Functional Thinking as a Route Into Algebra in the Elementary Grades

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Abstract This chapter explores how elementary teachers can use functional thinking to build algebraic reasoning into curriculum and instruction. In particular, we examine how children think about functions and how instructional materials and school activities can be extended to support students' functional thinking. Data are taken from a five-year research and professional development project conducted in an urban school district and from a graduate course for elementary teachers taught by the first author. We propose that elementary grades mathematics should, from the start of formal schooling, extend beyond the fairly common focus on recursive patterning to include curriculum and instruction that deliberately attends to how two or more quantities vary in relation to each other. We discuss how teachers can transform and extend their current resources so that arithmetic content can provide opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships between quantities, and we examine how teachers might embed this mathematics within the kinds of socio-mathematical norms that help children build mathematical generality.

The research reported here was supported in part by a grant from the U.S. Department of Education, Office of Educational Research and Improvement to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A600007-98). The opinions expressed herein do not necessarily reflect the position, policy, or endorsement of supporting agencies. This chapter is a revised version of an article published in *ZDM—International Reviews on Mathematical Education*, *37*(1), 34–42. DOI 10.1007/BF02655895.

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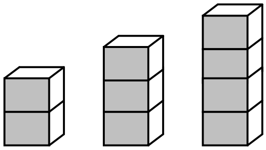
J. Cai, E. Knuth (eds.), *Early Algebraization*, Advances in Mathematics Education,
DOI 10.1007/978-3-642-17735-4_2, © Springer-Verlag Berlin Heidelberg 2011

Introduction

Current research is redefining what we understand about the kind of mathematics that young children can and should learn (National Research Council [NRC] 2001). Consider the following Towers of Cubes problem (see Fig. 1) taken from the National Council of Teachers of Mathematics [NCTM] *Principles and Standards for School Mathematics* (2000, p. 160):



What is the surface area of each tower of cubes (include the bottom)? As the tower gets taller, how does the surface area change? What is the surface area of a tower with fifty cubes?

Fig. 1 Towers of cubes

In the not so distant past, such a problem was mostly absent from typical elementary school[1] curricula and instruction in the United States. While it might have appeared as an enrichment task, it was likely marginalized by the press towards computational skills (Thompson et al. 1994) and procedures that children were (and are) compelled to memorize as a signal of their readiness for higher mathematical thinking. Or, it might have appeared in an abbreviated, arithmetic form as "What is the surface area of a tower built of 3 cubes?" However, algebraic reasoning as an activity of generalizing mathematical ideas, using literal symbolic representations, and representing functional relationships, all implicit in this task, is no longer reserved for secondary grades and beyond, but is an increasingly common thread in the fabric of ideas that constitute mathematical thinking at the elementary grades.

The Challenge of Curriculum and Instruction

Simply put, young children today need to learn a different kind of mathematics than their parents learned. Some argue that they need to be "algebra ready" (e.g., National Mathematics Advisory Panel 2008). But what experiences make them ready for algebra, and for what kind of algebra are they being made ready? Romberg and Kaput (1999) maintain that understanding the increasingly complex mathematics of the 21st century will require children to have a type of elementary school experience that goes beyond arithmetic and computational fluency to attend to the deeper underlying structure of mathematics. It will require experiences that help children learn to recognize and articulate mathematical structure and relationships and to

[1]Elementary school refers here to grades PreK-5.

use these insights *of* mathematical reasoning as objects *for* mathematical reasoning. This type of elementary school experience has come to be embodied in what many refer to as *early algebra*,[2] and because its underlying purpose is to deepen children's understanding of the structural form and generality of mathematics and not just provide isolated experiences in computation, scholars increasingly agree that it is the avenue through which young children can become mathematically successful in later grades. Thus, our perspective on "algebra readiness" is that experiences in building, expressing, and justifying mathematical generalizations—for us, the heart of algebra and algebraic thinking—should be a seamless process that begins at the start of formal schooling, not content for later grades for which elementary school children are "made ready" through a singular, myopic focus on arithmetic.

But changing the mathematics elementary school children learn—their daily curriculum—is only part of the solution. As Blanton and Kaput note, "most elementary teachers have little experience with the kinds of algebraic thinking that need to become the norm in schools and, instead, are often products of the type of school mathematics instruction that we need to replace" (2005). However, these very teachers are central to reforms in children's school mathematical experiences. Moreover, the instructional materials in most elementary schools today are basal texts, and even newer, standards-based materials are just beginning to incorporate systematic approaches to the development of algebraic reasoning (Kaput and Blanton 2005). These constraints represent the challenge of building algebraic thinking into curriculum and instruction.

There are two issues implicit in the above discussion that this article aims to address: (1) how opportunities for algebraic thinking can be integrated into the elementary grades to prepare students for more powerful mathematics in later years, and (2) how elementary teachers can transform their own resources and instruction in ways that effect (1).

Functional Thinking as a Route to Algebraic Thinking

Early algebra can occur in several interrelated forms in the classroom.[3] We focus here on *functional thinking* as a strand by which teachers can build generality into their curriculum and instruction. We broadly conceptualize functional thinking to

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[2]While there are multiple perspectives on early algebra, Lins and Kaput (2004) describe a general agreement among scholars that it involves "acts of deliberate generalization and expression of generality… [and] reasoning based on the forms of syntactically guided actions on those expressions".

[3]Kaput (2008) characterizes algebraic thinking as consisting of two core aspects: (1) making and expressing generalizations in increasingly formal and conventional symbol systems, and (2) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In turn, he argues that these core aspects cut across three longitudinal strands of school algebra: (1) Algebra as the study of structures and systems abstracted from computations and relations (e.g., algebra as generalized arithmetic); (2) Algebra as the study of functions, relations, and joint variation; and (3) Algebra as the application of a cluster of modeling languages to express and support reasoning about situations being modeled.

incorporate building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right. As the NCTM *Principles and Standards* (2000, p. 37) argues, children in the elementary grades should be able to

(1) Understand patterns, relations, and functions;
(2) Represent and analyze mathematical situations and structures using algebraic symbols;
(3) Use mathematical models to represent and understand quantitative relationships; and
(4) Analyze change in various contexts.

In addition, we use here three modes of analyzing patterns and relationships, outlined by Smith (2008), as a framework to discuss the kinds of functional thinking found in classroom data: (1) *recursive patterning* involves finding variation within a sequence of values; (2) *covariational thinking* is based on analyzing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function's description (e.g., "as x increases by one, y increases by three") (Confrey and Smith 1991); and (3) a *correspondence relationship* is based on identifying a correlation between variables (e.g., "y is 3 times x plus 2").

In what follows, we draw on data from a five-year research and professional development project in an urban school district (Kaput and Blanton 2005) and a subsequent graduate course for elementary teachers, taught by the first author, to examine how children think about functional relationships, its mathematical implications for later grades, and how instructional materials and school activities can be deepened and extended to support the development of functional thinking in the elementary grades.

Functional Thinking in the Elementary Grades

The idea of function has, for over a century, been regarded by mathematicians as a powerful, unifying idea in mathematics that merits a central place in the curriculum (Freudenthal 1982; Hamley 1934; Schwartz 1990). Indeed, the idea can be traced back to Leibniz (Boyer 1946). However, until very recently, the study of functions has been treated largely in the US as something to be learned in high school algebra, or even middle school mathematics. The perspective taken here is that the study of functions should be treated longitudinally and in its full richness beginning in early elementary school (NCTM 2000; Smith 2003).

But what capacity do young children have for functional thinking? Even though elementary school mathematics has more recently included recursive patterning, it has not attended pervasively to covariation or correspondence in functional thinking, especially in grades PreK-2. For instance, even NCTM (2000) suggests that, as late as fourth-grade, students might find a recursive pattern in the Towers of Cubes problem (see Fig. 1), and not until fifth-grade would they develop a correspondence relationship. Can elementary students, in fact, make the conceptual shift from simple recursive patterning to account for simultaneous changes in two or more variables? Moreover, at what grades can they do this? And can they, or in what ways can they, symbolize and operate on covariational or correspondence relationships in data?

Children's Capacity for Functional Thinking

Current research challenges the developmental constraints traditionally placed on young learners and their capacity for functional thinking. For example, researchers have found that elementary school children can develop and use a variety of representational tools to reason about functions, they can describe in words and symbols recursive, covarying, and correspondence relationships in data, and they can use symbolic language to model and solve equations with unknown quantities (e.g., Blanton 2008; Brizuela and Schliemann 2003; Brizuela et al. 2000; Carraher et al. 2008; Kaput and Blanton 2005; Moss et al. 2008; Schliemann and Carraher 2002; Schliemann et al. 2001).

While much of this research focuses on functional thinking in grades 3–5, we have found that students are not only capable of deeper functional analysis than previously thought, but that the genesis of these ideas appear at grades earlier than typically expected. In particular, we have found that the types of representations students use, the progression of mathematical language in their descriptions of functional relationships, the ways students track and organize data, the mathematical operations they use to interpret functional relationships, and how they express covariation and correspondence among quantities, can be scaffolded in instruction beginning with the very earliest grades, at the start of formal schooling (Blanton and Kaput 2004).

The following discussion draws on our research data to elaborate these capacities in children's functional thinking across elementary grades. We note that the data included here are intended to convey existence proofs of what is possible in children's thinking; our goal is not to examine the regularity with which functional thinking occurred in instruction.

The Development of Representational Infrastructure: Children's Use of Function Tables

Research, including early algebra research, suggests that students' flexibility with multiple representations both reflects and promotes deeper mathematical insights (Behr et al. 1983; Brizuela and Earnest 2008; Goldin and Shteingold 2001). Brizuela and Earnest note that "the connections between different representations help to resolve some of the ambiguity of isolated representations, [so] in order for concepts to be fully developed, children will need to represent them in various ways".

We found that teachers across the elementary grades were able to scaffold children's use of tables, graphs, pictures, words and symbols in gradually more sophisticated ways in order for them to make sense of data and interpret functional relationships (Blanton and Kaput 2004). For example, while students in grades PreK-1

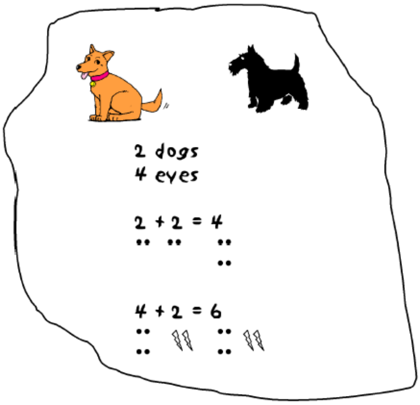


Fig. 2 Kindergarten students' representation for the numbers of eyes and eyes and tails for two dogs

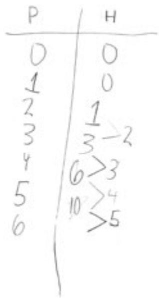


Fig. 3 A first-grader's t-chart for the Handshake Problem[5]

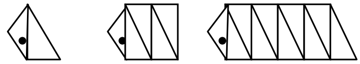
relied on counting visible objects or hand-written marks and registering their counts through inscriptions in t-charts[4] or through dots and hatch marks for eyes and tails (see Figs. 2 and 3), by second and third grade, students could routinely operate on data that had no iconic or tangible counterpart (e.g., tracking the number of eyes on ten dogs without pictures or physical objects). Moreover, while grades PreK-1 teachers typically led students in developing t-charts to organize their data, the responsibility for this began to shift to students during first grade. Figure 3 shows a t-chart, constructed by a first-grader, which records the total number of handshakes in a group of varying size (Blanton 2008).

We have found that the t-chart, or function table, becomes an important structure in children's mathematical reasoning. In the earlier grades (PreK-1), it provided a context to re-represent marks with numerals as children worked on the arithmetic of correspondence between quantity and numeral. But its introduction in these grades

[4]T-charts are teacher-termed function tables with a column of data for the independent variable followed by a column of data for the dependent variable.

[5]The Handshake Problem can be stated as follows: If 3 people are in a group, how many total handshakes would there be if every person shook hands with all people in the group once? How many handshakes would there be if there were 4 people in the group? Five people? Six people? Twenty people? Can you find a relationship between the number of people in the group and the total number of handshakes?

Fig. 4 The growing snake



as a tool for organizing covarying data also initiated its transformation from opaque to transparent object (Kaput 1995) in children's functional thinking as a representation that one could "look through" to "see" new relationships. The first grader's analysis of differences in the numbers of handshakes in Fig. 3 illustrates that as early as first grade, students can begin to transition beyond an understanding of t-charts as opaque objects—a place to record numbers—to a transparent object that can be used to determine relationships in data. We maintain that introducing such representational tools from the start of formal schooling can help spread the cognitive load across grades in a way that allows students in second and third grades (and beyond) to focus on more difficult tasks such as symbolizing correspondence and covariational relationships.

By second and third grades, we have found that students are able to use this tool transparently, as a mathematical object, in thinking about data. The following teacher narrative illustrates this algebraic reasoning with third-graders. The third-grade teacher who authored the narrative, Mrs. Gardiner, had designed a task in which students were to find the number of body parts a growing snake would have on day 10 and on day n, where each triangle equaled a body part. She drew the growing snake on the board for Days 1, 2, and 3 (see Fig. 4).

> The class worked on this problem for approximately 10 minutes. All organized their data with a t-chart. When I pulled the group together to discuss the problem, it was Karlie[6] who had her hand waving hard.... Karlie usually just sits and listens during math time, so her enthusiasm was very special. I called on her right away. 'I know that on day 10 the snake will have 101 body parts and I know that on day n the snake will have $n \times n + 1$. I know this because I used my t-chart and I looked for the relationship between n and body parts. This is the first time I saw the pattern, so please tell me I'm right!' she said excitedly.... The class had all come to pretty much the same answer.

This suggests to us that the t-chart helped structure Karlie's thinking about relationships between quantities. Unlike in the earlier elementary grades, where students were more likely to use t-charts opaquely as a storage system for numbers and were not yet able to attend to the meanings embedded in how data were positioned in the chart, the t-chart became the object, or tool, by which Karlie could compare data and find relationships. She was able to attend to how numbers were located in the chart and see through it to the relationships it made available to her. In this sense, we maintain that the t-chart had become transparent in how she used it to think about functions. Our point is that critical instruction in the earlier grades (PreK-1) can initiate the transition of representational tools from opaque to transparent objects in children's thinking so that children are able to shift their attention to more complex tasks in later elementary grades and beyond. This is exactly how

[6] All student names are pseudonyms.

mathematics has grown in power historically, as new representation systems were developed (including that of algebra itself) to increase the power of human thinking.

The Development of Students' Symbol Sense

One particularly vital aspect of early algebra is the transition from natural language to symbolic notational systems. If one's perspective is that development precedes learning, then the use of symbols as variables in elementary grades is, perhaps, not without controversy. However, we take the view here that learning promotes development and that it entails a pseudo-conceptual stage of concept formation in students' development of symbol sense. In describing the development of higher mental functioning in children, Vygotsky (1962) identified the notion of a *pseudo-concept* as an essential bridge in children's thinking to the final stage of concept formation. While the pseudo-concept a child possesses is phenotypically equivalent to that of an adult, it is psychologically different. As a result, the child is able to "operate with [the concept], to practice conceptual thinking, before he is clearly aware of the nature of these operations" (p. 69). This suggests that learning to think mathematically involves the acquisition of notational tools that are within the child's zone of proximal development, but not entirely owned by the child. In essence, it involves students' transition from an opaque to transparent use of symbols. Moreover, the dialectic between thought and language in learning (Vygotsky 1962) implies that symbolic notational systems are more fully conceptually formed in children's thinking as a result of children's interaction with them in meaningful contexts. In short, children can develop symbol sense as they have opportunity to use symbolic notation in meaningful ways (see also Brizuela et al. 2000).

We have found that, when curriculum and instruction provide opportunity for thinking about functional relationships, students can transition linguistically from iconic and natural language registers at grades PreK-1 to symbolic notational systems by grade 3 (Blanton and Kaput 2004). A first grade teacher described how one of her students made this transition while thinking about the Handshake Problem:

> I asked, 'Can I label one side [of the t-chart] 'people' and the other side 'handshake'?' One little boy said, 'Just write 'p' and 'h'.' I immediately stopped what I was doing. I asked, 'What did you say?' He continued to repeat what I heard him say. 'Awesome, how did you come up with that?' I probed. He continued, 'Well, 'people' begins with p and 'handshakes' begin with h.' (Blanton 2008, p. 43)

While this student's understanding of variable is certainly in its early stages (for example, care must be taken to ensure that the student does not confuse the variable as representing the object and not the quantity), the point here is that giving children opportunities to *begin* using symbolic representations can occur as early as first grade, and acquiring these more basic ideas in the early grades allows them greater cognitive room to explore more complex ideas in later elementary grades.

By third-grade, students can move beyond this more primitive act of symbolizing to describe and discuss functional relationships. We include the following teacher narrative to illustrate third-grade students using symbolic notation to think about the number of circle-shaped body parts on a growing caterpillar. In this vignette, Mrs. Gardiner has just described the Growing Caterpillar task to her students.[7]

> I showed my students my caterpillar example and all I wanted them to see was how I developed the problem. I had no idea that they would begin to solve the problem. I couldn't stop them. There were hands going up all over the place. Everyone wanted to tell me the pattern they saw when they looked at the growth of the caterpillar. I said, 'Guys, I haven't even asked you the question yet'. 'But I see the pattern, Mrs. Gardiner', yelled Jak. 'Okay, what do you think the pattern is?' I asked. 'I think it is x times 2 plus one', he said. 'How many of you agree with Jak?' I questioned. 'I don't know. I have to do a t-chart', explained Meg. 'Well, then let's do that together on the board', I said. With the students' help, we drew the following t-chart on the board (see Fig. 5):

Fig. 5 T-chart for growing caterpillar

x	y
1	2
2	5
3	10
4	17

> 'Now that we have that on the board, I don't agree with Jak', said Meg. 'Why is that Meg?' I asked. 'Because if it was x times 2 plus 1, then x would be one and y would be three. And, it's not. It's $x = 1$ and $y = 2$', she explained.... [If x equaled 1, then by Jak's formula, y would be $2(1) + 1$, or 3, not 2, as the t-chart indicated.] The class struggled with the pattern for a long time. Then Shane saw a pattern that I had not seen. He came up to the t-chart on the board and with a red marker highlighted the pattern. It looked like this (see Fig. 6):

Fig. 6 Shawn's Pattern for Growing Caterpillar task

x	y
1	2
2	5
3	10
4	17

> So, what Shane was saying is that if you add $1+2+2$, it equals 5. If you then add $2+5+3$, it equals 10. This ... didn't help him find the formula, but it did help Joe! 'I see it, I know

[7]Growing Caterpillar was similar to Growing Snake except for the shape of the body parts. The growth rates for the snake and caterpillar were the same. Mrs. Gardiner had given Growing Caterpillar to students two weeks prior to Growing Snake.

> the formula!' Joe cried out. 'Well, what is it?' I prodded. 'It's $x \times x + 1 = y$', he said. At that moment, a loud group of 'Oh yeah's' could be heard in the room.... I asked everyone why this was algebra. I think Jak put it best. He said, 'Because we have people looking for patterns and relationships and we have them developing a formula'.

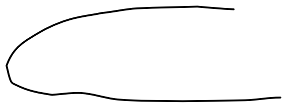
There are several points with respect to students' use of symbols that bear mentioning here. First, before any data were publicly recorded and without any prompting from the teacher, Jak proposed a symbolic relationship between an arbitrary day, x, and the number of caterpillar body parts. His spontaneous use of symbols conveys the generality with which he was beginning to think about functional relationships. Second, Meg was beginning to reason transparently with the t-chart and the symbolic relationship conjectured by Jak in order to refute his idea ("Now that we have that on the board, I don't agree with Jak"). That is, implicit in her refutation was her reasoning with both the meaning embedded in the structure of the t-chart, including the unique roles of dependent and independent variables, as well as the symbolic notation ("Because if it was x times 2 plus 1, then x would be one and y would be three. And, it's not. It's $x = 1$ and $y = 2$"). Meg's emerging transparent use of symbolic notation (as well as her evident understanding of equality, another critical issue in the development of children's algebraic thinking) is further indicated by her treatment of the expression 'x times 2 plus 1' and the dependent variable, y, as equivalent quantities.

Because the elementary grades often incorporate meaningful imagery and concrete experiences to support conceptual development, they, more so than secondary grades, can provide a rich, inquiry-based atmosphere for introducing symbolic notation. Thus, as with the development of representational infrastructure, we maintain that instruction should begin to scaffold students' thinking toward symbolic notation from the start of formal schooling so that students can transition from an opaque to transparent use of symbols as they progress through the elementary grades. Ultimately, elementary students who have learned to reason symbolically in meaningful ways will be much better prepared for the abstractions of more advanced mathematical thinking in later grades.

The Emergence of Thinking About Covariational and Correspondence Relationships

We have found it particularly compelling that, even as early as kindergarten, children can think about how quantities co-vary and, as early as first grade, can describe how quantities correspond (Blanton and Kaput 2004). For instance, in the task Cutting String (Blanton 2008; see also Cramer 2001), children are asked to look for a relationship between the number of cuts on a piece of string and the resulting number of pieces of string when the string is folded in a single loop (see Fig. 7). First graders were able to describe the relationship not only in recursive terms ("It gets two more each time"), but also in terms of a co-varying relationship "Every time you make one more snip it's two more" (Blanton 2008).

Fig. 7 Folded piece of string



In a task in which students were asked to describe the total number of eyes or the total number of eyes and tails for an increasing number of dogs, one kindergarten class described an additive covariational relationship between the numbers of eyes and dogs as "every time we add one more dog we get two more eyes". In first and second grade, students identified a multiplicative relationship of "doubles" and "triples" between the number of eyes and the number of eyes and tails, respectively, for an increasing number of dogs. The observation that the pattern "doubles" or "triples" suggests that students could attend to how quantities corresponded. For example, some quantity (in particular, the independent variable) needed to be doubled to get the total amount of eyes. Since data representing the total number of eyes (i.e., 2, 4, 6, 8...) were not doubled to get subsequent quantities of dog eyes (4 doubled does not yield the next value of 6; 6 doubled does not yield the next value of 8), this suggests that students were not looking for a recursive pattern such "add 2 every time" or "count by 2's", but a relationship between two quantities.

We recognize that some children might be responding to a known relationship without fully understanding its functional aspect. "Doubles", for example, is not uncommon in the vocabulary of early grades mathematics, and to say "it doubles" does not necessarily indicate a full conceptual understanding of correspondence or covariation, including an explicit understanding that the value of the independent variable is being doubled to obtain the value of the dependent variable. For some children, "doubles" could be code for a pattern recognized as adding by two's. However, these situations can prompt discussions that scaffold students' thinking about relationships between data, not just recursive patterning.

As the Growing Snake and Growing Caterpillar excerpts suggest, by third grade students can attend to how quantities co-vary and, moreover, symbolize relationships as a functional correspondence (e.g., "It's $x \times x + 1 = y$"). Thus, although the data on cutting string and dog eyes and tails illustrate a simple mathematical relationship for which some children used only natural language to describe covariational and correspondence relationships, we think this represents the critical kinds of experiences that children need in the earlier elementary grades in order to leverage deeper, more complex functional thinking in later elementary grades and beyond.

Implications of Children's Functional Thinking for Later Grades

The preceding discussion underscores how early algebra, and functional thinking in particular, can nurture the development of students' mathematical thinking in later grades. To begin with, it can help children build critical representational and linguistic tools for analyzing, describing and symbolizing patterns and relationships. Moreover, if teachers scaffold these ideas from the start of formal schooling, these experiences can provide a continuum of mathematical development whereby opaque symbols and tools can be transformed into transparent objects of functional thinking. T-charts and graphs become not just visual configurations, but structures embedded with meaning about relationships; symbols are no longer meaningless abstract inscriptions, but tools by which broader ideas can be mediated and communicated. Moreover, the elementary grades, because of its inclination towards concrete, tactile, and visual experiences in learning, can bridge the expression of mathematical ideas from natural, everyday language to symbolic notational systems in meaningful ways. For example, students in secondary grades are often given, *a priori*, a symbolic generalization about the commutative property of addition for real numbers a and b ($a + b = b + a$). In contrast, early algebra entails exploring this property through operations on particular numbers, then generalizing the property using everyday or symbolic language systems, where the symbolizing develops as a valid linguistic form of expression through children's interactions with number and operation (Carpenter et al. 2003).

All of these experiences—the development of representational and linguistic tools, the transformation of mathematical structures and symbols from opaque to transparent objects, and the integration of concrete, tactile, and visual experiences to support the development of mathematics with understanding—coalesce to build mathematical thinkers for whom abstract ideas are rooted in meaningful, concrete events. As a result, we argue that children for whom functional thinking is a routine part of mathematics in the elementary grades are better prepared than those who spend the first six or seven years of formal schooling fine-tuning arithmetic skills, procedures and facts.

Integrating Functional Thinking into Curriculum and Instruction

While much more could be said about children's capacity for functional thinking, our point thus far is that young children can identify and express functional relationships in progressively more symbolic ways and that instruction in the elementary grades that nurtures this kind of thinking can support students' mathematical thinking in later grades. Although this suggests a mandate for change in elementary school curricula, our reality is often working with teachers who have limited resources that, more often than not, focus on the development of children's arithmetic thinking. Moreover, curricular innovations alone, without the development of teachers' instructional and mathematical knowledge on how to build children's functional thinking, are not sufficient to produce real change in children's mathematical thinking. Smith (2003) notes that "elementary school teachers may create rich classroom experiences around patterns, yet not have a sense of how this topic ties into the ongoing mathematical development of their students, much less into the topic of functions" (p. 136). To address this, our early algebra work with teachers has involved three connected dimensions of change: (1) transforming teachers' instructional resource base, (2) using children's thinking to leverage teacher learning, and (3) creating classroom culture and practice to support algebraic thinking. In what follows, we address each of these and how they support the integration of functional thinking into curriculum and instruction.

Transforming Teachers' Resource Base to Support Students' Functional Thinking

In spite of limited resources or the lack of materials that integrate functional thinking in viable ways, elementary teachers can transform their existing instructional resource base to include the exploration of covariational and correspondence relationships. Our approach with teachers is to help them deliberately transform single-numerical-answer arithmetic problems to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships by varying the given parameters of a problem (Blanton and Kaput 2003). This is easily done with tasks such as the Telephone Problem, which might typically be posed as an arithmetic task with a single numerical answer:

How many telephone calls could be made among 5 friends if each person spoke with each friend exactly once on the telephone?

Stated this way, students simply need to compute a sum, although they might first draw a picture or diagram to keep track of the phone calls. Functional thinking can be introduced into the task by varying the number of friends in the group:

How many telephone calls would there be if there were 6 friends? Seven friends? Eight friends? Twenty friends? One hundred friends? Organize your data in a table. Describe any relationship you see between the number of phone calls and the number of friends in the group. How many phone calls would there be for *n* friends?

The tasks included here (e.g., Growing Snake, Growing Caterpillar, Towers of Cubes) are examples of this type of transformation; all are derived from single-numerical-answer tasks. For example, Towers of Cubes can be seen as an extension of the arithmetic problem "What is the surface area of a tower built of 3 one-inch cubes?" Similarly, Growing Snake can be seen as an extension of an arithmetic task in which students count the total number of body parts for a particular snake.

Varying Task Parameters Introduces Algebraic Thinking into the Curriculum

But how does this transformation lead to algebraic thinking or, specifically, functional thinking? First, varying a problem parameter enables students to generate a set of data that has a mathematical relationship, and using sufficiently large quantities for that parameter leads to the algebraic use of number. For example, in the Telephone Problem, finding the number of phone calls for a group whose size is large enough so that children cannot (or would not want to) model the problem and write down a corresponding sum to compute requires children to think about the structure in the numbers and how the numbers of phone calls for the various groups are related to the number of people in the group. From their analysis, children can identify a recursive pattern or conjecture a covariational or correspondence relationship between the total numbers of phone calls and the variations in the parameter that produces them. Moreover, depending on the grade and skill of the student, the teacher can scaffold students in describing their conjectures with symbolic notation. Children can then develop justifications for whether or not their conjectured relationships and patterns hold true. Finally, the mathematical generalizations that result, while important results in and of themselves, can become objects of mathematical reasoning as students become more sophisticated algebraic thinkers (Blanton 2008; Blanton and Kaput 2000). None of these processes occur if tasks remain purely arithmetic in scope.

As we describe elsewhere, our approach "recasts elementary mathematics in a profound way, not by ignoring its computational agenda, but by enlarging the agenda in ways that include the old in new forms that deliberately contextualize, deepen, and leverage the learning of basic skills and number sense by integrating them into the formulation of deeper mathematical understandings" (Kaput and Blanton 2005). In essence, a powerful result of transforming arithmetic tasks in this way is that children are doing many important things all at once, including building number sense, practicing number facts, building and recognizing patterns to model situations, and so forth (Blanton and Kaput 2003). In fact, this genre of tasks can provide large amounts of computational practice in a context that intrigues students and that avoids the mindlessness of numerical worksheets.

Transforming the Curriculum Empowers Teachers

Moreover, we have found that when teachers transform their own instructional resource base so that arithmetic tasks are extended to include opportunities for establishing and expressing mathematical generalizations, they are able to transcend constraints imposed by their existing school culture such as limited or inadequate resources, or even their own lack of experience with teaching algebraic thinking. Instead, they are able to see algebraic thinking as a fluid domain of thinking which permeates all of mathematics, not as a set of tasks or a prescribed curriculum. Thus, what we advocate, more so than an "early algebra curriculum" per se, involves the development of a habit of mind that transcends the particular resource being used and allows elementary teachers to see opportunities for algebraic thinking, and functional thinking in particular, in the mathematics they already teach, using the curriculum they have in place. After using only two functional thinking tasks with her students, one third-grade teacher wrote

> I had a new outlook on math. I knew I wanted to integrate algebraic thinking into every topic I did. The truth was that our curriculum was wonderful. It allowed plenty of ways to integrate this way of thinking. I just hadn't noticed up to this point (Blanton 2008, p. xii).

This sense of empowerment, as well as the development of an algebraic habit of mind, was later echoed by a first-grade teacher:

> [Functional thinking] activities at the beginning seemed like they were going to be hard to do, never mind creating my own. I've realized that they are a lot simpler to create and implement than I thought. I am really impressed with how these activities have shaped my way and my students' way of thinking algebraically. They have really opened my mind up about algebra and how, if we put it into a simple form, our students can do it! (Blanton 2008, p. 147)

Using Children's Functional Thinking to Leverage Teacher Learning

Integrating functional thinking into instruction does not rest solely on the particular materials the teacher chooses or develops. It requires an "algebra sense" by which teachers can identify occasions in children's thinking to extend conversations about arithmetic to those that explore mathematical generality. While the task one chooses can certainly support this, teachers also need the skills to interpret what children are writing about and talking about. In turn, a written or verbal record of student thinking can serve as a tool to engage teachers in thinking about content and practice. As teachers think collectively about how children make sense of data, whether and how they attend to how quantities relate, the kinds of meaning they derive from tables and graphs, and how they use symbols in describing and reasoning with mathematical ideas, they have the potential to build functional thinking into instruction in deeper and more compelling ways (Kaput and Blanton 2005).

The work of Cognitively-Guided Instruction (Carpenter and Fennema 1999) has been significant in bringing student thinking to the fore in how people conceptualize and engage in teacher professional development. More recently, researchers have extended this approach as a tool in the development of teachers' early algebraic thinking (Franke et al. 2001; Kaput and Blanton 2005). The assumption is that focusing on children's (algebraic) thinking in professional development builds teachers' capacity to identify classroom opportunities for generalization and to understand the representational, linguistic and symbolic tools that support this and the particular ways students use these to reason algebraically. Thus, if teachers are to build algebraic thinking into their instruction, they must become engaged in and by what students are saying, doing and writing as a catalyst for building their own classroom algebraic discourse. Moreover, they must be given occasions to use these classroom artifacts to negotiate mathematical and instructional knowledge within teacher communities of practice as a way to develop their *own* knowledge of algebra and teaching algebra. One fourth-grade teacher described her early experience in leading this work with her teacher peers:

> At our last professional development day, I told the other teachers that I have been doing algebra problems during my math workshop time. I told them the types of problems we did and how I have been implementing the problems in class. I told them it was a great way to get kids to look at numbers in different ways. I explained how it was more than algebra; it also helps kids practice basic arithmetic. I showed them samples of students' work. I even explained the importance of organizing data, finding a recursive pattern and finding a function. I talked so confidently about algebra that the teachers were intrigued. For the first time in my life, I was a math teacher! (Blanton 2008, p. 147)

Creating Classroom Culture and Practice to Support Functional Thinking

Building on children's functional thinking in instruction requires that a culture of practice that promotes this type of thinking exists. Classrooms in which children's functional thinking can thrive are those in which the teacher has established socio-mathematical norms of conjecturing, arguing, and generalizing in purposeful ways, where the arguments are taken seriously by students as ways of building reliable knowledge. Robust functional thinking requires children to interact with complex mathematical ideas, to negotiate new notational systems and to understand and use representational tools as objects for mathematical reasoning. It requires that the teacher respect and encourage these processes as standard practice on a daily basis, not as occasional enrichment treated as separate from the "regular" work of learning and practicing arithmetic.

The teacher narratives included here illustrate the kinds of classroom practice and culture that can support the development of children's functional thinking. For example, the Growing Caterpillar narrative depicts ways of doing mathematics in which the teacher (1) followed students' thinking in shaping a lesson's agenda ("I showed my students my caterpillar example and all I wanted them to see was how I developed the problem. I had no idea that they would begin to solve the problem. I couldn't stop them".), (2) placed the responsibility for conjecture, argumentation and justification with students ("Okay, what do you think the pattern is?", "How many of you agree with Jak?", "Why is that Meg?"), (3) cultivated children's use of representational structures as tools for reasoning (Meg: "I don't know. I have to do a t-chart"), (4) encouraged the use of symbolic notational systems as valid forms of mathematical expression (Jak: "I think it is x times 2 plus one"; Meg: "Because if it was x times 2 plus 1, then x would be one and y would be three. And, it's not. It's $x = 1$ and $y = 2$"; Joe: "I see it, I know the formula!.... It's $x \times x + 1 = y$"), and (5) used children's utterances to craft an idea-building, dialogic discourse that led to symbolizing a functional relationship. In short, these aspects of practice allowed children to construct a mathematical generalization about the caterpillar's growth.

Children's role in this process is critical; we are not advocating a form of practice in which children do not actively participate in the development of conjectures, the construction of arguments, the establishment of generalizations, or the use of notation, language, and tools for reasoning about functions. All of these experiences are critical components of the kind of classroom culture that makes functional thinking viable when it does occur.

Conclusion

This chapter elaborates the position that elementary school children are capable of functional thinking and that its study in the elementary grades can affect their success in mathematics in later grades. We propose that elementary grades mathematics extend beyond the fairly common, initial focus on recursive patterning to include curriculum and instruction that deliberately attends to how two or more quantities vary in relation to each other and that begins to scaffold these notions from the start of formal schooling. Because there is a fundamental conceptual shift that must occur in how teachers and students attend to data in recursive patterning as opposed to covariational or correspondence relationships, we speculate that the emphasis on recursive patterning that does occur in the early elementary grades curricula, could, if taught in isolation, impede the development of covariational and correspondence thinking about functions in later grades.

Children's capacity for functional thinking raises the issue of how it might be nurtured by curriculum and instruction in the elementary grades. We advocate here a habit of mind, not just curricular materials, whereby teachers understand both how to transform and extend their current resources so that the mostly arithmetic content of the elementary grades can be extended to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships and how to embed this mathematics within the kinds of socio-mathematical norms that allow children to build mathematical generality. Generalizing is a human activity and an innate, natural capacity that young children bring to the classroom (Mason 2008). Curriculum and instruction should build on these natural abilities to provide a deeper, more compelling mathematical experience for young children.

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Early Algebraization
A Global Dialogue from Multiple Perspectives
Cai, J.; Knuth, E. (Eds.)
2011, XXIV, 624 p., Hardcover
ISBN: 978-3-642-17734-7