# Contents

**Part I: Curricular Perspective**

## Preface to Part I

Jinfa Cai and Eric Knuth

3

### Functional Thinking as a Route Into Algebra in the Elementary Grades

Maria L. Blanton and James J. Kaput

5

- Introduction ........................................................................ 6
- The Challenge of Curriculum and Instruction ....................... 6
- Functional Thinking as a Route to Algebraic Thinking .......... 7
- Functional Thinking in the Elementary Grades ..................... 8
- Children’s Capacity for Functional Thinking ....................... 9
- Integrating Functional Thinking into Curriculum and Instruction . 16
- Transforming Teachers’ Resource Base to Support Students’ Functional Thinking ................................................. 17
- Using Children’s Functional Thinking to Leverage Teacher Learning .................................................................. 19
- Creating Classroom Culture and Practice to Support Functional Thinking ......................................................... 20

Conclusion ........................................................................ 20

References ......................................................................... 21

### Developing Students’ Algebraic Thinking in Earlier Grades: Lessons from China and Singapore

Jinfa Cai, Swee Fong Ng, and John C. Moyer

25

- Introduction ........................................................................ 26
- Features of the Chinese and Singaporean Curricula ............... 27
- Algebra Emphases in the Chinese and Singaporean Curricula ........................................................................... 27
- The Chinese Curriculum ...................................................... 28
- The Singaporean Curriculum ............................................. 32
<table>
<thead>
<tr>
<th>Chapter Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lessons from Chinese and Singaporean School Mathematics</td>
<td>34</td>
</tr>
<tr>
<td>Why Should Curricula Expect Students in Early Grades to Think Algebraically?</td>
<td>35</td>
</tr>
<tr>
<td>Are Young Children Capable of Thinking Algebraically?</td>
<td>36</td>
</tr>
<tr>
<td>How Can We Help Students to Think Arithmetically and Algebraically?</td>
<td>37</td>
</tr>
<tr>
<td>Are Authentic Applications Necessary for Students in Early Grades?</td>
<td>38</td>
</tr>
<tr>
<td>Conclusion</td>
<td>39</td>
</tr>
<tr>
<td>References</td>
<td>40</td>
</tr>
<tr>
<td>Developing Algebraic Thinking in the Context of Arithmetic</td>
<td>43</td>
</tr>
<tr>
<td>Susan Jo Russell, Deborah Schifter, and Virginia Bastable</td>
<td></td>
</tr>
<tr>
<td>Understanding the Behavior of the Operations</td>
<td>45</td>
</tr>
<tr>
<td>Generalizing and Justifying</td>
<td>51</td>
</tr>
<tr>
<td>1. Articulating General Claims</td>
<td>51</td>
</tr>
<tr>
<td>2. Developing a Mathematical Argument to Justify a General Claim</td>
<td>53</td>
</tr>
<tr>
<td>3. Representation-Based Proof: Tools for Proving in the Elementary Grades</td>
<td>56</td>
</tr>
<tr>
<td>Extending the Number System</td>
<td>59</td>
</tr>
<tr>
<td>Using Notation with Meaning</td>
<td>63</td>
</tr>
<tr>
<td>Connecting Arithmetic and Algebra</td>
<td>67</td>
</tr>
<tr>
<td>References</td>
<td>68</td>
</tr>
<tr>
<td>The Role of Theoretical Analysis in Developing Algebraic Thinking:</td>
<td>71</td>
</tr>
<tr>
<td>A Vygotskian Perspective</td>
<td></td>
</tr>
<tr>
<td>Jean Schmittau</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>71</td>
</tr>
<tr>
<td>Orienting Children to Theoretical Concepts</td>
<td>74</td>
</tr>
<tr>
<td>Role of Psychological Tools</td>
<td>76</td>
</tr>
<tr>
<td>The Part-Whole Relation</td>
<td>76</td>
</tr>
<tr>
<td>Concluding Remarks</td>
<td>84</td>
</tr>
<tr>
<td>References</td>
<td>85</td>
</tr>
<tr>
<td>The Arithmetic-Algebra Connection: A Historical-Pedagogical Perspective</td>
<td>87</td>
</tr>
<tr>
<td>K. Subramaniam and Rakhi Banerjee</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>87</td>
</tr>
<tr>
<td>Arithmetic and Algebra in the Indian Mathematical Tradition</td>
<td>91</td>
</tr>
<tr>
<td>Building on Students’ Understanding of Arithmetic</td>
<td>95</td>
</tr>
<tr>
<td>The Arithmetic Algebra Connection—A Framework</td>
<td>98</td>
</tr>
<tr>
<td>References</td>
<td>105</td>
</tr>
<tr>
<td>Tad Watanabe</td>
<td></td>
</tr>
<tr>
<td>School Algebra and Algebra in Early Grades</td>
<td>110</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Methodology</td>
<td>111</td>
</tr>
<tr>
<td>Algebra in Japanese Curriculum</td>
<td>112</td>
</tr>
<tr>
<td>Mathematical Expressions in Japanese Curriculum</td>
<td>114</td>
</tr>
<tr>
<td>Mathematical Expressions in Japanese Textbooks</td>
<td>114</td>
</tr>
<tr>
<td>Discussion</td>
<td>121</td>
</tr>
<tr>
<td>References</td>
<td>123</td>
</tr>
<tr>
<td><strong>Commentary on Part I</strong></td>
<td></td>
</tr>
<tr>
<td>Jeremy Kilpatrick</td>
<td></td>
</tr>
<tr>
<td>Algebra First</td>
<td>126</td>
</tr>
<tr>
<td>A Curriculum Topic</td>
<td>127</td>
</tr>
<tr>
<td>Numerical Patterns</td>
<td>128</td>
</tr>
<tr>
<td>Word Problems</td>
<td>128</td>
</tr>
<tr>
<td>Multiple Perspectives</td>
<td>129</td>
</tr>
<tr>
<td>References</td>
<td>129</td>
</tr>
<tr>
<td><strong>Part II: Cognitive Perspective</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Preface to Part II</strong></td>
<td>135</td>
</tr>
<tr>
<td>Eric Knuth and Jinfa Cai</td>
<td></td>
</tr>
<tr>
<td>Algebraic Thinking with and without Algebraic...</td>
<td>137</td>
</tr>
<tr>
<td>A Pathway for Learning</td>
<td></td>
</tr>
<tr>
<td>Murray S. Britt and Kathryn C. Irwin</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>138</td>
</tr>
<tr>
<td>Children’s Understanding of Generalities for...</td>
<td>139</td>
</tr>
<tr>
<td>Before Schooling</td>
<td></td>
</tr>
<tr>
<td>Algebraic Thinking and the New Zealand...</td>
<td>140</td>
</tr>
<tr>
<td>Students’ Algebraic Thinking in the Last Year of...</td>
<td>146</td>
</tr>
<tr>
<td>School (Age 11–12)</td>
<td></td>
</tr>
<tr>
<td>The Growth of Algebraic Thinking from Numbers to...</td>
<td>147</td>
</tr>
<tr>
<td>A Longitudinal Study</td>
<td></td>
</tr>
<tr>
<td>Discussion</td>
<td>152</td>
</tr>
<tr>
<td>A Pathway for Algebraic Thinking</td>
<td>153</td>
</tr>
<tr>
<td>References</td>
<td>157</td>
</tr>
<tr>
<td><strong>Examining Students’ Algebraic Thinking in a...</strong></td>
<td>161</td>
</tr>
<tr>
<td>A Longitudinal Study</td>
<td></td>
</tr>
<tr>
<td>Jinfa Cai, John C. Moyer, Ning Wang, and Bikai Nie</td>
<td></td>
</tr>
<tr>
<td>Standards-Based and Traditional Curricula in the...</td>
<td>162</td>
</tr>
<tr>
<td>LieCal Project</td>
<td>163</td>
</tr>
<tr>
<td>Highlights of the Differences between CMP and...</td>
<td></td>
</tr>
<tr>
<td>Curricula</td>
<td>164</td>
</tr>
<tr>
<td>Defining Variables</td>
<td>165</td>
</tr>
<tr>
<td>Defining Equations</td>
<td>165</td>
</tr>
<tr>
<td>Introducing Equation Solving</td>
<td>166</td>
</tr>
<tr>
<td>Using Mathematical Problems</td>
<td>168</td>
</tr>
</tbody>
</table>
Highlights of the Differences between CMP and Non-CMP
    Classroom Instruction .................................. 169
    Conceptual and Procedural Emphases .................. 170
    Instructional Tasks .................................... 171
Students’ Development of Algebraic Thinking: Methodological
    Considerations ........................................... 172
    The Focus of Algebraic Thinking ...................... 173
    Tasks and Data Analysis ................................ 174
Findings about the Development of Students’ Algebraic Thinking . 174
    Representing Situations ................................ 175
    Solving Equations ...................................... 177
    Making Generalizations ................................ 178
Conclusions and Instructional Implications .................. 180
References ................................................. 183

Years 2 to 6 Students’ Ability to Generalise: Models, Representations
    and Theory for Teaching and Learning .................. 187
    Tom J. Cooper and Elizabeth Warren
    Perspectives on the Mathematics of Early Algebra .... 188
    Representation and Generalisation .................... 190
        Models and Representations ......................... 191
        Generalisation ...................................... 191
        Focus of EATP ...................................... 193
        Focus of Chapter ................................... 194
    Design of EATP ......................................... 194
    Findings and Discussion ................................ 196
        Patterns ............................................. 197
        Change and Functions ............................... 198
        Equations and Equivalence ......................... 201
        Generalising Principles and Abstract Representations . 204
    Conclusions and Implications ......................... 206
        Models and Representations ......................... 206
        Generalisation ...................................... 207
        Theoretical Framework .............................. 209
    References ............................................. 211

Algebra in the Middle School: Developing Functional Relationships
    Through Quantitative Reasoning ........................ 215
    Amy B. Ellis
    What Is Quantitative Reasoning? ....................... 216
    The Importance of (and Difficulties with) Functional Thinking . 218
    An Alternative Approach to Function: Quantities and Covariation . 222
    A Flexible Understanding of Functions .................. 226
        Coordinating Covariation and Correspondence Approaches . 226
        Flexibility Across Forms ............................ 230
Fostering a Focus on Quantities ........................................... 234
References ........................................................................... 235

Representational Competence and Algebraic Modeling ............ 239
Andrew Izsák
Early Results on Students’ Understandings of Standard
Representations in Algebra .................................................... 241
Theoretical Accounts of Reasoning with External Representations . 241
Students’ Capacities to Reason with External Representations .... 243
First Result: Criteria for Evaluating External Representations .... 244
Second Result: Adaptive Interpretation .................................. 249
Conclusion ........................................................................... 253
References ........................................................................... 256

Middle School Students’ Understanding of Core Algebraic Concepts:
Equivalence & Variable ......................................................... 259
Eric J. Knuth, Martha W. Alibali, Nicole M. McNeil, Aaron Weinberg,
and Ana C. Stephens
Introduction ......................................................................... 260
Student Understanding of Equivalence & Variable ................. 261
Equivalence .......................................................................... 261
Variable .............................................................................. 262
Method ................................................................................. 262
Participants ........................................................................... 262
Data Collection ..................................................................... 263
Coding .................................................................................. 264
Results ................................................................................. 266
Interpretation of the Equal Sign ........................................... 266
Performance on the Equivalent Equations Problem ............... 267
Interpretation of a Literal Symbol ....................................... 270
Performance on the which Is Larger Problem ....................... 271
Discussion ............................................................................ 273
Equivalence Results ............................................................. 273
Variable Results ................................................................. 274
Concluding Remarks .......................................................... 275
References ........................................................................... 275

An Approach to Geometric and Numeric Patterning that Fosters Second
Grade Students’ Reasoning and Generalizing about Functions
and Co-variation ................................................................. 277
Joan Moss and Susan London McNab
Introduction ......................................................................... 277
Our Project ............................................................................ 279
Our Approach: Theoretical .................................................. 279
Instructional Sequence ......................................................... 281
Visual Representation: Geometric Growing Patterns .............. 281
Cognitive Issues Surrounding Pattern Generalization: What We Know from Various Theoretical Perspectives and Empirical Studies ........................................................................ 329

Clarifying the Definition of Pattern Generalization .......... 329
Types of Algebraic Generalization Involving Figural Patterns 330

Methodology .............................................................................. 331
Classroom Contexts from Years 1 to 3 of the Study .......... 331
Nature and Content of Classroom Teaching Experiments in Years 1 and 2 ................................................. 332
Nature and Content of Classroom Teaching Experiments in Year 3 ................................................................. 334
Nature and Content of Classroom Teaching Experiments in Years 1 and 2 ......................................................... 332
Data Collection and Analysis and Relevant Study Protocols . 335

Findings and Discussion Part 1: Accounting for Constructive and Deconstructive Generalizations .......... 338
Findings and Discussion Part 2: Understanding the Operations Needed in Developing a Pattern Generalization .......... 342
Findings and Discussion Part 3: Factors Affecting Students’ Ability to Develop CGs ........................................ 344
Findings and Discussion Part 4: A Three-Year Account of Classroom Mathematical Practices that Encouraged the Formation of Generalization Among Our Middle School Students ......................................................... 347
Year 1 Classroom Practices: From Figurally- to Numerically-Driven CSGs ......................................................... 348
Year 2 Practice: Continued Use of Numerically-Driven CSGs and a Refinement in the Case of Decreasing Linear Patterns ................................................................................. 351
Year 3 Practices: A Third Shift Back to Figural-based Generalization and the Consequent Occurrence of CSGs, CNGs, and DGs .......... 352
Findings and Discussion Part 5: Middle School Students’ Capability in Justifying CSGs ........................................ 354
Findings and Discussion Part 6: Middle School Students’ Capability in Constructing and Justifying CNGs and DGs .. 357
Conclusion ..................................................................................... 362
References ................................................................................... 363

Commentary on Part II ................................................................. 367
Bharath Sriraman and Kyeong-Hwa Lee
Introductory Remarks ................................................................. 367
Early Algebraization Versus Meaningful Arithmetic .......... 368
Generalized Arithmetic, Generalizing, Generalization .......... 369
From Haeckel to Lamarck to Early Algebraization .......... 370
References ................................................................................... 372
### Part III: Instructional Perspective

#### Preface to Part III

Eric Knuth and Jinfa Cai

377

#### Prospective Middle-School Mathematics Teachers’ Knowledge of Equations and Inequalities

Nerida F. Ellerton and M.A. (Ken) Clements

379

- The Context
- Mathematical Considerations Relating to the Teaching and Learning of Equations and Inequalities
- Student Misconceptions in Regard to Quadratic Equations
- Student Misconceptions with Regard to Linear Inequalities

- The Pre-Service Teachers Involved, and Tasks Used, in the Present Study
- “Clever” Tasks

- Developing the Pencil-and-Paper Instruments

- The Eight Equation/Algebraic Inequality Pairs

- Study Design, and Results
- Population and Sample Considerations
- Results

- Conclusions in Relation to the Prospective Teachers’ Knowledge of Algebraic Inequalities

- Prospective Teachers’ Knowledge in Relation to Quadratic Equations

- Bad News, Good News and Some Concluding Comments
- Bad News
- Good News
- Student Confidence Considerations
- Concluding Comments

References

395

401

402

406

407

#### The Algebraic Nature of Fractions: Developing Relational Thinking in Elementary School

Susan B. Empson, Linda Levi, and Thomas P. Carpenter

409

- What Is Relational Thinking?
- Use of Relational Thinking in Learning Fractions

- Understanding Fractional Quantities Through Relational Thinking

- Use of Relational Thinking to Make Sense of Operations Involving Fractions

- Discussion of Cases

- A Conjecture Concerning Relational Thinking as a Tool in Learning New Number Content
Professional Development to Support Students’ Algebraic Reasoning: An Example from the Problem-Solving Cycle Model
Karen Koellner, Jennifer Jacobs, Hilda Borko, Sarah Roberts, and Craig Schneider

Introduction
The Problem-Solving Cycle Model of Professional Development
The PSC as Implemented in the STAAR Project
Prior Research on the Development and Impact of the PSC
Impact of the PSC on Instructional Practice: A Case Study
Analysis
Methods
Ken Bryant
Data Sources
Data Analysis
Results and Discussion
Patterns Drawn from QMI Coding and Analysis
Vignette Analysis: Ken’s Skyscraper Windows Lesson
Conclusions
References

Using Habermas’ Theory of Rationality to Gain Insight into Students’ Understanding of Algebraic Language
Francesca Morselli and Paolo Boero

Introduction
Habermas’ Construct of Rational Behaviour
Adaptation of Habermas’ Construct of Rational Behavior to the Case of the Use of Algebraic Language
Epistemic Rationality
Teleological Rationality
Communicative Rationality
Relationships with Other Studies on Proving and Modeling and on the Teaching and Learning of Algebra
Proving
Modeling
Teaching and Learning of Algebra
Description and Interpretation of Student Behavior
Habermas’ Analytical Tool: Examples of Analysis of Student Behavior at Different School Levels
Habermas Analytical Tool: Analysis of a Teaching Experiment
The Context of the Study: Description of the Research Project
First Task: Choose a Number
Second Task: Representing the Game
Discussion
Theoretical Issues and Educational Strategies for Encouraging Teachers to Promote a Linguistic and Metacognitive Approach to Early Algebra

Annalisa Cusi, Nicolina A. Malara, and Giancarlo Navarra

Introduction ........................................... 483
In Europe ............................................. 484
From Traditional Algebra to Early Algebra ......... 485
Early Algebra as a Meta-Subject and the ArAl Project . 486
Socio-Constructive Teaching and Teacher Training .... 487
The Role of the Teacher’s Reflection ................. 488
The Role of the ArAl Glossary in Teacher Training .... 490
Algebraic Babbling ..................................... 492
Algebraic Babbling → Algebra as a Language ......... 493
Algebraic Babbling → Syntax, Semantics → Brioshi ... 494
Brioshi → Canonical/Non Canonical form of a
Number → ‘=’ ......................................... 495
The Multi-Commented Transcripts Methodology (MCTM) .... 496
From the Comments to a Classification of Attitudes .... 499
Example .............................................. 502
Concluding Remarks .................................. 504
References ............................................ 507

A Procedural Focus and a Relationship Focus to Algebra: How U.S. Teachers and Japanese Teachers Treat Systems of Equations

Margaret Smith

Background ............................................. 512
Algebraic Reasoning .................................. 512
TIMSS Video Studies ................................ 514
Data ..................................................... 515
Analysis ............................................... 515
Two Teachers’ Lessons ................................ 516
Discussion of Key Differences ................. 516
Conclusions ......................................... 526
References ........................................... 526

Teaching Algebraic Equations with Variation in Chinese Classroom

Jing Li, Aihui Peng, and Naiqing Song

Introduction ........................................... 529
The Source of the Data ............................... 531
Theoretical Framework .............................. 531
The Method of Research ............................ 533
Analysis of Data ..................................... 533
The Introduction of the Concept of Equation ....... 533
The Improvement of Understanding of Equation ........ 535
Equations Solving ........................................... 539
The Application of Equations ............................ 541
Discussion and Conclusion ............................... 545
Process of Teaching Algebra with Variation .......... 545
Operation of Teaching Algebra with Variation ......... 546
Final Comments .............................................. 548
References .................................................. 555

Commentary on Part III .................................... 557
John Mason
Introduction .................................................. 557
Systematics: Structure of Activity ....................... 558
What Is Algebra? ............................................ 559
What Is and What Could Be: Teaching Algebra as an Activity .. 560
Traditional Algebra Teaching ........................... 561
Envisioned Algebra Teaching ............................ 563
What Makes ‘Algebra’ Early? ............................ 566
Comparisons ................................................ 568
Transforming Algebra Teaching and Learning as an Activity ... 568
How Can Locally Successful Teaching Be Engineered for All? 569
What Is and Could Be Researched? ...................... 570
What Is Really Researched? ............................. 571
Conclusions ................................................ 574
References .................................................. 574

Overall Commentary on Early Algebraization: Perspectives for Research and Teaching .............................. 579
Carolyn Kieran
Shaping the Notion of Algebraic Thinking within Early Algebra . 580
Thinking about the General in the Particular ............ 581
Thinking Rule-Wise about Patterns ....................... 582
Thinking Relationally about Quantity, Number, and Numerical Operations ..................................... 583
Thinking Representationally about the Relations in Problem Situations ....................................... 585
Thinking Conceptually about the Procedural .......... 586
Anticipating, Conjecturing, and Justifying ............. 588
Gesturing, Visualizing, and Languaging ................ 590
The View of Algebraic Thinking that Emerges from this Volume . 591
References .................................................. 592

Author Index ................................................. 595
Subject Index ............................................... 609
Editors and Contributors ................................. 615
Early Algebraization
A Global Dialogue from Multiple Perspectives
Cai, J.; Knuth, E. (Eds.)
2011, XXIV, 624 p., Hardcover
ISBN: 978-3-642-17734-7