

Contents

1	Pappos's Theorem: Nine Proofs and Three Variations	3
1.1	Pappos's Theorem and Projective Geometry	4
1.2	Euclidean Versions of Pappos's Theorem	6
1.3	Projective Proofs of Pappos's Theorem	13
1.4	Conics	19
1.5	More Conics	22
1.6	Complex Numbers and Circles	24
1.7	Finally...	29

Part I Projective Geometry

2	Projective Planes	35
2.1	Drawings and Perspectives	36
2.2	The Axioms	40
2.3	The Smallest Projective Plane	43
3	Homogeneous Coordinates	47
3.1	A Spatial Point of View	47
3.2	The Real Projective Plane with Homogeneous Coordinates	49
3.3	Joins and Meets	52
3.4	Parallelism	55
3.5	Duality	56
3.6	Projective Transformations	58
3.7	Finite Projective Planes	64
4	Lines and Cross-Ratios	67
4.1	Coordinates on a Line	68
4.2	The Real Projective Line	69
4.3	Cross-Ratios (a First Encounter)	72
4.4	Elementary Properties of the Cross-Ratio	74

5 Calculating with Points on Lines..... 79

5.1 Harmonic Points 80

5.2 Projective Scales 82

5.3 From Geometry to Real Numbers..... 83

5.4 The Fundamental Theorem 86

5.5 A Note on Other Fields 88

5.6 Von Staudt’s Original Constructions 89

5.7 Pappos’s Theorem 91

6 Determinants 93

6.1 A “Determinantal” Point of View..... 94

6.2 A Few Useful Formulas..... 95

6.3 Plücker’s μ 96

6.4 Invariant Properties 99

6.5 Grassmann-Plücker relations 102

7 More on Bracket Algebra 109

7.1 From Points to Determinants... 109

7.2 ... and Back 112

7.3 A Glimpse of Invariant Theory 115

7.4 Projectively Invariant Functions 120

7.5 The Bracket Algebra 121

Part II Working and Playing with Geometry

8 Quadrilateral Sets and Liftings 129

8.1 Points on a Line 129

8.2 Quadrilateral Sets 131

8.3 Symmetry and Generalizations of Quadrilateral Sets 134

8.4 Quadrilateral Sets and von Staudt 136

8.5 Slope Conditions 137

8.6 Involutions and Quadrilateral Sets 139

9 Conics and Their Duals 145

9.1 The Equation of a Conic 145

9.2 Polars and Tangents 149

9.3 Dual Quadratic Forms 154

9.4 How Conics Transform 156

9.5 Degenerate Conics 157

9.6 Primal-Dual Pairs 159

10 Conics and Perspectivity 167

10.1 Conic through Five Points 167

10.2 Conics and Cross-Ratios 170

10.3 Perspective Generation of Conics 172

10.4 Transformations and Conics 175

10.5 Hesse’s “Übertragungsprinzip” 179

10.6 Pascal’s and Brianchon’s Theorems 184

10.7 Harmonic points on a conic 185

11 Calculating with Conics 189

11.1 Splitting a Degenerate Conic 190

11.2 The Necessity of “If” Operations 193

11.3 Intersecting a Conic and a Line 194

11.4 Intersecting Two Conics 196

11.5 The Role of Complex Numbers 199

11.6 One Tangent and Four Points 202

12 Projective d -space 209

12.1 Elements at Infinity 210

12.2 Homogeneous Coordinates and Transformations 211

12.3 Points and Planes in 3-Space 213

12.4 Lines in 3-Space 216

12.5 Joins and Meets: A Universal System 219

12.6 . . . And How to Use It 222

13 Diagram Techniques 227

13.1 From Points, Lines, and Matrices to Tensors 228

13.2 A Few Fine Points 231

13.3 Tensor Diagrams 232

13.4 How Transformations Work 234

13.5 The δ -tensor 236

13.6 ε -Tensors 237

13.7 The ε - δ Rule 239

13.8 Transforming ε -Tensors 241

13.9 Invariants of Line and Point Configurations 245

14 Working with diagrams 247

14.1 The Simplest Property: A Trace Condition 248

14.2 Pascal’s Theorem 250

14.3 Closed ε -Cycles 252

14.4 Conics, Quadratic Forms, and Tangents 256

14.5 Diagrams in \mathbb{RP}^3 259

14.6 The ε - δ -rule in Rank 4 262

14.7 Co- and Contravariant Lines in Rank 4 263

14.8 Tensors versus Plücker Coordinates 265

15 Configurations, Theorems, and Bracket Expressions 269

15.1 Desargues’s Theorem 270

15.2 Binomial Proofs 272

15.3 Chains and Cycles of Cross-Ratios 277

15.4 Ceva and Menelaus 279

15.5 Gluing Ceva and Menelaus Configurations 285
 15.6 Furthermore 291

Part III Measurements

16 Complex Numbers: A Primer 297
 16.1 Historical Background 298
 16.2 The Fundamental Theorem 301
 16.3 Geometry of Complex Numbers 302
 16.4 Euler’s Formula 304
 16.5 Complex Conjugation 307

17 The Complex Projective Line 311
 17.1 \mathbb{CP}^1 311
 17.2 Testing Geometric Properties 312
 17.3 Projective Transformations 315
 17.4 Inversions and Möbius Reflections 320
 17.5 Grassmann-Plücker relations 322
 17.6 Intersection Angles 324
 17.7 Stereographic Projection 326

18 Euclidean Geometry 329
 18.1 The points I and J 330
 18.2 Cocircularity 331
 18.3 The Robustness of the Cross-Ratio 333
 18.4 Transformations 334
 18.5 Translating Theorems 338
 18.6 More Geometric Properties 339
 18.7 Laguerre’s Formula 342
 18.8 Distances 345

19 Euclidean Structures from a Projective Perspective 349
 19.1 Mirror Images 350
 19.2 Angle Bisectors 351
 19.3 Center of a Circle 354
 19.4 Constructing the Foci of a Conic 356
 19.5 Constructing a Conic by Foci 360
 19.6 Triangle Theorems 362
 19.7 Hybrid Thinking 368

20 Cayley-Klein Geometries 375
 20.1 I and J Revisited 376
 20.2 Measurements in Cayley-Klein Geometries 377
 20.3 Nondegenerate Measurements along a Line 379
 20.4 Degenerate Measurements along a Line 386

20.5	A Planar Cayley-Klein Geometry	389
20.6	A Census of Cayley-Klein Geometries	393
20.7	Coarser and Finer Classifications	398
21	Measurements and Transformations	399
21.1	Measurements vs. Oriented Measurements	400
21.2	Transformations	401
21.3	Getting Rid of X and Y	407
21.4	Comparing Measurements	408
21.5	Reflections and Pole/Polar Pairs	413
21.6	From Reflections to Rotations	419
22	Cayley-Klein Geometries at Work	423
22.1	Orthogonality	424
22.2	Constructive versus Implicit Representations	427
22.3	Commonalities and Differences	429
22.4	Midpoints and Angle Bisectors	431
22.5	Trigonometry	437
23	Circles and Cycles	443
23.1	Circles via Distances	444
23.2	Relation to the Fundamental Conic	446
23.3	Centers at Infinity	448
23.4	Organizing Principles	450
23.5	Cycles in Galilean Geometry	459
24	Non-Euclidean Geometry: A Historical Interlude	465
24.1	The Inner Geometry of a Space	466
24.2	Euclid's Postulates	468
24.3	Gauss, Bolyai, and Lobachevsky	470
24.4	Beltrami and Klein	474
24.5	The Beltrami-Klein Model	476
24.6	Poincaré	479
25	Hyperbolic Geometry	483
25.1	The Staging Ground	483
25.2	Hyperbolic Transformations	485
25.3	Angles and Boundaries	487
25.4	The Poincaré Disk	489
25.5	\mathbb{CP}^1 Transformations and the Poincaré Disk	496
25.6	Angles and Distances in the Poincaré Disk	501
26	Selected Topics in Hyperbolic Geometry	505
26.1	Circles and Cycles in the Poincaré Disk	505
26.2	Area and Angle Defect	509
26.3	Thales and Pythagoras	514

- 26.4 Constructing Regular n -Gons 517
- 26.5 Symmetry Groups 519
- 27 What We Did Not Touch 525**
 - 27.1 Algebraic Projective Geometry 525
 - 27.2 Projective Geometry and Discrete Mathematics 531
 - 27.3 Projective Geometry and Quantum Theory 538
 - 27.4 Dynamic Projective Geometry 546
- References 557**
- Index 563**



<http://www.springer.com/978-3-642-17285-4>

Perspectives on Projective Geometry

A Guided Tour Through Real and Complex Geometry

Richter-Gebert, J.

2011, XXII, 571 p., Hardcover

ISBN: 978-3-642-17285-4