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Annalisa Cusi, Nicolina A. Malara, and Giancarlo Navarra

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Margaret Smith

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### Teaching Algebraic Equations with Variation in Chinese Classroom

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Early Algebraization
A Global Dialogue from Multiple Perspectives
Cai, J.; Knuth, E. (Eds.)
2011, XXIV, 624 p., Hardcover
ISBN: 978-3-642-17734-7