Chapter 2
Background Material

This chapter establishes the notational conventions used throughout while also providing results and computations needed for later analyses. Readers familiar with differential geometry may wish to skip this chapter and refer back when necessary.

2.1 Smooth Manifolds

Suppose M is a topological space. We say M is a *n-dimensional topological manifold* if it is Hausdorff, second countable and is 'locally Euclidean of dimension n' (i.e. every point $p \in M$ has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$). A *coordinate chart* is a pair (U, ϕ) where $U \subset M$ is open and $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a homeomorphism. If (U, ϕ) and (V, ψ) are two charts, the composition $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is called the *transition map* from ϕ to ψ. It is a homeomorphism since both ϕ and ψ are.

In order for calculus ideas to pass to the setting of manifolds we need to impose an extra smoothness condition on the chart structure. We say two charts (U, ϕ) and (V, ψ) are *smoothly compatible* if the transition map $\psi \circ \phi^{-1}$, as a map between open sets of $\mathbb{R}^n$, is a diffeomorphism.

We define a *smooth atlas* for M to be a collection of smoothly compatible charts whose domains cover M. We say two smooth atlases are *compatible* if their union is also a smooth atlas. As compatibility is an equivalence relation, we define a *differentiable structure* for M to be an equivalence class of smooth atlases. Thus a *smooth manifold* is a pair $(M, \mathscr{A})$ where M is a topological manifold and $\mathscr{A}$ is a smooth differentiable structure for M. When there is no ambiguity, we usually abuse notation and simply refer to a 'differentiable manifold M' without reference to the atlas. From here on, manifolds will always be of the differentiable kind.

B. Andrews and C. Hopper, *The Ricci Flow in Riemannian Geometry*, Lecture Notes in Mathematics 2011, DOI 10.1007/978-3-642-16286-2_2,

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2.1.1 Tangent Space

There are various equivalent ways of defining the tangent space of a manifold. For our purposes, we emphasis the construction of the tangent space as derivations on the algebra $C^\infty(M)$.

Definition 2.1. Let M be a smooth manifold with a point p. A $\mathbb{R}$-linear map $X : C^\infty(M) \to \mathbb{R}$ is called a *derivation at p* if it satisfies the Leibniz rule: $X(fg) = f(p)Xg + g(p)Xf$. The *tangent space* at p, denoted by T_pM, is the set of all derivations at p.

The tangent space T_pM is clearly a vector space under the canonical operations $(X + Y)f = Xf + Yf$ and $(\lambda X)f = \lambda(Xf)$ where $\lambda \in \mathbb{R}$. In fact T_pM is of finite dimension and isomorphic to $\mathbb{R}^n$. By removing the pointwise dependence in the above definition, we define:

Definition 2.2. A *derivation* is an $\mathbb{R}$-linear map $Y : C^\infty(M) \to C^\infty(M)$ which satisfies the Leibniz rule: $Y(fg) = f\,Yg + g\,Yf$.

We identify such derivations with vector fields on M (see Remark 2.10 below).

Remark 2.3. In the setting of abstract algebra, a derivation is a function on an algebra which generalises certain features of the derivative operator. Specifically, given an associative algebra $\mathscr{A}$ over a ring or field R, a R-derivation is a R-linear map $D : \mathscr{A} \to \mathscr{A}$ that satisfies the product rule: $D(ab) = (Da)b + a(Db)$ for $a, b \in \mathscr{A}$. In our case, the algebra $\mathscr{A} = C^\infty(M)$ and the field is $\mathbb{R}$.

2.2 Vector Bundles

Definition 2.4. Let F and M be smooth manifolds. A *fibre bundle* over M with fibre F is a smooth manifold E, together with a surjective submersion $\pi : E \to M$ satisfying a local triviality condition: For any $p \in M$ there exists an open set U in M containing p, and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ (called a local trivialization) such that $\pi = \pi_1 \circ \phi$ on $\pi^{-1}(U)$, where $\pi_1(x, y) = x$ is the projection onto the first factor. The fibre at p, denoted E_p, is the set $\pi^{-1}(p)$, which is diffeomorphic to F for each p.

Although a fibre bundle E is *locally* a product $U \times F$, this may not be true globally. The space E is called the total space, M the base space and π the projection. Occasionally we refer to the bundle by saying: 'let $\pi : E \to M$ be a (smooth) fibre bundle'. In most cases the fibre bundles we consider will be vector bundles in which the fibre F is a vector space and the local trivializations induce a well-defined linear structure on E_p for each p:

Definition 2.5. Let M be a differentiable manifold. A *smooth vector bundle* of rank k over M is a fibre bundle $\pi:\ E \to M$ with fibre $\mathbb{R}^k$, such that

1. The fibres $E_p = \pi^{-1}(p)$ have a k-dimensional vector space structure.
2. The local trivializations $\phi:\ \pi^{-1}(U) \to U \times \mathbb{R}^k$ are such that $\pi_2 \circ \phi|_{E_p}$ is a linear isomorphism for each $p \in U$, where $\pi_2(x, y) = y$.

One of the most fundamental vector bundles over a manifold M is the tangent bundle $TM = \bigcup_{p\in M} T_pM$. It is a vector bundle of rank equal to $\dim M$. Other examples include the tensor bundles constructed from TM (see Sect. 2.3.3).

Definition 2.6. A *section* of a fibre bundle $\pi : E \to M$ is a smooth map $X : M \to E$, written $p \mapsto X_p$, such that $\pi \circ X = \mathrm{id}_M$. If E is a vector bundle then the collection of all *smooth sections* over M, denoted by $\Gamma(E)$, is a real vector space under pointwise addition and scalar multiplication.

Definition 2.7. A *local frame* for a vector bundle E of rank k is a k-tuple (ξ_i) of pointwise linearly independent sections of E over open $U \subset M$, that is linear independent $\xi_1, \ldots, \xi_k \in \Gamma(E\big|_U)$.

Given such a local frame, any section α of E over U can be written in the form $\sum_{i=1}^k \alpha^i \xi_i$, where $\alpha^i \in C^\infty(U)$. Local frames correspond naturally to local trivializations, since the map $(p, \sum_{i=1}^k \alpha^i \xi_i(p)) \mapsto (p, \alpha^1(p), \ldots, \alpha^k(p))$ is a local trivialization, while the inverse images of a standard basis in a local trivialization defines a local frame. Moreover we recall the local frame criterion for smoothness of sections.

Proposition 2.8. *A section $\alpha \in \Gamma(E|_U)$ is a smooth if and only if its component functions α^i, with respect to (ξ_i), on U are smooth.*

Remark 2.9. In fact $\Gamma(E)$ is a module over the ring $C^\infty(M)$ since for each $X \in \Gamma(E)$ we define $fX \in \Gamma(E)$ by $(fX)(p) = f(p)X(p)$. For instance the space of sections of any tensor bundle is a module over $C^\infty(M)$.[1]

Remark 2.10. There is an important identification between derivations (as in Definition 2.2) and smooth sections of the tangent bundle:

Proposition 2.11. *Smooth sections of $TM \to M$ are in one-to-one correspondence with derivations of $C^\infty(M)$.*

[1] Recall that a module over a ring generalises the notion of a vector space. Instead of requiring the scalars to lie in a field, the 'scalars' may lie in an arbitrary ring. Formally a left R-module over a ring R is an Abelian group $(G, +)$ with scalar multiplication $\cdot : R \times G \to G$ that is associative and distributive.

Proof. If $X \in \Gamma(TM)$ is a smooth section, define a derivation $\mathcal{X} : f \to Xf$ by $(\mathcal{X}f)(p) = X_p(f)$. Conversely, given a derivation $\mathcal{Y} : C^\infty(M) \to C^\infty(M)$ define a section Y of TM by $Y_p f = (\mathcal{Y}f)(p)$. An easy exercise shows that this section is smooth. □

As a result, we can either think of a smooth section $X \in \Gamma(TM)$ as a smooth map $X : M \to TM$ with $X \circ \pi = \mathrm{id}_M$ or as a derivation – that is, a $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M)$ that satisfies the Leibniz rule. We call such an X a *vector field* and let the set of vector fields be denoted by $\mathscr{X}(M)$.

2.2.1 Subbundles

Definition 2.12. For a vector bundle $\pi : E \to M$, a *subbundle* of E is a vector bundle E' over M with an injective vector bundle homomorphism $i : E' \to E$ covering the identity map on M (so that $\pi_E \circ i = \pi_{E'}$, where π_E and $\pi_{E'}$ are the projections on E and E' respectively).

The essential idea of a subbundle of a vector bundle $E \to M$ is that it should be a smoothly varying family of linear subspaces E'_p of the fibres E_p that constitutes a vector bundle in their own right. However it is convenient to distinguish sections of the subbundle from sections of the larger bundle, and for this reason we use the definition above. One can think of the map i as an inclusion of E' into E.

Example 2.13. Let $f : M \hookrightarrow N$ be a smooth immersion between manifolds. The pushforward $f_* : TM \to TN$ (Sect. 2.8.2) over $f : M \to N$ induces a vector bundle mapping $i : TM \to f^*(TN)$ over M. On fibres over $p \in M$ this is the map $f_*|_p : T_pM \to T_{f(p)}N = (f^*TN)_p$ which is injective since f is an immersion. Hence, i exhibits TM as a subbundle of f^*TN over M.

It is also useful to note, using a rank type theorem, that:

Proposition 2.14. *If $f : E \to E'$ is a smooth bundle surjection over M. Then there exists a subbundle $j : E_0 \to E$ such that $j(E_0(p)) = \ker(f|_p)$ for each $p \in M$.*

In which case we have a well-defined subbundle $E_0 = \ker f$ inside E.

2.2.2 Frame Bundles

For a vector bundle $\pi : E \to M$ of rank k, there is an associated fibre bundle over M with fibre $\mathrm{GL}(k)$ called the *general linear frame bundle* $F(E)$. The fibre $F(E)_x$ over $x \in M$ consists of all linear isomorphisms $Y :\ \mathbb{R}^k \to E_x$, or equivalently the set of all ordered bases for E_x (by identifying the map Y with the basis (Y_a), where $Y_a = Y(e_a)$ for $a = 1, \ldots, k$). The group $\mathrm{GL}(k)$ acts on each fibre by composition, so that $\mathbb{A} \in \mathrm{GL}(k)$ acts on a frame $Y : \mathbb{R}^k \to E_x$ to give $Y^{\mathbb{A}} = Y \circ \mathbb{A} : \mathbb{R}^k \to E_x$ (alternatively, the basis $(Y_a) \in F(E)_x$ maps to the basis $(\mathbb{A}^a_b Y_a)$). From standard linear algebra, this action

$$\mathrm{GL}(k) \times F(E) \to F(E); \quad (\mathbb{A}, Y) \mapsto Y^{\mathbb{A}}$$

is simply transitive on each fibre (that is, for any $Y, Z \in F(E)_x$ there exists a unique $\mathbb{A} \in \mathrm{GL}(k)$ such that $Y^{\mathbb{A}} = Z$).

Note that a local trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ for E, together with a local chart $\eta\colon U \to \mathbb{R}^n$ for M, produces a chart for E compatible with the bundle structure: We take $(x, v) \mapsto \phi(x, v) = (x, \pi_2\phi(x, v)) \mapsto (\eta(x), \pi_2\phi(x, v)) \in \mathbb{R}^n \times \mathbb{R}^k$. Any such chart also produces a chart for $F(E)$ giving it the structure of a manifold of dimension $n + k^2$: We take $(x, Y) \mapsto (\eta(x), \pi_2 \circ \phi_x \circ Y) \in \mathbb{R}^n \times \mathrm{GL}(k) \subset \mathbb{R}^{n+k^2}$, where $\phi_x(\cdot) = \phi(x, \cdot)$. Similarly a local trivialization of $F(E)$ is defined by $(x, Y) \mapsto (x, \pi_2 \circ \phi_x \circ Y)$, giving $F(E)$ the structure of a fibre bundle with fibre $\mathrm{GL}(k)$ as claimed.

If the bundle E is equipped with a metric g (Sect. 2.4.4) – so that E_x is an inner product space – then one can introduce the *orthonormal frame bundle* $O(E)$. Specifically $O(E)$ is the subset of $F(E)$ defined by

$$O(E) = \{Y \in F(E) : g(Y_a, Y_b) = \delta_{ab}\}.$$

The orthogonal group $O(k)$ acts on $O(E)$ by

$$O(k) \times O(E) \to O(E); \quad (\mathbb{O}, Y) \mapsto Y^{\mathbb{O}}$$

where $Y^{\mathbb{O}}(u) = Y(\mathbb{O}u)$ for each $u \in \mathbb{R}^k$ and $O(E)$ is a fibre bundle over M with fibre $O(k)$.

Remark 2.15. A local frame for E consists of k pointwise linearly independent smooth sections of E over an open set U, say $p \mapsto \xi_i(p) \in E_p$ for $i = 1, \ldots, k$. This corresponds to a section Y of $F(E)$ over U, defined by $Y_p(u^i e_i) = u^i \xi_i(p)$ for each $p \in U$ and $u \in \mathbb{R}^k$.

2.3 Tensors

Let V be a finite dimensional vector space. A *covariant k-tensor* on V is a multilinear map $F : V^k \to \mathbb{R}$. Similarly, a *contravariant ℓ-tensor* is a multilinear map $F : (V^*)^\ell \to \mathbb{R}$. A mixed tensor of type $\binom{k}{\ell}$ or (k, ℓ) is a multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{\ell \text{ times}} \times \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

We denote the space of all k-tensors on V by $T^k(V)$, the space of contravariant ℓ-tensors by $T_\ell(V)$, and the space of all mixed (k,ℓ) tensors by $T^k_\ell(V)$.

The following canonical isomorphism is frequently useful:

Lemma 2.16. *The tensor space* $T^1_1(V)$ *is canonically isomorphic to* $\mathrm{End}(V)$, *where the (bases independent) isomorphism* $\Phi : End(V) \to T^1_1(V)$ *is given by*

$$\Phi(A) : (\omega, X) \mapsto \omega(A(X))$$

for all $A \in \mathrm{End}(V)$, $\omega \in V^*$, *and* $X \in V$.

Remark 2.17. Alternatively one could state this lemma by saying: If V and W are vector spaces then $V \otimes W^* \simeq \mathrm{End}(W,V)$ where the isomorphism $\Psi : V \otimes W^* \to \mathrm{End}(W,V)$ is given by $\Psi(v \otimes \xi) : w \to \xi(w)v$.

A general version of this identification is expressed as follows.

Lemma 2.18. *The tensor space* $T^k_{\ell+1}(V)$ *is canonically isomorphic to the space* $\mathrm{Mult}((V^*)^\ell \times V^k, V)$, *where the isomorphism* $\Phi : \mathrm{Mult}((V^*)^\ell \times V^k, V) \to T^k_{\ell+1}(V)$ *is given by*

$$\Phi(A) : (\omega^0, \omega^1, \ldots, \omega^\ell, X_1, \ldots, X_k) \mapsto \omega^0(A(\omega^1, \ldots, \omega^\ell, X_1, \ldots, X_k))$$

for all $A \in \mathrm{Mult}((V^*)^\ell \times V^k, V)$, $\omega_i \in V^*$, *and* $X_j \in V$.[2]

2.3.1 Tensor Products

There is a natural product that links the various tensor spaces over V. If $F \in T^k_\ell(V)$ and $G \in T^p_q(V)$, then the *tensor product* $F \otimes G \in T^{k+p}_{\ell+q}(V)$ is defined to be

$$\begin{aligned}(F \otimes G)&(\omega^1, \ldots, \omega^{\ell+q}, X_1, \ldots, X_{k+p}) \\ &= F(\omega^1, \ldots, \omega^\ell, X_1, \ldots, X_k)\, G(\omega^{\ell+1}, \ldots, \omega^{\ell+q}, X_{k+1}, \ldots, X_{k+p}).\end{aligned}$$

Moreover, if $(e_1, \ldots, e_n)$ is a basis for V and $(\varphi^1, \ldots, \varphi^n)$ is the corresponding dual basis, defined by $\varphi^i(e_j) = \delta^i_j$, then it can be shown that a *basis* for $T^k_\ell(V)$ takes the form

$$e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k},$$

where

$$e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}(\varphi^{s_1}, \ldots, \varphi^{s_\ell}, e_{r_1}, \ldots, e_{r_k}) = \delta^{s_1}_{j_1} \cdots \delta^{s_\ell}_{j_\ell}\, \delta^{i_1}_{r_1} \cdots \delta^{i_k}_{r_k}.$$

[2] Here $\mathrm{Mult}((V^*)^\ell \times V^k, V)$ is the set of multilinear maps from $(V^*)^\ell \times V^k$ to V.

Therefore any tensor $F \in T^k_\ell(V)$ can be written, with respect to this basis, as

$$F = F^{j_1,\dots,j_\ell}{}_{i_1,\dots,i_k} e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_k}$$

where $F^{j_1,\dots,j_\ell}{}_{i_1,\dots,i_k} = F(\varphi^{j_1}, \dots, \varphi^{j_\ell}, e_{i_1}, \dots, e_{i_k})$.

2.3.2 Tensor Contractions

A tensor contraction is an operation on one or more tensors that arises from the natural pairing of a (finite-dimensional) vector space with its dual.

Intuitively there is a natural notion of 'the trace of a matrix' $A = (A^i_j) \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ given by $\operatorname{tr} A = \sum_i A^i_i$. It is $\mathbb{R}$-linear and commutative in the sense that $\operatorname{tr} AB = \operatorname{tr} BA$. From the latter property, the trace is also cyclic (in the sense that $\operatorname{tr} ABC = \operatorname{tr} BCA = \operatorname{tr} CAB$). Therefore the trace is similarity-invariant, which means for any $P \in \mathrm{GL}(n)$ the trace $\operatorname{tr} P^{-1}AP = \operatorname{tr} PP^{-1}A = \operatorname{tr} A$. Whence we can extend tr over $\mathrm{End}(V)$ by taking the trace of a matrix representation – this definition is basis independent since different bases give rise to similar matrices – and so by Lemma 2.16 tr can act on tensors as well.

Naturally, we define the *contraction* of any $F \in T^1_1(V)$ by taking the trace of F as a linear map in $\mathrm{End}(V)$. In which case $\operatorname{tr} : T^1_1(V) \to \mathbb{R}$ is given by $\operatorname{tr} F = F(\varphi^i, e_i) = \sum_i F^i_i$, since $\Phi^{-1}(F) = (F(\varphi^i, e_j))^n_{i,j=1} \in \mathrm{End}(V)$. In general we define

$$\operatorname{tr} : T^{k+1}_{\ell+1}(V) \to T^k_\ell(V)$$

by

$$(\operatorname{tr} F)(\omega^1, \dots, \omega^\ell, X_1, \dots, X_k) = \operatorname{tr}\big(F(\omega^1, \dots, \omega^\ell, \,\cdot\,, X_1, \dots, X_k, \cdot\,)\big).$$

That is, we define $(\operatorname{tr} F)(\omega^1, \dots, \omega^\ell, X_1, \dots, X_k)$ to be the trace of the endomorphism $F(\omega^1, \dots, \omega^\ell, \,\cdot\,, X_1, \dots, X_k, \cdot\,) \in T^1_1(V) \simeq \mathrm{End}(V)$. In components this is equivalent to

$$(\operatorname{tr} F)^{j_1\dots j_\ell}{}_{i_1\dots i_k} = F^{j_1\dots j_\ell m}{}_{i_1\dots i_k m}.$$

It is clear that the contraction is linear and lowers the rank of a tensor by 2. Unfortunately there is no general notation for this operation! So it is best to explicitly describe the contraction in words each time it arises. We give some simple examples of how this might occur.

There are several variations of tr: Firstly, there is nothing special about which particular component pairs the contraction is taken over. For instance if $F = F_i{}^j{}_k \varphi^i \otimes e_j \otimes \varphi^k \in T_1^2(V)$ then one could take the contraction of F over the first two components: $(\operatorname{tr}_{12} F)_k = \operatorname{tr} F(\cdot, \cdot, e_k) = F_i{}^i{}_k$ or the last two components: $(\operatorname{tr}_{23} F)_k = \operatorname{tr} F(e_k, \cdot, \cdot) = F_k{}^i{}_i$.

Another variation occurs if one wants to take the contraction over multiple component pairs. For example if $F = F_{ij}{}^{k\ell} \varphi^i \otimes \varphi^j \otimes e_k \otimes e_\ell \in T_2^2(V)$, one could take the trace over the 1st and 3rd with the 2nd and 4th so that $\operatorname{tr} F = \operatorname{tr}_{13} \operatorname{tr}_{24} F = \operatorname{tr} F(\star, \cdot, \star, \cdot) = F_{pq}{}^{pq}$.

Furthermore, if one has a metric g then it is possible to take contractions over two indices that are either both vectors or covectors. This is done by taking a tensor product with the metric tensor (or its inverse) and contracting each of the two indices with one of the indices of the metric. This operation is known as *metric contraction* (see Sect. 2.4.3 for further details).

Finally, one of the most important applications arises when $F = \omega \otimes X \in T_1^1(V)$ for some (fixed) vector X and covector ω. In this case

$$\operatorname{tr} F = F(\partial_i, dx^i) = \omega(\partial_i) X(dx^i) = \omega_i X^i = \omega(X).$$

The idea can be extended as follows: If $F \in T_\ell^k(V)$ with $\omega^1, \ldots, \omega^\ell \in V^*$ and $X_1, \ldots, X_k \in V$ (fixed) then

$$F \otimes \omega^1 \otimes \cdots \otimes \omega^\ell \otimes X_1 \otimes \cdots \otimes X_k \in T_{2\ell}^{2k}(V).$$

So the contraction of this tensor over all indexes becomes

$$\begin{aligned} &\operatorname{tr}\left(F \otimes \omega^1 \otimes \cdots \otimes \omega^\ell \otimes X_1 \otimes \cdots \otimes X_k\right) \\ &= \omega_{j_1} \cdots \omega_{j_\ell} F^{j_1 \cdots j_\ell}{}_{i_1 \cdots i_k} X^{i_1} \cdots X^{i_k} \\ &= F(\omega^1, \ldots, \omega^\ell, X_1, \ldots, X_k). \end{aligned} \tag{2.1}$$

2.3.3 Tensor Bundles and Tensor Fields

For a manifold M we can apply the above tensor construction pointwise on each tangent space T_pM. In which case a (k, ℓ)-tensor at $p \in M$ is an element $T_\ell^k(T_pM)$. We define the *bundle of* (k, ℓ)*-tensors* on M by

$$T_\ell^k M = \bigcup_{p \in M} T_\ell^k(T_pM) = \bigcup_{p \in M} \otimes^k T_p^* M \otimes^\ell T_pM.$$

In particular, $T_1M = TM$ and $T^1M = T^*M$. An important subbundle of T^2M is $\operatorname{Sym}^2 T^*M$, the space of all symmetric $(2, 0)$-tensors on M. A (k, ℓ)-*tensor field* is an element of $\Gamma(T_\ell^k M) = \Gamma(\otimes^k T^*M \otimes^\ell TM)$ – we sometimes use the notation $\mathscr{T}_\ell^k(M)$ as a synonym for $\Gamma(T_\ell^k M)$.

To check that $T^k_\ell M$ is a vector bundle, let $\pi : T^k_\ell M \to M$ send $F \in T^k_\ell(T_pM)$ to the base point p. If (x^i) is a local chart on open $U \subset M$ around point p, then any tensor $F \in T^k_\ell(T_pM)$ can be expressed as

$$F = F^{j_1,\dots,j_\ell}{}_{i_1,\dots,i_k} \partial_{j_1} \otimes \cdots \otimes \partial_{j_\ell} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

The local trivialisation $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+\ell}}$ is given by

$$\phi : T^k_\ell(T_pM) \ni F \longmapsto (p, F^{j_1,\dots,j_\ell}{}_{i_1,\dots,i_k}).$$

2.3.4 Dual Bundles

If E is a vector bundle over M, the *dual bundle* E^* is the bundle whose fibres are the dual spaces of the fibres of E:

$$E^* = \{(p, \omega) : \ \omega \in E^*_p\}.$$

If (ξ_i) is a local frame for E over an open set $U \subset M$, then the map $\phi : \pi^{-1}_{E^*}(U) \to U \times \mathbb{R}^k$ defined by $(p, \omega) \mapsto (p, \omega(\xi_1(p)), \dots, \omega(\xi_k(p)))$ is a local trivialisation of E^* over U. The corresponding local frame for E^* is given by the sections θ^i defined by $\theta^i(\xi_j) = \delta^i_j$.

2.3.5 Tensor Products of Bundles

If $E_1, \dots, E_k$ are vector bundles over M, the tensor product $E_1 \otimes \cdots \otimes E_k$ is the vector bundle whose fibres are the tensor products $(E_1)_p \otimes \cdots \otimes (E_k)_p$. If U is an open set in M and $\{\xi^j_i : \ 1 \le i \le n_j\}$ is a local frame for E_j over U for $j = 1, \dots, k$, then $\left\{\xi^1_{i_1} \otimes \cdots \otimes \xi^k_{i_k} : \ 1 \le i_j \le n_j, \ 1 \le j \le k\right\}$ forms a local frame for $E_1 \otimes \cdots \otimes E_k$. Taking tensor products commutes with taking duals (in the sense of Sect. 2.3.4) and is associative. That is, $E^*_1 \otimes E^*_2 \simeq (E_1 \otimes E_2)^*$ and $(E_1 \otimes E_2) \otimes E_3 \simeq E_1 \otimes (E_2 \otimes E_3)$.

2.3.6 A Test for Tensorality

Let $E_1, \dots, E_k$ be vector bundles over M. Given a tensor field $F \in \Gamma(E^*_1 \otimes \cdots \otimes E^*_k)$ and sections $X_i \in \Gamma(E_i)$, Proposition 2.8 implies that the function on U defined by

$$F(X_1, \dots, X_k) : p \mapsto F_p(X_1\big|_p, \dots, X_k\big|_p),$$

is smooth, so that F induces a mapping $F : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to C^\infty(M)$. It can easily be seen that this map is multilinear over $C^\infty(M)$ in the sense that

$$F(f_1 X_1, \ldots, f_k X_k) = f_1 \cdots f_k F(X_1, \ldots, X_k)$$

for any $f_i \in C^\infty(M)$ and $X_i \in \Gamma(E_i)$. In fact the converse holds as well.

Proposition 2.19 (Tensor Test). *For vector bundles $E_1, \ldots, E_k$ over M, the mapping $F : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to C^\infty(M)$ is a tensor field, i.e. $F \in \Gamma(E_1^* \otimes \cdots \otimes E_k^*)$, if and only if F is multilinear over $C^\infty(M)$.*

By Lemma 2.18 we also have:

Proposition 2.20 (Bundle Valued Tensor Test). *For vector bundles E_0, $E_1, \ldots, E_k$ over M, the mapping $F : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to \Gamma(E_0)$ is a tensor field, i.e. $F \in \Gamma(E_1^* \otimes \cdots \otimes E_k^* \otimes E_0)$, if and only if F is multilinear over $C^\infty(M)$.*

Remark 2.21. This proposition leaves one to interpret

$$F \in \Gamma(E_1^* \otimes \cdots \otimes E_k^* \otimes E_0)$$

as an E_0-valued tensor acting on $E_1 \otimes \cdots \otimes E_k$.

The importance of Propositions 2.19 and 2.20 is that it allows one to work with tensors *without* referring to their pointwise attributes. For example, the metric g on M (as we shall see) can be consider as a pointwise inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ that smoothly depends on its base point. By our identification we can also think of this tensor as a map

$$g : \mathscr{X}(M) \times \mathscr{X}(M) \to C^\infty(M).$$

Therefore if X, Y and Z are vector fields, $g(X, Y) \in C^\infty(M)$ and so by Remark 2.10 we also have $Zg(X, Y) \in C^\infty(M)$.

2.4 Metric Tensors

An inner product on a vector space allows one to define lengths of vectors and angles between them. Riemannian metrics bring this structure onto the tangent space of a manifold.

2.4.1 Riemannian Metrics

A *Riemannian metric* g on a manifold M is a symmetric positive definite $(2,0)$-tensor field (i.e. $g \in \Gamma(\mathrm{Sym}^2 T^*M)$ and g_p is an inner product for each $p \in M$). A manifold M together with a given Riemannian metric g is called a *Riemannian manifold* (M, g).

In local coordinates (x^i), $g = g_{ij} dx^i \otimes dx^j$. The archetypical Riemannian manifold is $(\mathbb{R}^n, \delta_{ij})$. As the name suggests, the concept of the metric was first introduced by Bernhard Riemann in his 1854 habilitation dissertation.

2.4.1.1 Geodesics

We want to think of geodesics as length minimising curves. From this point of view, we seek to minimise the length functional

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_g dt$$

amongst all curves $\gamma : [0,1] \to M$. There is also a natural 'energy' functional:

$$E(\gamma) = \frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|_g^2 dt.$$

As $L(\gamma)^2 \leq 2E(\gamma)$, for any smooth curve $\gamma : [0,1] \to M$, the problem of minimising $L(\gamma)$ amongst all smooth curves γ is equivalent to minimising $E(\gamma)$. By doing so we find:

Theorem 2.22. *The Euler–Lagrange equations for the energy functional are*

$$\ddot{\gamma}^i(t) + \Gamma^i_{jk}(\gamma(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0, \tag{2.2}$$

where the connection coefficients Γ^i_{jk} are given by (2.9).

Hence any smooth curve $\gamma : [0,1] \to M$ satisfying (2.2) is called a geodesic. By definition they are critical points of the energy functional. Moreover, by the Picard–Lindelöf theorem we recall:

Lemma 2.23 (Short-Time Existence of Geodesics). *Suppose (M, g) is a Riemannian manifold. Let $p \in M$ and $v \in T_pM$ be given. Then there exists $\varepsilon > 0$ and precisely one geodesic $\gamma : [0, \varepsilon] \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$ and γ depends smoothly on p and v.*

2.4.2 The Product Metric

If $(M_1, g^{(1)})$ and $(M_2, g^{(2)})$ are two Riemannian manifolds then, by the natural identification $T_{(p_1,p_2)}M_1 \times M_2 \simeq T_{p_1}M_1 \oplus T_{p_2}M_2$, there is a canonical Riemannian metric $g = g^{(1)} \oplus g^{(2)}$ on $M_1 \times M_2$ defined by

$$g_{(p_1,p_2)}(u_1 + u_2, v_1 + v_2) = g^{(1)}_{p_1}(u_1, u_2) + g^{(2)}_{p_2}(v_1, v_2),$$

where $u_1, u_2 \in T_{p_1}M_1$ and $v_1, v_2 \in T_{p_2}M_2$. If $\dim M_1 = n$ and $\dim M_2 = m$, the product metric, in local coordinates $(x^1, \ldots, x^{n+m})$ about (p_1, p_2), is the block diagonal matrix:

$$(g_{ij}) = \begin{pmatrix} \boxed{g^{(1)}_{ij}} & \\ & \boxed{g^{(2)}_{ij}} \end{pmatrix}$$

where $(g^{(1)}_{ij})$ is an $n \times n$ block and $(g^{(2)}_{ij})$ is an $m \times m$ block.

2.4.3 Metric Contractions

As the Riemannian metric g is non-degenerate, there is a canonical g-dependent isomorphism between TM and T^*M.[3] By using this it is possible to take tensor contractions over two indices that are either both vectors or covectors.

For example, if h is a symmetric $(2, 0)$-tensor on a Riemannian manifold then $h^\sharp$ is a $(1, 1)$-tensor. In which case the trace of h with respect to g, denoted by $\mathrm{tr}_g h$, is

$$\mathrm{tr}_g\, h = \mathrm{tr}\, h^\sharp = h_i^{\ i} = g^{ij} h_{ij}.$$

Equivalently one could also write

$$\mathrm{tr}_g\, h = \mathrm{tr}_{13}\, \mathrm{tr}_{24}\, g^{-1} \otimes h = (g^{-1} \otimes h)(dx^i, dx^j, \partial_i, \partial_j) = g^{ij} h_{ij}.$$

2.4.4 Metrics on Bundles

A metric g on a vector bundle $\pi : E \to M$ is a section of $E^* \otimes E^*$ such that at each point p of M, g_p is an inner product on E_p (that is, g_p is symmetric and

[3] Specifically, the isomorphism $\sharp : T^*M \to TM$ sends a covector ω to $\omega^\sharp = \omega^i \partial_i = g^{ij}\omega_j \partial_i$, and $\flat : TM \to T^*M$ sends a vector X to $X^\flat = X_i dx^i = g_{ij} X^j dx^i$.

positive definite for each p: $g_p(\xi,\eta) = g_p(\eta,\xi)$ for all $\xi,\eta \in E_p$; $g_p(\xi,\xi) \geq 0$ for all $\xi \in E_p$, and $g_p(\xi,\xi) = 0 \Rightarrow \xi = 0$).

A metric on E defines a bundle isomorphism $\iota_g : E \to E^*$ given by $\iota_g(\xi) : \eta \mapsto g_p(\xi,\eta)$ for all $\xi,\eta \in E_p$.

2.4.5 Metric on Dual Bundles

If g is a metric on E, there is a unique metric on E^* (also denoted g) such that ι_g is a bundle isometry:

$$g(\iota_g(\xi), \iota_g(v)) = g(\xi,\eta)$$

for all $\xi,\eta \in E_p$; or equivalently $g(\omega,\sigma) = g(\iota_g^{-1}\omega, \iota_g^{-1}\sigma)$ for all $\omega,\sigma \in E_p^* = (E_p)^*$.

2.4.6 Metric on Tensor Product Bundles

If g_1 is a metric on E_1 and g_2 is a metric on E_2, then

$$g = g_1 \otimes g_2 \in \Gamma((E_1^* \otimes E_1^*) \otimes (E_2^* \otimes E_2^*)) \simeq \Gamma((E_1 \otimes E_2)^* \otimes (E_1 \otimes E_2)^*)$$

is the unique metric on $E_1 \otimes E_2$ such that

$$g(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) = g_1(\xi_1,\xi_2) g_2(\eta_1,\eta_2).$$

The construction of metrics on tensor bundles now follows. It is well-defined since the metric constructed on a tensor product of dual bundles agrees with that constructed on the dual bundle of a tensor product.

Example 2.24. Given tensors $S, T \in \mathscr{T}_\ell^k(M)$, the inner product, denoted by $\langle\cdot,\cdot\rangle$, at p is

$$\langle S,T\rangle = g^{a_1 b_1} \cdots g^{a_k b_k} g_{i_1 j_1} \cdots g_{i_\ell j_\ell} S_{a_1 \dots a_k}{}^{i_1 \dots i_\ell} T_{b_1 \dots b_k}{}^{j_1 \dots j_\ell} \ . \tag{2.3}$$

2.5 Connections

Connections provide a coordinate invariant way of taking directional derivatives of vector fields. In $\mathbb{R}^n$, the derivative of a vector field $X = X^i e_i$ in direction v is given by $D_v X = v(X^i) e_i$. Simply put, D_v differentiates the coefficient functions X^i and thinks of the basis vectors e_i as being held constant. However there is no canonical way to compare vectors from different vector spaces, hence there is no natural coordinate invariant analogy applicable to abstract manifolds.[4] To circumnavigate this, we impose an additional structure – in the form of a connection operator – that provides a way to 'connect' these tangent spaces.

Our method here is to directly specify how a connection acts on elements of $\Gamma(E)$ as a module over $C^\infty(M)$. They are of central importance in modern geometry largely because they allow a comparison between the local geometry at one point and the local geometry at another point.

Definition 2.25. A *connection* ∇ on a vector bundle E over M is a map

$$\nabla : \mathscr{X}(M) \times \Gamma(E) \to \Gamma(E),$$

written as $(X, \sigma) \mapsto \nabla_X \sigma$, that satisfies the following properties:

1. ∇ is $C^\infty(M)$-linear in X:

$$\nabla_{f_1 X_1 + f_2 X_2} \sigma = f_1 \nabla_{X_1} \sigma + f_1 \nabla_{X_1} \sigma$$

2. ∇ is $\mathbb{R}$-linear in σ:

$$\nabla_X(\lambda_1 \sigma_1 + \lambda_2 \sigma_2) = \lambda_1 \nabla_X \sigma_1 + \lambda_2 \nabla_X \sigma_2$$

and ∇ satisfies the product rule:

$$\nabla_X(f\sigma) = (Xf)\,\sigma + f \nabla_X \sigma$$

We say $\nabla_X \sigma$ is the *covariant derivative* of σ in the direction X.

Remark 2.26. Equivalently, we could take the connection to be

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

which is linear and satisfies the product rule. For if $\sigma \in \Gamma(E)$ then $\nabla \sigma \in \Gamma(T^*M \otimes E)$, where $(\nabla\sigma)(X) = \nabla_X \sigma$ is $C^\infty(M)$-linear in X by Property 1 of Definition 2.25. Thus Proposition 2.20 implies that $\nabla \sigma : \Gamma(TM) \to \Gamma(E)$ is an E-valued tensor acting on TM.

[4] There is however another generalisation of directional derivatives which is canonical: the Lie derivative. The Lie derivative evaluates the change of one vector field along the flow of another vector field. Thus, one must know both vector fields in an open neighbourhood. The covariant derivative on the other hand only depends on the vector direction at a single point, rather than a vector field in an open neighbourhood of a point. In other words, the covariant derivative is linear over $C^\infty(M)$ in the direction argument, while the Lie derivative is $C^\infty(M)$-linear in neither argument.

For a connection ∇ on the tangent bundle, we define the *connection coefficients* or *Christoffel symbols* of ∇ in a given set of local coordinates (x^i) by defining

$$\Gamma_{ij}{}^{k} = dx^k\left(\nabla_{\partial_i}\partial_j\right)$$

or equivalently,

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}{}^{k}\partial_k.$$

More generally, the connection coefficients of a connection ∇ on a bundle E can be defined with respect to a given local frame $\{\xi_\alpha\}$ for E by the equation

$$\nabla_{\partial_i}\xi_\alpha = \Gamma_{i\alpha}{}^{\beta}\xi_\beta.$$

2.5.1 Covariant Derivative of Tensor Fields

In applications one is often interested in computing the covariant derivative on the tensor bundles $T^k_\ell M$. This is a special case of a more general construction (see Sect. 2.5.3).

Proposition 2.27. *Given a connection ∇ on TM, there is a unique connection on the tensor bundle, also denoted by ∇, that satisfies the following properties:*

1. *On TM, ∇ agrees with the given connection.*
2. *On $C^\infty(M) = T^0M$, ∇ is the action of a vector as a derivation:*

$$\nabla_X f = Xf,$$

   for any smooth function f.
3. *∇ obeys the product rule with respect to tensor products:*

$$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G),$$

   for any tensors F and G.
4. *∇ commutes with all contractions:*

$$\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F),$$

   for any tensor F.

Example 2.28. We compute the covariant derivative of a 1-form ω with respect to the vector field X. Since $\operatorname{tr} dx^j \otimes \partial_i = \delta_i^j$, $\nabla_X(\operatorname{tr} dx^j \otimes \partial_i) = 0$. Thus $(\nabla_X dx^j)(\partial_i) = -dx^j(\nabla_X \partial_i)$ and so

$$\begin{aligned}(\nabla_X \omega)(\partial_k) &= \nabla_X \omega_k + \omega_j\, (\nabla_X dx^j)(\partial_k) \\ &= \nabla_X \omega_k - \omega_j dx^j(\nabla_X \partial_k) \\ &= X^i \partial_i \omega_k - \omega_j X^i\, \Gamma^j_{ik}.\end{aligned}$$

Therefore $\nabla_X \omega = (X^i \partial_i \omega_k - \omega_j X^i\, \Gamma^j_{ik}) dx^k$.

In general we have the following useful formulas.

Proposition 2.29. *For any tensor field $F \in \mathscr{T}^k_\ell(M)$, vector fields Y_i and 1-forms ω^j we have*

$$\begin{aligned}(\nabla_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) &= X(F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k)) \\ &- \sum_{j=1}^{\ell} F(\omega^1, \ldots, \nabla_X \omega^j, \ldots, \omega^\ell, Y_1, \ldots, Y_k) \\ &- \sum_{i=1}^{k} F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k).\end{aligned}$$

Proof. By (2.1) we have

$$\begin{aligned}\operatorname{tr}(F \otimes \omega^1 \otimes \cdots \otimes \omega^\ell \otimes Y_1 \otimes \cdots \otimes Y_k) &= \omega_{i_1} \cdots \omega_{i_k} F^{i_1 \cdots i_k}{}_{j_1 \cdots j_k} Y^{j_1} \cdots Y^{j_k} \\ &= F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k).\end{aligned}$$

Thus by Proposition 2.27,

$$\begin{aligned}&\nabla_X(F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k)) \\ &\quad= \operatorname{tr}\Big[(\nabla_X F) \otimes \omega^1 \otimes \cdots \otimes Y_k + F \otimes (\nabla_X \omega^1) \otimes \cdots \otimes Y_k \\ &\qquad + \cdots + F \otimes \omega^1 \otimes \cdots \otimes (\nabla_X Y_k)\Big] \\ &\quad= (\nabla_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) + F(\nabla_X \omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) \\ &\qquad + \cdots + F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, \nabla_X Y_k).\end{aligned}$$ □

As the covariant derivative is $C^\infty(M)$-linear over X, we define $\nabla F \in \Gamma(\otimes^{k+1} T^*M \otimes^\ell TM)$ by

$$(\nabla F)(X, Y_1, \ldots, Y_k, \omega^1, \ldots, \omega^\ell) = \nabla_X F(Y_1, \ldots, Y_k, \omega^1, \ldots, \omega^\ell),$$

for any $F \in \mathscr{T}^k_\ell(M)$. Hence (in this case) ∇ is an $\mathbb{R}$-linear map $\nabla : \mathscr{T}^k_\ell(M) \to \mathscr{T}^{k+1}_\ell(M)$ that takes a (k, ℓ)-tensor field and gives a $(k+1, \ell)$-tensor field.

2.5.2 *The Second Covariant Derivative of Tensor Fields*

By utilising the results of the previous section, we can make sense of the second covariant derivative ∇^2.

To do this, suppose the vector bundle $E = T^k_\ell M$ with associated connection ∇. By Remark 2.26, if $F \in \Gamma(E)$ then $\nabla F \in \Gamma(T^*M \otimes E)$ and so $\nabla^2 F \in \Gamma(T^*M \otimes T^*M \otimes E)$. Therefore, for any vector fields X, Y we find that

$$\begin{aligned}(\nabla^2 F)(X,Y) &= \big(\nabla_X(\nabla F)\big)(Y) \\ &= \nabla_X\big((\nabla F)(Y)\big) - (\nabla F)(\nabla_X Y) \\ &= \nabla_X(\nabla_Y F)) - (\nabla_{\nabla_X Y} F).\end{aligned} \tag{2.4}$$

Example 2.30. If $f \in C^\infty(M)$ is a $(0,0)$-tensor, then $\nabla^2 f$ is a $(2,0)$-tensor. In local coordinates:

$$\begin{aligned}\nabla^2_{\partial_i,\partial_j} f &= (\nabla_{\partial_i}(\nabla f))(\partial_j) \\ &= \partial_i((\nabla f)(\partial_j)) - (\nabla f)(\nabla_{\partial_i}\partial_j) \\ &= \partial_i(\partial_j f) - (\nabla f)(\Gamma^k_{ij}\partial_k) \\ &= \partial_i\,\partial_j f - \Gamma^k_{ij}\nabla_{\partial_k} f \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}.\end{aligned} \tag{2.5}$$

In general equation (2.4) amounts to the following useful formula:

Proposition 2.31. *If* ∇ *is a connection on* TM*, then*

$$\nabla^2_{Y,X} = \nabla_Y \circ \nabla_X - \nabla_{\nabla_Y X} : \mathscr{T}^k_\ell(M) \to \mathscr{T}^k_\ell(M) \tag{2.6}$$

where $X, Y \in \mathscr{X}(M)$ *are given vector fields.*

2.5.2.1 Notational Convention

It is important to note that we interpret

$$\nabla_X \nabla_Y F = \nabla^2_{X,Y} F = (\nabla\nabla F)(X, Y, \cdots) = (\nabla_X(\nabla F))(Y, \ldots),$$

whenever *no brackets* are specified; this is differs from $\nabla_X(\nabla_Y F)$ with brackets [KN96, pp. 124–125]. Furthermore, for notational simplicity we often write ∇_{∂_p} as just ∇_p and $(\nabla_{\partial_p} F)(dx^{i_1}, \ldots, dx^{i_\ell}, \partial_{j_1}, \ldots, \partial_{j_k})$ simply as $\nabla_p F^{i_1\cdots i_\ell}{}_{j_1\cdots j_k}$.

2.5.2.2 The Hessian

We define the Hessian of $f \in C^\infty(M)$ to be

$$\mathrm{Hess}(f) = \nabla df.$$

When applied to vector fields $X, Y \in \mathscr{X}(M)$, we find that $\mathrm{Hess}(f)(X,Y) = (\nabla df)(X,Y) = \nabla^2_{X,Y} f$. Also note that the Hessian is symmetric precisely when the connection is symmetric (see Example 2.30).

2.5.3 Connections on Dual and Tensor Product Bundles

So far we have looked at the covariant derivative on the tensor bundle $\bigotimes^k T^*M \otimes \bigotimes^\ell TM$. In fact much of the same structure works on a general vector bundle as well.

Proposition 2.32. *If ∇ is a connection on E, then there is a unique connection on E^*, also denoted by ∇, such that*

$$X(\omega(\xi)) = (\nabla_X \omega)(\xi) + \omega(\nabla_X \xi)$$

for any $\xi \in \Gamma(E)$, $\omega \in \Gamma(E^*)$ and $X \in \mathscr{X}(M)$.

Proposition 2.33. *If $\nabla^{(i)}$ is a connection on E_i for $i = 1, 2$, then there is a unique connection ∇ on $E_1 \otimes E_2$ such that*

$$\nabla_X(\xi_1 \otimes \xi_2) = (\nabla^{(1)}_X \xi_1) \otimes \xi_2 + \xi_1 \otimes (\nabla^{(2)}_X \xi_2)$$

for all $X \in \mathscr{X}(M)$ and $\xi_i \in \Gamma(E_i)$.

Propositions 2.32 and 2.33 define a canonical connection on any tensor bundle constructed from E by taking duals and tensor products. In particular, if $S \in \Gamma(E_1^* \otimes E_2)$ is an E_2-valued tensor acting on E_1, then $\nabla S \in \Gamma(T^*M \otimes E_1^* \otimes E_2)$ is given by

$$(\nabla_X S)(\xi) = {}^{E_2}\nabla_X \big(S(\xi)\big) - S\big({}^{E_1}\nabla_X \xi\big) \tag{2.7}$$

where $\xi \in \Gamma(E_1)$ and $X \in \mathscr{X}(M)$.

Moreover if we also have a connection $\widehat{\nabla}$ on TM, then $\nabla^2 S \in \Gamma(T^*M \otimes T^*M \otimes E_1^* \otimes E_2)$ – since we can construct this connection from the connections on TM, E_1 and E_2 by taking duals and tensor products. Explicitly,

$$\begin{aligned}(\nabla^2 S)(X, Y, \xi) &= {}^{E_2}\nabla_X\big((\nabla_Y S)(\xi)\big) - (\nabla_{\widehat{\nabla}_X Y} S)(\xi) - (\nabla_Y S)({}^{E_1}\nabla_X \xi) \\ &= \big(\nabla_X(\nabla_Y S)\big)(\xi) - (\nabla_{\widehat{\nabla}_X Y} S)(\xi) \end{aligned} \tag{2.8}$$

where $X, Y \in \mathscr{X}(M)$ and $\xi \in \Gamma(E_1)$.

2.5.4 The Levi–Civita Connection

When working on a Riemannian manifold, it is desirable to work with a particular connection that reflects the geometric properties of the metric. To do so, one needs the notions of compatibility and symmetry.

Definition 2.34. A connection ∇ on a vector bundle E is said to be *compatible* with a metric g on E if for any $\xi, \eta \in \Gamma(E)$ and $X \in \mathscr{X}(M)$,

$$X(g(\xi,\eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta).$$

Moreover, if ∇ is compatible with a metric g on E, then the induced connection on E^* is compatible with the induced metric on E^*. Also, if the connections on two vector bundles are compatible with given metrics, then the connection on the tensor product is compatible with the tensor product metric.

Unfortunately compatibility by itself is not enough to determine a unique connection. To get uniqueness we also need the connection to be symmetric.

Definition 2.35. A connection ∇ on TM is *symmetric* if its torsion vanishes.[5] That is, if $\nabla_X Y - \nabla_Y X = [X,Y]$ or equivalently $\Gamma_{ij}^k = \Gamma_{ji}^k$.

We can now state the fundamental theorem of Riemannian geometry.

Theorem 2.36. *Let (M,g) be a Riemannian manifold. There exists a unique connection ∇ on TM which is symmetric and compatible with g. This connection is referred to as* the Levi–Civita connection *of g.*

The reason why this connection has been anointed *the* Riemannian connection is that the symmetry and compatibility conditions are invariantly defined natural properties that force the connection to coincide with the tangential connection, whenever M is realised as a submanifold of $\mathbb{R}^n$ with the induced metric (which is always possible by the Nash embedding).

Proposition 2.37. *In local coordinate (x^i), the Christoffel symbols of the Levi–Civita connection are given by*

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}). \tag{2.9}$$

[5] The torsion τ of ∇ is defined by $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. τ is a $(2,1)$-tensor field since $\nabla_{fX}(gY) - \nabla_{gY} fX - [fX, gY] = fg(\nabla_X Y - \nabla_Y X - [X,Y])$.

2.6 Connection Laplacian

In its simplest form, the Laplacian Δ of $f \in C^\infty(M)$ is defined by $\Delta f = \operatorname{div}\operatorname{grad} f$. In fact, the Laplacian can be extended to act on tensor bundles over a Riemannian manifold (M, g). The resulting differential operator is referred to as the *connection Laplacian.* Note that there are a number of other second-order, linear, elliptic differential operators bearing the name Laplacian which have alternative definitions.

Definition 2.38. For any tensor field $F \in \mathscr{T}_\ell^k(M)$, the connection Laplacian

$$\Delta F = \operatorname{tr}_g \nabla^2 F \tag{2.10}$$

is the trace of the second covariant derivative with the metric g.

Explicitly,

$$\begin{aligned}
(\Delta F)^{j_1 \ldots j_\ell}{}_{i_1 \ldots i_k} &= (\operatorname{tr}_g \nabla^2 F)^{j_1 \cdots j_\ell}{}_{i_1 \ldots i_k} \\
&= \left(\operatorname{tr}_{13} \operatorname{tr}_{24} g^{-1} \otimes \nabla^2 F\right)^{j_1 \cdots j_\ell}{}_{i_1 \ldots i_k} \\
&= g^{pq} (\nabla_{\partial_p} \nabla_{\partial_q} F)(\partial_{j_1}, \ldots, \partial_{j_\ell}, dx^{i_1}, \ldots, dx^{i_k}).
\end{aligned}$$

Example 2.39. If the tensor bundle is $T^0 M = C^\infty(M)$, then (2.5) implies that

$$\begin{aligned}
\Delta f &= g^{ij} \nabla_{\partial_i} \nabla_{\partial_j} f \\
&= g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right).
\end{aligned}$$

2.7 Curvature

We introduce the curvature tensor as a purely algebraic object that arises from a connection on a vector bundle. From this we will look at the curvature on specific bundle structures.

2.7.1 Curvature on Vector Bundles

Definition 2.40. Let E be a vector bundle over M. If ∇ is a connection on E, then the curvature of the connection ∇ on the bundle E is the section $R_\nabla \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ defined by

$$R_\nabla(X, Y)\xi = \nabla_Y (\nabla_X \xi) - \nabla_X (\nabla_Y \xi) + \nabla_{[X,Y]} \xi. \tag{2.11}$$

In the literature, there is much variation in the sign convention; some define the curvature to be of opposite sign to ours.

2.7.2 Curvature on Dual and Tensor Product Bundles

The curvature on a dual bundle E^*, with respect to the dual connection, is characterised by the formula

$$0 = (R(X,Y)\omega)(\xi) + \omega(R(X,Y)\xi)$$

for all $X, Y \in \mathscr{X}(M)$, $\omega \in \Gamma(E^*)$ and $\xi \in \Gamma(E)$.

The curvature on a tensor product bundle $E_1 \otimes E_2$, with connection ∇ given by Proposition 2.33, can be computed in terms of the curvatures on each of the factors by the formula

$$R_\nabla(X,Y)(\xi_1 \otimes \xi_2) = (R_{\nabla^{(1)}}(X,Y)\xi_1) \otimes \xi_2 + \xi_1 \otimes (R_{\nabla^{(2)}}(X,Y)\xi_2)\,,$$

where $X, Y \in \mathscr{X}(M)$ and $\xi_i \in \Gamma(E_i)$, $i = 1, 2$.

Example 2.41. Of particular interest, the curvature on $E_1^* \otimes E_2$ (E_2-valued tensors acting on E_1) is given by

$$(R(X,Y)S)(\xi) = R_{\nabla^{(2)}}(X,Y)\big(S(\xi)\big) - S\big(R_{\nabla^{(1)}}(X,Y)\xi\big), \tag{2.12}$$

where $S \in \Gamma(E_1^* \otimes E_2)$, $\xi \in \Gamma(E_1)$ and $X, Y \in \mathscr{X}(M)$.

2.7.3 Curvature on the Tensor Bundle

One of the most important applications is the curvature of the tensor bundle. By (2.11) and Sect. 2.5.1, we have the following:

Proposition 2.42. *Let R be the curvature on the (k,ℓ)-tensor bundle. If $F, G \in \mathscr{T}_\ell^k(M)$ are tensors, then*

$$R(X,Y)(\operatorname{tr} F) = \operatorname{tr}(R(X,Y)F)$$
$$R(X,Y)(F \otimes G) = (R(X,Y)F) \otimes G + F \otimes (R(X,Y)G)$$

for any vector fields X and Y.

Moreover we also have the following important formulas.

Proposition 2.43. *Let R be the curvature on the (k,ℓ)-tensor bundle. If $F \in \mathscr{T}_\ell^k(M)$, then*

$$\begin{aligned} R(X,Y)\big(F(\omega^1,\dots,\omega^\ell,Z_1,\dots,Z_k)\big) &= (R(X,Y)F)(\omega^1,\dots,\omega^\ell,Z_1,\dots,Z_k) \\ &\quad + \sum_{j=1}^{\ell} F(\omega^1,\dots,R(X,Y)\omega^j,\dots,\omega^\ell,Z_1,\dots,Z_k) \\ &\quad + \sum_{i=1}^{k} F(\omega^1,\dots,\omega^\ell,Z_1,\dots,R(X,Y)Z_i,\dots,Z_k) \end{aligned}$$

for any vector fields X, Y, Z_i and 1-forms ω^j.

Proof. Let the vector bundle $E = T_\ell^k M$, so for any $\xi \in \Gamma(E)$ we find that

$$\begin{aligned} R(X,Y)\big(F(\xi)\big) &= R(X,Y)\big(\operatorname{tr} F \otimes \xi\big) \\ &= \operatorname{tr}\big[(R(X,Y)F) \otimes \xi + F \otimes (R(X,Y)\xi)\big] \\ &= (R(X,Y)F)(\xi) + F(R(X,Y)\xi). \end{aligned}$$

As ξ takes the form

$$\xi = \omega^1 \otimes \cdots \otimes \omega^\ell \otimes Z_1 \otimes \cdots \otimes Z_k,$$

a similar argument shows that $F(R(X,Y)\xi) = F(R(X,Y)\omega^1,\dots,Z_k) + \cdots + F(\omega^1,\dots,R(X,Y)Z_k)$ from which the result follows. □

Proposition 2.44. *Let R be the curvature on the (k,ℓ)-tensor bundle. If the connection ∇ on TM is symmetric, then*

$$R(X,Y) = \nabla^2_{Y,X} - \nabla^2_{X,Y} \tag{2.13}$$

and so $R(X,Y) : \mathscr{T}_\ell^k(M) \to \mathscr{T}_\ell^k(M)$.

Proof. For any (k,ℓ)-tensor F, we see by (2.6) that

$$\begin{aligned} \nabla^2_{Y,X}F - \nabla^2_{X,Y}F &= \nabla_Y(\nabla_X F) - \nabla_{\nabla_Y X}F - \nabla_X(\nabla_Y F) + \nabla_{\nabla_X Y}F \\ &= \nabla_Y(\nabla_X F) - \nabla_X(\nabla_Y F) + \nabla_{[X,Y]}F. \end{aligned}$$ □

Example 2.45. The curvature of $C^\infty(M) = T^0 M$ vanishes since

$$R(\partial_i,\partial_j)f = \nabla_{\partial_i}(\nabla_{\partial_j} f) - \nabla_{\partial_j}(\nabla_{\partial_i} f) + \cancel{\nabla_{[\partial_i,\partial_j]} f} = \partial_i\partial_j f - \partial_j\partial_i f = 0$$

for any $f \in C^\infty(M)$.

2.7.4 Riemannian Curvature

If (M, g) is a Riemannian manifold, the curvature $R \in \Gamma(\otimes^3 T^*M \otimes TM)$ of the Levi–Civita connection ∇ on TM is a $(3,1)$-tensor field that, in local coordinates (x^i), takes the form

$$R = R_{ijk}{}^{\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes \partial_\ell,$$

where $R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{\ell}\partial_\ell$. Accompanying this is the *Riemann curvature tensor*, also denoted by R. It is a covariant $(4,0)$-tensor field defined by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

for all $W, X, Y, Z \in \mathscr{X}(M)$. In local coordinates (x^i) it can be expressed as

$$R = R_{ijk\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell,$$

where $R_{ijk\ell} = g_{\ell p} R_{ijk}{}^{p}$.

Lemma 2.46. *In local coordinates (x^i), the curvature of the Levi–Civita connection can be expressed as follows:*

$$\begin{aligned} R_{ijk}{}^{\ell} &= \partial_j \Gamma^\ell_{ik} - \partial_i \Gamma^\ell_{jk} + \Gamma^m_{ik}\Gamma^\ell_{jm} - \Gamma^m_{jk}\Gamma^\ell_{im} \\ R_{ijk\ell} &= \frac{1}{2}\big(\partial_j\partial_k g_{i\ell} + \partial_i\partial_\ell g_{jk} - \partial_i\partial_k g_{j\ell} - \partial_j\partial_\ell g_{ik}\big) + g_{\ell p}(\Gamma^m_{ik}\Gamma^p_{jm} - \Gamma^m_{jk}\Gamma^p_{im}) \end{aligned}$$

2.7.4.1 Symmetries of the Curvature Tensor

The curvature tensor possesses a number of important symmetry properties. They are:

(a) Antisymmetric in first two arguments: $R_{ijk\ell} + R_{jik\ell} = 0$
(b) Antisymmetric in last two arguments: $R_{ijk\ell} + R_{ji\ell k} = 0$
(c) Symmetry between the first and last pair of arguments: $R_{ijk\ell} = R_{k\ell ij}$

In addition to this, there are also the ‘cyclic’ Bianchi identities:

(d) First Bianchi identity: $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$
(e) Second Bianchi identity: $\nabla_m R_{ijk\ell} + \nabla_k R_{ij\ell m} + \nabla_\ell R_{ijmk} = 0$

2.7.5 Ricci and Scalar Curvature

As the curvature tensor can be quite complicated, it is useful to consider various contractions that summarise some of the information contained in the curvature tensor. The first of these contractions is the *Ricci tensor*, denoted by Ric. It is defined as

$$\mathrm{Ric}(X,Y) = \mathrm{tr}_g\, R(X,\cdot,Y,\cdot) = (\mathrm{tr}_{14}\mathrm{tr}_{26}\, g^{-1} \otimes R)(X,Y).$$

In component form, $\mathrm{Ric}(\partial_i, \partial_j) = R_{ij} = R_{ikj}{}^{k} = g^{pq} R_{ipjq}$. From the symmetry properties of R it is clear that Ric is symmetric.

A further trace of the Ricci tensor gives a scalar quantity called the *scalar curvature*, denoted by Scal:

$$\mathrm{Scal} = \mathrm{tr}_g\, \mathrm{Ric} = \mathrm{Ric}_i{}^{i} = g^{ij} R_{ij}.$$

It is important to note that if the curvature tensor is defined with opposite sign, the contraction is defined so that the Ricci tensor matches the one given here. Hence the Ricci tensor has the same meaning for everyone. By Lemma 2.46, the Ricci tensor can be expressed locally as follows.

Lemma 2.47. *In local coordinates* (x^i)*, the Ricci tensor takes the form*

$$R_{ik} = \frac{1}{2} g^{j\ell} \left(\frac{\partial^2 g_{i\ell}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^\ell} - \frac{\partial^2 g_{j\ell}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^\ell} \right) + \Gamma_{ik}^m \Gamma_{jm}^j - \Gamma_{jk}^m \Gamma_{im}^j.$$

2.7.5.1 Contraction Commuting with Covariant Derivative

As we are working with a compatible connection, $\nabla g \equiv 0$. Thus one can commute covariant derivatives with metric contractions.

Proposition 2.48. *If* ∇ *is the Levi–Civita connection, then*

$$\nabla_k R_{ij} = g^{pq}\, \nabla_k R_{ipjq} \tag{2.14}$$

$$\nabla^2_{k,\ell} R_{ij} = g^{pq}\, \nabla^2_{k,\ell} R_{ipjq}. \tag{2.15}$$

Proof. To show (2.14), let $X, Y, Z \in \mathscr{X}(M)$ so that

$$\begin{aligned}
(\nabla_Z \mathrm{Ric})(X,Y) &= (\nabla_Z\, (\mathrm{tr}\, g^{-1} \otimes R))(X,Y) \\
&= \big(\mathrm{tr}\, \nabla_Z (g^{-1} \otimes R)\big)(X,Y) \\
&= \big(\mathrm{tr}\, \cancel{\nabla_Z g^{-1}} \otimes R + \mathrm{tr}\, g^{-1} \otimes \nabla_Z R\big)(X,Y) \\
&= (\mathrm{tr}_{14}\mathrm{tr}_{26}\, g^{-1} \otimes \nabla_Z R)(X,Y) \\
&= (\mathrm{tr}_g \nabla_Z R)(X,\cdot,Y,\cdot).
\end{aligned}$$

Similarly, to show (2.15) note that

$$\nabla^2\mathrm{Ric} = \nabla^2(\mathrm{tr}\, g^{-1} \otimes R) = \mathrm{tr}\left(\nabla^2 g^{-1} \otimes R + 2\,\nabla g^{-1} \otimes \nabla R + g^{-1} \otimes \nabla^2 R\right)$$
$$= \mathrm{tr}\, g^{-1} \otimes \nabla^2 R. \qquad \square$$

In later applications we will need the *contracted second Bianchi identity*:

$$g^{jk}\nabla_k \mathrm{Ric}_{ij} = \frac{1}{2}\nabla_i \mathrm{Scal}. \tag{2.16}$$

This follows easily from (2.14) and the second Bianchi identity, since

$$\begin{aligned} 0 &= g^{am}g^{bn}(\nabla_\ell R_{abmn} + \nabla_m R_{abn\ell} + \nabla_n R_{ab\ell m}) \\ &= g^{am}(\nabla_\ell \mathrm{Ric}_{am} - \nabla_m \mathrm{Ric}_{a\ell}) + g^{am}g^{bn}\nabla_n R_{ab\ell m} \\ &= \nabla_\ell \mathrm{Ric}_a{}^a - g^{am}\nabla_m \mathrm{Ric}_{a\ell} - g^{bn}\nabla_n R_{\ell m b}{}^m \\ &= \nabla_\ell \mathrm{Scal} - g^{am}\nabla_m \mathrm{Ric}_{a\ell} - g^{bn}\nabla_n \mathrm{Ric}_{\ell b} \end{aligned}$$

from which (2.16) now follows.

2.7.6 Sectional Curvature

Suppose (M, g) is a Riemannian manifold. If Π is a two-dimensional subspace of T_pM, we define the *sectional curvature* K of Π to be

$$K(\Pi) = R(e_1, e_2, e_1, e_2),$$

where $\{e_1, e_2\}$ is an orthonormal basis for Π. By a rotation or reflection in the plane, one can show K is independent of the choice of basis. We refer to the oriented plane generated from e_i and e_j by the notation $e_i \wedge e_j$ (cf. Sect. C.3). Furthermore if $\{u, v\}$ is any basis for the 2-plane Π, one has

$$K(u \wedge v) = \frac{R(u, v, u, v)}{|u|^2|v|^2 - g(u, v)^2}.$$

If $U \subset T_pM$ is a neighbourhood of zero on which $\exp_p$ is a diffeomorphism, then $S_\Pi := \exp_p(\Pi \cap U)$ is a 2-dimensional submanifold of M containing p, called the *plane of section* determined by Π. That is, it is the surface swept out by geodesics whose initial tangent vectors lie in Π. One can geometrically interpret the sectional curvature of M associated to Π to be the Gaussian curvature of the surface S_Π at p with the induced metric.

By computing the sectional curvature of the plane $\frac{1}{2}(e_i + e_k) \wedge (e_j + e_\ell)$ one can show:

Proposition 2.49. *The curvature tensor R is completely determined by the sectional curvature. In particular,*

$$\begin{aligned} R_{ijk\ell} &= \frac{1}{3}K\Big(\frac{(e_i + e_k) \wedge (e_j + e_\ell)}{2}\Big) + \frac{1}{3}K\Big(\frac{(e_i - e_k) \wedge (e_j - e_\ell)}{2}\Big) \\ &\quad - \frac{1}{3}K\Big(\frac{(e_j + e_k) \wedge (e_i + e_\ell)}{2}\Big) - \frac{1}{3}K\Big(\frac{(e_j - e_k) \wedge (e_i - e_\ell)}{2}\Big) \\ &\quad - \frac{1}{6}K(e_j \wedge e_\ell) - \frac{1}{6}K(e_i \wedge e_k) + \frac{1}{6}K(e_i \wedge e_\ell) + \frac{1}{6}K(e_j \wedge e_k). \end{aligned}$$

One can also show that the scalar curvature $\mathrm{Scal} = \mathrm{Ric}_j{}^j = \sum_{j \neq k} K(e_j \wedge e_k)$, where (e_i) is orthonormal basis for T_pM.

Each of the model spaces $\mathbb{R}^n$, S^n and $\mathbb{H}^n$ has an isometry group that acts transitively on orthonormal frames, and so acts transitively on 2-planes in the tangent bundle. Therefore each has a *constant sectional curvature* – in the sense that the sectional curvatures are the same for all planes at all points.

It is well known that the Euclidean space $\mathbb{R}^n$ has constant zero sectional curvature (this is geometrically intuitive as each 2-plane section has zero Gaussian curvature). The sphere S^n of radius 1 has constant sectional curvature equal to 1 and the hyperbolic space $\mathbb{H}^n$ has constant sectional curvature equal to -1.

2.7.7 Berger's Lemma

A simple but important result is the so-called lemma of Berger [Ber60b, Sect. 6]. Following [Kar70], we show the curvature can be bounded whenever the sectional curvature is bounded from above and below. That is, if a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ has sectional curvature bounds $\delta = \min_{u,v \in T_pM} K(u \wedge v)$ and $\Delta = \max_{u,v \in T_pM} K(u \wedge v)$ with the assumption $\delta \geq 0$, we prove the following bounds on the curvature tensor:

Lemma 2.50 (Berger). *For orthonormal $u, v, w, x \in T_pM$, one can bound the curvature tensor by*

$$|R(u, v, w, v)| \leq \frac{1}{2}(\Delta - \delta) \tag{2.17}$$

$$|R(u, v, w, x)| \leq \frac{2}{3}(\Delta - \delta). \tag{2.18}$$

Proof. From the symmetries of the curvature tensor we find that:

$$\begin{aligned}4R(u,v,w,v) &= R(u+w,v,u+w,v) - R(u-w,v,u-w,v)\\ 6R(u,v,w,x) &= R(u,v+x,w,v+x) - R(u,v-x,w,v-x)\\ &\quad - R(v,u+x,w,u+x) + R(v,u-x,w,u-x).\end{aligned}$$

Using the definition of the sectional curvature together with the first identity gives

$$\begin{aligned}2|R(u,v,w,v)| &= \frac{1}{2}\Big|K\big((u+w)\wedge v\big)\big(|u+w|^2|v|^2 - \cancel{\langle u+w,v\rangle}\big)\\ &\quad - K\big((u-w)\wedge v\big)\big(|u-w|^2|v|^2 - \cancel{\langle u-w,v\rangle}\big)\Big|\\ &= \big|K\big((u+w)\wedge v\big) - K\big((u-w)\wedge v\big)\big|\\ &\le (\Delta - \delta)\end{aligned}$$

which is identity (2.17). To prove (2.18), use the second identity and apply (2.17) to the four terms – whilst taking into consideration $|v\pm x|^2 = |u\pm x|^2 = 2$, for orthonormal u, v, w and x. □

2.8 Pullback Bundle Structure

Let M and N be smooth manifolds, let E be a vector bundle over N and f be a smooth map from M to N.

Definition 2.51. The pullback bundle of E by f, denoted f^*E, is the smooth vector bundle over M defined by $f^*E = \{(p,\xi) : \ p \in M, \xi \in E, \pi(\xi) = f(p)\}$. If $\xi_1, \ldots, \xi_k$ are a local frame for E near $f(p) \in N$, then $\Xi_i(p) = \xi_i(f(p))$ are a local frame for $f^*(E)$ near p.

Lemma 2.52. *Pullbacks commute with taking duals and tensor products:*

$$(f^*E)^* = f^*(E^*) \qquad \textit{and} \qquad (f^*E_1)\otimes(f^*E_2) = f^*(E_1\otimes E_2).$$

2.8.1 Restrictions

The *restriction* $\xi_f \in \Gamma(f^*E)$ of $\xi \in \Gamma(E)$ to f is defined by

$$\xi_f(p) = \xi(f(p)) \in E_{f(p)} = (f^*E)_p,$$

for all points $p \in M$.

Example 2.53. Suppose g is a metric on E. Then $g \in \Gamma(E^* \otimes E^*)$, and by restriction we obtain $g_f \in \Gamma((f^*E)^* \otimes (f^*E)^*)$, which is a metric on f^*E (the 'restriction of g to f'): If $\xi, \eta \in (f^*E)_p = E_{f(p)}$, then $(g_f(p))(\xi,\eta) = (g(f(p))(\xi,\eta)$.

Remark 2.54. In using this terminology one wants to distinguish the restriction of a tensor field on E (which is a section of a tensor bundle over f^*E) with the pullback of a tensor on the tangent bundle, which is discussed below. Thus the metric in the above example should not be called the 'pullback metric'. Notice that we can restrict both covariant and contravariant tensors, in contrast to the situation with pullbacks.

2.8.2 Pushforwards

If $f: M \to N$ is smooth, then for each $p \in M$, we have the linear map $f_*(p): T_pM \to T_{f(p)}N = (f^*TN)_p$. That is, $f_*(p) \in T_p^*M \otimes (f^*TN)_p$, so f_* is a smooth section of $T^*M \otimes f^*TN$. Given a section $X \in \Gamma(TM) = \mathscr{X}(M)$, the *pushforward* of X is the section $f_*X \in \Gamma(f^*TN)$ given by applying f_* to X.

2.8.3 Pullbacks of Tensors

By duality (combined with restriction) we can define an operation taking $(k,0)$-tensors on N to $(k,0)$-tensors on M, which we call the *pullback* operation: If S is a $(k,0)$-tensor on N (i.e. $S \in \Gamma(\otimes^k T^*N)$), then by restriction we have $S_f \in \Gamma(\otimes^k(f^*T^*N))$, and we define $f^*S \in \Gamma(\otimes^k T^*M)$ by

$$f^*S(X_1,\ldots,X_k) = S_f(f_*X_1,\ldots,f_*X_k) \tag{2.19}$$

where X_i are vector fields on M.

Example 2.55. If f is an embedding and g is a Riemannian metric on N, then f^*g is the pullback metric on M (often called the 'induced metric').

This definition can be extended a little to include bundle-valued tensors: Suppose $S \in \Gamma(\otimes^k T^*N \otimes E)$ is an E-valued $(k,0)$-tensor field on N. Then restriction gives $S_f \in \Gamma(\otimes^k(f^*T^*N) \otimes f^*E)$, and we will denote by f^*S the f^*E-valued k-tensor on M defined by

$$f^*S(X_1,\ldots,X_k) = S_f(f^*X_1,\ldots,f^*X_k).$$

That is, the same formula as before except now both sides are f^*E-valued.

2.8.4 The Pullback Connection

Let ∇ be a connection on E over N, and $f : M \to N$ a smooth map.

Theorem 2.56. *There is a unique connection ${}^f\nabla$ on f^*E, referred to as the pullback connection, such that*

$$ {}^f\nabla_v(\xi_f) = \nabla_{f_* v}\xi $$

for any $v \in TM$ and $\xi \in \Gamma(E)$.

Remark 2.57. To justify the term 'pullback connection': If $\xi \in \Gamma(E)$, then $\nabla\xi \in \Gamma(T^*N \otimes E)$, so the pullback gives

$$ {}^f\nabla\xi_f := f^*(\nabla\xi) \in \Gamma(T^*M \otimes f^*E). $$

To define ${}^f\nabla_v\xi$ for arbitrary $\xi \in \Gamma(f^*E)$, we fix $p \in M$ and choose a local frame $\sigma_1, \dots, \sigma_k$ about $f(p)$ for E. Then we can write $\xi = \sum_{i=1}^k \xi^i\,(\sigma_i)_f$ with each ξ^i a smooth function defined near p, so the rules for a connection together with the pullback connection condition give

$$ \begin{aligned} {}^f\nabla_v\xi &= {}^f\nabla_v(\xi^i\,(\sigma_i)_f) \\ &= \xi^i\,{}^f\nabla_v(\sigma_i)_f + v(\xi^i)\,(\sigma_i)_f \\ &= \xi^i\nabla_{f_* v}\sigma_i + v(\xi^i)\,(\sigma_i)_f. \end{aligned} \tag{2.20} $$

Note that pullback connections on duals and tensor products of pullback bundles agree with those obtained by applying the previous constructions to the pullback bundles on the factors.

There are two other important properties of the pullback connection:

Proposition 2.58. *If g is a metric on E and ∇ is a connection on E compatible with g, then ${}^f\nabla$ is compatible with the restriction metric g_f.*

Proof. As ∇ is compatible with g if and only if $\nabla g = 0$, we therefore must show that ${}^f\nabla g_f = 0$ if $\nabla g = 0$. However this is immediate, since ${}^f\nabla_v(g_f) = \nabla_{f_* v}g = 0$. □

Proposition 2.59. *The curvature of the pullback connection is the pullback of the curvature of the original connection. That is,*

$$ R_{{}^f\nabla}(X,Y)\xi_f = (f^*R_\nabla)(X,Y)\xi $$

where $X, Y \in \mathscr{X}(M)$ and $\xi \in \Gamma(E)$. Note that $R_\nabla \in \Gamma(T^*N \otimes T^*N \otimes E^* \otimes E)$ here, so that

$$ \begin{aligned} f^*(R_\nabla) &\in \Gamma(T^*M \otimes T^*M \otimes f^*(E^* \otimes E)) \\ &= \Gamma(T^*M \otimes T^*M \otimes (f^*E)^* \otimes f^*E). \end{aligned} $$

Proof. Since curvature is tensorial, it is enough to check the formula for a basis. Choose a local frame $\{\sigma_p\}_{p=1}^k$ for E, so that $\{(\sigma_p)_f\}$ is a local frame for f^*E. Also choose local coordinates $\{y^\alpha\}$ for N near $f(p)$ and $\{x^i\}$ for M near p, and write $f^\alpha = y^\alpha \circ f$. Then

$$\begin{aligned} R_{^f\nabla}(\partial_i, \partial_j)(\sigma_p)_f &= {}^f\nabla_{\partial_j}\left({}^f\nabla_{\partial_i}(\sigma_p)_f\right) - (i \leftrightarrow j) \\ &= {}^f\nabla_j\left(\nabla_{f_*\partial_i}\sigma_p\right) - (i \leftrightarrow j) \\ &= {}^f\nabla_j\left(\partial_i f^\alpha \nabla_\alpha \sigma_p\right) - (i \leftrightarrow j) \\ &= (\partial_j\partial_i f^\alpha)\nabla_\alpha\sigma_p + \partial_i f^\alpha\, {}^f\nabla_j\left((\nabla_\alpha\sigma_p)_f\right) - (i \leftrightarrow j) \\ &= \partial_i f^\alpha \nabla_{f_*\partial_j}\left(\nabla_\alpha\sigma_p\right) - (i \leftrightarrow j) \\ &= \partial_i f^\alpha \partial_j f^\beta\left(\nabla_\beta\left(\nabla_\alpha\sigma_p\right) - (\alpha \leftrightarrow \beta)\right) \\ &= \partial_i f^\alpha \partial_j f^\beta R_\nabla(\partial_\alpha, \partial_\beta)\sigma_p \\ &= R_\nabla(f_*\partial_i, f_*\partial_j)\sigma_p. \end{aligned}$$

□

When pulling back a tangent bundle, there is another important property:

Proposition 2.60. *If ∇ is a symmetric connection on TN, then the pull-back connection ${}^f\nabla$ on f^*TN is symmetric in the sense that*

$${}^f\nabla_U(f_*V) - {}^f\nabla_V(f_*U) = f_*([U, V])$$

for any $U, V \in \Gamma(TM)$.

Proof. As before choose local coordinates x^i for M near p, and y^α for N near $f(p)$. By writing $U = U^i\partial_i$ and $V = V^j\partial_j$ we find that

$$\begin{aligned} &{}^f\nabla_U(f_*V) - {}^f\nabla_V(f_*U) \\ &\quad = {}^f\nabla_U\left(V^j\partial_j f^\alpha\partial_\alpha\right) - (U \leftrightarrow V) \\ &\quad = U^i\partial_i\left(V^j\partial_j f^\alpha\right)\partial_\alpha + V^j\partial_j f^\alpha\, {}^f\nabla_U\partial_\alpha - (U \leftrightarrow V) \\ &\quad = (U^i\partial_i V^j - V^i\partial_i U^j)\partial_j f^\alpha\partial_\alpha + U^iV^j(\partial_i\partial_j f^\alpha - \partial_j\partial_i f^\alpha)\partial_\alpha \\ &\qquad + V^jU^i\partial_j f^\alpha\partial_i f^\beta\left(\nabla_\beta\partial_\alpha - \nabla_\alpha\partial_\beta\right) \\ &\quad = f_*\left([U, V]\right). \end{aligned}$$

□

2.8.5 Parallel Transport

Parallel transport is a way of using a connection to compare geometrical data at different points along smooth curves:

Let ∇ be a connection on the bundle $\pi : E \to M$. If $\gamma : I \to M$ is a smooth curve, then a *smooth section along* γ is a section of γ^*E. Associated to this is the pullback connection ${}^{\gamma}\nabla$. A section V along a curve γ is *parallel along* γ if ${}^{\gamma}\nabla_{\partial_t} V \equiv 0$. In a local frame (e_j) over a neighbourhood of $\gamma(t_0)$:

$$
{}^{\gamma}\nabla_{\partial_t} V(t_0) = \big(\dot{V}^k(t_0) + \Gamma^k_{ij}(\gamma(t_0))V^j(t_0)\dot{\gamma}^i(t_0)\big)e_k,
$$

where $V(t) = V^j(t)e_j$ and Γ^k_{ij} is the Christoffel symbol of ∇ in this frame.

Theorem 2.61. *Given a curve $\gamma : I \to M$ and a vector $V_0 \in E_{\gamma(0)}$, there exists a unique parallel section V along γ such that $V(0) = V_0$. Such a V is called the* parallel translate *of V_0 along γ.*

Moreover, for such a curve γ there exists a unique family of linear isomorphisms $P_t : E_{\gamma(0)} \to E_{\gamma(t)}$ such that a vector field V along γ is parallel if and only if $V(t) = P_t(V_0)$ for all t.

2.8.6 Product Manifolds' Tangent Space Decomposition

Given manifolds M_1 and M_2, let $\pi_j : M_1 \times M_2 \to M_j$ be the standard smooth projection maps. The pushforward $(\pi_j)_* : T(M_1 \times M_2) \to TM_j$ over $\pi_j : M_1 \times M_2 \to M_j$ induces, via the pullback bundle, a smooth bundle morphism

$$
\Pi_j : T(M_1 \times M_2) \to \pi_j^*(TM_j)
$$

over $M_1 \times M_2$. In which case one has the bundle morphism

$$
\Pi_1 \oplus \Pi_2 : T(M_1 \times M_2) \to \pi_1^*(TM_1) \oplus \pi_2^*(TM_2)
$$

over $M_1 \times M_2$. On fibres over (x_1, x_2) this is simply the pointwise isomorphism $T_{(x_1,x_2)}(M_1 \times M_2) \simeq T_{x_1}M_1 \oplus T_{x_2}M_2$, so $\Pi_1 \oplus \Pi_2$ is in fact a smooth bundle isomorphism. Furthermore, $\Pi_j : T(M_1 \times M_2) \to \pi_j^*(TM_j)$ is bundle surjection as π_j is a projection. Thus by Proposition 2.14 there exists a well-defined subbundle E_1 inside $T(M_1 \times M_2)$ given by

$$
E_1 = \ker \Pi_1 = \{v \in T(M_1 \times M_2) : \Pi_1(v) = 0\}.
$$

We observe that this is in fact equal to $\pi_2^*(TM_2)$, since E_1 consists of all vectors such that Π_1 projection vanishes. Therefore one has isomorphic vector bundles

$$
T(M_1 \times M_2) \simeq \ker \Pi_1 \oplus \ker \Pi_2.
$$

Example 2.62. If we let $M_1 = M$ and $M_2 = \mathbb{R}$ with $\pi_1 = \pi$ projection from $M \times \mathbb{R}$ onto M and $\pi_2 = t$ be the projection from $M \times \mathbb{R}$ onto $\mathbb{R}$, then on fibres over $(x_1, x_2) = (x_0, t_0)$ we have that

$$\Pi_2(v) = t_*(v) = dt(v)$$

since t is a $\mathbb{R}$-valued function on $M \times \mathbb{R}$. So by letting $\mathfrak{S} = \ker dt = \{v \in T(M \times \mathbb{R}) : dt(v) = 0\}$ we have that

$$T(M \times \mathbb{R}) \simeq \mathfrak{S} \oplus \mathbb{R}\,\partial_t, \tag{2.21}$$

since $(\pi_2^* T\mathbb{R})_{(x_0,t_0)} = (t^* T\mathbb{R})_{(x_0,t_0)} = T_{t_0}\mathbb{R} = \mathbb{R}\,\partial_t|_{t_0}$ where $\partial_t|_{t_0}$ is the standard coordinate basis for $T_{t_0}\mathbb{R}$.

2.8.7 Connections and Metrics on Subbundles

Suppose F is a subbundle of a vector bundle E over a manifold M, as defined in Definition 2.12. If E is equipped with a metric g, then there is a natural metric induced on F by the inclusion: If $\iota: F \to E$ is the inclusion of F in E, then the induced metric on F is defined by $g_F(\xi, \eta) = g(\iota(\xi), \iota(\eta))$.

There is not in general any natural way to induce a connection on F from a connection ∇ on E. We will consider only the following special case:[6]

Definition 2.63. A subbundle F of a vector bundle E is called *parallel* if F is invariant under parallel transport, i.e. for any smooth curve $\sigma: [0,1] \to M$, and any parallel section ξ of $\sigma^* E$ over $[0,1]$ with $\xi(0) \in F_{\sigma(0)}$, we have $\xi(t) \in F_{\sigma(t)}$ for all $t \in [0,1]$.

One can check that a subbundle F is parallel if and only if the connection on E maps sections of F to F, i.e. $\nabla_u(\iota\xi) \in \iota(F_p)$ for any $u \in T_pM$ and $\xi \in \Gamma(F)$. If F is parallel there is a unique connection ∇^F on F such that

$$\iota \nabla^F_u \xi = \nabla_u (\iota\xi)$$

for every $u \in TM$ and $\xi \in \Gamma(F)$. Note also that if ∇ is compatible with a metric g on E, and F is a parallel subbundle of E, then ∇^F is compatible with the induced metric g_F.

Example 2.64. An important example which will reappear later (cf. Sect. 7.5) is the following: Let E be a vector bundle with connection ∇. Then the bundle of symmetric 2-tensors on E is a parallel subbundle of the bundle of 2-tensors

[6] Although natural constructions can be done much more generally, for example when a pair of complementary subbundles is supplied.

on E. To prove this we need to check that $\nabla_U T$ is symmetric whenever T is a symmetric 2-tensor. But this is immediate: We have for any $X, Y \in \Gamma(E)$ and $U \in \mathscr{X}(M)$,

$$\begin{aligned}(\nabla_U T)(X,Y) &= U(T(X,Y)) - T(\nabla_U X, Y) - T(X, \nabla_U Y)\\ &= U(T(Y,X)) - T(\nabla_U Y, X) - T(Y, \nabla_U X)\\ &= (\nabla_U T)(Y,X),\end{aligned}$$

so $\nabla_U T$ is symmetric as required.

2.8.8 The Taylor Expansion of a Riemannian Metric

As an application of the pullback structure seen in this section, we compute the Taylor expansion of the Riemannian metric in exponential normal coordinates. In particular the curvature tensor is an obstruction to the existence of local coordinates in which the second derivatives of the metric tensor vanish.

Theorem 2.65. *Let (M, g) be a Riemannian manifold. With respect to a geodesic normal coordinates system about $p \in M$, the metric g_{ij} may be expressed as:*

$$g_{ij}(u^1, \ldots, u^n) = \delta_{ij} - \frac{1}{3} R_{ikj\ell} u^k u^\ell + O(\|u\|^3).$$

Remark 2.66. The proof produces a complete Taylor expansion about $u = 0$ (see also [LP87, pp. 60-1]).

Proof. Consider $\varphi : \mathbb{R}^2 \to M$ defined by $\varphi(s,t) = \exp_p(tV(s))$ where $V(s) \in S^{n-1} \subset T_p M$. Let $V(0) = u$, $V'(0) = v$.

As $\varphi_* : T_{(s,t)}\mathbb{R}^2 \to (\varphi^* TM)_{(s,t)}$, $\varphi_* \partial_s = (\exp_p)_*(tV'(s)) = t\,\partial_s V^i(s)(\partial_i)_\varphi$. Thus we find that the pullback metric of g via φ is

$$\begin{aligned}(\varphi^* g)(\partial_s, \partial_s)\big|_{(s,t)} &= g_\varphi(\varphi_*\partial_s, \varphi_*\partial_s)\big|_{(s,t)}\\ &= t^2\, \partial_s V^i \partial_s V^j g_{ij}(\varphi(s,t)). \qquad (2.22)\end{aligned}$$

Note that ${}^\varphi\nabla g_\varphi \in \Gamma(T\mathbb{R}^2 \otimes (\varphi^* TM)^* \otimes (\varphi^* TM)^*)$, so by Proposition 2.58:

$$\begin{aligned}0 &= ({}^\varphi\nabla g_\varphi)(\partial_t, \varphi_*\partial_s, \varphi_*\partial_s)\\ &= \partial_t g_\varphi(\varphi_*\partial_s, \varphi_*\partial_s) - g_\varphi({}^\varphi\nabla_{\partial_t}\varphi_*\partial_s, \varphi_*\partial_s) - g_\varphi(\varphi_*\partial_s, {}^\varphi\nabla_{\partial_t}\varphi_*\partial_s).\end{aligned}$$

So by taking $(\partial_t)^k$ derivatives of (2.22), we find that

$$
\begin{aligned}
&(\partial_t)^k g_\varphi(\varphi_*\partial_s, \varphi_*\partial_s) \\
&\quad = \sum_{\ell=0}^{k} \binom{k}{\ell} g_\varphi({}^\varphi\nabla^{(k-\ell)}_{\partial_t}\varphi_*\partial_s, {}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s) \\
&\quad = \Big(k(k-1)g^{(k-2)}_{ij}(t) + 2k\,t g^{(k-1)}_{ij}(t) + t^2 g^{(k)}_{ij}(t)\Big)\partial_s V^i \partial_s V^j.
\end{aligned}
$$

Evaluating this expression at $(s,t) = (0,0)$ gives

$$
k(k-1)g^{(k-2)}_{ij}(0)v^i v^j = \sum_{\ell=0}^{k} \binom{k}{\ell} g_\varphi({}^\varphi\nabla^{(k-\ell)}_{\partial_t}\varphi_*\partial_s, {}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s). \tag{2.23}
$$

Note that $\varphi_*\partial_s\big|_{(0,0)} = 0$, $\varphi_*\partial_t\big|_{(0,0)} = u$, ${}^\varphi\nabla_{\partial_t}\varphi_*\partial_t\big|_{(0,0)} = 0$ and

$$
{}^\varphi\nabla_{\partial_t}\varphi_*\partial_s\big|_{(0,0)} = (\partial_t(\varphi_*\partial_s)^i)(\partial_i)_\varphi\big|_{(0,0)} = v.
$$

We now claim:

Claim 2.67. Under the assumption ${}^\varphi\nabla_{\partial_t}\varphi_*\partial_t \equiv 0$,

$$
\begin{aligned}
&{}^\varphi\nabla^{(k)}_{\partial_t}\varphi_*\partial_s = \\
&\quad \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} (\nabla^{(k-2-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t, \ldots, \varphi_*\partial_t}_{k-2-\ell \text{ times}}, {}^\varphi\nabla^{(\ell)}_{\partial_t}(\varphi_*\partial_s), \varphi_*\partial_t\big)\varphi_*\partial_t.
\end{aligned}
$$

Proof of Claim. The case $k = 2$ is proved as follows:

$$
\begin{aligned}
{}^\varphi\nabla^{(2)}_{\partial_t}\varphi_*\partial_s &= {}^\varphi\nabla_{\partial_t}({}^\varphi\nabla_{\partial_s}\varphi_*\partial_t) && \text{by Proposition 2.60} \\
&= {}^\varphi\nabla_{\partial_s}({}^\varphi\nabla_{\partial_t}\varphi_*\partial_t) + {}^{\varphi\nabla}R(\partial_s, \partial_t)(\varphi_*\partial_t) && \text{by definition of } {}^{\varphi\nabla}R \\
&= {}^{\varphi\nabla}R(\partial_s, \partial_t)(\varphi_*\partial_t) && \text{by assumption} \\
&= {}^{\nabla}R(\varphi_*\partial_s, \varphi_*\partial_t)(\varphi_*\partial_t) && \text{by Proposition 2.59}
\end{aligned}
$$

as required. For the inductive step we suppose the identity is true for $k = j$, and differentiate. Since ${}^\varphi\nabla_{\partial_t}\varphi_*\partial_t = 0$, we find:

$$
\begin{aligned}
&{}^\varphi\nabla_{\partial_t}\left(\sum_{\ell=0}^{j-2} \binom{j-2}{\ell} (\nabla^{(j-2-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t, \ldots, \varphi_*\partial_t}_{j-2-\ell \text{ times}}, {}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s, \varphi_*\partial_t\big)\varphi_*\partial_t\right) \\
&= \sum_{\ell=0}^{j-2} \binom{j-2}{\ell} {}^\varphi\nabla_{\partial_t}(\nabla^{(j-2-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t, \ldots, \varphi_*\partial_t}_{j-2-\ell \text{ times}}, {}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s, \varphi_*\partial_t\big)\varphi_*\partial_t
\end{aligned}
$$

$$+\sum_{\ell=0}^{j-2}\binom{j-2}{\ell}(\nabla^{(j-2-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t,\ldots,\varphi_*\partial_t}_{j-2-\ell\text{ times}},{}^\varphi\nabla^{(\ell+1)}_{\partial_t}\varphi_*\partial_s,\varphi_*\partial_t\big)\varphi_*\partial_t$$

$$=\sum_{\ell=0}^{j-2}\binom{j-2}{\ell}(\nabla^{(j-1-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t,\ldots,\varphi_*\partial_t}_{j-1-\ell\text{ times}},{}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s,\varphi_*\partial_t\big)\varphi_*\partial_t$$

$$+\sum_{\ell=0}^{j-2}\binom{j-2}{\ell}(\nabla^{(j-2-\ell)}R)_\varphi(\underbrace{\varphi_*\partial_t,\ldots,\varphi_*\partial_t}_{j-2-\ell\text{ times}},{}^\varphi\nabla^{(\ell+1)}_{\partial_t}\varphi_*\partial_s,\varphi_*\partial_t)\varphi_*\partial_t$$

$$=\sum_{\ell=0}^{j-1}\left(\binom{j-2}{\ell}+\binom{j-2}{\ell-1}\right)(\nabla^{(j-1-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t,\ldots,\varphi_*\partial_t}_{j-1-\ell\text{ times}},{}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s,\varphi_*\partial_t\big)\varphi_*\partial_t$$

$$=\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(\nabla^{(j-1-\ell)}R)_\varphi\big(\underbrace{\varphi_*\partial_t,\ldots,\varphi_*\partial_t}_{j-1-\ell\text{ times}},{}^\varphi\nabla^{(\ell)}_{\partial_t}\varphi_*\partial_s,\varphi_*\partial_t\big)\varphi_*\partial_t$$

completing the induction. Here we used the identity ${}^\varphi\nabla_{\partial_t}(\nabla^{(j-2-\ell)}R)_\varphi = \nabla_{\varphi_*\partial_t}\left(\nabla^{(j-2-\ell)}R\right)$ to get from the first equality to the second – this is the characterisation of the pullback connection in Theorem 2.56. □

Finally, we compute the Taylor expansion of $t \mapsto g_{ij}(\varphi(0,t))$ around $t=0$ (so that $g_{ij}(\gamma_u(t))=g_{ij}(0)+g^{(1)}_{ij}(0)+\frac{1}{2}g^{(2)}_{ij}(0)+\ldots$). The 0-order term $g_\varphi(\partial_i,\partial_j)\big|_{(0,0)}=g_{ij}(\gamma_u(0))=g_{ij}(p)=\delta_{ij}$ as we are working in normal coordinates. The 1st order vanishes and by (2.23) we find that $12\,g^{(2)}_{ij}(0)v^iv^j=8\,g_\varphi$ $(R_\varphi(v,u)u,v)$, so $g^{(2)}_{ij}(0)=-\frac{2}{3}R_{ikj\ell}u^ku^\ell$. The theorem now follows. □

2.9 Integration and Divergence Theorems

If (M,g) is an oriented Riemannian manifold with boundary and $\widetilde{g}$ is the induced Riemannian metric on ∂M, then we define the volume form of $\widetilde{g}$ by $d\sigma_{\widetilde{g}} = \iota_\nu d\mu_g|_{\partial M}$. In particular if X is a smooth vector field, we have $\iota_X d\mu_g|_{\partial M} = \langle X,\nu\rangle_g\, d\sigma_{\widetilde{g}}$. In light of this, we define the divergence $\mathrm{div}X$ to be the quantity that satisfies:

$$d(\iota_X d\mu) = \mathrm{div}X\,d\mu. \tag{2.24}$$

Theorem 2.68 (Divergence theorem). *Let (M,g) be a compact oriented Riemannian manifold. If X is a vector field, then*

$$\int_M \mathrm{div}X\,d\mu = \int_{\partial M}\langle X,\nu\rangle_g\,d\sigma.$$

In particular, if M is closed then $\int_M \mathrm{div}X\,d\mu = 0$.

Proof. Define the $(n-1)$-form α by $\alpha = \iota_X d\mu$. So by Stokes' theorem,

$$\int_M \operatorname{div} X \, d\mu = \int_M d\alpha = \int_{\partial M} \alpha = \int_{\partial M} \iota_X d\mu = \int_{\partial M} \langle X, \nu \rangle \, d\sigma. \qquad \square$$

From the divergence theorem we have the following useful formulas:

Proposition 2.69 (Integration by parts). *On a Riemannian manifold (M, g) with $u, v \in C^\infty(M)$ the following holds:*

(a) On a closed manifold,

$$\int_M \Delta u \, d\mu = 0.$$

(b) On a compact manifold,

$$\int_M (u\Delta v - v\Delta u) d\mu = \int_{\partial M} \left(u \frac{\partial u}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\sigma.$$

In particular, on a closed manifold $\int_M u\Delta v \, d\mu = \int_M v \Delta u \, d\mu$.

(c) On a compact manifold,

$$\int_M u \Delta v \, d\mu + \int_M \langle \nabla u, \nabla v \rangle \, d\mu = \int_{\partial M} \frac{\partial v}{\partial \nu} u \, d\sigma.$$

In particular, on a closed manifold $\int_M \langle \nabla u, \nabla v \rangle \, d\mu = - \int_M u \Delta v \, d\mu$.

2.9.1 Remarks on the Divergence Expression

We seek a local expression for the divergence, defined by (2.24), and show it is equivalent to the trace of the covariant derivative. That is,

$$\operatorname{div} X = \operatorname{tr} \nabla X = \operatorname{tr} (\nabla X)(\,\cdot\,,\,\cdot\,) = (\nabla_i X)(dx^i) \tag{2.25}$$

Lemma 2.70. *The divergence* $\operatorname{div} X$ *of a vector field X, defined by* (2.24), *can be expressed in local coordinates by*

$$\operatorname{div}(X^i \partial_i) = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}). \tag{2.26}$$

Proof. By Cartan's formula[7] and (2.24) we have

$$\operatorname{div}(X) d\mu = d \circ \iota_X d\mu = (d \circ \iota_X + \iota_X \circ d) d\mu = \mathcal{L}_X d\mu.$$

[7] Which states that $\mathcal{L}_X \omega = \iota_X(d\omega) + d(\iota_X \omega)$, for any smooth vector field X and any smooth differential form ω.

From the left-hand side we find that $(\mathrm{div}X\,d\mu)(\partial_1,\ldots,\partial_n) = \mathrm{div}X\,\sqrt{\det g}$ and from the right-hand side we find that

$$\begin{aligned}(\mathcal{L}_X d\mu)(\partial_1,\ldots,\partial_n) &= \mathcal{L}_X(\sqrt{\det g}) - d\mu(\ldots,\mathcal{L}_X\partial_i,\ldots)\\ &= X(\sqrt{\det g}) + d\mu(\ldots,(\partial_i X^j)\partial_j,\ldots)\\ &= X(\sqrt{\det g}) + (\partial_i X^j)\delta^i_j\sqrt{\det g}\\ &= \partial_i(X^i\sqrt{\det g})\end{aligned}$$

□

Claim 2.71. The definitions of div$X$ given by (2.24) and (2.25) coincide.

Proof. As $(\nabla_X dx^j)(\partial_i) = -dx^j(\nabla_X\partial_i)$, equation (2.25) implies that

$$(\nabla_i X)(dx^i) = \partial_i X^i - X(\nabla_i dx^i) = \partial_i X^i + \Gamma^i_{ij}X^j \tag{2.27}$$

On the other hand, (2.26) implies that $\mathrm{div}X = \partial_i X^i + \frac{1}{\sqrt{\det g}}X^i\partial_i(\sqrt{\det g})$, where by the chain rule

$$\begin{aligned}\partial_i(\sqrt{\det g}) = \frac{1}{2}\frac{1}{\sqrt{\det g}}\frac{\partial\det g}{\partial g_{pq}}\frac{\partial g_{pq}}{\partial x^i} &= \frac{1}{2}\sqrt{\det g}g^{pq}\frac{\partial g_{pq}}{\partial x^i}\\ &= g^{pq}\Gamma^\ell_{ip}g_{\ell q}\sqrt{\det g} = \Gamma^p_{ip}\sqrt{\det g}.\end{aligned}$$

□

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http://www.springer.com/978-3-642-16285-5

The Ricci Flow in Riemannian Geometry
A Complete Proof of the Differentiable 1/4-Pinching
Sphere Theorem
Andrews, B.; Hopper, C.
2011, XVIII, 302 p. 13 illus., 2 illus. in color., Softcover
ISBN: 978-3-642-16285-5