an analytic disc in $\overline{\Omega}$ with boundary in Ω must be contained in Ω . Oka showed that in such case the function $\Omega \ni z \mapsto -\log \text{dist}(z, b\Omega)$ is plurisubharmonic on Ω . Clearly this function blows up at $b\Omega$, so by adding the term $|z|^2$ we get a strongly plurisubharmonic exhaustion function on Ω . Similarly, Levi pseudoconvexity of a domain $\Omega \Subset \mathbb{C}^n$ easily implies that the function $-\log \text{dist}(\cdot, b\Omega)$ is plurisubharmonic on Ω .

Could this be a characterization of domains of holomorphy?

This Levi problem was solved in the affirmative by Oka in 1942 for domains in \mathbb{C}^2 [385]; the higher dimensional case followed ten years later. In summary we have the following result [267, Theorem 2.6.7].

Theorem 2.1.4. [385, 387, 388, 54, 378] *The following conditions are equivalent for a domain Ω in \mathbb{C}^n (or a domain over \mathbb{C}^n):*

- (a) Ω is a domain of holomorphy.
- (b) Ω is Hartogs pseudoconvex.
- (c) The function $-\log \text{dist}(\cdot, b\Omega)$ is plurisubharmonic.
- (d) There exists a (strongly) plurisubharmonic exhaustion function on Ω .

A domain $\Omega \subset \mathbb{C}^n$ with \mathcal{C}^2 boundary is a domain of holomorphy if and only if it is Levi pseudoconvex.

Every domain in (or over) \mathbb{C}^n admits an envelope of holomorphy which can be constructed by ‘pushing analytic discs’ countably many times. A construction of the envelope in one step for domains in \mathbb{C}^2 , and also in any two dimensional Stein manifold, was given by B. Jöricke in 2009 [283]. For results in this direction see also Merker and Porten [354].

2.2 Stein Manifolds and Stein Spaces

In 1951 Karl Stein introduced the following class of complex manifolds.

Definition 2.2.1. (K. Stein [460]) *A complex manifold X is said to be a Stein manifold (or a holomorphically complete manifold) if the following hold:*

- (a) *For every pair of distinct points $x \neq y$ in X there is a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.*
- (b) *For every point $p \in X$ there exist functions $f_1, \dots, f_n \in \mathcal{O}(X)$, $n = \dim X$, whose differentials df_j are \mathbb{C} -linearly independent at p .*
- (c) *X is holomorphically convex (see Def. 2.1.2).*

Property (b) means that global holomorphic functions provide local charts at each point. Here are some observation and examples:

- An open set in \mathbb{C}^n is Stein if and only if it is a domain of holomorphy. (This follows from the Cartan-Thullen Theorem 2.1.3.)
- A Stein manifold does not contain any compact complex subvariety of positive dimension. (Apply axiom (a) and the maximum principle.)
- The Cartesian product $X \times Y$ of a pair of Stein manifolds is Stein.
- A closed complex submanifold X of \mathbb{C}^N is Stein. (Use coordinate functions restricted to X . For the converse see Theorem 2.2.8 below.) More generally, every closed complex submanifold of a Stein manifold is Stein.
- An open Riemann surface is a Stein manifold. (This nontrivial result is due to Behnke and Stein [40, 41].)

- If $X \rightarrow Y$ is a holomorphic covering space and Y is Stein then X is Stein. (This is due to K. Stein [461].)
- If $E \rightarrow X$ is a holomorphic vector bundle over a Stein base X then the total space E is also Stein. However, fiber bundles with fiber \mathbb{C}^n ($n > 1$) and with a nonlinear transition group may fail to be Stein; see §4.21.

The notion of a Stein space was first introduced by Hans Grauert in 1955 [220]. The standard definition is the following one.

Definition 2.2.2. *A second countable complex space X is a Stein space if it satisfies properties (a), (c) in Def. 2.2.1 and also*

(b') *Every local ring $\mathcal{O}_{X,x}$ is generated by functions in $\mathcal{O}(X)$.*

Condition (b') means that there is a holomorphic map $X \rightarrow \mathbb{C}^N$ which embeds a neighborhood of x as a local complex subvariety of \mathbb{C}^N . Grauert showed in [220] that one gets an equivalent definition by keeping (c) and replacing (a) and (b) (resp. (b')) by the following property.

Definition 2.2.3. *A complex space X is called K-complete if for every point $x \in X$ there is a holomorphic map $f: X \rightarrow \mathbb{C}^N$ (with $N = N_x$) such that x is an isolated point of the fiber $f^{-1}(f(x))$.*

It is immediate that axiom (a) implies K -completeness. In summary:

Theorem 2.2.4. [220] *A complex space X is a Stein space if and only if it is holomorphically convex and it satisfies one of the following two properties:*

- Holomorphic functions separate points on X (axiom (a) in Def. 2.2.1).*
- X is K -complete in the sense of Def. 2.2.3.*

For further characterizations of Stein spaces see [228, p. 152].

The following *Oka-Weil theorem* generalizes the classical Runge theorem. The analogous result holds for sections of holomorphic vector bundles and, more generally, for sections of coherent analytic sheaves (see §2.4).

Theorem 2.2.5. *If X is a Stein space and K is a compact $\mathcal{O}(X)$ -convex subset of X then every holomorphic function in an open neighborhood of K can be approximated uniformly on K by functions in $\mathcal{O}(X)$.*

Theorem 2.2.5 was first proved for domains of holomorphy by Oka [382] using his *Oka lemma* [267, Lemma 2.7.5]. It is immediate that an $\mathcal{O}(X)$ -convex set K can be approximated from the outside by *analytic polyhedra*, i.e., by Stein open sets of the form

$$U = \{x \in X: |h_j(x)| < 1, j = 1, \dots, m\}, \quad h_1, \dots, h_m \in \mathcal{O}(X).$$

By adding more functions we can insure that $h = (h_1, \dots, h_m): X \rightarrow \mathbb{C}^m$ embeds U properly in the polydisc $\mathbb{D}^m \subset \mathbb{C}^m$. The key point proved by Oka is that for any function $f \in \mathcal{O}(U)$ there is a $g \in \mathcal{O}(\mathbb{D}^m)$ such that $g \circ h = f$. (This is a special case of Cartan's extension theorem, Corollary 2.4.3.) By expanding g in power series and approximating it by Taylor polynomials $P \in \mathbb{C}[z_1, \dots, z_m]$ we get functions $P \circ h \in \mathcal{O}(X)$ approximating f on K . Another proof can be given by L^2 -methods (see [267]).

Definition 2.2.6. *A domain Ω in a complex space X is Runge in X if every holomorphic function $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compacts in Ω by functions in $\mathcal{O}(X)$; equivalently, if the subalgebra $\{f|_\Omega: f \in \mathcal{O}(X)\}$ of $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(\Omega)$.*

Theorem 2.2.7. [267, p. 91] *A Stein domain Ω in a Stein space X is Runge in X if and only if for every compact set $K \subset \Omega$ we have $\widehat{K}_{\mathcal{O}(\Omega)} = \widehat{K}_{\mathcal{O}(X)}$.*

An important characterization of Stein manifolds is that they are embeddable in Euclidean spaces. It is an immediate consequence of Definition 2.2.1 that for every relatively compact domain Ω in a Stein manifold X there is a holomorphic map $f: X \rightarrow \mathbb{C}^N$ for a big enough N such that $f|_\Omega: \Omega \rightarrow \mathbb{C}^N$ is an injective holomorphic immersion. In 1956 Reinhold Remmert proved a substantially stronger result that every Stein manifold admits a *proper* holomorphic embedding in some Euclidean space \mathbb{C}^N [411]. In 1960–61, Errett Bishop and Raghavan Narasimhan independently showed that if $\dim_{\mathbb{C}} X = n$ then the number N in Remmert's theorem can be taken to be $2n + 1$.

Theorem 2.2.8. ([48], [366, Theorem 5])

- (a) *If X is a Stein manifold of dimension n then the set of proper holomorphic embeddings of X into \mathbb{C}^{2n+1} is dense in $\mathcal{O}(X)^{2n+1}$.*
- (b) *If X is a Stein space of dimension n then the set of holomorphic maps $X \rightarrow \mathbb{C}^{2n+1}$ which are proper, injective and regular on the regular part X_{reg} is dense in $\mathcal{O}(X)^{2n+1}$.*
- (c) *If X is a Stein space of dimension n and of finite embedding dimension m then for $N = \max\{n+m, 2n+1\}$ the set of proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ is dense in $\mathcal{O}(X)^N$.*

More precise embedding theorems for Stein manifolds and Stein spaces are proved in §8.2 – §8.4, and for Riemann surfaces in §8.9 – §8.10. Since every real analytic manifold admits a Stein complexification [224], we get the following consequence:

Corollary 2.2.9. [224, Theorem 3] *Every real analytic manifold admits a proper real analytic embedding into a Euclidean space \mathbb{R}^N .*

Since Stein manifolds are complex submanifolds of Euclidean spaces, it is not surprising that they can be approximated by affine algebraic manifolds. It was proved by E. L. Stout [472] that any relatively compact domain in a Stein manifold is biholomorphically equivalent to a domain in an affine algebraic manifold. (For the real algebraic case see Nash [371].) More precise algebraic approximation results were obtained by Demailly, Lempert and Schiffman [105, 336] and by Lisca and Matič [343] (see Theorem 9.8.1 on p. 434).

2.3 Characterization by Plurisubharmonic Functions

It is a fundamental fact that Stein manifolds and Stein spaces are characterized by plurisubharmonicity (Theorem 2.3.2 below). Often the most efficient way to show that a complex space is Stein is to produce a strongly plurisubharmonic exhaustion function on it. For example, this is how Y.-T. Siu proved in 1976 [446] that a Stein subvariety of any complex space has a basis of open Stein neighborhoods (Theorem 3.1.1 on p. 57). Stein neighborhood constructions often allow us to transfer a problem on a complex space to Euclidean space where it becomes tractable; Chapter 3 focuses on such methods.

It follows from holomorphic convexity that every Stein space X is exhausted by an increasing sequence of compacts $K_1 \subset K_2 \subset \cdots \subset \cup_j K_j = X$ such that $K_j = \widehat{K}_j$ for every j . Using such exhaustions and axioms (a), (b') one can easily find strongly plurisubharmonic exhaustion functions of the form

$$\rho = \sum_{j=1}^{\infty} |f_j|^2: X \rightarrow \mathbb{R}_+, \quad f_j \in \mathcal{O}(X), \quad j = 1, 2, \dots$$

By a more precise argument one obtains the following result approximating $\mathcal{O}(X)$ -convex sets by sublevel sets of strongly plurisubharmonic functions (see [267, Theorem 5.1.5, p. 117]).

Proposition 2.3.1. *If K is a compact $\mathcal{O}(X)$ -convex set in a Stein space X then for every open set $U \subset X$ containing K there exists a smooth strongly plurisubharmonic function $\rho: X \rightarrow \mathbb{R}$ such that $\rho < 0$ on K and $\rho > 1$ on $X \setminus U$. Furthermore, there exists a plurisubharmonic exhaustion function $\rho: X \rightarrow \mathbb{R}_+$ such that $\rho^{-1}(0) = K$ and ρ is strongly plurisubharmonic on $X \setminus K = \{\rho > 0\}$.*

Note that the function $\rho_a: \mathbb{C}^N \rightarrow \mathbb{R}_+$ given by $\rho_a = |z - a|^2$ is strongly plurisubharmonic on any complex subvariety $X \subset \mathbb{C}^N$; if X is closed then this is an exhaustion function on X . Furthermore, if X is smooth then $\rho_a|_X$ is a Morse function on X for most choices of the point $a \in \mathbb{C}^N$.

These observations show that a Stein space admits plenty of smooth strongly plurisubharmonic exhaustion functions. The following converse is the most useful characterization of Stein manifolds and Stein spaces.

Theorem 2.3.2. (a) [224, 113] *A complex manifold which admits a strongly plurisubharmonic exhaustion function is a Stein manifold.*

(b) [368, 145] *A complex space which admits a strongly plurisubharmonic exhaustion function is a Stein space.*

Furthermore, if $\rho: X \rightarrow \mathbb{R}$ is a strongly plurisubharmonic exhaustion function then each sublevel set $\{x \in X: \rho(x) \leq c\}$ is $\mathcal{O}(X)$ -convex.

Corollary 2.3.3. *For every compact set K in a Stein space X the $\mathcal{O}(X)$ -hull of K coincides with its plurisubharmonic hull:*

$$\widehat{K}_{\mathcal{O}(X)} = \widehat{K}_{\text{Psh}(X)}.$$

Hence every holomorphic function in a neighborhood of a compact $\text{Psh}(X)$ -convex set $K = \widehat{K}_{\text{Psh}(X)}$ is a uniform limit on K of functions in $\mathcal{O}(X)$.

An efficient proof of these results is given by the L^2 -method for solving nonhomogeneous $\bar{\partial}$ -equations with weights of the form $e^{-\rho}$ with $\rho \in \text{Psh}(X)$ [266, 267]. Theorem 2.3.2 implies the following solution of the Levi problem.

Corollary 2.3.4. *A domain Ω in a Stein space X which admits a plurisubharmonic exhaustion function $\rho: \Omega \rightarrow \mathbb{R}$ is Stein. In particular, every Levi (or Hartogs) pseudoconvex domain in a Stein manifold is Stein.*

The existence of a Morse strongly plurisubharmonic function implies that every Stein manifold X is homotopy equivalent to a CW complex of real dimension at most $n = \dim_{\mathbb{C}} X$ (the Lefschetz theorem; see §3.11). The same holds for any finite dimensional Stein space.

There exist several notions of *ambient holomorphic convexity* of a compact set (see [474]). We shall use the following properties.

Definition 2.3.5. *Assume that K is a compact set in a complex space X .*

- (i) *K is a Stein compactum if it admits a basis of Stein neighborhoods in X .*
- (ii) *K is holomorphically convex if it admits an open Stein neighborhood Ω in X such that K is $\mathcal{O}(\Omega)$ -convex.*

Proposition 2.3.1 and Theorem 2.3.2 imply the following.

Proposition 2.3.6. *A compact set K in a Stein space X is holomorphically convex if and only if there exists a plurisubharmonic function $\rho: U \rightarrow \mathbb{R}_+$ in an open neighborhood U of K such that $\rho^{-1}(0) = K$ and ρ is strongly plurisubharmonic on $U \setminus K = \{\rho > 0\}$.*

The sets $\Omega_c = \{x \in U: \rho(x) < c\}$ for small $c > 0$ then form a basis of Stein neighborhoods of K such that K is $\mathcal{O}(\Omega_c)$ -convex.

Proposition 2.3.7. *Let K be a compact set in a complex space X . Assume that there exist a neighborhood $U \subset X$ of K , a strongly plurisubharmonic function $\rho: U \rightarrow \mathbb{R}$, and a weakly plurisubharmonic function $\tau: U \rightarrow \mathbb{R}_+$ such that $K = \{\tau = 0\}$. Then K is a Stein compactum.*

Proof. Pick an open neighborhood $V \Subset U$ of K and choose a fast growing convex increasing function $\chi: \mathbb{R} \rightarrow \mathbb{R}$ such that the strongly plurisubharmonic function $\phi = \rho + \chi \circ \tau: U \rightarrow \mathbb{R}$ satisfies $\phi|_K < 0$ and $K \subset V_c = \{\phi < c\} \Subset V$ for some $c > 0$. By Theorem 2.3.2 the domain V_c is Stein. \square

The closure of a smooth weakly pseudoconvex domain $D \Subset \mathbb{C}^n$ need not be a Stein compactum; an example is the *worm domain* [110]. For the existence of bounded strongly plurisubharmonic exhaustion functions on certain weakly Levi pseudoconvex domains see [111].

2.4 Cartan-Serre Theorems A & B

The famous Theorems A and B of Henri Cartan were proved in his seminar in 1951–52. It would be impossible to overstate the importance of these results to the development of analytic and algebraic geometry.

Theorem 2.4.1. (Theorems A and B; [75, 77, 228].) *For every coherent analytic sheaf \mathcal{F} on a Stein space (X, \mathcal{O}_X) the following hold:*

- (A) *The stalks \mathcal{F}_x are generated as $\mathcal{O}_{X,x}$ -modules by global sections of \mathcal{F} .*
- (B) *$H^p(X; \mathcal{F}) = 0$ for all $p = 1, 2, \dots$*

The corresponding results hold for every coherent algebraic sheaf over an affine algebraic variety $X \subset \mathbb{C}^N$ [440, p. 237, Théorème 2]. For coherent sheaves with continuous boundary values on strongly pseudoconvex domains see [262, 330].

We recall the relevant notions; a comprehensive account is available in [229]. An *analytic sheaf* (or \mathcal{O}_X -sheaf) on a complex space X is a sheaf \mathcal{F} of \mathcal{O}_X -modules; that is, a sheaf whose stalk \mathcal{F}_x over any point $x \in X$ is a module over the ring $\mathcal{O}_{X,x}$. The sheaf \mathcal{F} is *locally finitely generated* if for every point $x_0 \in X$ there exist an open neighborhood $U \subset X$ and finitely many sections $f_1, \dots, f_k \in \mathcal{F}(U) = \Gamma(U, \mathcal{F})$ whose germs at any point $x \in U$ generate \mathcal{F}_x as an $\mathcal{O}_{X,x}$ -module. The simplest example is \mathcal{O}_X^k , the direct sum of k copies of the structure sheaf \mathcal{O}_X ; this is the sheaf of holomorphic sections of the trivial bundle $X \times \mathbb{C}^k \rightarrow X$. An analytic sheaf is *coherent* if it is locally finitely generated and if for any set of local sections $f_1, \dots, f_k \in \mathcal{F}(U)$ the corresponding *sheaf of relations* $\mathcal{R} = \mathcal{R}(f_1, \dots, f_k)$ is also locally finitely generated. The latter sheaf has stalks

$$\mathcal{R}_x = \left\{ g_{1,x}, \dots, g_{k,x} \in \mathcal{O}_{X,x} : \sum_{j=1}^k g_{j,x} f_{j,x} = 0 \right\}, \quad x \in U. \quad (2.4)$$

From the above description we see that on small open sets $U \subset X$ we have a short exact sequence of analytic sheaf homomorphisms

$$\mathcal{O}_U^m \xrightarrow{\alpha} \mathcal{O}_U^k \xrightarrow{\beta} \mathcal{F}|_U \longrightarrow 0, \quad (2.5)$$

where $\beta(g_{1,x}, \dots, g_{k,x}) = \sum_{j=1}^k g_{j,x} f_{j,x}$. Hence β maps the standard basis sections $e_j = (0, \dots, 1, \dots, 0)$ of \mathcal{O}_U^k onto the generators f_j of $\mathcal{F}|_U$ and $\mathcal{R} = \ker \beta = \text{im } \alpha$ is the sheaf of relations (2.4). If X is a Stein space then such resolution exists over any relatively compact open subset $U \Subset X$.

Here are the main examples of coherent sheaves on a complex space X :

- The structure sheaf \mathcal{O}_X ([386], [229, p. 59]).
- The sheaf of ideals \mathcal{O}_A of a complex subvariety A in X (Cartan's coherence theorem, [74], [79, p. 631], [229, p. 84]).
- A locally free analytic sheaf. (Locally free analytic sheaves are sheaves of holomorphic sections of holomorphic vector bundles.)
- The Whitney sum $\mathcal{E} \oplus \mathcal{F}$ and the tensor product $\mathcal{E} \otimes \mathcal{F}$ of coherent sheaves.
- The sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ of \mathcal{O}_X -homomorphisms $\mathcal{E} \rightarrow \mathcal{F}$ between a pair of coherent analytic sheaves. In particular, the dual \mathcal{E}^* of a coherent sheaf.
- The kernel $\ker \beta$ and the image $\text{im } \beta$ of an \mathcal{O}_X -homomorphism $\beta: \mathcal{F} \rightarrow \mathcal{G}$ of coherent analytic sheaves. In summary, given a short exact sequence of homomorphisms of \mathcal{O}_X -analytic sheaves

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0, \quad (2.6)$$

if two sheaves are coherent then so is the third [229, p. 236].

- The direct image of an \mathcal{O}_X -coherent sheaf by a proper holomorphic map $X \rightarrow Y$ of complex spaces is a coherent \mathcal{O}_Y -sheaf [229, p. 207].

Each coherent analytic sheaf \mathcal{F} can be represented as the sheaf of germs of fiberwise linear holomorphic functions on a linear space $\pi: L \rightarrow X$ [140]. More precisely, there is a contravariant equivalence between the category of coherent analytic sheaves and the category of linear spaces such that locally free sheaves correspond to vector bundles. The sheaf of germs of holomorphic sections $X \rightarrow L$ of any linear space is also coherent [140, p. 53, Corollary].

We now mention some applications of Theorems A and B; see [228, Chapter V] for more on this subject.

Corollary 2.4.2. *Let $\mathcal{F} \xrightarrow{\beta} \mathcal{G}$ be an epimorphism of analytic sheaves over a Stein space X . If the kernel $\mathcal{E} = \ker \beta$ is coherent then the induced map on sections $\mathcal{F}(X) \rightarrow \mathcal{G}(X)$, $f \mapsto \beta(f)$, is surjective.*

Proof. Since $H^1(X; \mathcal{E}) = 0$ by Theorem B, the conclusion follows from the exact cohomology sequence $\mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow H^1(X; \mathcal{E}) = 0$. \square

Applying this to the exact sequence $0 \rightarrow \mathcal{J}_A \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J}_A \rightarrow 0$ where A is a closed complex subvariety of X one obtains

Corollary 2.4.3. (Cartan's extension theorem.) *Every holomorphic function on a closed complex subvariety in a Stein space X extends to a holomorphic function on X .*

Corollary 2.4.4. (Cartan's division theorem.) *If \mathcal{F} is a coherent analytic sheaf on a Stein space X and if $f_1, \dots, f_k \in \mathcal{F}(X)$ generate each stalk \mathcal{F}_x , then every section $f \in \mathcal{F}(X)$ is of the form $f = \sum_{j=1}^k g_j f_j$ for some $g_j \in \mathcal{O}(X)$.*

Proof. Consider the exact sequence $0 \rightarrow \mathcal{R} \rightarrow \mathcal{O}^k \xrightarrow{\beta} \mathcal{F} \rightarrow 0$ as in (2.5). Since $\mathcal{R} = \ker \beta$ is coherent, the conclusion follows from Corollary 2.4.2. \square

Corollary 2.4.5. *Given a short exact sequence (2.6) of analytic sheaves on a Stein space, if \mathcal{G} is locally free then there exists a sheaf homomorphism $\phi: \mathcal{G} \rightarrow \mathcal{F}$ such that $\beta \circ \phi = \text{Id}_{\mathcal{G}}$. In particular, a short exact sequence of homomorphisms of holomorphic vector bundles over a Stein space splits.*

Proof. Consider the induced exact sequence

$$0 \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{E}) \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F}) \xrightarrow{\beta} \mathcal{H}om(\mathcal{G}, \mathcal{G}) \longrightarrow 0.$$

Surjectivity of β is due to \mathcal{G} being locally free. By Theorem B we have $H^1(X; \mathcal{H}om(\mathcal{G}, \mathcal{E})) = 0$ and hence β is surjective also on the level of sections. Hence $\text{Id}_{\mathcal{G}}$ lifts to a homomorphism $\phi: \mathcal{G} \rightarrow \mathcal{F}$ with $\beta \circ \phi = \text{Id}_{\mathcal{G}}$. \square

Theorem 2.4.6. *On any Stein manifold X the Dolbeault cohomology groups vanish: $H_{\bar{\partial}}^{p,q}(X) = 0$ for all $p \geq 0$, $q \geq 1$.*

Proof. The sheaf Ω_p of holomorphic p -forms on X admits a resolution

$$0 \rightarrow \Omega_p \hookrightarrow \mathcal{E}_{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}_{p,1} \xrightarrow{\bar{\partial}} \mathcal{E}_{p,2} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}_{p,n} \rightarrow 0.$$

Since the sheaves $\mathcal{E}_{p,q}$ of smooth (p, q) -forms on X are fine, their cohomology vanishes. Leray's theorem implies that $H_{\bar{\partial}}^{p,q}(X) \cong H^q(X; \Omega_p)$. Since the sheaf Ω_p is coherent analytic, these groups are zero. \square

J.-P. Serre proved that each element of a de Rham cohomology group $H^p(X; \mathbb{C})$ of a Stein manifold is represented by a closed holomorphic p -form (see [439, Theorem 1], [228, p. 155]). The de Rham cohomology of an affine algebraic manifold is represented by algebraic forms.

The space of sections of a coherent analytic sheaf with the topology of uniform convergence on compacts is a Fréchet space [228, p. 167]. We have the Runge approximation theorem [228, p. 170]:

Theorem 2.4.7. *Let \mathcal{F} be a coherent analytic sheaf over a Stein space X . If K is a compact $\mathcal{O}(X)$ -convex set in X then any section over an open neighborhood of K can be approximated uniformly on K by sections in $\mathcal{F}(X)$.*

2.5 The $\bar{\partial}$ -Problem

The $\bar{\partial}$ -problem asks for a solution of the equation $\bar{\partial}u = f$ for a given $\bar{\partial}$ -closed form f . By Theorem 2.4.6 this problem is always solvable on a Stein manifold. A more direct proof which also gives L^2 estimates is provided by the L^2 theory of Hörmander [266, 267], Andreotti and Vesentini [22] and Kohn [304, 305]. (A comprehensive account of this subject can be found in [83].) We quote the following result for $(0, 1)$ -forms that is used in the present text. Let $d\lambda$ denote the Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Theorem 2.5.1. [267, Theorem 4.4.2, p. 94] *Let Ω be a pseudoconvex (Stein) domain in \mathbb{C}^n and ϕ a plurisubharmonic function in Ω . For every $(0, 1)$ -form $f = \sum f_j d\bar{z}_j$ such that $f_j \in L^2_{loc}(\Omega)$ and $\bar{\partial}f = 0$ (in the weak sense) there exists $u \in L^2_{loc}(\Omega)$ such that*

$$\bar{\partial}u = f \quad \text{and} \quad \int_{\Omega} \frac{|u|^2}{(1 + |z|^2)^2} e^{-\phi} d\lambda \leq \int_{\Omega} \sum_{j=1}^n |f_j|^2 e^{-\phi} d\lambda.$$

By taking Ω bounded and $\phi = 0$ we get the estimate

$$\bar{\partial}u = f \quad \text{and} \quad \int_{\Omega} |u|^2 d\lambda \leq C \int_{\Omega} \sum_j |f_j|^2 d\lambda \tag{2.7}$$

where the constant C depends only on the radius of Ω and on the dimension n . The analogous results hold on relatively compact domains in Stein manifolds.

To pass from L^2 to C^k estimates one needs the following well known lemma which follows from the Bochner-Martinelli formula [189, Lemma 3.2].

Lemma 2.5.2. (Interior elliptic regularity estimates.) *Let \mathbb{B}^n denote the open unit ball in \mathbb{C}^n . For each integer $s \in \mathbb{Z}_+$ there is a constant $c_s > 0$ such that if $f \in C^{s+1}(\epsilon\mathbb{B})$ for some $\epsilon > 0$ and $\alpha \in \mathbb{Z}_+^{2n}$ is a multiindex with $|\alpha| = s$ then*

$$c_s |\partial^\alpha f(0)| \leq \epsilon^{-n-s} \|f\|_{L^2(\epsilon\mathbb{B})} + \sum_{|\beta| \leq s} \epsilon^{|\beta|+1-s} \|\partial^\beta(\bar{\partial}f)\|_{L^\infty(\epsilon\mathbb{B})}.$$

In particular we have the sup-norm estimate

$$c_0 |f(0)| \leq \epsilon^{-n} \|f\|_{L^2(\epsilon\mathbb{B})} + \epsilon \|\bar{\partial}f\|_{L^\infty(\epsilon\mathbb{B})}.$$

On bounded strongly pseudoconvex domains in Stein manifolds the $\bar{\partial}$ -equation can also be solved by means of integral formulas with holomorphic kernels. This kernel method gives optimal Hölder estimates. The first results of this type were obtained by G. Henkin [253] and R. de Arellano (see [256]). We shall use the following result due to Range and Siu [408] and Lieb and Range [339, Theorem 1]; see also [356, Theorem 1'] and [338]. We denote by $\mathcal{C}_{p,q-1}^l$ the space of (p, q) -forms with coefficients of class \mathcal{C}^l .

Theorem 2.5.3. *For every bounded strongly pseudoconvex domain D with \mathcal{C}^2 boundary in a Stein manifold there exists a linear operator $T: \mathcal{C}_{0,1}^0(D) \rightarrow \mathcal{C}^{1/2}(D)$ such that, if $f \in \mathcal{C}_{0,1}^0(\bar{D}) \cap \mathcal{C}_{0,1}^1(D)$ and $\bar{\partial}f = 0$ in D then*

$$\bar{\partial}(Tf) = f, \quad \|Tf\|_{\mathcal{C}^{1/2}(\bar{D})} \leq c_D \|f\|_{\mathcal{C}_{0,1}^0(\bar{D})}.$$

The constant c_D can be chosen uniform for all domains sufficiently \mathcal{C}^2 -close to D . If D has boundary of class \mathcal{C}^ℓ for some $\ell \in \{2, 3, \dots\}$ then there exists a linear operator $T: \mathcal{C}_{0,1}^0(D) \rightarrow \mathcal{C}^0(D)$ satisfying the following properties:

- (i) If $f \in \mathcal{C}_{0,1}^0(\bar{D}) \cap \mathcal{C}_{0,1}^1(D)$ and $\bar{\partial}f = 0$ then $\bar{\partial}(Tf) = f$.
- (ii) If $f \in \mathcal{C}_{0,1}^0(\bar{D}) \cap \mathcal{C}_{0,1}^r(D)$ for some $r \in \{1, \dots, \ell\}$ then

$$\|Tf\|_{\mathcal{C}^{1,1/2}(\bar{D})} \leq C_{l,D} \|f\|_{\mathcal{C}_{0,1}^l(\bar{D})}, \quad l = 0, 1, \dots, r. \quad (2.8)$$

Although these results are stated in the original papers for domains with \mathcal{C}^∞ boundaries, one only needs \mathcal{C}^ℓ boundary to get estimates up to order ℓ ; this is implicitly contained in the paper by Michel and Perotti [356].

In [189, Theorem 3.1] the authors constructed special integral kernels for solving the $\bar{\partial}$ -equation on thin tubular neighborhoods of totally real submanifolds. The precise result is the following.

Theorem 2.5.4. *Let $M \subset \mathbb{C}^n$ be a totally real submanifold of class \mathcal{C}^1 and $c \in (0, 1)$. Denote by $M(\delta)$ the tube of radius $\delta > 0$ around M . Then there is a number $\delta_0 > 0$ and for each integer $l \geq 1$ a constant $C_l > 0$ such that the following hold for $0 < \delta \leq \delta_0$, $p \geq 0$, $q \geq 1$: For each $u \in \mathcal{C}_{p,q}^l(M(\delta))$ with $\bar{\partial}u = 0$ there is a $v \in \mathcal{C}_{p,q-1}^l(M(\delta))$ satisfying $\bar{\partial}v = u$ in $M(c\delta)$ and the following estimate for all $\alpha \in \mathbb{Z}_+^{2n}$ with $|\alpha| \leq l$:*

$$\|\partial^\alpha v\|_{L^\infty(M(c\delta))} \leq C_l \left(\delta \|\partial^\alpha u\|_{L^\infty(M(\delta))} + \delta^{1-|\alpha|} \|u\|_{L^\infty(M(\delta))} \right). \quad (2.9)$$

In particular, we have the sup-norm estimate

$$\|v\|_{L^\infty(M(c\delta))} \leq C\delta \|u\|_{L^\infty(M(\delta))}.$$

The integral kernels used to prove this theorem are of Henkin-de Arellano type, but are made especially for tubes around totally real submanifolds. This theorem was used in [189] to find optimal results on approximating diffeomorphisms between totally real submanifolds by biholomorphisms of their tubular neighborhoods (see §4.13).



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