Chapter 2
Critical Mass and Efficiency

Abstract This chapter forms the heart of this book. After deriving the properties of neutron travel through materials, a detailed analysis is presented of how the critical mass of a fissile material, in both "bare" and "tampered" configurations, can be calculated. The calculations are applied to both uranium-235 and plutonium-239. Analytic expressions are developed for estimating bomb energy yield and efficiency. A numerical simulation is developed to analyze conditions of pressure, fission rate, expansion, and energy yield within a fissioning bomb core, and is applied to the Hiroshima *Little Boy* bomb. Spreadsheets for performing the calculations are made available to interested users through a supporting website.

Every atom of separated uranium or plutonium in the Manhattan Project was precious, so estimating the amount of fissile material needed to make a workable nuclear weapon – the so-called critical mass – was a crucial issue for the developers of *Little Boy* and *Fat Man*. Equally important was being able to estimate what efficiency one might expect for a fission bomb. For various reasons, not all of the fissile material in a bomb core actually undergoes fission during a nuclear explosion; if the expected efficiency were to prove so low that one might just as well use a few conventional bombs to achieve the same energy release, there would be no point in taking on the massive engineering challenges involved in making nuclear weapons. In this chapter we investigate these issues.

The concept of critical mass involves two competing effects. As nuclei fission they emit secondary neutrons. A fundamental empirical law of nuclear physics, derived in Sect. 2.1, demands that a certain fraction of these neutrons reach the surface of the mass and escape while the remainder are consumed in fissioning other nuclei. However, if on average more than one neutron is emitted per fission we can afford to let some escape since only one is required to initiate a subsequent fission. For a small sample of material the escape probability is high; as the size of the sample increases, the escape probability declines and at some point will reach a value such that the number of neutrons that fail to escape will number enough to fission every nucleus in the sample. Thus, there is a minimum size (hence mass) of

B.C. Reed, *The Physics of the Manhattan Project*,
DOI 10.1007/978-3-642-14709-8_2,

material at which every nucleus will in principle be fissioned even while some neutrons escape.

The above description of critical mass should be regarded as a purely qualitative one. Technically, the important issue is known as *criticality*. Criticality is said to obtain when the number of free neutrons in a bomb core is increasing with time. A full understanding of criticality demands familiarity with time-dependent diffusion theory. Application of diffusion theory to this problem requires understanding a concept known as the *mean free path* (MFP) for neutron travel, so this is developed in Sect. 2.1. Section 2.2 takes up a time-dependent diffusion theory treatment of criticality. Section 2.3 addresses the effect of surrounding the fissile core with a *tamper*, a metallic casing which has the effects of decreasing the critical mass and improving the efficiency of the explosion. Sections 2.4 and 2.5 take up the issue of bomb efficiency through analytic approximations and a numerical simulation, respectively. Section 2.6 presents an alternate treatment of untamped criticality that has an interesting historical connection.

For readers interested in further sources, an excellent account of the concept of critical mass appears in Logan (1996); see also Bernstein (2002).

2.1 Neutron Mean Free Path

See Fig. 2.1. A thin slab of material of thickness s (ideally, one atomic layer) and cross-sectional area Σ is bombarded by incoming neutrons at a rate R_o neutrons/(m^2 s).

Let the bulk density of the material be ρ g/cm^3. In nuclear reaction calculations, however, density is usually expressed as a *number density* of nuclei in the material, that is, the number of nuclei per cubic meter. In terms of ρ this is given by

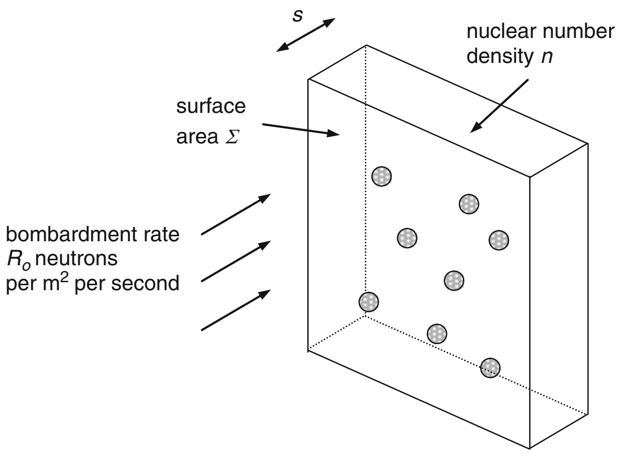

Fig. 2.1 Neutrons penetrating a thin target foil

$$n = 10^6 \left(\frac{\rho N_A}{A} \right), \tag{2.1}$$

where N_A is Avogadro's number and A is the atomic weight (g/mol) of the material; the factor of 10^6 arises from converting cm^3 to m^3.

Assume that each nucleus presents a total reaction cross-section of σ square meters to the incoming neutrons. Cross-sections are usually measured in barns (bn), where 1 bn $= 10^{-28}$ m^2. The first question we address is: "How many reactions will occur per second as a consequence of the bombardment rate R_o?" The volume of the slab is Σs, hence the number of nuclei contained in it will be Σsn. If each nucleus presents an effective cross-sectional area σ to the incoming neutrons, then the total area presented by all nuclei would be $\Sigma sn\sigma$. The *fraction* of the surface area of the slab that is available for reactions to occur is then $(\Sigma sn\sigma/\Sigma) = sn\sigma$. The rate of reactions R (reactions/s) can then sensibly be assumed to be the rate of bombarding particles over the entire surface area of the slab times the fraction of the surface area available for reactions:

$$\begin{pmatrix} Reactions \\ per\ \mathrm{second} \end{pmatrix} = \begin{pmatrix} incident\ neutron \\ flux/\mathrm{second} \end{pmatrix} \begin{pmatrix} fraction\ of\ surface\ area \\ occupied\ by\ cross - \mathrm{section} \end{pmatrix}$$

or

$$R = (R_o \Sigma)(s\, n\, \sigma). \tag{2.2}$$

The *probability* P that an individual incident neutron precipitates a reaction is then

$$P_{react} = \frac{\begin{pmatrix} reactions \\ per\ second \end{pmatrix}}{\begin{pmatrix} incident\ neutron\ flux \\ per\ second \end{pmatrix}} = s\, n\, \sigma. \tag{2.3}$$

For our purposes, more directly useful is not the probability that a neutron will be consumed in a reaction, but rather that it will pass through the slab to escape out the back side:

$$P_{escape} = 1 - P_{react} = 1 - s\, n\, \sigma. \tag{2.4}$$

Now consider a block of material of macroscopic thickness x. As shown in Fig. 2.2, we can imagine this to be comprised of a large number of thin slabs each of thickness s placed back-to-back.

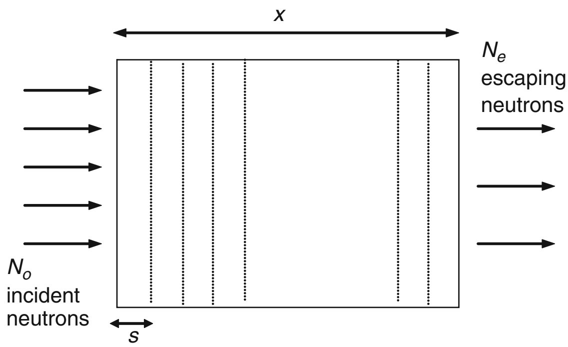

Fig. 2.2 Neutrons penetrating a thick target

The number of slabs is $\eta = x/s$. If N_o neutrons are incident on the left side of the block, the number that would survive to emerge from the first thin slab would be N_oP, where P is the probability in (2.4). These neutrons are then incident on the second slab, and the number that would emerge unscathed from that passage would be $(N_oP)P = N_oP^2$. These neutrons would then strike the third slab and so on. The number that survive passage through the entire block to escape from the right side would be N_oP^η, or

$$N_{esc} = N_o(1 - s\,n\,\sigma)^{x/s}. \tag{2.5}$$

Define $z = -sn\sigma$. The number that escape can then be written as

$$N_{esc} = N_o(1 + z)^{-\sigma n x/z} = N_o\left[(1 + z)^{1/z}\right]^{-\sigma n x}. \tag{2.6}$$

Now, ideally, s is very small, which means that $z \to 0$. The definition of the base of the natural logarithms, e, is $e = \lim\limits_{z \to 0}(1 + z)^{1/z}$, so we have

$$N_{esc} = N_o e^{-\sigma n x},$$

or

$$P_{direct\ escape} = \frac{N_{esc}}{N_o} = e^{-\sigma n x}. \tag{2.7}$$

Equation (2.7) is the fundamental escape probability law. In words, it says that the probability that a bombarding neutron will pass through a slab of material of thickness x depends exponentially on x, on the number density of nuclei in the slab, and on the reaction cross-section of those nuclei to incoming neutrons. If $\sigma = 0$, all of the incident particles will pass through unscathed. If $(\sigma\,n\,x) \to \infty$, *none* of the incident particles will make it through.

In practice, (2.7) is used to experimentally establish values for cross-sections by bombarding a slab of material with a known number of incident particles and then seeing how many emerge from the other side; think of (2.7) as effectively *defining* σ. Due to quantum-mechanical effects, the cross-section is not the geometric area of the nucleus.

The total cross section had in mind here can be broken down into a sum of cross-sections for individual processes such as fission, elastic scattering, inelastic scattering, non-fission capture and the like:

$$\sigma_{total} = \sigma_{fission} + \sigma_{elastic\ scatter} + \sigma_{inelastic\ scatter} + \sigma_{capture} + \ldots \quad (2.8)$$

In practice, cross-sections can depend very sensitively on the energy of the incoming neutrons; such energy-dependence plays a crucial role in the contrast between how nuclear reactors and nuclear weapons function. As an example, Fig. 2.3 (see also Fig. 1.9) shows the variation of the fission cross-section for ^{235}U under neutron bombardment for neutrons in the energy range 1–10 eV; note the dramatic resonance effects at certain energies. This graph shows only a small fraction of the energy range over which the cross-section for the $^{235}U(n, f)$ reaction has been measured; measurements from 10^{-5} eV to 20 MeV are available from the source listed in the figure caption.

A very important result that derives from this escape-probability law is an expression for the *average* distance that an incident neutron will penetrate into the slab before being involved in a reaction. Look at Fig. 2.4, where we now have a slab of thickness L and where x is a coordinate for any position within the slab. Imagine also a small slice of thickness dx whose front edge is located at position x.

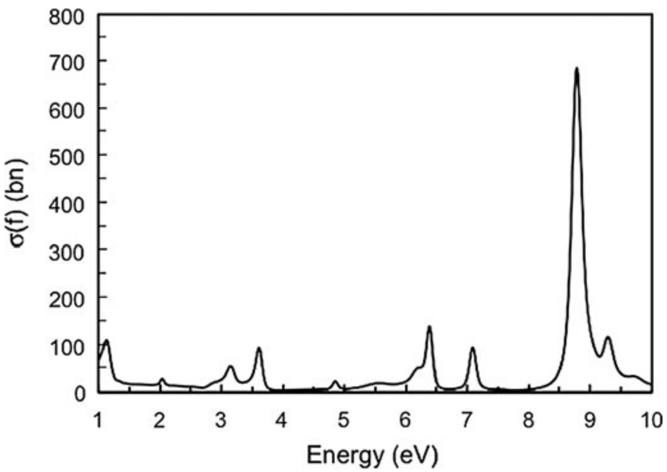

Fig. 2.3 Cross-section for the $^{235}U(n, f)$ reaction over the energy range 1–10 eV. At 0.01 eV, the cross-section for this reaction is about 930 bn. Data from National Nuclear Data Center

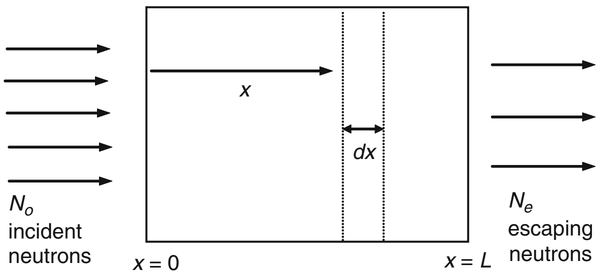

Fig. 2.4 Neutrons penetrating a target of thickness L

From (2.7), the probability that a neutron will penetrate through the entire slab to emerge from the face at $x = L$ is $P_{emerge} = e^{-\sigma nL}$. This means that the probability that a neutron will be involved in a reaction and *not* travel through to the face at $x = L$ will be $P_{react} = 1 - e^{-\sigma nL}$. If N_o neutrons are incident at the $x = 0$ face then the number that will be consumed in reactions within the slab will be $N_{react} = N_o(1 - e^{-\sigma nL})$. We will use this result in a moment.

Also from (2.7), the number of neutrons that penetrate to x and $x + dx$, respectively, is give by

$$N_x = N_o e^{-\sigma n x} \tag{2.9}$$

and

$$N_{x+dx} = N_o e^{-\sigma n(x+dx)}. \tag{2.10}$$

Some of the neutrons that reach x will be involved in reactions before reaching $x + dx$, that is, $N_x > N_{x+dx}$. The number of neutrons consumed between x and $x + dx$, designated as dN_x, is given by

$$dN_x = N_x - N_{x+dx} = N_o e^{-\sigma n x}\left(1 - e^{-\sigma n dx}\right). \tag{2.11}$$

If dx is infinitesimal, then $(\sigma n dx)$ will be very small. This means that we can write $e^{-\sigma n (dx)} \sim 1 - \sigma n(dx)$, and hence write dN_x as

$$dN_x = N_o e^{-\sigma n x}(\sigma n dx), \tag{2.12}$$

a result equivalent to differentiating (2.7) above.

Now, these dN_x neutrons penetrated distance x into the slab before being consumed in a reaction, so the total travel distance accumulated by all of them in doing so would be $(x dN_x)$. The average distance that a neutron destined to be consumed in a reaction will travel before being consumed is given by integrating accumulated travel distances over the length of the slab and dividing by the number of neutrons consumed in reactions within the slab, that is, $N_{react} = N_o(1 - e^{-\sigma n L})$ from above:

$$\langle x \rangle = \frac{1}{N_{react}} \int_0^L x\, dN_x = \frac{1}{N_o(1 - e^{-\sigma n L})} \int_0^L (N_o \sigma n)\, x\, e^{-\sigma n x} dx$$

$$= \frac{1}{\sigma n} \left[\frac{1 - e^{-\sigma n L}(1 + \sigma n L)}{1 - e^{-\sigma n L}} \right]. \tag{2.13}$$

If we have a slab of infinite thickness, or, more generally, one such that the product $\sigma n L$ is large, then $e^{-\sigma n L}$ will be small and we will have

$$\langle x \rangle_{(\sigma n L) large} \to \frac{1}{\sigma n}. \tag{2.14}$$

This quantity is known as the *characteristic length* or *mean free path* for the particular reaction quantified by σ. This quantity will figure prominently in Sects. 2.2 through 2.6. If it is computed for an individual cross section such as $\sigma_{fission}$ or $\sigma_{capture}$, one speaks of the mean free path for fission or capture. Such lengths are often designated by the symbol λ. As an example, consider fission in ^{235}U. The nuclear number density n is 4.794×10^{28} m^{-3}, and the fission cross section is $\sigma_f = 1.235$ bn $= 1.235 \times 10^{-28}$ m^2. These numbers give $\lambda_f = 16.9$ cm, or about 6.65 in.

Finally, it should be emphasized that the derivations in this section do not apply to bombarding particles that are *charged*, in which case one has very complex ionization issues to deal with.

2.2 Critical Mass: Diffusion Theory

We now consider critical mass per se. Qualitatively, the concept of critical mass derives from the observation that some species of nuclei fission upon being struck by a bombarding neutron and consequently release secondary neutrons. In a sample of fissile material these secondary neutrons can potentially go on to induce other fissions, resulting in a chain reaction. However, the development in the preceding section indicates that we can expect a certain number of neutrons to reach the surface of the sample and escape, particularly if the sample is small. If the density of neutrons within the sample is increasing with time, *criticality* is said to obtain. Whether or not this condition is fulfilled depends on quantities such as the density of the material, its cross-section for fission, and the number of neutrons emitted per fission, which is designated by the symbol ν.

To explore the time-dependence of the number of neutrons in the core requires the use of time-dependent *diffusion theory*. In this section we use this theory to calculate the critical masses of so-called “bare” spherical assemblies of ^{233}U, ^{235}U, ^{237}Np, ^{239}Pu, and ^{241}Am, the five isotopes that one can feasibly consider for use in nuclear weapons. Of these, ^{235}U and ^{239}Pu are used in practice. The term “bare” refers to an *untamped* core. More correctly, we compute critical *radii* which can be transformed into equivalent critical *masses* upon knowing the density of the material involved.

The development presented here is based on the development in Appendix G of a differential equation which describes the spatiotemporal behavior of the neutron number density *N*, that is, the number of neutrons per cubic meter within the core. The derivation in Appendix G depends upon on some material developed in Sect. 3.5; it is consequently recommended that both those sections be read in advance of this one. Also, be sure not to confuse *n* and *N*; the former is the number density of fissile *nuclei* while the latter is the number density of *neutrons*; both play roles in what follows. Note also that the definition of *N* here differs from that in the previous section, where it represented a number of neutrons.

Before proceeding, an important limitation of this approach needs to be made clear. Following Serber et al. (1992), I model neutron flow within the bomb core by use of a diffusion equation. A diffusion approach is appropriate if neutron scattering is isotropic. Even if this is not so, a diffusion approach will still be reasonable if neutrons suffer a large enough number of scatterings so as to effectively erase non-isotropic angular effects. Unfortunately, neither of these conditions are fulfilled in the case of a uranium core: fast neutrons elastically scattering against uranium show a strong forward-peaked effect, and the mean free path of a fast neutron in ^{235}U, about 3.6 cm, is only about half of the 8.4-cm bare critical radius (see Table 2.1 below). I adopt a diffusion-theory approach for a number of reasons, however. As much of the physics of this area remains classified or at least not easily accessible, we are forced to settle for an approximate model; diffusion theory has the advantage of being analytically tractable at an upper-undergraduate level. Also, despite these various limitations, a comparison of critical radii as predicted by diffusion theory

Table 2.1 Threshold critical radii and masses (untamped)

Quantity	Unit	^{235}U	^{239}Pu	^{233}U	^{237}Np	^{241}Am
A	g/mol	235.04	239.05	233.04	237.05	241.06
ρ	g/cm^3	18.71	15.6	18.55	20.25	13.67
σ_f	bn	1.235	1.800	1.946	1.335	1.378
σ_{el}	bn	4.566	4.394	4.447	4.965	4.833
ν	–	2.637	3.172	2.755	2.81	2.5
n	10^{22} cm^{-3}	4.794	3.930	4.794	5.144	3.415
$\lambda_{fission}$	cm	16.89	14.14	10.72	14.56	21.25
$\lambda_{elastic}$	cm	4.57	5.79	4.69	3.92	6.06
λ_{total}	cm	3.60	4.11	3.26	3.09	4.71
R_O	cm	8.37	6.346	5.676	6.736	11.307
M_O	kg	45.9	16.7	14.2	25.92	82.8

with those of an openly-published more exact treatment shows that the two agree within about 5% for the range of fissility parameters of interest here (Reed 2008).

Central to any discussion of critical radius are the *fission* and *transport* mean free paths for neutrons, respectively symbolized as λ_f and λ_t. These are given by (2.14) as

$$\lambda_f = \frac{1}{\sigma_f n} \tag{2.15}$$

and

$$\lambda_t = \frac{1}{\sigma_t n}, \tag{2.16}$$

where σ_t is the so-called transport cross-section. If neutron scattering is isotropic (which we assume), the transport cross-section is given by the sum of the fission and elastic-scattering cross-sections:

$$\sigma_t = \sigma_f + \sigma_{el}. \tag{2.17}$$

We do not consider here the role of *in*elastic scattering, which affects the situation only indirectly in that it lowers the mean neutron velocity. To keep the treatment simple we will also not deal at this point with the effect of any external tamper/neutron reflector.

In a spherical fissioning bomb core, the diffusion theory of Appendix G provides the following differential equation for the neutron number density:

$$\frac{\partial N}{\partial t} = \frac{v_{neut}}{\lambda_f}(\nu - 1)N + \frac{\lambda_t v_{neut}}{3}\left(\nabla^2 N\right), \tag{2.18}$$

where v_{neut} is the average neutron velocity and the other symbols are as defined earlier.

Now, let r represent the usual spherical radial coordinate. Upon assuming a solution for $N(t, r)$ of the form $N(t, r) = N_t(t)N_r(r)$, (2.18) can be separated as

$$\frac{1}{N_t}\left(\frac{\partial N_t}{\partial t}\right) = \left(\frac{\nu - 1}{\tau}\right) + \frac{D}{N_r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial N_r}{\partial r}\right)\right], \tag{2.19}$$

where D is the so-called diffusion coefficient,

$$D = \frac{\lambda_t v_{neut}}{3}, \tag{2.20}$$

and where τ is the mean time that a neutron will travel before causing a fission:

$$\tau = \frac{\lambda_f}{v_{neut}}. \tag{2.21}$$

If the separation constant for (2.19) is defined as α/τ (that is, the constant to which both sides of the equation must be equal), then the solution for the time-dependent part of the neutron density emerges directly as

$$N_t(t) = N_o e^{(\alpha/\tau)t} \tag{2.22}$$

where N_o represents the neutron density at $t = 0$. N_o would be set by whatever device is used to initiate the chain-reaction. Note that we could have called the separation constant just α, but this form will prove a little more convenient for subsequent algebra. With this definition of the separation constant, the radial part of (2.19) appears as

$$\left(\frac{\nu - 1}{\tau}\right) + \frac{D}{N_r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r}{\partial r}\right)\right] = \frac{\alpha}{\tau}. \tag{2.23}$$

The first and last terms in (2.23) can be combined (this is why the separation constant was defined as α/τ); on dividing (2.23) by D, we find

$$\frac{1}{d^2} + \frac{1}{N_r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r}{\partial r}\right)\right] = 0, \tag{2.24}$$

where

$$d = \sqrt{\frac{\lambda_f\,\lambda_t}{3\,(-\alpha + \nu - 1)}}. \tag{2.25}$$

Now define a new dimensionless coordinate x according as

$$x = \frac{r}{d}. \tag{2.26}$$

This brings (2.24) to the form

$$\frac{1}{N_r}\left[\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial N_r}{\partial x}\right)\right] = -1. \tag{2.27}$$

Aside from a normalization constant, the solution of this differential equation can easily be verified to be

$$N_r(r) = \left(\frac{\sin x}{x}\right). \tag{2.28}$$

To determine a critical radius R_C, we need a boundary condition to apply to (2.28). As explained in Appendix G, this takes the form

$$N(R_C) = -\frac{2\,\lambda_t}{3}\left(\frac{\partial N}{\partial r}\right)_{R_C} = -\frac{2\,\lambda_t}{3\,d}\left(\frac{\partial N}{\partial x}\right)_{R_C}. \tag{2.29}$$

On applying this to (2.28), one finds that the critical radius is given by solving the transcendental equation

$$x\cot(x) + x/\eta - 1 = 0, \tag{2.30}$$

where

$$\eta = \frac{2\lambda_t}{3d} = 2\sqrt{\frac{\lambda_t(-\alpha + \nu - 1)}{3\lambda_f}}. \tag{2.31}$$

With fixed values for the density and nuclear constants for some fissile material, (2.30) and (2.31) contain two variables: the core radius r and the exponential factor α, and they can be solved in two different ways. For both approaches, assume that we are working with material of "normal" density, which we designate as ρ_o. For the first approach, start by looking back at (2.22). If $\alpha = 0$, the neutron number density is neither increasing nor decreasing with time; in this case one has what is called *threshold criticality*. To determine the so-called threshold bare critical radius R_o, set $\alpha = 0$ in (2.25) and (2.31), set the density to ρ_o, solve (2.30) for x, and then get r ($=R_o$) from (2.26). The corresponding threshold bare critical mass M_o then follows from $M_o = (4\pi/3)R_o^3\rho_o$. It is this mass that one usually sees referred to as *the* critical mass; this quantity will figure prominently in the discussion of efficiency in Sects. 2.4 and 2.5.

The second type of solution begins with assuming that one has a core of some radius $r > R_o$. In this case one will find that (2.30) will be satisfied by some value of $\alpha > 0$, with α increasing as r increases. That is, since $x/\eta = 3r/2\lambda_t$ in (2.30) is independent of α, we can set r to some desired value; (2.30) can then be solved for x, which gives d from (2.26), and hence α from (2.25). With $\alpha > 0$ the reaction will grow exponentially in time until all of the fissile material is used up, a situation known as "supercriticality." To see why increasing the radius demands that α must increase, implicitly differentiate (2.30) to show that $d\eta/dx = (\eta/x)^2\left(1 - x^2/\sin^2 x\right)$, which demands $d\eta/dx < 0$ for all values of x. From the definition of x, an increase in r (and/or in the density, for that matter) will cause x to increase. To keep (2.30) satisfied means that η must decrease, which, from (2.31), can happen only if α increases.

We come now to a very important point. This is that the condition for threshold criticality can in general be expressed as a constraint on the product ρr where ρ is the mass density of the fissile material and r is the core radius. The factor η in (2.30) is independent of the density, depending only on the cross-sections and secondary neutron number ν. Hence, for $\alpha = 0$, (2.30) will be satisfied by some unique value of x which will be characteristic of the material being considered. Since $x = r/d$ and d itself is proportional to $1/\rho$ [see (2.25)], we can equivalently say that the solution of (2.30) demands a unique value of ρr for a given combination of σ and ν values. If as above R_o is the bare threshold critical radius for material of normal density ρ_o, then any combination of r and ρ such that $\rho r = \rho_o R_o$ will also be threshold critical, and any combination with $\rho r > \rho_o R_o$ will be supercritical. For a sphere of material of mass M, the mass, density, and radius relate as $M \propto \rho r^3$, which means that the "criticality product" ρr can be written as $\rho r \propto M/r^2$. This relationship underlies the concept of *implosion* weapons. If a sufficiently strong implosion can be achieved, then one can get away with having less than a "normal" critical mass by starting with a sphere of material of normal density and crushing it to high density by implosion; such weapons are thus inherently more efficient than those that depend on a non-implosive "gun" mechanism to assemble subcritical components. As described in Sect. 4.2, the implosion technique also helps to overcome pre-detonation issues with spontaneous fission. The key point here is that there is no *unique* critical mass for a given fissile material.

Table 2.1 shows calculated critical radii and masses for five nuclides usually considered for use in nuclear weapons; due to short alpha or beta half-lives and/or high spontaneous fission rates, no nuclides beyond those listed in the Table are likely to be suitable candidates for weapons materials.

Sources for the fission and elastic-scattering cross-sections appearing in the Table are given in Appendix B; the values quoted therein are used as they are averaged over the fission-energy spectra of the nuclides. The ν values were adopted from the Evaluated Nuclear Data Files (ENDF) maintained by the National Nuclear Data Center at Brookhaven National Laboratory (http://www.nndc.bnl.gov). For ^{235}U and ^{239}Pu, the ν values are for prompt neutrons of energy 2 MeV, about the average energy of fission neutrons. The ν value for ^{233}U refers to neutrons of energy 2.5 MeV; that for ^{237}Np was adopted from Hyde (1964) for neutrons of energy 1.4 MeV, and that for ^{241}Am is assumed. The densities for ^{235}U and ^{233}U are respectively (235/238) and (233/238) times the density of natural uranium, 18.95 g/cm^3.

Spreadsheet **CriticalityAnalytic.xls**[1] allows users to carry out these calculations for themselves. This spreadsheet is actually used for the calculations developed in this section and in Sects. 2.3 and 2.4. In its simplest use – corresponding to this section – the user enters the relevant parameters: the core density, atomic weight, fission and scattering cross-sections, and the number of secondary neutrons per fission. The "Goal Seek" function then allows one to solve (2.30) and (2.31) for x (assuming $\alpha = 0$), from which the bare critical radius and mass are computed.

In practice, having available only a single critical mass of fissile material will not produce much of an explosion. The reason for this is that fissioning nuclei give rise to fission products with tremendous kinetic energies. The core consequently very rapidly – within microseconds – heats up and expands, causing its density to drop below that necessary to maintain criticality. In a core comprised of but a single critical mass this will happen at the moment fissions begin, so the chain reaction will quickly fizzle as α falls below zero. To get an explosion of appreciable efficiency one must start with more than a single critical mass of fissile material or implode an initially subcritical mass to high density before initiating the explosion. If the core is surrounded by a massive tamper that is imploded to crush the

[1] All Excel sheets are available at http://www.manhattanphysics.com

core, the tamper will resist the expansion to some extent and can also serve to reflect some of the escaped neutrons back into the core to cause more fissions. The issue of using more than one critical mass to enhance weapon efficiency is examined in more detail in Sects. 2.4 and 2.5.

To close this section, it is interesting to look briefly at a famous *mis*calculation of critical mass on the part of Werner Heisenberg. At the end of World War II a number of prominent German physicists including Heisenberg were interned for 6 months in England and their conversations secretly recorded. This story is detailed in Bernstein (2001); see also Logan (1996) and Bernstein (2002). On the evening of August 6, 1945, the internees were informed that an atomic bomb had been dropped on Hiroshima and that the energy released was equivalent to about 20,000 tons of TNT (in actuality, the yield was about 13,000 tons). Heisenberg then estimated the critical mass based on this information and a subtly erroneous model of the fission process.

We saw in Sect. 1.6 that complete fission of 1 kg of ^{235}U liberates energy equivalent to about 17 kt TNT. Heisenberg predicated his estimate of the critical mass on the basis of assuming that about 1 kg of material did in fact fission. One kilogram of ^{235}U corresponds to about $\Omega \sim 2.56 \times 10^{24}$ nuclei. Assuming that on average $\nu = 2$ neutrons are liberated per fission, then the number of fission generations G necessary to fission the entire kilogram would be $\nu^G = \Omega$. Solving for G gives $G = \ln(\Omega)/\ln(\nu) \sim 81$, which Heisenberg rounded to 80. So far, this calculation is fine. Heisenberg then argued that as neutrons fly around in the bomb core they will randomly bounce between nuclei, traveling a mean distance λ between each collision; λ here is the mean free path between fissions as in (2.15) above. From Table 2.1 we have $\lambda \sim 17$ cm for U-235, but, at the time, Heisenberg took $\lambda \sim 6$ cm. Since a random walk of G steps where each is of length λ will take one a distance $r \sim \lambda\sqrt{G}$ from the starting point, he estimated a critical radius of $r \sim (6cm)\sqrt{80} \sim 54$ cm. This would correspond to a mass of some 12,500 kg, roughly 13 U.S. tons! Given that only one kilogram actually underwent fission, this would be a fantastically inefficient weapon. Such a bomb and its associated tamper, casing, and instrumentation would represent an unbearably heavy load for a World War II-era bomber.

The problem with Heisenberg's calculation was that he imagined the fission process to be created by a single neutron that randomly bounces throughout the bomb core, begetting secondary neutrons along the way. Further, his model is too stringent; there is no need for every neutron to cause a fission; many neutrons escape. In the days following August 6 Heisenberg revised his model, arriving at the diffusion theory approach described in this section.

2.3 Effect of Tamper

In the preceding section we saw how to calculate the critical mass of a sphere of fissile material. In that development we neglected the effect of any surrounding *tamper*. In this section we develop a simple model to account for the presence of a tamper. The discussion here draws from the preceding section and from Bernstein (2002), Serber (1992), and especially Reed (2009).

The idea behind a tamper is to surround the fissile core with a shell of dense material, as suggested in Fig. 2.5. This serves two purposes: (i) it reduces the critical mass, and (ii) it slows the inevitable expansion of the core, allowing more time for fissions to occur until the core density drops to the point where criticality no longer holds. The reduction in critical mass occurs because the tamper will reflect some escaped neutrons back into the core; indeed, the modern name for a tamper is "reflector", but I retain the historical terminology here. This effect is explored in this section; estimating the distance over which the core expands before criticality no longer holds in taken up in the next section. This slowing effect is difficult to model analytically, but can be treated with an approximate numerical model, which is done in Sect. 2.5.

The discussion here parallels that in Sect. 2.2. Neutrons that escape form the core will diffuse into the tamper. To describe the behavior of neutrons in the tamper we can use (2.18) without the term corresponding to production of neutrons, that is, the first term on the right side of (2.18); we are assuming that the tamper is not made of fissile material:

$$\frac{\partial N_{tamp}}{\partial t} = \frac{\lambda_{trans}^{tamp} v_{neut}}{3} \left(\nabla^2 N_{tamp} \right), \tag{2.32}$$

where N_{tamp} is the number density of and λ_{trans}^{tamp} the transport mean free path for neutrons in the tamper. v_{neut} is the average neutron speed within the tamper, which we will later assume for sake of simplicity to be the same as that within the core. We are assuming that the tamper does not absorb neutrons; otherwise, we would have to add a term to (2.32) represent that effect.

Superscripts and subscripts *tamp* will be used liberally here as it will be necessary to join *tamper* physics to *core* physics via suitable boundary conditions.

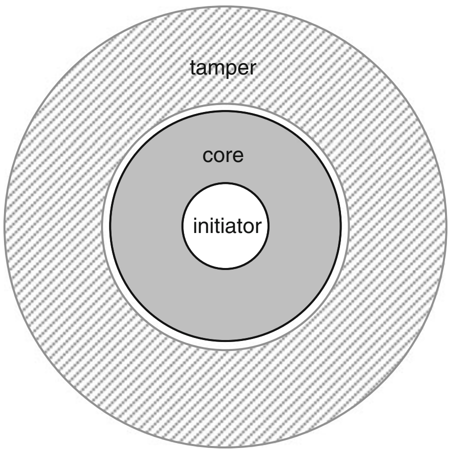

Fig. 2.5 Tamped bomb core

As was done in Sect. 2.2, take a trial solution for N_{tamp} of the form $N_{tamp}(t,r) = N_t^{tamp}(t)\,N_r^{tamp}(r)$ where $N_t^{tamp}(t)$ and $N_r^{tamp}(r)$ are respectively the time-and space dependences of N_{tamp}; r is the usual spherical radial coordinate. Upon substituting this into (2.32) we find, in analogy to (2.19),

$$\frac{1}{N_t^{tamp}}\left(\frac{\partial N_t^{tamp}}{\partial t}\right) = \left(\frac{\lambda_{trans}^{tamp} v_{neut}}{3}\right)\frac{1}{N_r^{tamp}}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r^{tamp}}{\partial r}\right)\right]. \tag{2.33}$$

Define the separation constant here to be δ/τ where τ is the mean time that a neutron will travel *in the core* before causing a fission, that is, as defined in (2.21):

$$\tau = \frac{\lambda_{fiss}^{core}}{v_{neut}}. \tag{2.34}$$

This choice renders (2.33) as

$$\frac{1}{N_t^{tamp}}\left(\frac{\partial N_t^{tamp}}{\partial t}\right) = \left(\frac{\lambda_{trans}^{tamp} v_{neut}}{3}\right)\frac{1}{N_r^{tamp}}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial N_r^{tamp}}{\partial r}\right)\right] = \frac{\delta}{\tau}. \tag{2.35}$$

It may seem strange to invoke a *core* quantity when dealing with diffusion in the *tamper*, but we can define the separation constant however we please. In principle, δ may be different from the exponential factor α of Sect. 2.2, but we will find that boundary conditions demand that they be equal. This choice of separation constant is advantageous in that the neutron velocity v_{neut}, which we assume to be the same in both materials, cancels out.

The solution of (2.35) depends on whether δ is positive, negative, or zero; the latter choice corresponds to threshold criticality in analogy to $\alpha = 0$ in Sect. 2.2. The situations of practical interest will be $\delta \geq 0$, in which case the solutions have the form

$$N_{tamp} = \begin{cases} \dfrac{A}{r} + B & (\delta = 0) \\[2ex] e^{(\delta/\tau)t}\left\{A\dfrac{e^{r/d_{tamp}}}{r} + B\dfrac{e^{-r/d_{tamp}}}{r}\right\} & (\delta > 0), \end{cases} \tag{2.36}$$

where A and B are constants of integration (different for the two cases), and where

$$d_{tamp} = \sqrt{\frac{\lambda_{trans}^{tamp}\ \lambda_{fiss}^{core}}{3\,\delta}}. \tag{2.37}$$

The situation we now have is that the neutron density in the core is described by (2.22) and (2.28) as

$$N_{core} = A_{core} e^{(\alpha/\tau)t} \frac{\sin(r/d_{core})}{r}, \tag{2.38}$$

with d_{core} given by (2.25):

$$d_{core} = \sqrt{\frac{\lambda_{fiss}^{core} \lambda_{trans}^{core}}{3(-\alpha + v - 1)}}, \tag{2.39}$$

while that in the tamper is given by (2.36) and (2.37).

The physical question is: "What boundary conditions apply in order that we have a physically reasonable solution?" Let the core have radius R_{core} and let the outer radius of the tamper be R_{tamp}; we assume that the inner edge of the tamper is snug against the core. First consider the core/tamper interface. If no neutrons are created or lost at this interface then it follows that both the density and flux of neutrons across the interface must be continuous. That is, we must have

$$N_{core}(R_{core}) = N_{tamp}(R_{core}) \tag{2.40}$$

and, from (6.72) of Appendix G,

$$\lambda_{trans}^{core} \left(\frac{\partial N_{core}}{\partial r} \right)_{R_{core}} = \lambda_{trans}^{tamp} \left(\frac{\partial N_{tamp}}{\partial r} \right)_{R_{core}}. \tag{2.41}$$

Equation (2.41) accounts for the effect of any neutron reflectivity of the tamper via λ_{trans}^{tamp}.

In addition, we must consider what is happening at the outer edge of the tamper. If there is no "backflow" of neutrons from the outside, then the situation is analogous to the boundary condition of (2.29) that was applied to the outer edge of the untampered core:

$$N_{tamp}(R_{tamp}) = -\frac{2}{3} \lambda_{trans}^{tamp} \left(\frac{\partial N_{tamp}}{\partial r} \right)_{R_{tamp}}. \tag{2.42}$$

Applying (2.40)–(2.42) to (2.36)–(2.39) results, after some tedious algebra, in the following constraints:

$$\left[1 + \frac{2R_{thresh} \lambda_{trans}^{tamp}}{3R_{tamp}^2} - \frac{R_{thresh}}{R_{tamp}} \right] \left[\left(\frac{R_{thresh}}{d_{core}} \right) \cot \left(\frac{R_{thresh}}{d_{core}} \right) - 1 \right] + \frac{\lambda_{trans}^{tamp}}{\lambda_{trans}^{core}} = 0, \quad (\delta = 0) \tag{2.43}$$

and, for $\delta > 0$,

$$e^{2\,(x_{ct}-x_t)}\left[\frac{x_c\cot x_c - 1 - \lambda\,(x_{ct}-1)}{R_{tamp}+2\lambda_{trans}^{tamp}(x_t-1)/3}\right] = \left[\frac{x_c\cot x_c - 1 + \lambda\,(x_{ct}+1)}{R_{tamp}-2\lambda_{trans}^{tamp}(x_t+1)/3}\right], \tag{2.44}$$

where

$$\left.\begin{aligned} x_{ct} &= R_{core}/d_{tamp} \\ x_c &= R_{core}/d_{core} \\ x_t &= R_{tamp}/d_{tamp} \\ \lambda &= \lambda_{trans}^{tamp}/\lambda_{trans}^{core} \end{aligned}\right\}. \tag{2.45}$$

It is also necessary to demand that $\alpha = \delta$, else the fact that (2.40)–(2.42) must also hold as a function of *time* would be violated. Some comments on these results follow.

(i) Equation (2.43) corresponds to *tamped threshold criticality*, where $\alpha = \delta = 0$. Once values for the d's and λ's are given, the only unknown is R_{thresh}, the threshold critical radius for a tamped core.

(ii) To use (2.44) and (2.45), proceed as follows: (i) Decide on the number of tamped threshold critical masses C (>1) of material for your bomb core. This will have radius $R_{core} = C^{1/3}R_{thresh}$, where R_{thresh} comes from solving (2.43). (ii) Pick a value for R_{tamp}, the *outer* radius of the tamper. (iii) Solve (2.44) numerically for α (=δ), which enters the d's and x's of (2.44) and (2.45) through (2.37) and (2.39).

The value of knowing α will become clear when the efficiency and yield calculations of Sects. 2.4 and 2.5 are developed; for the present, our main concern is with R_{thresh}.

A special-case application of (2.43) can be used to get an approximate sense of how dramatically the presence of a tamper decreases the threshold critical mass. Suppose that the tamper is very thick, $R_{tamp} >> R_{thresh}$. In this case (2.43) reduces to

$$(R_{thresh}/d_{core})\ \cot\left(R_{thresh}/d_{core}\right) = 1 - \left(\lambda_{trans}^{tamp}/\lambda_{trans}^{core}\right). \tag{2.46}$$

Now consider two sub-cases. The first is that the tamper is in fact a vacuum. Since empty space would have essentially zero cross-section for neutron scattering, this is equivalent to specifying $\lambda_{trans}^{tamp} = \infty$, in which case (2.46) becomes

$$(R_{thresh}/d_{core})\ \cot\left(R_{thresh}/d_{core}\right) = -\infty. \tag{2.47}$$

This can only be satisfied if

$$\left(\frac{R_{thresh}}{d_{core}}\right)_{vacuum\ tamper} = \pi. \tag{2.48}$$

The second sub-case is more realistic in that we imagine a thick tamper with a non-zero transport mean free path. For simplicity, assume that $\lambda_{trans}^{core} \sim \lambda_{trans}^{tamp}$, that is, that the neutron-scattering properties of the tamper are much like those of the core. In this case (2.46) becomes

$$(R_{thresh}/d_{core}) \cot(R_{thresh}/d_{core}) = 0. \tag{2.49}$$

The solution in this case is

$$\left(\frac{R_{Thresh}}{d_{core}}\right)_{thick\ tamper\ finite\ cross-\text{section}} = \frac{\pi}{2}, \tag{2.50}$$

one-half the value of the vacuum-tamper case. To summarize: With an infinitely-thick tamper of finite transport mean free path, the threshold critical radius is one-half of what it would be if no tamper were present at all. A factor of two in radius means a factor of eight in mass, so the advantage of using a tamper is dramatic, even aside from the issue of any retardation of core expansion. This factor of two is predicated on an unrealistic assumption for the tamper thickness and so we cannot expect such a dramatic effect in reality, but we will see that the effect is dramatic enough.

What sort of critical-mass reduction can one expect in practice? In a website devoted to design details of nuclear weapons, Sublette (2007) records that the Hiroshima *Little Boy* bomb used tungsten-carbide (WC) as its tamper material. Tungsten has five naturally-occurring isotopes, ^{180}W, ^{182}W, ^{183}W, ^{184}W, and ^{186}W, with abundances 0.0012, 0.265, 0.1431, 0.3064, and 0.2843, respectively. The KAERI table-of-nuclides site referenced in Appendix B gives elastic-scattering cross sections for the four most abundant of these as (in order of increasing weight) 4.369, 3.914, 4.253, and 4.253 bn. Neglecting the small abundance of ^{180}W, the abundance-weighted average of these is 4.235 bn. Adding the 2.352 bn elastic-scattering cross-section for ^{12}C gives a total of 6.587 bn; the cross-sections must be added, not averaged, since we are considering the tungsten-carbide molecules to be "single" scattering centers of atomic weight equal to the sum of the individual atomic weights for W and C, 183.84 + 12.00 = 195.84. The bulk density of tungsten-carbide is 14.8 g/cm^3. Assuming an outer radius for the tamper of 17.5 cm (the choice of this value is explained below), (2.43) indicates that the tamped threshold critical radius of ^{235}U in this configuration is 6.20 cm, equivalent to a mass of 18.7 kg, about 60% less than the untamped value of 45.9 kg (Table 2.1). Figure 2.6 shows how the tamped threshold critical mass for a U-235 core depends on the outer radius of a surrounding tungsten-carbide tamper. The mass of the tamper would be about 38 kg for an outer radius of 10 cm and just over 950 kg for an outer radius of 25 cm.

A shell of tungsten carbide of outer radius 17.5 cm and thickness 11.3 cm has a mass of 317 kg. The 17.5 cm outer radius was chosen as Sublette records that the *Little Boy* tamper had a mass of about 311 kg and that its core comprised about 64 kg of ^{235}U in a cylindrical shape surrounded by a cylindrical WC tamper of diameter and length 13 in. (see also Coster-Mullen (2010)). Assuming for simplicity spherical geometry, a 64-kg core at a density of 18.71 g/cm^3 would have an outer radius of 9.35 cm; a 311-kg tamper would then require an outer radius of about 18 cm. For the mass of its tamper, therefore, *Little Boy* utilized about 3.5 threshold critical masses of fissile material.

Spreadsheet **CriticalityAnalytic.xls** allows users to carry out these calculations for themselves. In addition to the core parameters entered for the calculations of Sect. 2.2, the user enters the density, atomic weight, scattering cross-section and outer radius of the tamper. The "Goal Seek" function is then to determine the tamped threshold critical radius and mass from (2.43).

Why was tungsten-carbide used as the *Little Boy* tamper material? As one of the purposes of the tamper is to briefly retard core expansion, denser tamper materials are preferable; tungsten-carbide is fairly dense and has a low neutron absorption cross-section. In this sense it would seem that depleted uranium, which the Manhattan Project possessed in abundance, would be an ideal tamper material. (*Depleted* is the term given to the uranium that remains after one has extracted its fissile U-235. The term may sound strange in that the remains are actually enriched in U-238, but the term is used in the sense of the material having been depleted of U-235.) The reason that it was not used may be that it has a fairly high spontaneous fission rate, about 675 per kg/s (see Sect. 4.2). Over the approximately 100 μs required to assemble the core of a Hiroshima gun-type bomb, a 300 kg depleted-U tamper would have a fairly high probability of suffering a spontaneous fission and hence of initiating a predetonation. Further, as discussed in Sect. 1.9, U-238 has a significant inelastic-scattering cross-section: fast neutrons striking it tend to be slowed to the point that they become likely to be captured and hence lost to the possibility of being reflected back into the core. Former weapons designer Theodore Taylor has pointed out that beryllium is one of the best neutron reflectors known: its fission-spectrum

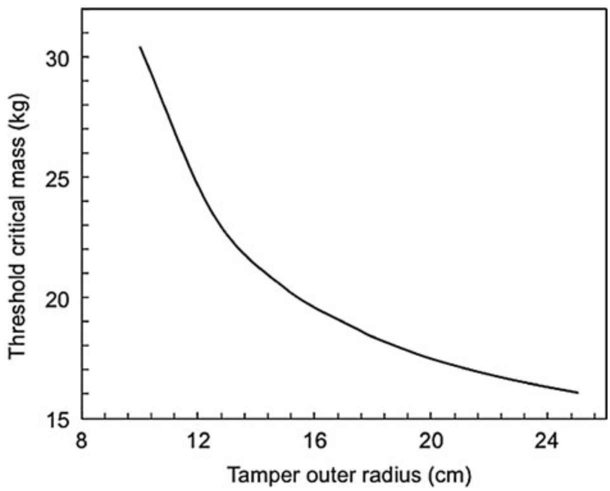

Fig. 2.6 Threshold tamped critical mass of a pure ^{235}U core as a function of the outer radius of a tamper of tungsten-carbide (A = 195.84 g/mol, ρ = 14.8 g/cm^3, $\sigma_{elastic}$ = 6.587 bn)

averaged elastic scattering cross section is about 2.7 bn, while its inelastic-scattering cross-section is only about 40 μbn (McPhee 1974).

2.4 Estimating Bomb Efficiency: Analytic

Material in this section is adopted from a publication elsewhere by the author (Reed 2007).

In the preceding sections we examined how to estimate the critical mass for bare and tamped cores of fissile material. The analysis in Sect. 2.2 revealed that the threshold bare critical mass of ^{235}U is about 46 kg. In Sect. 1.6, however, we saw that complete fission of 1 kg of ^{235}U liberates energy equivalent to that of about 17 kt of TNT. Given that the *Little Boy* uranium bomb that was dropped on Hiroshima used about 64 kg of ^{235}U and is estimated to have had an explosive yield of only about 13 kt, we can infer that it must have been rather inefficient. The purpose of this section is to explore what factors dictate the efficiency of a fission weapon and to show how one can estimate that efficiency.

This section is the first of two devoted to the question of weapon efficiency and yield. In this section these issues are examined purely analytically. The advantage of an analytic approach is that it is helpful for establishing a sense of how the efficiency depends on the various parameters involved: the mass and density of the core and the various nuclear constants. However, conditions inside an exploding bomb core evolve very rapidly as a function of time, and this evolution cannot be fully captured with analytic approximations, elegant as they may be. To do so, one really needs to numerically integrate the core conditions as a function of time, tracking core size, expansion rate, pressure, neutron density and energy release along the way. Such a numerical integration is the subject of the next section; these two sections therefore closely complement each other and should be read as a unit. In the present section, we consider only *untamped* cores for sake of simplicity; tamped cores are considered in the following section.

To begin, it is helpful to appreciate that the efficiency of a nuclear weapon involves three distinct time scales. The first is mechanical in nature: the time required to assemble the subcritical fissile components into a critical assembly before fission is initiated. In principle, this time can be as long as desired, but in practice it is constrained by the occurrence of spontaneous fission. We do not want spontaneous fissions to be likely during the time required to assemble the core lest stray neutrons trigger a predetonation.

What is the order of magnitude of the assembly time? In a simple "gun-type" bomb, the idea is that a "projectile" piece of fissile material is fired like a shell inside an artillery barrel toward a mating "target" piece of fissile material, as sketched in Fig. 2.7. In World War II, the highest velocity that could be achieved for an artillery shell was about 1,000 m/s. If a projectile piece of length ~ 10 cm is shot toward a mating target piece at this speed, the time required for it to become fully engaged with the target piece from the time that the leading edge of the projectile meets the target piece will be ~ (10 cm)/(10^5 cm/s) ~ 10^{-4} s ~ 100 μs.

This type of assembly mechanism was used in the Hiroshima *Little Boy* bomb, which explains its cylindrical shape as illustrated in the photograph in Fig. 2.8. As shown in the cross-sectional schematic in Fig. 2.9, the projectile piece was fired from the tail end of the bomb and traveled most of the approximately 10-foot length of the weapon toward the nose.

As we will see in a more detailed analysis presented in Sect. 4.2, spontaneous fission was not an issue for assembling a uranium bomb over a time of 100 μs, but was such a serious issue with plutonium that it necessitated development of the implosion mechanism for triggering those weapons. So far as the present section is concerned, however, the essential idea is that if the spontaneous fission probability can be kept negligible during the assembly time (which we assume), the efficiency of the weapon is dictated by the other two time scales.

The first of these other two time scales is nuclear in nature. Once fission has been initiated, how much time is required for all of the fissile material to be consumed? This time we call $t_{fission}$. The other is again mechanical. As soon as fissions have been initiated, the core will begin to expand due to the extreme gas pressure of the fission fragments. As we will see, this expansion leads after a time $t_{criticality}$ to loss of criticality, after which the reaction rate will diminish. Weapon efficiency will depend on how these times compare: if $t_{criticality} > t_{fission}$ then in principle all of the core material will undergo fission and the efficiency would be 100%.

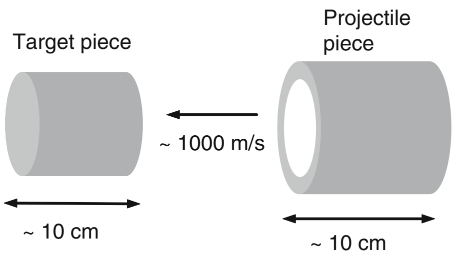

Fig. 2.7 Assembly timescale for a gun-type fission weapon

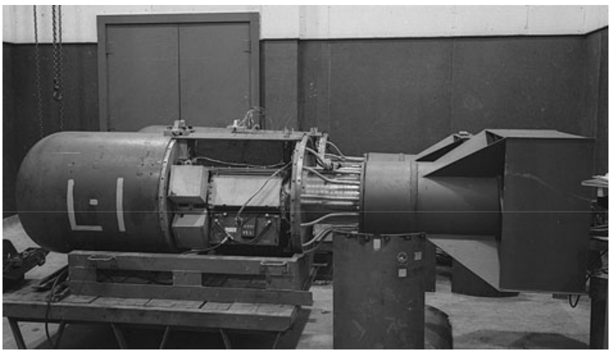

Fig. 2.8 *Little Boy* test units. *Little Boy* was 126 in. long, 28 in. in diameter, and weighed 8,900 pounds when fully assembled (Sublette 2007). Photo courtesy Alan Carr, Los Alamos National Laboratory

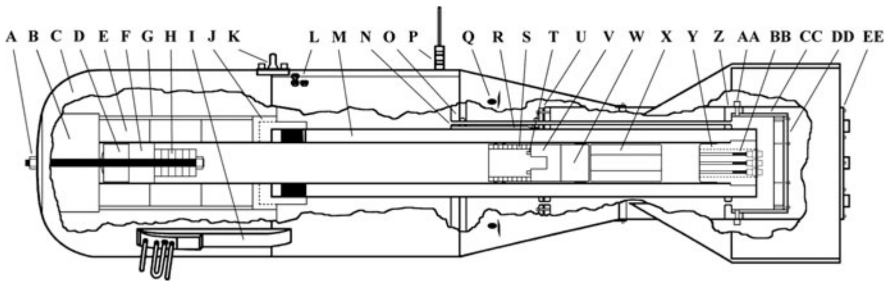

Fig. 2.9 Cross-section drawing of Y-1852 *Little Boy* showing major components. Not shown are radar units, clock box with pullout wires, barometric switches and tubing, batteries, and electrical wiring. Numbers in parentheses indicate quantity of identical components. Drawing is to scale. Copyright by and used with kind permission of John Coster-Mullen

A. Front nose elastic locknut attached to 1-in. diameter Cd-plated draw bolt
B. 15.125-in. diameter forged steel nose nut
C. 28-in. diameter forged steel target case
D. Impact-absorbing anvil with shim
E. 13-in. diameter 3-piece WC tamper liner assembly with 6.5-in. bore
F. 6.5-in. diameter WC tamper insert base
G. 14-in. diameter K-46 steel WC tamper liner sleeve
H. 4-in. diameter U-235 target insert discs (6)
I. Yagi antenna assemblies (4)
J. Target-case to gun-tube adapter with four vent slots and 6.5-in. hole
K. Lift lug
L. Safing/arming plugs (3)
M. 6.5-in. bore gun
N. 0.75-in. diameter armored tubes containing priming wiring (3)
O. 27.25-in. diameter bulkhead plate
P. Electrical plugs (3)
Q. Barometric ports (8)
R. 1-in. diameter rear alignment rods (3)
S. 6.25-in. diameter U-235 projectile rings (9)
T. Polonium–beryllium initiators (4)
U. Tail tube forward plate
V. Projectile WC filler plug
W. Projectile steel back
X. 2-pound Cordite powder bags (4)
Y. Gun breech with removable inner breech plug and stationary outer bushing
Z. Tail tube aft plate
(AA) 2.25-in. long 5/8–18 socket-head tail tube bolts (4)
(BB) Mark-15 Mod 1 electric gun primers with AN-3102-20AN receptacles (3)
(CC) 15-in. diameter armored inner tail tube
(DD) Inner armor plate bolted to 15-in. diameter armored tube
(EE) Rear plate with smoke puff tubes bolted to 17-in. diameter tail tube

First consider $t_{criticality}$. This involves two key ideas: (i) that a fissioning bomb core will rapidly (within about a microsecond) heat up, melt, vaporize, and thereafter behave as an expanding gas with the expansion driven by the gas pressure in a $P\Delta V$ manner, and (ii) that the vast majority of energy liberated in fission reactions can be assumed to go into the kinetic energy of the fission products. Our approach here will be to establish the range of radius (and hence time) over which the core can expand before the expansion lowers the density of the fissile material to subcriticality. Fission reactions will continue to happen after this time, of course, but it is this "criticality shutdown timescale" that fundamentally sets the efficiency scale of the weapon.

As in the preceding sections, let $N(r, t)$ represent the number density of neutrons within the core; our concern here is with the time-dependence of this quantity. From (2.22), the time-evolution of the number-density of neutrons within the core is given by

$$N(t) = N_o e^{(\alpha/\tau)\,t}, \tag{2.51}$$

where N_o is the neutron density at $t = 0$. N_o is set by the number of neutrons released by some "initiator" at the bomb core, and α is given by solving (2.25), (2.30), and (2.31) for the core at hand. Recall that for threshold criticality $\alpha = 0$ and that for a core of more than one critical mass we will have $\alpha > 0$, an issue to which we will return in a moment.

On average, a neutron will cause another fission after traveling for a time given by $\tau = \lambda_f / v_{neut}$ where λ_f is the mean free path for fission and v_{neut} is the average neutron velocity. Inverting this, we can say that a single neutron will lead to a subsequent fission at a *rate* of $1/\tau$ per second. Hence the rate of fissions as a function of time is given by

$$fissions/\text{sec} = \left(\frac{N_o V}{\tau}\right) e^{(\alpha/\tau)\,t}. \tag{2.52}$$

Equation (2.52) is actually more complicated than it looks because α is really a function of time. To see this, consider a core of some general radius r and density ρ. Both r and ρ will vary in time as the core expands. In Sect. 2.2 we saw that the condition for criticality can be expressed as $\rho r \geq K$ where K is a constant characteristic of the material being used, and that, for a core of some mass M, this condition is expressible as $\rho r \propto M/r^2$. As the core expands, the value of ρr will decrease and must eventually fall below the level needed to maintain criticality; we call this situation "criticality shutdown." This is also known in the technical literature as *second criticality*. For a single critical mass of normal-density material, this will happen as soon as the expansion begins. One way to (briefly) circumvent this is to provide a tamper to momentarily retard the expansion and so to give the reaction time to build up to a significant degree. Another is to start with a core of more than one critical mass of material of normal density, and this is what is assumed here. The effect of a tamper and the detailed time-evolution of $\alpha(t)$ is dealt with in the following section.

Thus, assume that we have a core of C (> 1) *untamped* threshold critical masses of material of normal density; the initial radius of such a core will be $r_i = C^{1/3}R_o$. We can then solve the diffusion-theory criticality equations, (2.30) and (2.31), for the value of α that just satisfies those equations upon setting the radius to be $C^{1/3}$ times the threshold critical radius determined in Table 2.1. But as the core expands due to the momentum acquired by fission fragments, α will decline from this initial value down to zero at the moment of criticality shutdown, hence the remark above that α is a function of time. To avoid having to deal with this complexity, we take α to be an "effective" α given by the average of these two extreme values, that is, $\alpha_{initial}/2$; this is done automatically in the **CriticalityAnalytic.xls** spreadsheet. This assumption is not strictly valid as the core expands exponentially as opposed to linearly in time, but the intent here is to get a sense of how the efficiency depends on the various parameters at hand.

Now consider the energy released by these fissions. If each fission liberates energy E_f, then the rate of energy liberation throughout the entire volume V of the core will be

$$\frac{dE}{dt} = \left(\frac{N_o V E_f}{\tau}\right) e^{(\alpha/\tau)\,t}. \tag{2.53}$$

Integrating this from time $t = 0$ to some general time t gives the energy liberated to that time:

$$E(t) = \left(\frac{N_o V E_f}{\tau}\right) \int_0^t e^{(\alpha/\tau)\,t} dt = \left(\frac{N_o V E_f}{\alpha}\right) e^{(\alpha/\tau)\,t}, \tag{2.54}$$

where it has been assumed that $e^{(\alpha/\tau)t} >> 1$ for the timescale of interest, an assumption to be investigated *a posteriori*. The *energy density* corresponding to $E(t)$ is given by $U(t) = E(t)/V$, and corresponding to this, we know from thermodynamics that there will be a growth in pressure given by $P(t) = \gamma\, U(t)$. The choice of γ depends on whether gas pressure ($\gamma = 2/3$) or radiation pressure ($\gamma = 1/3$) is dominant; in the case of a "gas" of uranium nuclei of standard density of that metal, radiation pressure dominates for per-particle energies greater than about 2 keV (see Problem 2.12). Thus

$$P(t) = \left(\frac{\gamma N_o E_f}{\alpha}\right) e^{(\alpha/\tau)\,t} = P_o\, e^{(\alpha/\tau)\,t}, \tag{2.55}$$

where $P_o = \left(\gamma N_o E_f/\alpha\right)$ is the pressure at $t = 0$.

For simplicity, we model the bomb core as an expanding sphere of radius $r(t)$ with every atom in it moving at speed v. Do not confuse this velocity with the average neutron speed, which enters into τ. If the sphere is of density $\rho(t)$ and total mass M, its total kinetic energy will be

$$K_{core} = \frac{1}{2}Mv^2 = \left(\frac{2\pi}{3}\right)\rho\, v^2 r^3. \tag{2.56}$$

Now invoke the work-energy theorem in its thermodynamic formulation $W = P(t)\, dV$ and equate the work done by the gas (or radiation) pressure in changing the core volume by dV over time dt to the change in the core's kinetic energy over that time:

$$P(t)\,\frac{dV}{dt} = \frac{dK_{core}}{dt}. \tag{2.57}$$

To formulate this explicitly, write $dK_{core}/dt = (2\pi/3)\rho r^3(2v dv/dt)$, $dV/dt = 4\pi r^2(dr/dt)$, and incorporate (2.55) to give

$$\frac{dv}{dt} = \left(\frac{3P_o}{\rho\, r}\right) e^{(\alpha/\tau)\, t}. \tag{2.58}$$

To solve this for the radius of the core as a function of time we face the problem of what to do about the fact that both ρ and r are functions of time. We deal with this by means of an approximation.

Review the discussion about core expansion following (2.52) above. As the core expands, its density when it has any general radius r will be $\rho(r) = C\rho_o(R_o/r)^3$, and criticality will hold until such time as $\rho r = \rho_o R_o$, or, on eliminating ρ, $r = C^{1/2}R_o$. We can then define Δr, the range of radius over which criticality holds:

$$\Delta r = \left(C^{1/2} - C^{1/3}\right) R_o, \tag{2.59}$$

a result we will use in a moment.

Now, since $r_i = C^{1/3}R_o$, $(\rho r)_{\text{initial}} = C^{1/3}(\rho_o R_o)$. For $C = 2$ (for example), this gives $(\rho r)_{\text{initial}} = 1.26(\rho_o R_o)$. At criticality shutdown we will have $(\rho r)_{\text{crit}} = (\rho_o R_o)$, so $(\rho r)_{\text{crit}}$ and $(\rho r)_{\text{initial}}$ do not differ greatly. In view of this, we assume that the product ρr in (2.58) can be replaced with a mean value given by the average of the initial and final (loss-of-criticality) radii:

$$\langle \rho r \rangle = \frac{1}{2}\left(1 + C^{1/3}\right) \rho_o R_o. \tag{2.60}$$

We can now integrate (2.58) from time $t = 0$ to some general time t to determine the velocity of the expanding core at that time:

$$v(t) = \left(\frac{3P_o}{\langle\rho\, r\rangle}\right) \int_0^t e^{(\alpha/\tau)\, t} dt = \left(\frac{3P_o\, \tau}{\langle\rho\, r\rangle\, \alpha}\right) e^{(\alpha/\tau)\, t}, \tag{2.61}$$

where it has again been assumed that $e^{(\alpha/\tau)t} >> 1$.

The stage is now set to compute the amount of time that the core will take to expand through the distance Δr of (2.59). Writing $v = dr/dt$ and integrating (2.61) from r_i to $r_i + \Delta r$ for time $= 0$ to $t_{criticality}$ gives

$$t_{crit} \sim \left(\frac{\tau}{\alpha}\right) \ln \left[\frac{\Delta r\, \alpha^2 \langle\rho\, r\rangle}{3P_o \tau^2}\right] = \left(\frac{\tau}{\alpha}\right) \ln \left[\frac{\Delta r\, \alpha^3 \langle\rho\, r\rangle}{3\,\gamma\, \tau^2 N_o E_f}\right], \tag{2.62}$$

again assuming $e^{(\alpha/\tau)t} >> 1$ and using $P_o = \gamma\, N_o E_f/\alpha$. Notice that we cannot determine t_{crit} without knowing the initial neutron density N_o.

We now define efficiency. Equation (2.54) gives the total energy liberated up to time t. If all of the nuclei were to fission, then total energy $E_f\, n\, V$ would be liberated, where n and V are the initial nuclear number density and volume of the core. We define efficiency as the ratio of the total energy liberated up to time t_{crit} to the total possible that can be liberated if all nuclei fission:

$$Efficiency = \frac{\left(\frac{E_f N_o\, V}{\alpha}\right)}{\left(E_f\, n\, V\right)} \exp\left[(\alpha/\tau)\, t_{crit}\right] = \frac{\Delta r\, \alpha^2 \langle\rho\, r\rangle}{3\,\gamma\, n\, \tau^2 E_f}, \tag{2.63}$$

where we again substituted for P_o. *Note that the efficiency does not depend on the initial neutron density.*

The yield of the weapon is given by the product of this efficiency times the core mass (in kilograms) times the energy liberated per kilogram of fissioned nuclei, $E_f N_A(1000/A)$, where A is the atomic weight in g/mol.

To help determine what value of γ to use, we can compute the total energy liberated to time t_{crit} as in (2.63), and then compute the energy per particle by dividing by the number of nuclei in the core, nV. The result is

$$\begin{pmatrix} \textit{energy per nucleus} \\ \textit{at time } t_{crit} \end{pmatrix} = (\textit{efficiency}) E_f. \tag{2.64}$$

Even if the efficiency is very low, say 0.1%, then for $E_f = 180$ MeV the energy per nucleus would be 180 keV, much higher than the ~2 keV per-particle energy where radiation pressure dominates over gas pressure; it would thus seem reasonable to take $\gamma = 1/3$.

Further, it can be shown by substituting (2.62) into (2.55) and (2.61) that at the time of criticality shutdown the core velocity is given by

$$v\left(t_{crit}\right) = \frac{\alpha\, \Delta r}{\tau}, \tag{2.65}$$

and that the pressure within the core is given by

$$P(t_{crit}) = \frac{\alpha^2\,\Delta r\,\langle \rho r\rangle}{3\,\tau^2}. \tag{2.66}$$

Curiously, this pressure does not depend on the value of γ.

To determine t_{crit} explicitly requires adopting a number of "initial" neutrons to be distributed throughout the volume of the core. But since t_{crit} depends logarithmically on N_o, it is not particularly sensitive to the choice made for that number; presumably the *minimum* sensible value is one initial neutron.

We can also estimate the timescale to fission the entire core by demanding that the integral of (2.52) from time zero to time t_{fiss} equals the total number of nuclei within the core, nV:

$$nV = \left(\frac{N_o V}{\tau}\right)\int_0^{t_{fiss}} e^{(\alpha/\tau)\,t}dt \Rightarrow t_{fiss} = \left(\frac{\tau}{\alpha}\right)\ \ln\left[\frac{\alpha\, n}{N_o}\right]. \tag{2.67}$$

Numbers for uranium and plutonium cores of $C = 2$ bare threshold critical masses appear in Table 2.2. Secondary neutrons are assumed to have $E = 2$ MeV, and it is assumed that the initial number of neutrons is one.

The timescales and pressures involved in the detonation process are remarkable: Neutrons travel for a time of only $\tau \sim 1/100$ µs between fissions, and criticality shuts down after only 1–2 µs. A pressure of 10^{15} Pa is equivalent to about 10 *billion* atmospheres. In the case of ^{235}U, changing the initial number of neutrons to 1,000 changes the fission and criticality timescales by only about 10%, down to 1.81 and 1.64 µs, respectively. Since $(\alpha/\tau)t_{crit} \sim 50$, the assumption that $e^{(\alpha/\tau)t} >> 1$ is quite reasonable. Even though $t_{crit}/t_{fiss} \sim 0.9$, the efficiencies are low: small changes in an exponential argument lead to large changes in the results.

Spreadsheet **CriticalityAnalytic.xls** carries out the efficiency and yield calculations for an untamped core as developed above. In addition to the parameters already entered for the calculations of the preceding two sections, the user need only

Table 2.2 Criticality and efficiency parameters for $C = 2$, $E_f = 180$ MeV, $\gamma = 1/3$. Initial number of neutrons = 1. Secondary neutron energy = 2 MeV

Quantity	Unit	Physical meaning	^{235}U	^{239}Pu
$\alpha_{initial}/2$	–	Effective value of α	0.246	0.304
R_O	cm	Threshold critical radius	8.37	6.346
τ	ns	Neutron travel time between fissions	8.64	7.23
Δr	cm	Expansion distance to crit shutdown	1.29	0.98
Efficiency	%	Efficiency	1.34	1.71
$P(t_{crit})$	10^{15} Pa	Pressure at crit shutdown	6.20	6.47
Yield	kt	Explosive yield	21.7	9.9
t_{fiss}	µs	Time to fission all nuclei	2.08	1.39
t_{crit}	µs	Time to crit shutdown	1.93	1.29

additionally specify an initial number of neutrons, a value for γ, and the mass of the core. The "Goal Seek" function is then run a third time, to solve (2.30) and (2.31) for the value of α. The spreadsheet then computes and displays quantities such as the expansion distance to second criticality, the fission and criticality timescales, the pressure within and velocity of the core at second criticality, and the efficiency and yield.

When applied to a 64 kg ^{235}U core ($C = 1.39$), **CriticalityAnalytic.xls** indicates that the expansion distance to second criticality is $\Delta r = 0.53$ cm and that the yield will be only 1.6 kt. This is not directly comparable to the ~13 kt yield of *Little Boy*, however, as that device was tamped; a more realistic simulation of *Little Boy* is given in the next section.

It is important to emphasize that the above calculations cannot be applied to a tamped core; that is, one cannot simply solve (2.44) and (2.45) for a core of some specified mass and tamper of some size (outer radius) and use the value of α so obtained in the time and efficiency expressions established above. The reason for this has to do with the distance through which the core can expand before second criticality, (2.59) above:

$$\Delta r = \left(C^{1/2} - C^{1/3} \right) R_o. \tag{2.68}$$

This expression derived from the fact that the criticality equation for the untamped case involves the density and radius of the core in the combination ρr; in the tamped case the criticality condition admits no such combination of parameters, so the subsequent calculations of criticality timescale and efficiency do not simply transform to using a tamped critical radius. Efficiency in the case of a tamped core can only be established numerically, which is the subject of the next section, where we will see that, typically, $\Delta r_{tamped} > \Delta r_{bare}$.

2.5 Estimating Bomb Efficiency: Numerical

In this section, a numerical approach to estimating weapon efficiency and yield is developed. The essential physics necessary for this development was established in the preceding three sections; what is new here is how that physics is used. The analysis presented in this section is adopted from a publication elsewhere by the author (Reed 2010).

The approach taken here is one of standard numerical integration: The parameters of a bomb core and tamper are specified, along with a timestep Δt. At each timestep, the energy released from the core is computed, from which the acceleration of the core at that moment can be computed. The velocity and radius of the core can then be tracked until such time as second criticality occurs, after which the rate of fissions will drop drastically and very little additional energy will be liberated.

The simulation developed here is realized via a spreadsheet where rows correspond to time steps and the columns are used to track various quantities.

This spreadsheet, **CriticalityNumerical.xls**, is very similar to that developed in the preceding sections, **CriticalityAnalytic.xls**.

Specifically, the integration process involves eight steps:

(i) Fundamental parameters are specified: the mass of the core, its atomic weight, initial density, and nuclear characteristics σ_f, σ_{el}, and v. For the tamper, its atomic weight, density, initial outer radius (effectively, its mass) and elastic-scattering cross-section are specified. The energy release per fission E_f and gas/radiation pressure constant γ are also specified. A timestep Δt also needs to be set; from the discussion in the preceding section, this will be on the order of nanoseconds.

(ii) Elapsed time, the speed of the core, and the total energy released are initialized to zero; the core radius is initialized according as its mass and initial density.

(iii) The exponential neutron-density growth parameter α is determined by numerical solution of (2.44) and (2.45).

(iv) The rate of fissions at a given time is given by (2.52):

$$fissions/\sec = \left(\frac{N_o V}{\tau}\right) e^{(\alpha/\tau)\, t}. \tag{2.69}$$

(v) The amount of energy released during time Δt is computed from (2.53):

$$\Delta E = \left(\frac{N_o V E_f}{\tau}\right) e^{(\alpha/\tau)\, t} (\Delta t). \tag{2.70}$$

(vi) The total energy released to time t is updated, $E(t) \to E(t) + \Delta E$, and the pressure at time t is given by [see the discussion preceding (2.55)]

$$P_{core}(t) = \frac{\gamma E(t)}{V_{core}(t)}. \tag{2.71}$$

I use the core volume here on the rationale that the fission products which cause the gas/radiation pressure will likely largely remain within the core.

(vii) A key step is computing the change in the speed of the core over the elapsed time Δt due to the energy released during that time. In the discussion leading up to (2.58), this was approached by invoking the work-energy theorem:

$$P(t) \frac{dV_{core}}{dt} = \frac{dK_{core}}{dt}. \tag{2.72}$$

To improve the veracity of the simulation, it is desirable to account, at least in some approximate way, for the retarding effect of the tamper on the expansion of the core. To do this, I treat the dK/dt term in (2.72) as involving the speed of the core but with the mass involved being that of the core *plus* that of the tamper. The dV/dt term is taken to apply to the core only. I treat the tamper as being of constant density but with an outer radius that is recomputed at each step to keep its mass as specified at the outset; the inner edge of the tamper is assumed to remain snug against the expanding core. With r as the radius and v the speed of the core, we have

$$\frac{\gamma E(t)}{V_{core}(t)}\left(\frac{dV_{core}}{dt}\right) = \frac{dK_{total}}{dt}$$

$$\Rightarrow \frac{\gamma E(t)}{V_{core}(t)}\left(4\pi r^2 \frac{dr}{dt}\right) = \frac{1}{2}M_{c+t}\left(2v\frac{dv}{dt}\right),$$

from which we can compute the change in expansion speed of the core over time Δt as

$$\Delta v = \left[\frac{4\pi r^2 \gamma E(t)}{V_{core} M_{c+t}}\right](\Delta t). \tag{2.73}$$

With this, the expansion speed of the core and its outer radius can be updated according as $v(t) \rightarrow v(t) + \Delta v$ and $r(t) \rightarrow r(t) + v(t)\Delta t$. The outer radius of the tamper is then adjusted on the assumption that its density and mass remain constant.

(viii) Return to step (iii) to begin the next timestep; continue until second criticality is reached when $\alpha = 0$.

The assumption that the density of the tamper remains constant is probably not realistic: nuclear engineers speak of the "snowplow" effect, where high-density tamper material piles up just ahead of the expanding core/tamper interface. But the point here is an order-of-magnitude pedagogical model.

CriticalityNumerical.xls consists of three interlinked sheets. The first is essentially a copy of **CriticalityAnalytic.xls**, where the user inputs the fundamental data of step (i) above. As before, the Excel "Goal Seek" function is then run three times, to establish values for (1) the bare threshold critical radius, (2) the tamped threshold critical radius, and (3) the value of α corresponding to the chosen core mass. The radii (and corresponding masses) in (1) and (2) are computed for reference; the tamped threshold critical radius is also used in computing a "normalized" radius as described below.

A significant complexity in carrying out this simulation is that one apparently needs to solve (2.44) and (2.45) for the value of α at each time-stepped core radius: the fission rate, energy generation rate, and pressure all depend on α as a function of time. I have found, however, that α is usually quite linear as a function of core radius. This behavior can be used to greatly simplify the programming of the simulation. Sheet 2 of the spreadsheet allows the user to establish parameters for this linear behavior for the values of the various parameters that were input on Sheet 1. Sheet 2 consists of rows representing radii running from the initial core radius to 1.4 times the value of the second-criticality radius for a *bare* core of the mass chosen by the user on Sheet 1; this range appears to be suitable to establish the behavior of α. For convenience, Sheet 2 utilizes a "normalized" radius defined as

$$r_{norm} = \frac{r - C^{1/3} R_{tamp}^{thresh}}{\left(C^{1/2} - C^{1/3}\right) R_{tamp}^{thresh}}, \tag{2.74}$$

where C is now defined as the number of tamped threshold critical masses. $r_{norm} = 1$ corresponds to the second criticality radius one would compute from (2.59) if it applied as well to a tamped core. Sheet 2 tracks the changing mass density, nuclear number density, and fission and total mean free paths within the core as a function of r. By running the Goal Seek function on each of 28 radii between 1.0 and 1.4 normalized radii, the user adjusts α in each case to render (2.44) equal to zero. The behavior of $\alpha(r)$ is displayed in an automatically-generated graph. On a separate line with α fixed to a value very near zero (10^{-10} is built-in), the user adjusts the radius to once again render (2.44) equal to zero, thus establishing the radius of second criticality for the parameters of the system. The slope and intercept of a linear $\alpha(r)$ fit are then automatically computed in preparation for the next step. While one could use just the initial and final radii to establish the linear relationship, it is probably wise to check the extent of linearity with all 28 radial points.

The actual time-dependent simulation occurs on Sheet 3. The simulation is set up to involve 500 timesteps, one per row. The initial core radius is transferred from Sheet 1 for $t = 0$. Because much of the energy release in a nuclear weapon occurs during the last few generation of fissions before second criticality, Sheet 3 allows the user to set up two different timescales: an "initial" one (dtinit) intended for use in the first few rows of the Sheet when a larger timestep can be tolerated without much loss of accuracy, and a later one (dtlate), to be chosen considerably smaller and used for the majority of the rows. In this way a user can optimize the 500 rows to both capture sufficient accuracy in the last few fission generations while arranging for $\alpha(r)$ to just approach zero at the last steps of the process. Typical choices for dtinit and dtlate might be a few tenths of a microsecond and a few tenths of a nanosecond, respectively. At each radius, Sheet 3 computes the value of $\alpha(r)$ from the linear approximation of Sheet 2, the core volume, mass density, nuclear number densities and mean free paths within the core, τ, rates of fission and energy generation, pressure, and total energy liberated to that time. The core speed and radius are updated depending upon the timestep in play, and the updated radius is transferred to the subsequent row to seed the next step. The user is automatically presented with graphs of $\alpha(r)$, the fission rate, pressure, and total energy liberated (in kilotons equivalent) as functions of time.

2.5.1 A Simulation of the Hiroshima Little Boy Bomb

Using the parameters for the *Little Boy* bomb given in Sect. 2.3 (64 kg core of radius 9.35 cm plus a 311 kg tungsten-carbide tamper of outer radius 18 cm), the following results were obtained with **CriticalityNumerical.xls**.

Figure 2.10 shows the run of $\alpha(r)$ for this situation: it is sensibly linear over the expansion of the core to second criticality at a radius of 12.31 cm, with $\alpha(r) \sim -18.53r + 2.28$. This represents an expansion distance of $\Delta r = 2.96$ cm from the initial core radius of 9.35 cm; for an *untamped* 64 kg core, (2.59) predicts a value for Δr of only 0.53 cm; the effect of the tamper is significant.

Figures 2.11 and 2.12 show α, the integrated energy release, and the fission rate and pressure as functions of time. The number of initial neutrons is taken to be one. Notice that α actually remains close to its initial value until just before second criticality. The brevity and violence of the detonation are astonishing. The vast majority of the energy is liberated within an interval of about 0.1 μs. The pressure peaks at close to 5×10^{15} Pa, or about 50 *billion* atmospheres, equivalent to about one-fifth of that at the center of the Sun. The fission rate peaks at about 3.6×10^{31} per second. The core acceleration peaks at about 1.4×10^{12} m/s^2 at $t \sim 0.9$ μs, and second criticality occurs at $t \sim 1.07$ μs, at which time the core expansion velocity is about 270 km/s. These graphs dramatically illustrate what Robert Serber wrote in *The Los Alamos Primer*: "Since only the last few generations will release enough energy to produce much expansion, it is just possible for the reaction to occur to an interesting extent before it is stopped by the spreading of the active material".

The predicted yield of *Little Boy* from this simulation is 11.9 kt. This result is in surprisingly good agreement with the estimated ~12-kt yield published by Penney et al. (1970). At a fission yield of 17.59 kt/kg of pure U-235 (at 180 MeV/fission), this represents an efficiency of only about 1.1% for the 64-kg core. While some of this agreement must be fortuitous in view of the approximations incorporated in the present model, it is encouraging to see that it gives results of the correct order of magnitude. That the yield estimate needs to be taken with some skepticism is demonstrated by the fact that increasing the initial number of neutrons to 10 increases the yield to 12.5 kt. However, this change does not much affect the

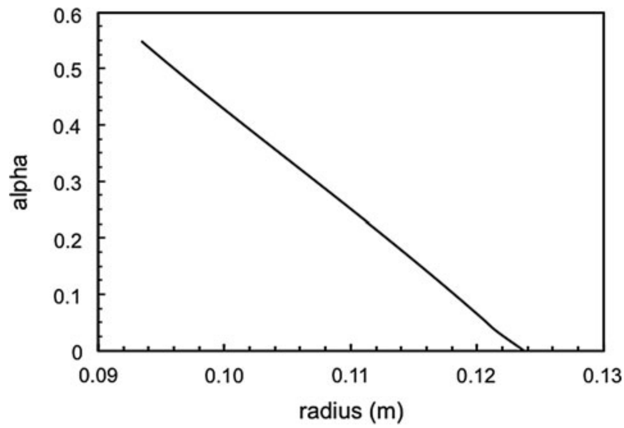

Fig. 2.10 Neutron density exponential growth parameter α vs. core radius for a simulation of the *Little Boy* bomb: 64 kg core plus 311 kg tungsten-carbide tamper

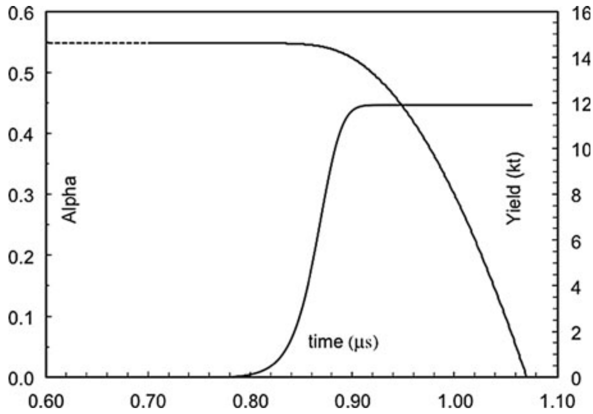

Fig. 2.11 Neutron density exponential growth parameter α (*descending curve*, *left scale*) and integrated energy release in kilotons (*ascending curve*, *right scale*) vs. time for a simulation of the *Little Boy* bomb

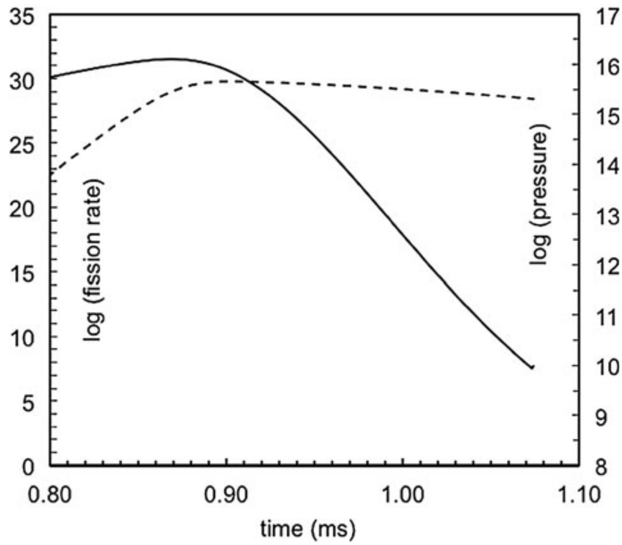

Fig. 2.12 Logarithm (base 10) of fission rate (*solid curve*, left scale) and logarithm of pressure (*dashed curve*, right scale) vs. time for a simulation of the *Little Boy* bomb

timescale or the peak pressure and fission rates. A 1952 Los Alamos report on the yield of the Hiroshima bomb, http://www.fas.org/sgp/othergov/doe/lanl/la-1398.pdf, gives a yield of 18.5 ± 5 kt for *Little Boy*; published yield estimates are clearly subject to considerable uncertainty.

Figure 2.13 shows how the simulated yield of the 64-kg core varies as a function of tamper mass; the points are the results of simulations for initial tamper outer radii

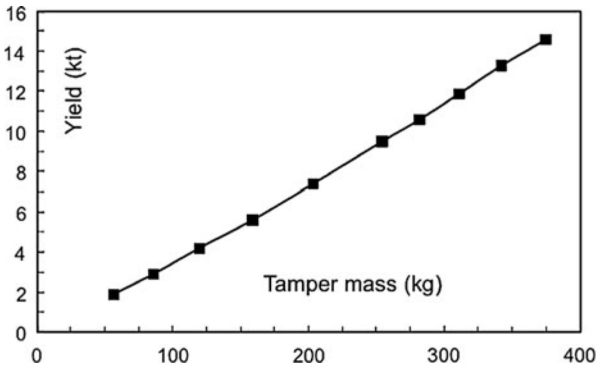

Fig. 2.13 Yield of a 64-kg U-235 core vs. mass of surrounding tungsten-carbide tamper. The curve is interpolated. The *Little Boy* tamper had a mass of about 310 kg

of 12, 13, ... 17, 17.5, 18, 18.5, and 19 cm. In the latter case the mass of the tamper would be about 375 kg, or just over 800 pounds. As the tamper mass increases so does the efficiency of the weapon as measured by the number of kilotons of yield per kilogram of fissile material.

2.6 Another Look at Untamped Criticality: Just One Number

In Sect. 2.2, we saw that the criticality condition for an untamped core is

$$x\cot(x) + \gamma x - 1 = 0, \tag{2.75}$$

where, for threshold criticality ($\alpha = 0$),

$$\gamma = \frac{1}{2}\sqrt{\frac{3\lambda_f}{\lambda_t(\nu - 1)}} = \frac{1}{2}\sqrt{\frac{3\sigma_t}{\sigma_f(\nu - 1)}}. \tag{2.76}$$

Once the nuclear parameters σ_f, σ_{el}, and ν are set, (2.75) is solved numerically for x, from which the critical radius R follows from (again with $\alpha = 0$)

$$R = dx = \sqrt{\frac{\lambda_f\lambda_t}{3\,(\nu - 1)}}x = \frac{1}{n}\sqrt{\frac{1}{3\sigma_f\,\sigma_t(\nu - 1)}}x, \tag{2.77}$$

where n is again the nuclear number density. The critical radius is fundamentally set by σ_f, σ_{el}, ν, and n; our concern here will be with the first three of these variables.

Since these quantities will be different for different fissile isotopes, it would appear that there is no "general" statement one can make regarding critical radii.

The purpose here, however, is to show how σ_f, σ_{el}, and v can be combined into one convenient dimensionless variable that largely dictates the critical radius in any particular case – the "just one number" of the title of this section.

As formulated, (2.75) and (2.76) are convenient in that both x and γ are dimensionless, but are awkward in that γ is not conveniently bounded: if v is very large γ will approach zero, but as $v \to 1$, it will diverge to infinity. It would be handy to some combination of σ_f, σ_{el}, and v that is finitely bounded.

Such a combination was developed by Peierls (1939), in a paper which was the first published in English to explore what he termed "criticality conditions in neutron multiplication." He defined a quantity ξ given by

$$\xi^2 = \frac{\sigma_f (v-1)}{\sigma_{el} + v\,\sigma_f}. \tag{2.78}$$

For $1 \leq v \leq \infty$, $0 \leq \xi \leq 1$. Note that it is the elastic-scattering cross-section σ_{el} that appears in the denominator of the definition of ξ, not the transport cross-section $\sigma_t = \sigma_{el} + \sigma_f$.

If (2.76) and (2.78) are both solved for $(v - 1)$ and the results equated, the relationship between γ and ξ emerges as

$$\gamma = \sqrt{\frac{3}{4}\left(\frac{1}{\xi^2} - 1\right)}. \tag{2.79}$$

Similarly, if the definition of d in (2.77) is solved for $(v - 1)$, then one finds

$$d = \sqrt{\frac{1}{3}\left(\frac{1}{\xi^2} - 1\right)}\,\lambda_t. \tag{2.80}$$

A general formulation of critical radii can now be made as follows: For a range of values of ξ between zero and one, (2.75) and (2.79) can be solved for x. For each solution, (2.77) and (2.80) then show that the value of R/λ_t is purely a function of ξ:

$$\frac{R}{\lambda_t} = x(\xi)\,d = x(\xi)\sqrt{\frac{1}{3}\left(\frac{1}{\xi^2} - 1\right)}. \tag{2.81}$$

In other words, a graph of $x(\xi)\,d(\xi) \equiv R/\lambda_t$ vs. ξ can be used to immediately indicate the ratio of the untamped threshold critical radius to the transport mean free path for any fissile isotope whose σ_f, σ_{el}, and v values are specified. The advantage of this approach is that the graph need only be constructed once.

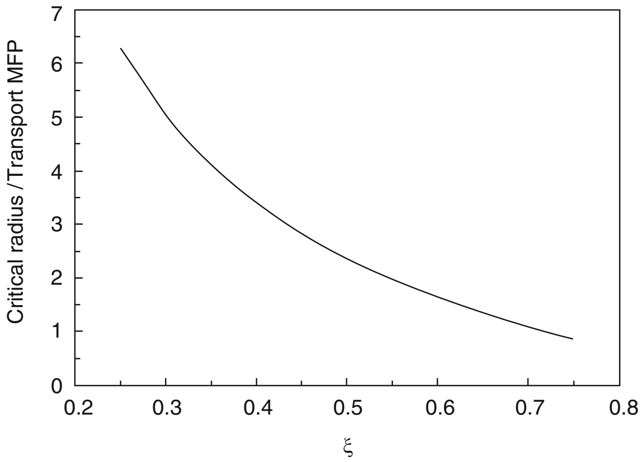

Fig. 2.14 Ratio of untamped threshold critical radius to transport mean free path as a function of Peierls' ξ parameter of (2.78)

Figure 2.14 shows R/λ_t as a function of ξ. For ^{235}U and ^{239}Pu, $\xi \sim 0.5084$ and 0.6221, and $R/\lambda_t \sim 2.33$ and 1.54, respectively. It is intuitively sensible that for small values of ξ (that is, for $v \to 1$), the critical radius will be large, and vice-versa.

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