Preface

Stochastic differential equations are playing an increasingly important role in applications to finance, numerical analysis, physics, and biology. In the finite-dimensional case, there are two definitive books: one by Gikhman and Skorokhod [25], which studies the existence and uniqueness problem along with probabilistic properties of solutions, and another by Khasminskii [39], which studies the asymptotic behavior of the solutions using the Lyapunov function method. Our object in this book is to study these topics in the infinite-dimensional case. The two main problems one faces are the invalidity of the Peano theorem in the infinite-dimensional case and the appearance of unbounded operators if one wants to apply finite-dimensional techniques to stochastic partial differential equations (SPDEs).

Motivated by these difficulties, we discuss the theory in the deterministic case from two points of view. The first method (see Pazy [63] and Butzer and Berens [6]) involves semigroups generated by unbounded operators and results in constructing mild solutions. The other sets up the equation in a Gelfand triplet $V \hookrightarrow H \hookrightarrow V^*$ of Hilbert spaces with the space V as the domain of the unbounded operator and V^* its continuous dual. In this case variational solutions are produced. This approach is studied by Agmon [1] and Lions [48], who assume that either the injection $V \hookrightarrow H$ is compact and the unbounded operator is coercive or that the unbounded operator is coercive and monotone.

The systematic study of the first approach to SPDEs was first undertaken by Ichikawa [32, 33] and is explained in the timely monographs of Da Prato and Zabczyk [11, 12]. The approach of J.P. Lions was first used by Viot [75] (see also Metivier [56] and Metivier and Viot [58]). Working under the assumption that the embedding $V \hookrightarrow H$ is compact and that the coefficients of the equation are coercive, the existence of a weak solution was proven. These results were generalized by Pardoux [62], who assumed coercivity and monotonicity, and used the crucial deterministic result of Lions [48] to produce the strong solution. Later, Krylov, and Rozovskii [42] established the above-mentioned result of Lions in the stochastic case and also produced strong solutions. The initial presentation was given by Rozovskii in [66]. However, rather rigorous and complete exposition in a slightly general form is provided by Prévôt and Röckner [64].

In addition to presenting these results on SPDEs, we discuss the work of Leha and Ritter [46, 47] on SDEs in $\mathbb{R}^\infty$ with applications to interacting particle systems and the related work of Albeverio et al. [2, 3], and also of Gawarecki and Mandrekar [20, 21] on the equations in the field of Glauber dynamics for quantum lattice systems. In both cases the authors study infinite systems of SDEs.

We do not present here the approach used in Kalliapur and Xiong [37], as it requires introducing additional terminology for nuclear spaces. For this type of problem (referred to as "type 2" equations by K. Itô in [35]), we refer the reader to [22, 23], and [24], as well as to [37].

A third approach, which involves solutions being Hida distribution is presented by Holden et al. in the monograph [31].

The book is divided into two parts. We begin Part I with a discussion of the semigroup and variational methods for solving PDEs. We simultaneously develop stochastic calculus with respect to a Q-Wiener process and a cylindrical Wiener process, relying on the classic approach presented in [49]. These foundations allow us to develop the theory of semilinear partial differential equations. We address the case of Lipschitz coefficients first and produce unique mild solutions as in [11]; however, we then extend our research to the case where the equation coefficients depend on the entire "past" of the solution, invoking the techniques of Gikhman and Skorokhod [25]. We also prove Markov and Feller properties for mild solutions, their dependence on the initial condition, and the Kolmogorov backward equation for the related transition semigroup. Here we have adapted the work of B. Øksendal [61], S. Cerrai [8], and Da Prato and Zabczyk [11].

To go beyond the Lipschitz case, we have adapted the method of approximating continuous functions by Lipschitz functions $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ from Gikhman and Skorokhod [25] to the case of continuous functions $f : [0, T] \times H \to H$ [22]. This technique enabled us to study the existence of weak solutions for SDEs with continuous coefficients, with the solution identified in a larger Hilbert space, where the original Hilbert space is compactly embedded. This arrangement is used, as we have already mentioned, due to the invalidity of the Peano theorem. In addition, we study martingale solutions to semilinear SDEs in the case of a compact semigroup and for coefficients depending on the entire past of the solution.

The variational method is addressed in Chap. 4, where we study both the weak and strong solutions. The problem of the existence of weak variational solutions is not well addressed in the existent literature, and our original results are obtained with the help of the ideas presented in Kallianpur et al. [36]. We have followed the approach of Prévôt and Röckner in our presentation of the problem of the existence and uniqueness of strong solutions.

We conclude Part I with an interesting problem of an infinite system of SDEs that does not arise from a stochastic partial differential equation and serves as a model of an interacting particle system and in Glauber dynamics for quantum lattice systems.

In Part II of the book, we present the asymptotic behaviors of solutions to infinite-dimensional stochastic differential equations. The study of this topic was undertaken for specific cases by Ichikawa [32, 33] and Da Prato and Zabczyk [12] in the case of mild solutions. A general Lyapunov function approach for strong solutions in a

Gelfand triplet setting for exponential stability was originated in the work of Khasminskii and Mandrekar [40] (see also [55]). A generalization of this approach for mild and strong solutions involving exponential boundedness was put forward by R. Liu and Mandrekar [52, 53]. This work allows readers to study the existence of invariant measure [52] and weak recurrence of the solutions to compact sets [51]. Some of these results were presented by K. Liu in a slightly more general form in [50].

Although we have studied the existence and uniqueness of non-Markovian solutions, we do not investigate the ergodic properties of these processes, as the techniques in this field are still in development [28].

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