Chapter 2
Hamiltonian Formalism

2.1 Derivation of Hamiltonian Equations

As we have discussed, Lagrangian formulation of classical mechanics is based on Euler–Lagrange (Newton) equations of motion, which represent a system of second-order differential equations, written for a set of variables that describe the position of a physical system of interest. Hamiltonian formulation suggests an equivalent description in terms of first-order equations written for independent variables describing the position and velocity of the system. The aim of this section is to establish an equivalence of the two descriptions.

2.1.1 Preliminaries

Hamiltonian equations can be obtained from Lagrangian ones by successive application of two well-known procedures in a theory of differential equations: reduction of order and change of variables. Both procedures are intended to obtain an equivalent system of equations from a given system. So, we recall here some elementary facts from the theory of ordinary differential equations, which will be used below.

Reduction of the order of a system. A second-order system of \( n \) equations for \( n \) independent variables \( q^a(\tau) \),

\[
F^a(q^a, \dot{q}^b, \ddot{q}^c) = 0, \tag{2.1}
\]

is equivalent to the first-order system of \( 2n \) equations for \( 2n \) independent variables \( q^a(\tau), v^b(\tau) \)

\[
\dot{q}^a = v^a, \quad F^a(q^a, v^b, \dot{v}^c) = 0, \tag{2.2}
\]

in the following sense:

(a) If \( q^a(\tau) \) obeys Eq. (2.1), then the functions \( q^a(\tau), v^a(\tau) \equiv \dot{q}^a(\tau) \) obey Eq. (2.2);

(b) If the functions \( q^a(\tau), v^a(\tau) \) obey Eq. (2.2), then \( q^a(\tau) \) obeys Eq. (2.1).
In other words, there is a one-to-one correspondence among solutions to the systems. The system (2.2) is referred to as the first-order form of the system (2.1).

Normal form of a system. We restrict ourselves to the first-order system

\[ G^i(z^j, \dot{z}^k) = 0. \]  

It is said to be presented in the normal form if all the equations are solved algebraically with respect to higher derivatives

\[ \dot{z}^i = g^i(z^j). \]  

Any system with \( \text{det} \frac{\partial G^i}{\partial \dot{z}^j} \neq 0 \) can (locally) be rewritten in the normal form. According to the theory of differential equations, a normal system has well established properties. In particular, under known restrictions to functions \( g^i \), the theorem for the existence and uniqueness of a solution holds: let \( z^i_0 \) be given numbers, then locally there exists a unique solution \( z^i(\tau) \) of the system (2.4) that obeys the initial conditions \( z^i(0) = z^i_0 \). Physically it means the causal dynamics and, in turn, a possibility of interpretation of the system (2.3) as the equations of motion for some physical system of classical mechanics.

Change of variables. Let \( q^i(z^j) \) be given functions, with the property

\[ \text{det} \frac{\partial q^i}{\partial z^j} \neq 0. \]  

Starting from original parametrization \( z^i \) of the configuration space for the system (2.3), functions \( q^i(z^j) \) can be used to define another parametrization \( z'^i \), namely

\[ z'^i = q^i(z^j). \]  

We use the same letter \( z'^i \) to denote the function and the new coordinate, as long as this does not lead to any misunderstanding. According to the condition (2.5), change of variables \( z^i \to z'^i \) is invertible: the expressions (2.6) can be resolved with relation to \( z^i \), with the result being

\[ z^i = z^i(z'^j). \]  

Once the functions \( q^i(z^j) \) have been chosen, we can use the new coordinates to analyze the system (2.3). Namely, the system

\[ G^i(z^j(z'^k), \dot{z}^j(z'^k)) = 0, \]  

where \( \dot{z}^j(z'^k) = \frac{\partial z^i}{\partial z'^k} \dot{z}'^k \), is equivalent to the initial system (2.3): if \( z'^i(\tau) \) obeys the system (2.3), then \( z'^i(\tau) \equiv z'^i(z^j(\tau)) \) obeys (2.8), and vice versa.
Below we prefer to use the notation
\[ G^i (z^j, \dot{z}^j) \bigg|_{z = z'(\tau)} = 0, \] (2.9)
instead of (2.8), since sometimes caution is needed in making use of the substitution, see, for example, Eqs. (2.27) and (2.28) below.

**Hamiltonian system.** Let \( q^a, p_a, a = 1, 2, \ldots, n \) be independent variables. The normal system
\[ \dot{q}^a = Q^a(q, p, \tau), \quad \dot{p}_a = P_a(q, p, \tau), \] (2.10)
with the given functions \( Q, P \) is called the *Hamiltonian system*, if there is a function \( H(q, p, \tau) \), such that
\[ Q^a = \frac{\partial H}{\partial p_a}, \quad P_a = -\frac{\partial H}{\partial q^a}. \] (2.11)
In accordance with this, the Hamiltonian system can be written in the form
\[ \dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}. \] (2.12)
Equation (2.11) implies the necessary conditions for the system to be a Hamiltonian one
\[ \frac{\partial Q^a}{\partial q^b} = -\frac{\partial P_b}{\partial p_a}, \quad \frac{\partial Q^a}{\partial p_b} = \frac{\partial Q^b}{\partial p_a}, \quad \frac{\partial P_a}{\partial q^b} = \frac{\partial P_b}{\partial q^a}. \] (2.13)

**2.1.2 From Lagrangian to Hamiltonian Equations**

Let \( q^a, a = 1, 2, \ldots, n \) represent generalized coordinates of the configuration space for a mechanical system with the Lagrangian being \( L(q^a, \dot{q}^a) \). Then the dynamics is governed by the second-order Euler-Lagrange equation
\[ \frac{d}{d\tau} \left( \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a} \right) - \frac{\partial L(q, \dot{q})}{\partial q^a} = 0. \] (2.14)
For any Lagrangian system there is an equivalent Hamiltonian system. We demonstrate this mathematically notable fact for the particular case of a *nonsingular Lagrangian*
\[ \det \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^a \partial \dot{q}^b} \neq 0. \] (2.15)
In this case, the system (2.14) can be rewritten in the first-order normal form. Then in specially chosen coordinates it acquires the Hamiltonian form. It basically gives the Hamiltonian formulation of mechanics.

Our presentation below is somewhat more detailed as compared with other textbooks. This has been done for two reasons. First, equivalence, which represents one of the basic facts of classical mechanics, will be manifest in our discussion. Second, our treatment of the subject turns out to be useful for singular Lagrangians, revealing an algebraic structure of the Hamiltonian formulation for these systems [24].

Computing the derivative with respect to $\tau$ in Eq. (2.14), the latter can be written as

$$M_{ab}\ddot{q}^b = K_a,$$

where it was denoted

$$M_{ab}(q, \dot{q}) = \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^a \partial \dot{q}^b}, \quad K_a(q, \dot{q}) = \frac{\partial L}{\partial q^a} - \frac{\partial^2 L}{\partial \dot{q}^a \partial q^b} \dot{q}^b.$$ (2.17)

Let us start with construction of the first-order form for the system (2.16). We find it instructive to present here a less formal reasoning, as compared to that of Sect. 2.1.1. We introduce $2n$-dimensional configuration-velocity space parameterized by independent coordinates $q^a, v^b$ (sometimes the coordinates $v^b$ are called generalized velocities). Let us define evolution in this space according to the equations

$$\ddot{q}^a = v^a, \quad \ddot{v}^b = \bar{K}_a,$$ (2.18)

with $M(q, \dot{q}), K(q, \dot{q})$ given by Eq. (2.17). As before, time dependence of the coordinates $q^a(\tau)$ is determined by Lagrangian equations (2.16), while $v^a(\tau)$ accompanies $\dot{q}^a(\tau)$: $v^a(\tau)$ is determined from the known $q^a(\tau)$, taking its derivative. Evidently, systems (2.16) and (2.18) are equivalent. Further, we can use one of the equations of the system in other equations, obtaining an equivalent system. Substitution of the second equation from (2.18) into the first one gives the desired first order system

$$\dot{q}^a = v^a, \quad \ddot{v}^b = \bar{K}_a,$$ (2.19)

where $\bar{M}, \bar{K}$ are obtained from (2.17) by the replacement $\dot{q} \to v$, for example

$$\bar{M}_{ab} = M_{ab}(q, \dot{q})|_{\dot{q} \to v} = \frac{\partial L(q, v)}{\partial v^a \partial v^b}.$$ (2.20)

According to Eq. (2.15), the matrix $\bar{M}$ is invertible. Applying the inverse matrix $\tilde{M}$, the Eqs. (2.19) can be presented in the normal form $\dot{q} = v, \dot{v} = \tilde{M}\bar{K}$. The right-hand sides of these equations do not obey Eq. (2.13). So in terms of the variables $q, v$ the system is not a Hamiltonian one.
2.1 Derivation of Hamiltonian Equations

Making the variable change \( q \rightarrow q(q', v') \), \( v \rightarrow v(q', v') \) in Eq. (2.19), we could look for the new variables that imply the Hamiltonian form of the system. The point here is that there is a wide class of so-called canonical transformations that preserve the Hamiltonian form of an arbitrary Hamiltonian system (see Sect. 2.7 below). Hence the variables under discussion are not unique.\(^1\) The remarkable observation made by W. R. Hamilton was that it is sufficient to make the change of variables of the form (with \( v' \) conventionally denoted as \( p \))

\[
(q^a, \quad p^b) \leftrightarrow (q'^a, \quad p^b = \frac{\partial L(q, v)}{\partial q^b}) \tag{2.21}
\]
to transform the system (2.19) into the Hamiltonian one. Due to Eq. (2.15) we have \( \det \frac{\partial p^b(q, v)}{\partial v^c} \neq 0 \). The latter condition guarantees invertibility of the transformation (2.21). Let us denote the inverse transformation as

\[
v^a = v^a(q, p). \tag{2.22}
\]

This implies the identities

\[
\frac{\partial L}{\partial v^a} \bigg|_{v(q,p)} = p_a, \quad \frac{\partial p_a}{\partial v^b} = \frac{\partial^2 L}{\partial v^a \partial v^b} = M_{ab}(q, v). \tag{2.23}
\]

Let us confirm that in terms of the variables \( q, p \) the system (2.19) acquires the Hamiltonian form.

According to Sect. 2.1.1, the dynamics for the new variables is obtained from (2.19) by substitution \( v \rightarrow v(q, p) \). We have

\[
\dot{q}^a = v^a(q, p), \tag{2.24}
\]

\[
\tilde{M}_{ab} \big|_{v(q,p)} \frac{\partial v^b}{\partial q^c} \ddot{q}^c = \tilde{K}_a \big|_{v(q,p)} - \tilde{M}_{ab} \big|_{v(q,p)} \frac{\partial v^b}{\partial q^c} v^c(q, p) = \frac{\partial L(q, v)}{\partial q^a} \bigg|_{v(q,p)} - \frac{\partial^2 L(q, v)}{\partial q^c \partial v^a} \frac{\partial v^b}{\partial q^c} v^c(q, p). \tag{2.25}
\]

In the last equation, the left hand side is just \( p_a \), as is implied\(^2\) by Eq. (2.23), while the expression inside the brackets vanishes since it is \( \frac{\partial}{\partial q^c} \left( \frac{\partial L}{\partial v^a} \bigg|_{v(q,p)} \right) = \frac{\partial p_a}{\partial q^c} = 0 \). Then the Eqs. (2.24) and (2.25) acquire the form

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1. If the change \( q(q', v'), \; v(q', v') \) transforms the system (2.19) into the Hamiltonian one, and \( q(q'', v''), \; v(q'', v'') \) is the canonical transformation, then the change \( q(q''(q''', v'''), v''(q'', v'')) \), \( v(q''(q''', v''''), v''(q'', v''')) \) transforms (2.19) into the Hamiltonian system as well.

2. Recall that the Jacobi matrices of direct and inverse transformations are opposites: from the identity \( z^i(z'^j(z^{k})) = z^i \) we have \( \frac{\partial z^i}{\partial z'^j} \bigg|_{z^j(z)} \frac{\partial z^j}{\partial z^{k}} = \delta^i_k \). See also Exercise 2.1.2 on page 83.
\[ \dot{q}^a = v^a(q, p), \quad \dot{p}_a = \frac{\partial \tilde{L}(q, v)}{\partial q^a} \bigg|_{v(q, p)}. \]  

(2.26)

To substitute \(v(q, p)\) in the last equation, let us compute

\[ \frac{\partial}{\partial q^a} L(q, v(q, p)) = \frac{\partial L(q, v)}{\partial q^a} \bigg|_{v(q, p)} + \frac{\partial L(q, v)}{\partial v^b} \bigg|_{v(q, p)} \frac{\partial v^b}{\partial q^a} \]

which implies

\[ \frac{\partial L(q, v)}{\partial q^a} \bigg|_{v(q, p)} = -\frac{\partial}{\partial q^a} \left( p_b v^b(q, p) - L(q, v(q, p)) \right). \]  

(2.28)

Let us denote

\[ H(q, p) = p_b v^b(q, p) - L(q, v(q, p)), \]  

(2.29)

where \(v(q, p)\) is given in implicit form by Eq. (2.21). Then the expression (2.28) reads

\[ \frac{\partial \tilde{L}(q, v)}{\partial q^a} \bigg|_{v(q, p)} = -\frac{\partial H(q, p)}{\partial q^a}. \]  

(2.30)

The function \(H(q, p)\) is called the Hamiltonian of the physical system. To complete the derivation of the Hamiltonian equations, note the following property of the Hamiltonian:

\[ \frac{\partial H}{\partial p_a} = v^a(q, p) + p_b \frac{\partial v^b}{\partial p_a} - \frac{\partial L(q, v)}{\partial v^b} \bigg|_{v(q, p)} \frac{\partial v^b}{\partial p_a} = v^a(q, p). \]  

(2.31)

Using these results, the equations of motion (2.26) acquire the Hamiltonian form

\[ \dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}, \]  

(2.32)

and are known as Hamiltonian equations of motion. Note that the first equation is the Eq. (2.24) written in another notation.

The coordinates \(p_a\) defined by Eq. (2.21) are called conjugated momenta for \(q^a\). The configuration-velocity space parameterized by the coordinates \(q^a, p_b\) is referred to as the phase space of the system.

The passage (2.29) from \(L(q, v)\) to \(H(q, p)\) is known as Legendre transformation. Its basic properties are presented by Eqs. (2.31), (2.30). Note its meaning: if
the variable change \( v^a \rightarrow p_b \) (the variables \( q^a \) are considered as parameters) is “generated” by the function \( L(v) \) according to Eq. (2.21), \( p_a = \frac{\partial L}{\partial v^a} \), then the Legendre transformation gives the generating function \( H \) of the inverse transformation (2.22), \( v^a = \frac{\partial H}{\partial p_a} \). See also Exercise 5 below.

To sum up, in this section we have demonstrated that for the case of a nonsingular system, the Lagrangian equations of motion (2.14) for the configuration space variables \( q^a \) are equivalent to the Hamiltonian equations (2.32) for independent phase-space variables \( q^a, p_b \). According to our procedure, the Hamiltonian formulation of mechanics is the first order form of the Lagrangian formulation, further rewritten using the special coordinates \( q^a, p_b \) of the configuration-velocity space. Schematically we write

\[
q^a \rightarrow (q^a, v^b) \leftrightarrow (q^a, p_b).
\]  

**Exercises**

1. Check that the function \( v^a(q, p) \) defined by (2.21) obeys the equation

\[
\frac{\partial v^a(q, p)}{\partial p_c} = \tilde{M}^b c (q, v) \bigg|_{v(q, p)},
\]

where \( \tilde{M} \) is the inverse matrix for \( M \).

2. Derive the identity

\[
\frac{\partial v^a}{\partial q^c} = -\tilde{M}^{ab} \frac{\partial^2 \tilde{L}}{\partial v^b \partial q^c} \bigg|_{v(q, p)}.
\]

3. Work out the Lagrangian equations (2.18) from the Hamiltonian ones (2.32) and (2.29).

4. Confirm that all the results of this section remain true for the time-dependent Lagrangian \( L(q, \dot{q}, \tau) \).

5. Legendre transformation.

   (a) Let the functions \( f_i(x^j) \) be generated by the function \( F(x^j) \), that is \( f_i = \frac{\partial F}{\partial x^j} \), and let \( g_i \) be the inverse function of \( f_i \). Verify that \( g_i(x^j) \) is generated by \( x^i g_i - F(g) \), \( g_i = \frac{\partial}{\partial x^j} (x^i g_i - F(g)) \).

   (b) Observe that for a one-dimensional case the Legendre transformation gives a simple formula for the indefinite integral of the inverse function.

   (c) If \( F \) depends on the parameters \( y^d \), and \( f_i(x, y) = \frac{\partial F(x, y)}{\partial x^i} \), then derivatives of the generating functions with respect to \( y \) are the same, \( \frac{\partial F}{\partial y^d} \bigg|_{x \rightarrow g(x, y)} = \frac{\partial}{\partial y^d} (x^i g_j - F(g, y)) \).

### 2.1.3 Short Prescription for Hamiltonization Procedure, Physical Interpretation of Hamiltonian

The passage from a Lagrangian to a Hamiltonian description of a system is referred to as the *Hamiltonization procedure*. Note that the resulting Hamiltonian equation

\[
q^a \rightarrow (q^a, v^b) \leftrightarrow (q^a, p_b).
\]
(2.32) do not contain the velocities $v^a$. Then we expect the existence of a formal recipe for the Hamiltonization procedure that, in particular, does not mention the velocities. Let

$$S = \int d\tau L(q^a, \dot{q}^a),$$

be the Lagrangian action of some nonsingular system. Inspection of the previous section allows us to formulate the recipe as follows.

(1) Write the conjugated momenta for the variables $q^a$ according to the equations (see Eq. (2.21))

$$p_a = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a}.$$  

(2.35)

(2) Resolve the equations algebraically in relation to $\dot{q}^a$: $\dot{q}^a = v^a(q, p)$, and find the Hamiltonian (see Eq. (2.29))

$$H(q, p) = \left( p_b \dot{q}^b - L(q, \dot{q}) \right)\bigg|_{\dot{q} = v(q, p)}.$$  

(2.36)

(3) Write the Hamiltonian equations (2.32).

According to the previous section, the resulting equations are equivalent to the Lagrangian equations of motion for the action (2.34).

Note that the function $H(q, p)$ turns out to be a basic object of Hamiltonian formalism. To reveal the physical interpretation of the Hamiltonian, let us consider a particle in the presence of a potential $U(x)$. The corresponding action is

$$S = \int d\tau \left[ \frac{1}{2} m (\dot{x}^a)^2 - U(x^a) \right].$$  

(2.37)

To construct the Hamiltonian formulation, we have the momenta $p_a = m \dot{x}^a$. This implies $\dot{x}^a = \frac{1}{m} p_a$, and leads to the Hamiltonian $H(x, p) = \frac{1}{2m} (p^a)^2 + U(x)$. Making the inverse change, we obtain the position-velocity function: $E(x, \dot{x}) \equiv H(x, p)|_{p = m \dot{x}} = \frac{1}{2} m (\dot{x}^a)^2 + U(x^a)$ which represents the total energy of the particle. The reasoning works equally for a system of particles. Thus the Hamiltonian of nonsingular Lagrangian theory in Cartesian coordinates represents the total energy of a system written in terms of the phase space variables.³

### Exercise

Bearing in mind the ambiguity presented in the Hamiltonization procedure (see the discussion just before Eq. (2.21)), let us define momenta for the model

³ The case of generalized coordinates will be discussed below; see Exercise 3 on page 140.
(2.37) according to the rule $\dot{x}^a = \frac{1}{m} p_a + A_a(x)$, where $A_a(x)$ is a given function. Write the Hamiltonian equations and work out conditions for $A_a$ which imply their canonical form (that is the form (2.32) with a function $\tilde{H}$). Write the corresponding Hamiltonian $\tilde{H}$. Does it have an interpretation as the energy of the particle? Derive the Lagrangian equations from the Hamiltonian ones.

2.1.4 Inverse Problem: From Hamiltonian to Lagrangian Formulation

Let $H(q, p)$ be the Hamiltonian of some non-singular Lagrangian system. The problem is to restore the corresponding Lagrangian, that is, to construct a function $L(q, \dot{q})$ which would lead to the given $H(q, p)$ after the Hamiltonization procedure. For this purpose we have the phase-space expression (2.29), which determines the desired $L$ as a function of $q, p$: $L(q, v(q, p)) = p_a v^a - H(q, p)$. According to Sect. 2.1.2, phase space and configuration-velocity space quantities are related by the change of variables (2.21) and (2.22). Then $L$, as a function of $q, v$, is obtained by making this change in the previous expression

$$L(q, v) = \left( p_a v^a - H(q, p) \right) \big|_{p(q,v)}.$$  \hspace{1cm} (2.38)

To find the transition functions $p(q, v)$, it is sufficient to recall Eq. (2.31), which determines the inverse functions: $v^a(q, p) = \frac{\partial H(q,p)}{\partial p_a}$. Thus we resolve the equations $v^a = \frac{\partial H(q,p)}{\partial p_a}$ in relation of $p$: $p_a = p_a(q, v)$, which gives the desired transition functions.

The resulting formal prescription can be formulated without mentioning the velocities: starting from a given $H(q, p)$, solve the part of Hamiltonian equations $\ddot{q}^a - \frac{\partial H(q,p)}{\partial p_a} = 0$ with respect to $p$: $p = p(q, \dot{q})$. Then $L(q, \dot{q}) = \left[ p_a \dot{q}^a - H(q, p) \right] \big|_{p(q,\dot{q})}$.

2.2 Poisson Bracket and Symplectic Matrix

Here we introduce standard notation and conventions used to deal with Hamiltonian equations. Let $\{A(q, p), B(q, p), \ldots\}$ be a set of phase-space functions.

Definition 1 The Poisson bracket is an application that with any two phase-space functions $A, B$ associates a third function denoted $\{A, B\}$, according to the rule

$$\{A, B\} = \frac{\partial A}{\partial q^a} \frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial q^a} \frac{\partial A}{\partial p_a}. \hspace{1cm} (2.39)$$
The definition implies the following properties of the Poisson bracket:

(a) antisymmetry

\[
\{A, B\} = -\{B, A\}; \tag{2.40}
\]

(b) linearity with respect to both arguments, as a consequence of (2.40). linearity with respect to second argument is

\[
\{A, \lambda B + \eta C\} = \lambda \{A, B\} + \eta \{A, C\}, \quad \lambda, \eta = \text{const}; \tag{2.41}
\]

(c) Leibnitz rule

\[
\{A, BC\} = \{A, B\} C + B \{A, C\}; \tag{2.42}
\]

(d) Jacobi identity

\[
\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{2.43}
\]

**Exercise**

Verify (2.43) by direct computations. Hint: consider separately all the terms involving, for example, two derivatives of \(B\).

Poisson brackets among phase-space variables are called *fundamental brackets*. They are:

\[
\{q^a, p_b\} = \delta^a_b, \quad \{q^a, q^b\} = 0, \quad \{p_a, p_b\} = 0. \tag{2.44}
\]

Poisson brackets can be used to rewrite Hamiltonian equations in the form:

\[
\dot{q}^a = \{q^a, H\}, \quad \dot{p}_a = \{p_a, H\}. \tag{2.45}
\]

Hence the Poisson bracket of \(q, p\) with the Hamiltonian determines their rate of variation with time. Moreover, the same is true for any phase-space function: if \(q^a(\tau), p_b(\tau)\) is a solution to the Hamiltonian equations, the rate of variation of the function \(A(q(\tau), p(\tau))\) can be computed as:

\[
\dot{A}(q, p) = \frac{\partial A}{\partial q^a} \dot{q}^a + \frac{\partial A}{\partial p_a} \dot{p}_a = \frac{\partial A}{\partial q^a}(q^a, H) + \frac{\partial A}{\partial p_a}(p_a, H)
\]

\[
= \{A, H\}. \tag{2.46}
\]

Thus \(\{A(q, p), H\} = 0\) implies that \(A\) is a *conserved quantity*, that is, it has a fixed value throughout any given solution. As an example, let us apply this result to
compute the rate of variation of a Hamiltonian. We have $\dot{H} = \{H, H\} = 0$, due to the antisymmetry of the Poisson bracket. Hence the Hamiltonian is the conserved quantity, which gives a further argument in support of its interpretation as the total energy.

Below it will be convenient to work with phase-space quantities by using the following notation. For the phase-space coordinates we use the unique symbol: $(q^a, p_b) \equiv z^i$, $i = 1, 2, \ldots, 2n$, or, equivalently, for $a, b = 1, 2, \ldots, n$ we have $z^a = q^a$ and $z^{n+a} = p_b$. Thus Latin indices from the middle of the alphabet run from 1 to $2n$. Let us also introduce the $2n \times 2n$-dimensional symplectic matrix composed of four $n \times n$ blocks

$$
\omega^{ij} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

In more detail, for $a, b = 1, 2, \ldots, n$ one writes $\omega^{ab} = 0$, $\omega^{a,n+b} = \delta^{ab}$, $\omega^{n+a,b} = -\delta^{ab}$, $\omega^{n+a,n+b} = 0$. The symplectic matrix is antisymmetric: $\omega^{ij} = -\omega^{ji}$ and invertible, with the inverse matrix being

$$
\omega_{ij} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
$$

In this notation the Poisson brackets (2.39) and (2.44) acquire a more compact form

$$
\{A, B\} = \frac{\partial A}{\partial z^i} \omega^{ij} \frac{\partial B}{\partial z^j}, \quad \{z^i, z^j\} = \omega^{ij},
$$

while the Hamiltonian equations can be written as

$$
\dot{z}^i = \omega^{ij} \frac{\partial H}{\partial z^j}, \quad \text{or} \quad \dot{z}^i = \{z^i, H\}.
$$

**Exercise**

Verify the Jacobi identity with use of the representation (2.49).

### 2.3 General Solution to Hamiltonian Equations

As a first application of Hamiltonian formalism, we find here a general solution to Hamiltonian equations with an arbitrary time-independent Hamiltonian in terms of power series with respect to $\tau$.

As a preliminary step, consider the differential operator defined by the formal series
\[ e^{a \partial_x} = 1 + a \partial_x + \frac{1}{2} a \partial_x (a \partial_x) + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} (a \partial_x)^n, \quad a = \text{const.} \quad (2.51) \]

This obeys the properties \( e^{a \partial_x} x = x + a, \) \( e^{a \partial_x} G(x) = G(e^{a \partial_x} x), \) as can be verified by expansion in power series of both sides of these equalities. There is a generalization of the last equality for the case of a function \( a(x) \)

\[ e^{a(x) \partial_x} G(x) = G \left( e^{a(x) \partial_x} x \right). \quad (2.52) \]

**Exercise**

Verify the validity of Eq. (2.52) up to the third order of power expansion.

Due to the identity (2.52), the function \( f(\tau, x) = e^{\tau a(x) \partial_x} x \) turns out to be a formal solution to the equation

\[ \frac{\partial f}{\partial \tau} = a(f). \quad (2.53) \]

Besides, this obeys the initial condition \( f(0, x) = x. \) This observation can be further generalized for the case of several variables, the functions \( f^i(\tau, x^i) = e^{\tau a^i(x^i) \partial_k x^i} \) obey the system

\[ \frac{\partial f^i}{\partial \tau} = a^i(f^j). \quad (2.54) \]

Note that the Hamilton equations \( \dot{z}^i = \{z^i, H\} \) represent a system of this type. So its solution is

\[ z^i(\tau) = e^{\tau \{z^0_0, H(z_0)\} \frac{\partial}{\partial z^0_0} z^i_0}. \quad (2.55) \]

In particular, the position of a system as a function of \( 2n \) constants is given by

\[ q^a(\tau, q_0, p_0) = e^{\tau \{z^0_0, H(z_0)\} \frac{\partial}{\partial z^0_0} q^a_0}. \quad (2.56) \]

For the Hamiltonian \( H = \frac{\hat{p}^2}{2m} + U(q) \) it implies

\[ q^a(\tau, q_0, p_0) = e^{\tau \left( \frac{\hat{p}^0_0}{m} \hat{\nabla} U \cdot \hat{\nabla} p_0 \right) \frac{\partial}{\partial \hat{p}^0_0} q^a_0}. \quad (2.57) \]

We illustrate the formula (2.55) with several examples.
2.3 General Solution to Hamiltonian Equations

1. For a free particle with the Hamiltonian \( H = \frac{\vec{p}^2}{2m} \) we have

\[
x^a(t) = e^{\frac{i}{\hbar} \vec{p}_0 \cdot \vec{\nabla}} x_0^a = \left( 1 + t \frac{1}{m} \vec{p}_0 \cdot \vec{\nabla} + \frac{t^2}{2m^2} (\vec{p}_0 \cdot \vec{\nabla})(\vec{p}_0 \cdot \vec{\nabla}) + \ldots \right) x_0^a
\]
\[
= x_0^a + \frac{1}{m} p_0 \alpha t,
\]
\[
p_\alpha(t) = e^{\frac{i}{\hbar} \vec{p}_0 \cdot \vec{\nabla}} p_\alpha = \left( 1 + t \frac{1}{m} \vec{p}_0 \cdot \vec{\nabla} + \frac{t^2}{2m^2} (\vec{p}_0 \cdot \vec{\nabla})(\vec{p}_0 \cdot \vec{\nabla}) + \ldots \right) p_\alpha = p_\alpha. \tag{2.58}
\]

2. For a one-dimensional harmonic oscillator with the Hamiltonian \( H = \frac{\vec{p}^2}{2m} + \frac{1}{2}kx^2 \) we obtain

\[
x(t) = e^{\left( \frac{1}{m} p_0 \partial_x - kx \partial_{p_x} \right)} x_0 = \left( 1 + t \frac{1}{m} \vec{p}_0 \cdot \vec{\nabla} + \frac{t^2}{2m^2} (\vec{p}_0 \cdot \vec{\nabla})(\vec{p}_0 \cdot \vec{\nabla}) + \ldots \right) x_0 + p_0 \frac{1}{m} t - x_0 \frac{k}{m} t^2 + x_0 \frac{k^2}{m^2} t^3 + \ldots. \tag{2.59}
\]

Bringing together even and odd degrees of \( t \), it gives the expected result

\[
x(t) = x_0 \left( 1 - \frac{k}{m^2} t^2 + \frac{k^2}{m^4} t^4 - \ldots \right) + p_0 \left( \frac{1}{m} t - \frac{k}{m^2} t^3 + \frac{k^2}{m^3} t^5 - \ldots \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \sqrt{\frac{k}{m}} t \right)^{2n} + \frac{p_0}{\sqrt{km}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} \left( \sqrt{\frac{k}{m}} t \right)^{2n+1}
\]
\[
= x_0 \cos \sqrt{\frac{k}{m}} t + \frac{p_0}{\sqrt{km}} \sin \sqrt{\frac{k}{m}} t. \tag{2.60}
\]

3. Kepler’s problem. Consider a particle with the initial position and velocity being \( x^a(0) = x_0^a, \dot{x}^a(0) = \dot{v}_0^a \), subject to a central field potential with origin at the center of a coordinate system. As we have seen in Sect. 1.6, trajectory of motion is a second-order plane curve (ellipse, hyperbola or parabola) with the polar equation being (see Eq. (1.146) )

\[
r = \frac{p}{1 + e \cos(\theta + \theta_0)}. \tag{2.61}
\]
Here the constant \( p \), the eccentricity \( e \), and the inclination angle \( \theta_0 \) of the semi-axis are determined through initial values of the problem. We reproduce this result using the formula (2.55).

The system is described by the action

\[
S = \int dt \left( \frac{m}{2} \ddot{x}^2 + \frac{\alpha}{|\dot{x}|} \right).
\]

(2.62)

Remember that due to the conservation of angular momentum, the particle trajectory lies on the plane of vectors \( \vec{x}_0, \vec{v}_0 \). Choosing the cartesian axis \( x^1, x^2 \) on the plane, we discard the third variable in the action, \( x^3(t) = 0 \). To proceed further, let us rewrite the action in polar coordinates, \( x^1 = r \cos \theta, x^2 = r \sin \theta \)

\[
S = \int dt \left( \frac{m}{2} \dot{r}^2 + \frac{m}{2} r^2 \dot{\theta}^2 + \frac{\alpha}{r} \right).
\]

(2.63)

Denoting conjugated momenta for \( r, \theta \) as \( p_r, p_{\theta} \), the Hamiltonian reads

\[
H = \frac{1}{2m} p_r^2 + \frac{1}{2mr^2} p_{\theta}^2 - \frac{\alpha}{r}.
\]

(2.64)

It leads to the equations

\[
\dot{r} = \frac{1}{m} p_r, \quad \dot{\theta} = \frac{1}{mr^2} p_{\theta} - \frac{\alpha}{r^2}, \quad \dot{p}_r = \frac{1}{mr^2} p_{\theta}, \quad \dot{p}_{\theta} = 0.
\]

(2.65)

The last equation implies \( p_{\theta} = l = \text{const} \). We are interested in finding a form of trajectory, \( r(\theta), p(\theta) \). The relevant equations can be obtained from (2.65). Considering \( r = r(t), p = p(t), \theta = \theta(t) \) as parametric equations of the trajectory, we write \( r' = \frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}}, p' = \frac{dp}{d\theta} = \frac{\dot{p}}{\dot{\theta}} \). Using (2.65) in these expressions, we obtain the equations

\[
r' = \frac{r^2}{l} p, \quad p' = \frac{l}{r} - \frac{\alpha m}{l}.
\]

(2.66)

They acquire a more simple form if we introduce the variable \( q = \frac{1}{r} \). Then

\[
q' = -\frac{1}{l} p, \quad p' = lq - \frac{\alpha m}{l}.
\]

(2.67)

They form a Hamiltonian system, with the Hamiltonian being

\[\text{Notice that (2.66) is not a Hamiltonian system. The regular way to construct Hamiltonian equations for a trajectory will be discussed in Sect. 6.1.2 below.}\]
\[ H(r, p) = -\frac{1}{2l}p^2 - \frac{l}{2}q^2 + \frac{\alpha m}{l}q. \]  

(2.68)

Using Eq. (2.55) with this Hamiltonian, we obtain the solution

\[
q(\theta) = e^{\theta(-\frac{p_0}{l}\partial_\theta + (q_0 - \frac{\alpha m}{l^2}\partial_\theta))}q_0
\]

\[
= q_0 - \frac{1}{l}p_0\theta - \left(q_0 - \frac{\alpha m}{l^2}\right)\frac{\theta^2}{2!} + \frac{1}{l}p_0\frac{\theta^3}{3!} - \left(q_0 - \frac{\alpha m}{l^2}\right)\frac{\theta^4}{4!}
\]

\[
= \frac{\alpha m}{l^2} + \left(q_0 - \frac{\alpha m}{l^2}\right)\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} - \frac{p_0}{l}\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}
\]

\[
= \frac{\alpha m}{l^2} + \left(q_0 - \frac{\alpha m}{l^2}\right)\cos \theta - \frac{p_0}{l} \sin \theta.
\]  

(2.69)

Returning to the variable \(r = \frac{1}{q}\), we have

\[
\frac{l^2}{\alpha mr(\theta)} = 1 + \left(\frac{l^2}{\alpha mr_0} - 1\right)\cos \theta - \frac{l p_0}{\alpha m} \sin \theta
\]

\[
= 1 + A \cos \theta - B \sin \theta
\]

\[
= 1 + \sqrt{A^2 + B^2} \left(\frac{A}{\sqrt{A^2 + B^2}} \cos \theta - \frac{B}{\sqrt{A^2 + B^2}} \sin \theta\right).
\]  

(2.70)

Comparing \(A^2 + B^2\) with the Hamiltonian (2.64), we obtain \(A^2 + B^2 = 1 + \frac{2l^2E}{\alpha^2m}\), where \(E = \frac{p_0^2}{2m} + \frac{l^2}{2mr_0^2} - \frac{\alpha}{r_0}\) represents the total energy. Besides, since

\[
\left(\frac{A}{\sqrt{A^2 + B^2}}\right)^2 + \left(\frac{B}{\sqrt{A^2 + B^2}}\right)^2 = 1,
\]

there is an angle \(\theta_0\) such that \(\frac{A}{\sqrt{A^2 + B^2}} = \cos \theta_0\). \(\frac{B}{\sqrt{A^2 + B^2}} = \sin \theta_0\). Taking this into account, the equation of the trajectory acquires the form (2.61)

\[
r(\theta) = \frac{l^2(\alpha m)^{-1}}{1 + \sqrt{1 + \frac{2l^2E}{\alpha^2m} \cos(\theta + \theta_0)}}.
\]  

(2.71)

### 2.4 Picture of Motion in Phase Space

Here we illustrate some advantages of Hamiltonian formalism as compared with the Lagrangian one. In particular it will be seen that a general solution to Hamiltonian equations has useful interpretations in the framework of hydrodynamics and differential geometry.

**General solution as the phase-space flux.** Hamiltonian equations (2.50) represent a normal system of \(2n\) first-order differential equations for \(2n\) variables \(z'(\tau)\).
According to the general theory of differential equations, the theorem of existence and uniqueness of a solution holds for the case: for given numbers $z^i_0$, locally there is a unique solution $z^i(\tau)$ of the system, which obeys the initial conditions: $z^i(0) = z^i_0$.

Let us recall also the definition of a general solution: $2n$ functions of $2n + 1$ variables $z^i(\tau, c^j)$ are called a general solution of the system (2.50), if: (a) they obey the system for all $c^j$; (b) for given initial conditions $z^i_0$, there are numbers $\tilde{c}^j$ such that $z^i(0, \tilde{c}^j) = z^i_0$.

Owing to the above-mentioned theorem, a general solution to the normal system contains all particular solutions (trajectories) of the system, any one of them appearing after the appropriate fixation of the constants $c^j$.

These results imply a remarkable picture of motion in phase space: trajectories of the Hamiltonian system (2.50) do not intercept each other. To confirm this, let us suppose that two trajectories have interception at some point $z^i_0$. These numbers can be taken as initial conditions of the problem (2.50), and, according to the theorem, there is only one trajectory which passes through $z^i_0$, contrary to the initial supposition. Thus, trajectories of a Hamiltonian system in phase space form a flow, similarly to the picture of the motion of a fluid. Moreover, the “fluid” turns out to be incompressible, see Sect. 4.4.1. Note that it is very different from the corresponding picture of motion in the configuration space; see Fig. 2.1 on page 92.

**Geometric interpretation of the symplectic matrix.** In contrast to Lagrangian equations, Hamiltonian ones have a simple interpretation in the framework of differential geometry. Let us consider the right-hand sides of Hamiltonian equations as components $H^i$ of a vector field in the phase space: $H^i(z^k) \equiv \omega^{ij} \frac{\partial H}{\partial z^j}$. Then the Hamiltonian equations $\dot{z}^i = H^i(z)$ state that any solution to equations of motion is a trajectory of this vector field (according to differential geometry, a line is the trajectory of a given vector field, if vectors of the field are tangent vectors to the line at each point). Hamiltonian vector field $H^i$ also has certain interpretation. Let $H(z)$ = const represent a surface of constant energy. Then the vector field $H_i = \frac{\partial H}{\partial z^i} \equiv (\text{grad } H)_i|_{H=\text{const}}$ is normal to the surface at each point. The scalar product of $H^i$ with the vector grad $H$ vanishes: $H^i (\text{grad } H)_i = \partial_j H \omega^{ij} \partial_i H = 0$.

![Fig. 2.1](image-url) Trajectory flows on configuration and phase spaces
2.5 Conserved Quantities and the Poisson Bracket

Fig. 2.2 Solutions lie on the surfaces of constant energy of the phase space. They are trajectories of the Hamiltonian vector field, constructed from $H$: $H^j(z^k) = \omega^j_{ij}(\text{grad } H)j$

that is, the Hamiltonian vector field $H^i$ is tangent to the surface. Hence each trajectory $z^i(\tau)$ lies on one of the surfaces of constant energy, as should be the case, see Fig. 2.2 on page 93. Now, observe the remarkable role played by the symplectic matrix $\omega^{ij}$. There is a whole hyperplane of the vectors, which are normal to $\text{grad } H$ at a given point. It is the matrix $\omega$ that transforms the normal vector $\text{grad } H$ into the tangent vector to a trajectory!

Note that in terms of the coordinates $z^i$ the vector field $H^i$ is divergenceless: $\partial_i H^i = 0$. Now, let us consider the field $H^i(z^k)$ in the coordinates: $z_j \equiv z^i_{ij}$. First, the Hamiltonian, as a function of $z_i$, is $H(z_i) \equiv H(z^i(z_j))$.

**Exercise**

Write $H(z^i) = q - p^2$ in terms of the variables $z_i$.

Further, since $\omega^{ik} \frac{\partial}{\partial z^k} z_j = \delta^i_j$, the derivative associated with $z_j$ is $\partial^i = \frac{\partial}{\partial z^i} \equiv \omega^{ik} \frac{\partial}{\partial z^k}$. The Hamiltonian vector field $H^i(z_k)$ in these coordinates is $H^i(z_k) = \partial^i H(z_k)$, and turns out to be curl-free (conservative): $\partial^j H^i - \partial^i H^j = 0$. This fact will be explored in Chap. 4.

2.5 Conserved Quantities and the Poisson Bracket

**Definition 2** A function $Q(z^i, \tau)$ is called an integral of motion, if for any solution $z^i(\tau)$ of the Hamiltonian equations, $Q$ retains a constant value:

$$Q(z(\tau), \tau) = c, \quad \text{or} \quad \frac{d}{d\tau} Q = 0 \quad \text{on-shell.} \quad (2.72)$$
Here “on-shell” stands for “for an arbitrary solution to equations of motion”, while “off-shell” means “for an arbitrary function \( z^i(\tau) \)”. Of course \( c \) may change when we pass from one trajectory to another. In the current literature, integral of motion is also referred to as first integral, constant of motion, conserved quantity, conservation law, charge or dynamical invariant – according to taste. Hereafter we use the term (conserved) charge, as the shortest among these expressions.

There is an important necessary and sufficient condition for a quantity \( Q \) to be a conserved charge.

**Assertion** \( Q(z^i, \tau) \) represents a conserved charge if and only if

\[
\frac{\partial Q}{d\tau} + \{Q, H\} = 0 \quad \text{off-shell.} \tag{2.73}
\]

In particular, the quantity \( Q(z^i) \) (without manifest dependence on \( \tau \)) is conserved if and only if its bracket with the Hamiltonian vanishes

\[
\{Q, H\} = 0. \tag{2.74}
\]

**Proof** We write identically

\[
\frac{dQ}{d\tau} = \frac{\partial Q}{\partial \tau_0} + \{Q, H\} + \frac{\partial Q}{\partial z^i}(\dot{z}^i - \{Q, H\}), \tag{2.75}
\]

so the condition (2.73) implies (2.72). Conversely, supposing that (2.72) is true, we have \( \frac{dQ}{d\tau} + \{Q, H\} = 0 \), on-shell. Given a phase-space point \( z^i_0 \) and a value \( \tau_0 \), let \( z^i(\tau) \) represent the trajectory that passes through \( z^i_0 \) at the instant \( \tau_0 \). Inserting the solution into the equation and taking \( \tau = \tau_0 \) we obtain \( \frac{\partial Q(\tau_0, z_0)}{\partial \tau} + \{Q(\tau_0, z_0), H(z_0, \tau_0)\} = 0 \) for any \( \tau_0, z_0 \), as has been stated.

An example of a charge is the Hamiltonian of a conservative system (see page 87). The search for the charges turns out to be an important task. From a pragmatic point of view, knowledge of them allows us to simplify (sometimes to solve) equations of motion of a system (it is sufficient to recall that conservation of angular momentum allows us to reduce the three-dimensional Kepler problem to a two-dimensional one). Let us point out also that in quantum theory the concept of a trajectory does not survive and is replaced by an abstract state space associated with the system. But the notion of conserved charges survives, and they play a crucial role in the interpretation of the state space, establishing a correspondence between the states and physical particles.
A powerful method for obtaining charges for a system which exhibits certain symmetries is provided by the Noether theorem, which is discussed in Chap. 7. Here we describe some general properties of a set of charges.

If $Q$ is a charge, an arbitrary function $f(Q)$ will also be a charge. If $Q_1, Q_2$ are charges, their product and linear combinations with numerical coefficients also represent charges. It is convenient to introduce the notion of independent charges as follows: the charges $Q_\alpha(z^i, \tau), \alpha = 1, 2, \ldots, k \leq 2n$ are called functionally independent, if

$$
\text{rank} \frac{\partial Q_\alpha}{\partial z^i} = k. \tag{2.76}
$$

This implies that the expressions $Q_\alpha(z^i, \tau) = c_\alpha$ can be resolved with respect to $k$ variables $z^\alpha$ among $z^i$:

$$
z^\alpha = G^\alpha(z^a, c_\alpha, \tau), \tag{2.77}
$$

where $z^a$ are the remaining variables of the set $z^i$. As will be discussed in Sect. 7.9, knowledge of $k$ functionally independent charges immediately reduces the order of equations of motion by $k$ units (that is, there is an equivalent system with the total number of derivatives being $2n - k$).

It is a simple matter to confirm the existence of $2n$ independent charges for a given dynamical system. Let the functions $f^i(\tau, c_j)$ represent a general solution to the Hamiltonian equations. This implies, in particular, that $\det \frac{\partial f^i}{\partial c_j} \neq 0$. If we write the equations $z^i = f^i(\tau, c_j)$, they can be resolved with respect to $c$: $Q_j(z^i, \tau) = c_j$, giving the functions $Q_j(z^i, \tau)$. By construction, substitution of any solution $z^i(\tau)$ into $Q$ turns them into constants. Hence we have obtained $2n$ independent combinations of $z^i, \tau$, namely $Q_j(z^i, \tau)$, which do not depend on time for their solutions, and thus represent the conserved charges.

Of course, in practice the problem is the opposite: it is interesting to reveal as many charges as possible by independent methods, and use them to search for a general solution to equations of motion. In particular, inverting the previous discussion, we conclude that the knowledge of $2n$ independent charges is equivalent to knowledge of the general solution.

The set of charges is endowed with a remarkable algebraic structure in relation to the Poisson bracket: the bracket of two charges is also a charge. This is proved by direct computation

$$
\frac{d}{d\tau} \{Q_1, Q_2\} = \frac{\partial}{\partial \tau} \{Q_1, Q_2\} + \{Q_1, Q_2, H\} = \\
\left\{ \frac{\partial Q_1}{\partial \tau}, Q_2 \right\} + \left\{ Q_1, \frac{\partial Q_2}{\partial \tau} \right\} - \{Q_2, H, Q_1\} - \{H, Q_1, Q_2\} = \\
\left\{ \frac{\partial Q_1}{\partial \tau} + \{Q_1, H\}, Q_2 \right\} + \left\{ Q_1, \frac{\partial Q_2}{\partial \tau} + \{Q_2, H\} \right\} = 0. \tag{2.78}
$$
Here the Jacobi identity was used for the transition from the first to the second line. The last line is equal to zero since $Q_1, Q_2$ obey Eq. (2.73). Thus $Q_3 \equiv \{Q_1, Q_2\}$ is conserved. Of course, it can be identically null or can be functionally dependent on $Q_1, Q_2$. If not, the Poisson bracket can be used to generate new charges from the known ones.

As an illustration, consider a free-moving particle, with the Hamiltonian $H = \frac{1}{2m}(p^i)^2, i = 1, 2, 3$, and the corresponding Hamiltonian equations $\dot{x}^i = \frac{1}{m}p^i, \dot{p}^i = 0$. Besides the Hamiltonian, the conserved charges are the momenta $p^i = c^i = \text{const}$ (as follows from their equations), and angular momentum $L^i = \epsilon^{ijk}x^j p^k = d^i = \text{const}$ (since on-shell $\dot{L}^i = \frac{1}{2m}\epsilon^{ijk}p^j p^k \equiv 0$). $H$ can be omitted, since it forms a functionally dependent set with $p^i$. As to the remaining six charges, only five of them are functionally independent (imagine that all they are independent. Then it would be possible to solve the equations $Q_i(z^i) = c_i$, obtaining a general solution to the equations of motion in the form $z^i = f^i(c_i)$, and arriving at the rather strange result that the particle cannot move! Of course, their dependence can be verified by direct computation of the corresponding Jacobian). By choosing $p^i$ and $L^2, L^3$ as independent quantities, we find the dynamics of $p^i, x^2, x^3$ in terms of $x^1$: $p^i = c^i, x^2 = \frac{c^2}{c^1}x^1 - \frac{d^3}{c^1}, x^3 = \frac{c^3}{c^1}x^1 + \frac{d^2}{c^1}$. Thus, to find a general solution to the equations of motion, we need to solve only one of them, namely $\dot{x}^1 = \frac{c^1}{2m}$, which gives the time-dependent charge $x^1 = \frac{c^1}{2m} \tau + b$.

**Exercises**

1. Compute the number of functionally independent charges for the case of a free particle in $n$-dimensional space, $n > 3$.
2. Confirm the Poisson bracket algebra of the charges:

\[
\{L^i, L^j\} = \epsilon^{ijk}L^k, \quad \{L^i, p^j\} = \epsilon^{ijk}p^k. \quad (2.79)
\]

2.6 Phase Space Transformations and Hamiltonian Equations

In many interesting cases, the Lagrangian equations can be solved with use of the coordinate transformations $q \rightarrow q'(q)$ in the configuration space. In particular, if the system in question exhibits certain symmetries, they can be taken into account to search for adapted coordinates. This often leads to separation of variables in Lagrangian equations, much simplifying the problem. Well-known examples are the use of polar coordinates in the Kepler problem and the use of center-of-mass variables in the two-body problem. The Hamiltonian formulation gives supplementary possibilities due to the fact that a set of transformations in the phase space is much larger, allowing us to mix position and velocity variables: $q \rightarrow q'(q, p), p \rightarrow p'(q, p)$. In this section we find out how Hamiltonian equations transform
under phase-space transformations. It will be seen that an arbitrary transformation spoils the canonical form of the Hamiltonian equations. So it is reasonable to choose the subset which preserves their form. Transformations of this subset are called canonical transformations, and are discussed in the next section.

Let $\varphi^i(z^j, \tau)$ represent $2n$ given functions of $2n + 1$ variables, with the property

$$\det \frac{\partial \varphi^i}{\partial z^j} \neq 0, \quad \text{for any} \quad \tau. \quad (2.80)$$

Starting from the original parametrization $z^i$ of phase space, functions $\varphi^i$ with fixed $\tau_0$ can be used to define another parametrization $z'^i$, namely

$$z'^i = \varphi^i(z^j, \tau_0). \quad (2.81)$$

According to the condition (2.80), transformation of the coordinates $z^i \to z'^i$ is invertible: the expressions (2.81) can be resolved with relation to $z^i$, with the result being

$$z^i = \psi^i(z'^j, \tau_0). \quad (2.82)$$

By construction, there are identities

$$\varphi^i(\psi(z', \tau_0), \tau_0) \equiv z'^i, \quad \psi^i(\varphi(z, \tau_0), \tau_0) \equiv z^i. \quad (2.83)$$

From this we obtain more identities

$$\begin{align*}
\left. \frac{\partial \varphi^k(z, \tau)}{\partial z^i} \right|_{z=\psi(z', \tau)} & = \delta^k_j, \\
\left. \frac{\partial \varphi^i(z, \tau)}{\partial \tau} \right|_{z=\psi(z', \tau)} & = - \left. \frac{\partial \varphi^j(z, \tau)}{\partial z^j} \right|_{z=\psi(z', \tau)} \frac{\partial \psi^j(z', \tau)}{\partial \tau}, \\
\left. \frac{\partial \psi^i(z', \tau)}{\partial \tau} \right|_{z'=\phi(z, \tau)} & = - \left. \frac{\partial \psi^j(z', \tau)}{\partial z'^j} \right|_{z'=\phi(z, \tau)} \frac{\partial \varphi^j(z, \tau)}{\partial \tau}. \quad (2.84)
\end{align*}$$

The first identity relates Jacobi matrices of inverse and direct transformations: the matrices turn out to be opposites. The second identity relates derivatives with respect to $\tau$ of the direct ($\varphi$) and the inverse ($\psi$) transformations. The third identity differs from the second one by changing $\varphi \leftrightarrow \psi$, as should be the case (it is a matter of convenience which transformation is called the “direct” and the “inverse” one).

---

5 In configuration space, transformations of the type (2.81) with manifest dependence on $\tau$ are well-known and have a clear physical interpretation. For example, a Galilean transformation $x'^i = x^i + v^i \tau + a^i$ gives the relationship between the coordinates of inertial frames, with relative velocity being $v^i$. 
Coordinate transformation (2.81) implies an *induced map* in the space of functions (curves) \( z^i(\tau) \)

\[
\varphi^i : z^i(\tau) \longrightarrow z''^i(\tau) \equiv \varphi^i(z^j(\tau), \tau).
\]  

(2.85)

So, changing the description in terms of \( z^i(\tau) \) to a description in terms \( z'^i(\tau) \), it is said that a *phase-space transformation* has been carried out: \( z^i \rightarrow z''^i \). The function \( \psi \) gives an inverse transformation

\[
\psi^i : z''^i(\tau) \longrightarrow z^i(\tau) \equiv \psi^i(z'^j(\tau), \tau),
\]  

(2.86)

and we have \( \varphi^i(\psi(z'(\tau), \tau), \tau) \equiv z''^i(\tau), \psi^i(\varphi(z(\tau), \tau), \tau) \equiv z'^i(\tau) \).

*Comment* As has already been mentioned, for a fixed \( \tau \) the expression (2.81) has a clear geometric interpretation as a phase-space transformation \( z^i \rightarrow z''^i \). Although it is not strictly necessary, it is convenient to discuss two different interpretations of (2.81), (2.85) with a varying \( \tau \).

(A) A geometric interpretation of (2.81) can be obtained in the *extended phase space* \( \mathbb{R}^{2n+1} = \mathbb{R}(\tau) \otimes \mathbb{R}^{2n}(z^i) \) with coordinates \( (\tau, z^i) \). In this space we can change the parametrization as follows

\[
\begin{pmatrix} \tau \\ z^i \end{pmatrix} \leftrightarrow \begin{pmatrix} \tau' \\ z''^i \end{pmatrix}, \quad \begin{cases} \tau' = f(z, \tau) \\ z''^i = \varphi^i(z, \tau). \end{cases}
\]  

(2.87)

Now, Eq. (2.81) is a particular case of this coordinate transformation, with \( \tau' \equiv \tau \). Let \( \tau = \tau(s), z'^i = z^i(s) \) represent parametric equations of a curve in \( \mathbb{R}^{2n+1} \). The particular case is \( \tau = s, z'^i = z^i(s) \), where one of the coordinates, namely \( \tau \), was chosen as the parameter. In this sense the functions \( z^i(\tau) \) can be considered as parametric equations of a curve in \( \mathbb{R}^{2n+1} \). Then \( z''^i = \varphi^i(z(\tau), \tau) \) are equations of this curve in the coordinate system \( (\tau, z''^i) \).

(B) Sometimes it will be useful to treat certain combinations of the functions \( \varphi^i(z^j, \tau) \) (2.81) as components of a (time dependent) vector field on the phase space \( z^i \). In particular, we discuss below curl-free vector fields, see Eq. (3.14).

Starting from Hamilton equations

\[
\dot{z}^i = \omega^{ij} \frac{\partial H(z^k)}{\partial z^j}, \quad i, j = 1, 2, \ldots, 2n,
\]  

(2.88)

let us ask, what is the form they acquire in parametrization (2.81). In other words, if \( z(\tau) \) obeys (2.88), what equations hold for \( z'^i(\tau) \) defined by Eq. (2.85)? The answer is given by the following:

**Affirmation** Let \( z'^i = \varphi^i(z^j, \tau) \) be a phase-space transformation. The system (2.88) and the following one
are equivalent in the following sense:

(a) if \( z^i(\tau) \) obeys (2.88) then \( z'^i(\tau) \equiv \varphi^i(z(\tau), \tau) \) obeys (2.89). (b) if \( z'^i(\tau) \) obeys (2.89) then \( z^i(\tau) \equiv \psi^i(z'(\tau), \tau) \) obeys (2.88).

Let us prove item (a). According to (2.8), equations for \( z' \) arise after substitution of \( z \) in the form (2.86) into (2.88). This leads immediately to Eq. (2.89). The same result can be obtained by direct computation of the derivative (bearing in mind the identity \( \dot{z}_i(\tau) \equiv \omega_{ij} \frac{\partial H(z)}{\partial z_j} \bigg|_{z(\tau)} \))

\[
\dot{z}_k' = \left( \frac{\partial \varphi^k}{\partial z^i} \omega_{ij} \frac{\partial \varphi^l}{\partial z^j} \frac{\partial H(\psi(z', \tau))}{\partial z^l} + \frac{\partial \varphi^k}{\partial \tau} \right) \bigg|_{z'=\psi(z', \tau)}, \tag{2.89}
\]

For the transition to the last line we have used the equality: \( z(\tau) = \psi(z'(\tau), \tau) = \psi(z', \tau) \bigg|_{z'(\tau)} \). Item (b) can be proved in a similar fashion.

Hereafter we use simplified notation, similar to that used in differential geometry. Instead of \( z'^i = \varphi^i(z^j, \tau) \) and \( z^i = \psi^i(z'^j, \tau) \) we write

\[
z'^i = z'^i(z^j, \tau), \quad z^i = z^i(z'^j, \tau), \tag{2.91}
\]

Thus the new coordinate (value of function) and the transition function itself are denoted by the same symbol. The notation for partial derivatives is

\[
\frac{\partial}{\partial z^i} \equiv \partial_i, \quad \omega_{ij} \frac{\partial}{\partial z^j} \equiv \partial^i, \quad \frac{\partial}{\partial z'^i} \equiv \partial'_i, \quad \omega_{ij} \frac{\partial}{\partial z'^j} \equiv \partial'^i, \quad \frac{\partial}{\partial \tau} \equiv \partial_\tau. \tag{2.92}
\]

Also, we sometimes omit the operation of substitution:

\(6 \) \( \partial^i \) represents the usual partial derivative with respect to variable \( z_l \equiv z^k \omega_{kl} \), since \( \partial^i (z^k \omega_{kl}) = \delta_l^i \).
A(z)|_{z(z')} \rightarrow A(z) \text{ or } A(z). \tag{2.93}

If the left and right hand sides of an expression have wrong “balance of variables”, we need to substitute $z(z')$ on the left or on the right hand side. In this notation we can write, for example

\[ z'^{i}(z', \tau_0, \tau_0) \equiv z'^{i} \text { instead of } (2.83). \tag{2.94} \]

The identities (2.84) can now be written as follows

\[
\begin{align*}
\frac{\partial z'^{k}}{\partial z'^{l}} & = \frac{\partial z'^{l}}{\partial z'^{j}} = \delta^{k}_{j}, \\
\frac{\partial z'^{i}(z', \tau)}{\partial \tau} & = - \frac{\partial z'^{i}}{\partial z'^{j}} \frac{\partial z^{j}(z', \tau)}{\partial \tau}, \\
\frac{\partial z^{i}(z', \tau)}{\partial \tau} & = - \frac{\partial z^{i}}{\partial z'^{j}} \frac{\partial z'^{j}(z', \tau)}{\partial \tau}, \tag{2.95}
\end{align*}
\]

where, for example, the last equation implies substitution of $z'(z, \tau)$ on l.h.s. and in the first term on r.h.s. Equivalently, we can substitute $z(z', \tau)$ in the last term on r.h.s. Hamiltonian equations for $z'^{i}$ (2.89) acquire the form

\[
\dot{z}'^{k} = \{z'^{k}, z'^{l}\} \bigg|_{z(z', \tau)} \frac{\partial H(z(z', \tau))}{\partial z'^{l}} + \frac{\partial z'^{k}(z, \tau)}{\partial \tau} \bigg|_{z(z', \tau)}, \tag{2.96}
\]

where $\{z'^{k}, z'^{l}\} \big|_{z}$ is the Poisson bracket computed with respect to $z$.

### 2.7 Definition of Canonical Transformation

From comparison of Eqs. (2.88) and (2.89) we conclude that phase-space transformation generally does not preserve the initial form of Hamiltonian equations. It justifies the following

**Definition 3** The transformation $z'^{i} = \varphi^{i}(z^{j}, \tau)$ is called *canonical* if for any Hamiltonian system the corresponding induced transformation (2.85) preserves the canonical form of the Hamiltonian equations:

\[
\dot{z}^{i} = \omega^{ij} \frac{\partial H(z, \tau)}{\partial z^{j}} \rightarrow \dot{z}'^{i} = \omega^{ij} \frac{\partial \tilde{H}(z', \tau)}{\partial z'^{j}}, \text{ any } H, \text{ some } \tilde{H}. \tag{2.97}
\]

It will be seen below that $\tilde{H}$ is related to $H$ according to a simple rule (in particular, for the case of time-independent canonical transformation, we have $\tilde{H}(z') = cH(z')$, $c = \text{const}$).
Transformations that do not alter a given Hamiltonian system are called *canonoid transformations.*

By construction, the composition of canonical transformations is also a canonical transformation: if \( z \rightarrow z' = z'(z, \tau) \), and \( z' \rightarrow z'' = z''(z', \tau) \) are canonical, then \( z \rightarrow z'' = z''(z'(z, \tau), \tau) \) is a canonical transformation. The set of canonical transformations form a group, with a product defined by this law of composition. This allows us to describe the ambiguity present in the Hamiltonization procedure: besides Eq. (2.21), any change of the form

\[
(q, v) \rightarrow (q', p(q, v), \tau), \quad p(q, v) = \frac{\partial L}{\partial v}
\]

where \( p(q, v) = \frac{\partial L}{\partial v} \) and \( q'(q, p, \tau) \) is a canonical transformation, transforms Eq. (2.19) into the Hamiltonian system.

From Eqs. (2.96) and (2.97) it follows that the canonical transformation \( z'(z, \tau) \) obeys

\[
\{z'^k, z'^l\} \big|_{z'(z, \tau)} \partial'_l H(z'(z, \tau)) + \partial'_k z'^k(z, \tau) \big|_{z'(z, \tau)} = \omega^{kl} \partial'_l \tilde{H}(z', \tau), \quad \forall H \text{ and some } \tilde{H}.
\]  

From this expression we immediately obtain two useful consequences. First, taking derivative \( \partial'_k \) of Eq. (2.98) we have

\[
\partial'_k \left( \{z'^k, z'^l\}_z \big|_{z(z', \tau)} \partial'_l H(z'(z, \tau)) + \partial'_k \partial'_l z'^k(z, \tau) \big|_{z(z', \tau)} \right) = 0.
\]  

Since this is true for any \( H \), the first and second terms vanish separately. In particular, the derivative of the Poisson bracket must be zero, hence \( \{z'^k, z'^l\}_z \big|_{z(z', \tau)} = c^{kl}(\tau) \), where \( c^{kl} \) does not depend on \( z' \). So, the substitution of \( z(z', \tau) \) can be omitted, and we have

\[
\{z'^k, z'^l\}_z = c^{kl}(\tau).
\]  

Second, denoting the left hand side of Eq. (2.98) by \( J'^k \), it can be written as \( J'^k = \partial'^k \tilde{H} \). From this it follows:

\[
\partial'^l J'^j = \partial'^j J'^l.
\]  

In greater detail, the identity is

\[
\partial'^i \left( \partial'_t z'^j(z, \tau) \big|_{z(z', \tau)} \right) - (i \leftrightarrow j) = \left( \partial'^j W^{ij} - (i \leftrightarrow j) \right) \partial'^l H - \left( W^{ik} \omega^{jl} - (i \leftrightarrow j) \right) \partial'^l H = 0.
\]  

(2.102)
where

\[ W^{ij} \equiv \{ z'^i, z'^j \}_{\bar{z}(z', \tau)}. \tag{2.103} \]

Since this is true for any \( H \), we write separately

\[
\begin{align*}
\partial^a z'^i (\partial^\tau z'^j (z, \tau)) \bigg|_{z(z', \tau)} - (i \leftrightarrow j) &= 0, \\
\partial^a W^{bd} - \partial^b W^{ad} &= 0, \\
W^{ik} \omega^{jl} - W^{jk} \omega^{il} + W^{il} \omega^{jk} - W^{jl} \omega^{ik} &= 0. \tag{2.104}
\end{align*}
\]

The Eqs. (2.100) and (2.104) hold for an arbitrary canonical transformation and will be the starting point for our analysis below. In particular, it will be shown in Chap. 4, that the system (2.104) is equivalent to a simple statement that the symplectic matrix is invariant under the canonical transformation (disregarding the constant \( c \)):

\[
\frac{\partial z'^k}{\partial z'^i} \frac{\partial z'^l}{\partial z'^j} = c \omega^{kl}, \quad c = \text{const}, \tag{2.105}
\]

Transformations with \( c = 1 \) are called \textit{univalent canonical transformations}.

### 2.8 Generalized Hamiltonian Equations: Example of Non-canonical Poisson Bracket

Here we discuss the form that the Hamiltonian equations acquire in an arbitrary parametrization of the configuration-velocity space.

In Sect. 2.2 the Hamiltonian equations were written in terms of the Poisson bracket

\[
\dot{z}^i = \{ z^i, H \}, \quad \{ z^i, z^j \} = \omega^{ij}, \tag{2.106}
\]

with the numeric matrix \( \omega^{ij} \), see Eq. (2.47). According to Eq. (2.89), after the time-independent transformation \( z^i \rightarrow z'^i = z'^i(z^j) \), the equations read

\[
\dot{z}'^i = W^{ij} \frac{\partial \tilde{H}(z')}{\partial z'^j} \equiv \{ z'^i, \tilde{H}(z') \}^{(W)}, \tag{2.107}
\]

where \( \tilde{H}(z') \equiv H(z(z')) \), and \( W \) is now a \( z' \)-dependent matrix

\[
W^{ij} = \frac{\partial z'^i}{\partial z'^k} \omega^{kl} \frac{\partial z'^j}{\partial z'^l} \bigg|_{z(z')} . \tag{2.108}
\]
This was used in Eq. (2.107) to define a non-canonical Poisson bracket

\[
\{A(z'), B(z')\}^{(W)} \equiv \frac{\partial A}{\partial z'^i} W^{ij} \frac{\partial B}{\partial z'^j}.
\] (2.109)

It can be shown that this obeys all the properties (2.40), (2.41), (2.42), and (2.43) of the Poisson bracket. The Eqs. (2.107) with the non-canonical Poisson bracket are known as generalized Hamiltonian equations.

As an example, let us discuss the transformation

\[
(q^a, p^b) \rightarrow (q'^a = q^a, p'_b = p_b + b_b(q)),
\] (2.110)

where \(b_b(q)\) is a given function. The Eqs. (2.106) acquire the form

\[
\begin{align*}
\dot{q}^a &= \frac{\partial \tilde{H}(q, p')}{\partial p'_a}, \\
\dot{p}'_a &= -\frac{\partial \tilde{H}(q, p')}{\partial q^a} + \left(\frac{\partial b_a}{\partial q^b} - \frac{\partial b_b}{\partial q^a}\right) v^b(q^a, p'_b - b_b).
\end{align*}
\] (2.111)

where

\[
\tilde{H}(q, p') \equiv H(q^a, p'_b - b_b(q)) = \left.(p'_a - b_a) v^a - L(q, v)\right|_{v(q, p' - b)}.
\] (2.112)

Let us point out that the same result appears if we carry out the Hamiltonization procedure of the Eqs. (2.19) using the variable change

\[
p'_a = \frac{\partial L(q, v)}{\partial v^a} + b_a(q),
\] (2.113)

instead of the standard one (2.21).

The normal system (2.111) is equivalent to the Euler-Lagrange equations for any given function \(b_a(q)\).

In this case, the symplectic form of the non-canonical bracket (2.109) is given by the expression

\[
W^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & W_{ab} \end{pmatrix}, \quad W_{ab}(q) \equiv \frac{\partial b_a}{\partial q^b} - \frac{\partial b_b}{\partial q^a}.
\] (2.114)

This implies the fundamental brackets

\[
\{q^a, q^b\}^{(W)} = 0, \quad \{q^a, p'_b\}^{(W)} = \delta_{ab}, \quad \{p'_a, p'_b\}^{(W)} = W_{ab}(q),
\] (2.115)

which obey the properties (2.40), (2.41), (2.42), and (2.43). The Eqs. (2.111) acquire
the form
\[ \dot{z}^i = \{z^i, \tilde{H}(z)\}^{(W)}, \quad \text{where} \quad z^i = (q^a, p'_b), \] (2.116)

with the Hamiltonian (2.112).

Non-canonical brackets (2.109), (2.114), and (2.115) naturally appear in the description of a system with velocity-dependent interactions. As an example, consider the Lagrangian action of non-relativistic particle on an external electromagnetic background (see Sect. 1.12.2)

\[ S = \int d\tau \left( \frac{1}{2} (\dot{q}^a)^2 + \dot{q}^a A_a(q) \right). \] (2.117)

The standard definition of momentum \( p_a = \frac{\delta L}{\delta \dot{q}^a} = \dot{q}^a + A_a(q) \) leads to the Hamiltonian

\[ H(q, p) = \frac{1}{2} (p_a - A_a)^2, \] (2.118)

which implies the Hamiltonian equations

\[ \dot{q}^a = p_a - A_a \equiv \{q^a, H\}, \quad \dot{p}_a = (p_b - A_b) \frac{\partial A_b}{\partial q^a} \equiv \{p_a, H\}, \] (2.119)

with the canonical Poisson bracket.

Now, using Eq. (2.113) as a definition of momentum: \( p'_a = \dot{q}_a + A_a(q) + b_a(q) \), it is natural to take \( b_a = -A_a \), which leads to the expression \( p'_a = \dot{q}_a \). Then Eq. (2.112) gives the Hamiltonian

\[ H(q, p) = \frac{1}{2} (p'_a)^2. \] (2.120)

According to Eq. (2.111) the Hamiltonian equations are

\[ \dot{q}^a = p_a \equiv \{q^a, H\}', \quad \dot{p}'_a = -F_{ab} p'_b \equiv \{p'_a, H\}', \] (2.121)

with the non-canonical Poisson bracket

\[ \{q^a, q^b\}' = 0, \quad \{q^a, p'_b\}' = \delta_{ab}, \quad \{p'_a, p'_b\}' = F_{ab}(q). \] (2.122)

Here \( F \) is a field strength of the vector potential: \( F_{ab} = \frac{\partial A_a}{\partial q^b} - \frac{\partial A_b}{\partial q^a} \). It is easy to see that both (2.119) and (2.121) imply the same Lagrangian equations \( \ddot{q}^a = -F_{ab} \dot{q}^b \).

Note that the Hamiltonian (2.120) formally coincides with the free-particle one. In this sense, in the second formulation the interaction is encoded in the non-
canonical Poisson bracket. Inclusion of the velocity-dependent interactions into the bracket was suggested in [6].

Let us return to the Eqs. (2.110) and ask about the existence of a function $b_b(q)$ that preserves the canonical form of Hamiltonian equations. The Eqs. (2.111) will be in the canonical form if the last term vanishes.

**Exercise**

Show that $\left(\frac{\partial b_b}{\partial q^b} - \frac{\partial b_b}{\partial q^a}\right) v^b = 0$ implies $\frac{\partial b_a}{\partial q^b} - \frac{\partial b_b}{\partial q^a} = 0$.

In turn, the latter equation implies that $b_a = \frac{\partial g}{\partial q^a}$ for a function $g$. So, for a change of the form

$$q'^a = q^a, \quad p'_a = p_a + \frac{\partial g(q)}{\partial q^a},$$

we obtain the canonical equations with the Hamiltonian

$$H' = H \left(q'^a, p'_b - \frac{\partial g}{\partial q^b}\right).$$

According to the terminology of Sect. 2.7, Eq. (2.123) represents an example of a canonical transformation.

**Exercises**

1. Show that the symplectic form (2.114) is invertible; find the inverse matrix $W_{ij}$. Show that the latter obeys the equation

$$\partial_{[k} W_{ij]} \equiv \partial_k W_{ij} + \partial_i W_{jk} + \partial_j W_{ki} = 0.$$ (2.125)

A two-form with this property is called a **closed form**.

---

7 While in the second formulation the vector potential $A_a$ is not explicitly presented (Eqs. (2.120), (2.121), and (2.122) depend on $F$, that is an electric $\vec{E}$ and magnetic $\vec{B}$ fields), it nevertheless reappears in the course of quantization. In fact, the operators, which reproduce the brackets (2.122), should contain $A$: $q'^a \rightarrow \hat{q}^a = q^a, \ p'_a \rightarrow \hat{p}_a = \frac{\partial}{\partial q^a} + A_a$. This leads to the same Schrödinger equation as in the initial formulation, with explicit dependence on $A$. In turn, dependence of the Schrödinger equation on $A$ has interesting consequences. Contrary to the conclusions of classical mechanics, four-vector electromagnetic potential can affect the motion of charged particles, even in the region where the electric and the magnetic fields vanish. This effect [25, 26], known as the Aharonov-Bohm effect, has been confirmed by experiment.
2. Let $W_{ij}(z)$ be an antisymmetric invertible matrix, with the inverse matrix obeying the Eq. (2.125). Show that the bracket (2.109), constructed from this $W$, obeys properties (2.40), (2.41), (2.42), and (2.43).

3. Show that the bracket (2.109) with $W$ given by Eq. (2.108) obeys properties (2.40), (2.41), (2.42), and (2.43).

4. Find the non-canonical bracket and the generalized Hamiltonian equations in the initial parametrization $(q^a, v^b)$ of the configuration-velocity space (see Sect. 2.1.2).

2.9 Hamiltonian Action Functional

Similarly to Lagrangian equations, Hamiltonian ones can be obtained from the principle of least action applied to an appropriately chosen action functional. The Hamiltonian action functional is given by

$$S_H = \int d\tau \left( p_a \dot{q}^a - H(q^a, p_b, \tau) \right). \quad (2.126)$$

A variational problem is formulated as follows. We look for a curve $z^i(\tau)$ with fixed initial and final positions $q^a(\tau_1) = q^a_1$, $q^a(\tau_2) = q^a_2$ and arbitrary momenta, that would give a minimum for the functional (see Fig. 2.3 on page 106). The variation of the functional is

$$\delta S_H = \int d\tau \left( \left( \dot{q}^a - \frac{\partial H}{\partial p_a} \right) \delta p_a - \left( \dot{p}_a + \frac{\partial H}{\partial q^a} \right) \delta q^a + (p_a \delta q^a)^{t_2}_{t_{1}} \right). \quad (2.127)$$

Owing to the boundary conditions we have: $\delta q^a(\tau_1) = \delta q^a(\tau_2) = 0$, so the last term vanishes. Therefore $\delta S_H = 0$ implies the Hamiltonian Eqs. (2.32).

![Fig. 2.3 Variational problem for the Hamiltonian action functional](image)
Exercises

1. The addition of a total derivative term to the Lagrangian does not alter the Lagrangian equations of motion. Is the same true for the Hamiltonian action? See also the exercise on page 140.

2. Disregarding the boundary term, the Hamiltonian action can be written in the form

\[ S_H = \int d\tau \left( \frac{1}{2} \dot{z}^i \omega_{ij} \dot{z}^j - H(z^i) \right). \]  

(2.128)

Is it possible to formulate a consistent variational problem for this functional, which should lead to the Hamiltonian equations?

2.10 Schrödinger Equation as the Hamiltonian System

Hamiltonian action appears in applications more often than one might expect. As an example, consider the quantum mechanics of a particle subject to the potential \( V(t, x^i) \). The Schrödinger equation for the complex wave function \( \Psi(t, x^i) \)

\[ i\hbar \dot{\Psi} = -\frac{\hbar^2}{2m} \Delta \Psi + V(\Psi), \]  

(2.129)

is equivalent to the system of two equations for two real functions (the real and imaginary parts of \( \Psi \), \( \Psi = \varphi + ip \)). We have

\[ \hbar \dot{\varphi} = -\left( \frac{\hbar^2}{2m} \Delta - V \right) p, \]  

(2.130)

\[ \hbar \dot{p} = \left( \frac{\hbar^2}{2m} \Delta - V \right) \varphi. \]  

(2.131)

Remind the notation \( \Delta = \frac{\partial^2}{\partial x^i \partial x^i}, \quad \tilde{\nabla} = \frac{\partial}{\partial x^i}, \quad \dot{\varphi} \equiv \frac{\partial \varphi(t, x^i)}{\partial t}, \quad \dot{p} \equiv \frac{\partial p(t, x^i)}{\partial t}. \) We can treat \( \varphi(t, x^i) \) and \( p(t, x^i) \) as coordinate and conjugate momenta of the field \( \varphi \) at the space point \( x^i \). Then the system has the Hamiltonian form \( \dot{\varphi} = \{\varphi, H\}, \quad \dot{p} = \{p, H\}, \) with the Hamiltonian being

\[ H = \frac{1}{2\hbar} \int d^3x \left( \frac{\hbar^2}{2m} (\tilde{\nabla} \varphi \tilde{\nabla} \varphi + \tilde{\nabla} p \tilde{\nabla} p) + V(\varphi^2 + p^2) \right). \]  

(2.132)

Hence the Eqs. (2.130) and (2.131) arise from the variation problem with the Hamiltonian action obtained according to Eq. (2.126)
\[ S_H = \int dt d^3x \left[ \pi \dot{\phi} - \frac{1}{2\hbar} \left( \frac{\hbar^2}{2m} (\nabla \phi \nabla \phi + \nabla p \nabla p) + V(\phi^2 + p^2) \right) \right]. \] (2.133)

Disregarding the boundary term (see Exercise 2 in this relation), this can also be rewritten in terms of the wave function \( \Psi \) and its complex conjugate \( \Psi^* \)

\[ S_H = \int dt d^3x \left[ \frac{i\hbar}{2} (\Psi^* \dot{\Psi} - \dot{\Psi}^* \Psi) - \frac{\hbar^2}{2m} \nabla \Psi^* \nabla \Psi - V \Psi^* \Psi \right]. \] (2.134)

### 2.10.1 Lagrangian Action Associated with the Schrödinger Equation

Due to the Hamiltonian nature of the Schrödinger equation, it is natural to search for a Lagrangian formulation of the system (2.130) and (2.131), that is a second-order equation with respect to the time derivative\(^8\) for the real function \( \varphi(t, x^i) \). According to Sect. 2.1.4, we need to solve (2.130) with respect to \( p \) and then to substitute the result either in Eq. (2.131) or into the Hamiltonian action (2.133). This leads immediately to the rather formal non-local expression \( \dot{p} = \hbar \left( -\frac{\hbar^2}{2m} \Delta - V \right)^{-1} \partial_t \varphi \).

So, the Schrödinger system cannot be obtained starting from a (nonsingular) Lagrangian. Nevertheless, for the case of time-independent potential \( V(x^i) \), there is a Lagrangian field theory with the property that any solution to the Schrödinger equation can be constructed from a solution to this theory. To find it let us look for solutions of the form

\[ \Psi = - \left( \frac{\hbar^2}{2m} \Delta - V \right) \phi + i \hbar \dot{\phi}, \] (2.135)

where \( \phi(t, x^i) \) is a real function. Inserting (2.135) into (2.129) we conclude that \( \Psi \) will be a solution to the Schrödinger equation if \( \phi \) obeys the equation

\[ \hbar^2 \ddot{\phi} + \left( \frac{\hbar^2}{2m} \Delta - V \right)^2 \phi = 0, \] (2.136)

which follows from the Lagrangian action

---

\(^8\) In fact, the problem has already been raised by Schrödinger [27]. Equation (2.136) below was tested by Schrödinger as a candidate for the wave function equation and then abandoned.
\[ S = \int dt d^3x \left[ \frac{\hbar}{2} \dot{\phi}^2 - \frac{1}{2\hbar} \left( \frac{\hbar^2}{2m} \Delta - V \right) \phi \left( \frac{\hbar^2}{2m} \Delta - V \right) \phi \right]. \quad (2.137) \]

This can be treated as the classical theory of field \( \phi \) on the given external background \( V(x^i) \). The action contains Planck’s constant as a parameter. After the rescaling \((t, x^i, \phi) \rightarrow (\hbar t, \hbar x^i, \sqrt{\hbar} \phi)\) it appears in the potential only, \( V(\hbar x^i) \), and thus plays the role of a coupling constant of the field \( \phi \) with the background.

The formula (2.135) implies that after introduction of the field \( \phi \) into the formalism, its mathematical structure becomes analogous to that of electrodynamics. The dynamics of the magnetic \( \vec{B} \) and electric \( \vec{E} \) fields is governed by first-order Maxwell equations with respect to the time variable. Equivalently, we can use the vector potential \( A_a \), which obeys the second-order equations following from the Lagrangian action discussed in Sect. 1.12.2. \( A_a \) represents the potential for magnetic and electric fields, generating them according to \( \vec{B} = [\vec{\nabla}, \vec{A}], \vec{E} = -\frac{i}{\epsilon} \partial_t \vec{A} \). Similarly to this, the field \( \phi \) turns out to be a potential for the wave function, generating its real and imaginary parts according to Eq. (2.135), see also Fig. 2.4 on the page 109.

In quantum mechanics the quantity \( \Psi^* \Psi \) has an interpretation as a probability density, that is the expression \( \Psi^* \Psi(t, x^i) \Psi(t, x^i) d^3x \) represents the probability of finding a particle in the volume \( d^3x \) around the point \( x^i \) at the instant \( t \). According to the formula (2.135), we write

\[
\Psi^* \Psi = \hbar^2 (\dot{\phi})^2 + \left[ \left( -\frac{\hbar^2}{2m} \Delta + V \right) \phi \right]^2 = 2\hbar E, \quad (2.138)
\]

<table>
<thead>
<tr>
<th>Electrodynamics</th>
<th>Quantum mechanics</th>
</tr>
</thead>
<tbody>
<tr>
<td>There is the Lagrangian formulation in terms of ( A_a )</td>
<td>The same in terms of ( \phi )</td>
</tr>
<tr>
<td>( A_a ) represents the potential for magnetic and electric fields, one has ( \vec{B} = \vec{\nabla} \times \vec{A}, \vec{E} = -\frac{i}{\epsilon} \partial_t \vec{A} )</td>
<td>( \Psi = \varphi + ip = -(\Delta - V)\phi + i\hbar \partial_t \phi )</td>
</tr>
<tr>
<td>While the Maxwell equations are written in terms of ( \vec{B}, \vec{E} ), the field ( \vec{E} ) is the conjugate momenta for ( \vec{A} ) but not for ( \vec{B} )</td>
<td>While the Schrödinger equation is written in terms of ( \varphi, p ), the field ( p ) is the conjugate momenta for ( \phi ) but not for ( \varphi )</td>
</tr>
<tr>
<td>Maxwell equations form the generalized Hamiltonian system with the Hamiltonian ( \sim \vec{E}^2 + \vec{B}^2 )</td>
<td>Schrödinger equation forms the generalized Hamiltonian system with the Hamiltonian ( \sim p^2 + \varphi^2 )</td>
</tr>
</tbody>
</table>

Fig. 2.4 Real field \( \phi \) as the wave function potential
where $E = T + U$ is the energy density of the field $\phi$. Eq. (2.138) states that the probability density is the energy density of the wave potential $\phi$. So the preservation of probability is just an energy conservation law of the theory (2.137).

It is instructive to compare also the Hamiltonian equations of the theory (2.137)

\begin{equation}
\hbar \dot{\phi} = p, \quad \hbar \dot{p} = -\left( \frac{\hbar^2}{2m} \triangle - V \right) \phi, \tag{2.139}
\end{equation}

with the Schrödinger system. Note the following correspondence among solutions to these systems: (a) If the functions $\phi, p$ obey Eq. (2.130), (2.131), then the functions $\phi' = \phi, -\left( \frac{\hbar^2}{2m} \triangle - V \right) p$ obey Eq. (2.139). (b) If the functions $\phi, p$ obey Eqs. (2.139), then $\phi = -\left( \frac{\hbar^2}{2m} \triangle - V \right) \phi, p$ obey Eqs. (2.130) and (2.131). The kernel of the map $(\phi, p) \rightarrow (\phi', p)$ is composed of pure imaginary time-independent wave functions $\Psi = i \Pi(x^i)$, where $\Pi$ is any solution to the stationary Schrödinger equation $\left( \frac{\hbar^2}{2m} \triangle - V \right) \Pi = 0$.

Any solution to the field theory (2.137) determines a solution to the Schrödinger equation according to Eq. (2.135). We should ask whether an arbitrary solution to the Schrödinger equation can be presented in the form (2.135). An affirmative answer can be obtained as follows.

Let $\Psi = \phi + ip$ be a solution to the Schrödinger equation. Consider the expression (2.135) as an equation for determining $\phi$.

\begin{equation}
\dot{\phi} = \frac{1}{\hbar} p, \tag{2.140}
\end{equation}

\begin{equation}
\left( \frac{\hbar^2}{2m} \triangle - V \right) \phi = -\phi, \tag{2.141}
\end{equation}

Here the right-hand sides are known functions. Take Eq. (2.141) at $t = 0$, $\left( \frac{\hbar^2}{2m} \triangle - V \right) \phi = -\phi(0, x^i)$. The elliptic equation can be solved (at least for the analytic function $\varphi(x^i)$); let us denote the solution as $C(x^i)$. Then the function

\begin{equation}
\phi(t, x^i) = \frac{1}{\hbar} \int_0^t d\tau p(\tau, x^i) + C(x^i), \tag{2.142}
\end{equation}

obeys the Eqs. (2.140) and (2.141). They imply the desired result: any solution to the Schrödinger equation can be presented through the field $\phi$ and its momenta according to (2.135). Finally, note that Eqs. (2.140) and (2.141) together with Eqs. (2.130) and (2.131) imply that $\phi$ obeys Eq. (2.136).

Let us finish this section with one more comment. As we have seen, treating a Schrödinger system as a Hamiltonian one, it is impossible to construct the corresponding Lagrangian formulation owing to the presence of the spatial derivatives of momentum in the Hamiltonian. To avoid this problem, we can try to treat the
Schrödinger system as a generalized Hamiltonian system. We rewrite (2.130) in the form

\[ \dot{\varphi} = \{ \varphi, H' \}' \quad \dot{p} = \{ p, H' \}' \quad (2.143) \]

where \( H' \) is the “free field” generalized Hamiltonian

\[ H' = \int d^3x \frac{1}{2\hbar} (p^2 + \varphi^2) = \int d^3x \frac{1}{2\hbar} \psi^* \psi, \quad (2.144) \]

and the non-canonical Poisson bracket is specified by

\[ \{ \varphi, \varphi \}' = \{ p, p \}' = 0, \quad \{ \varphi(t, x), p(t, y) \}' = -\left( \frac{\hbar^2}{2m} \Delta - V \right) \delta^3(x - y). \quad (2.145) \]

In contrast to \( H \), the Hamiltonian \( H' \) does not contain the spatial derivatives of momentum.

A non-canonical bracket turns out to be a characteristic property of singular Lagrangian theories discussed in Chap. 8. There we obtain a more systematic treatment of the observations made above: there is a singular Lagrangian theory subject to second class constraints underlying both the Schrödinger equation and the classical field theory (2.137).

### 2.10.2 Probability as a Conserved Charge Via the Noether Theorem

In quantum mechanics the quantity \( \psi^* \psi \) has an interpretation of a probability density, that is the expression \( \psi^* \psi(t, x^i) \psi(t, x^i) d^3x \) represents the probability of finding a particle in the volume \( d^3x \) around the point \( x^i \) at the instant \( t \). Consistency of the interpretation implies that, for the given solution \( \psi(t, x^i) \), the probability of finding the particle anywhere in space, \( P(t) = \int d^3x \psi^* \psi \) must be the same number at any instant (the number can be further normalized to be 1), or \( \frac{dP}{dt} = 0 \) for any solution. That is, \( P \) must be the conserved charge of the theory. In Chap. 7 we will discuss the Noether theorem that gives a deep relationship among the symmetry properties of an action and the existence of conserved charges for the corresponding equations of motion. Here we obtain this relationship for a particular example of the Schrödinger equation, showing that the preservation of probability can be considered as a consequence of a symmetry presented in the Hamiltonian action (2.134).

Given the number \( \theta \), let us make the following substitution

\[ \psi \rightarrow e^{i\theta} \psi, \quad (2.146) \]
in the Hamiltonian action (2.134). Since this involves only products of a wave function with its complex conjugate, $\Psi^* \Psi$, the action does not change

$$S_H[e^{i\theta} \Psi] - S_H[\Psi] = 0. \quad (2.147)$$

According to Sect. 1.6, the action is invariant under (2.146). The symmetry transformation has a simple geometric interpretation as a rotation through the angle $\theta$ of a two-dimensional vector space spanned by $(\varphi, p)$.

What are the consequences of the invariance? Take an expansion of $e^{i\theta} \Psi$ in the power series at $\theta = 0$, keeping only a linear term, $e^{i\theta} \Psi = \Psi + \delta \Psi$, where $\delta \Psi = i \Psi \theta$. Then Eq. (2.147) implies (confirm that!)

$$S_H[\Psi + \delta \Psi]|_{O(\theta)} - S_H[\Psi] = 0. \quad (2.148)$$

That is variation of the action vanishes as well. On other hand the variation can be computed according to the known formula (1.105), so we have

$$\delta S_H = \int dt d^3 x \left[ \left( i \hbar \dot{\Psi} + \frac{\hbar^2}{2m} \Delta \Psi - V \Psi \right) \delta \Psi^* + (c.c) \delta \Psi + \frac{i \hbar}{2} \partial_t (\Psi^* \delta \Psi - \delta \Psi^* \Psi) + \frac{-\hbar^2}{2m} \partial_i (\delta \Psi^* \vec{\nabla} \Psi + \vec{\nabla} \Psi^* \delta \Psi) \right] = 0, \quad (2.149)$$

where (c.c.) stands for a complex conjugation of the previous bracket. Supposing $\Psi$ obeys the Schrödinger equation, the first and the second terms vanish, and (omitting the factor $-\hbar$) we conclude that

$$\partial_t J + \partial_i J^i = 0, \quad (2.150)$$

where

$$J = \Psi^* \Psi,$$

$$J^i = \frac{-i \hbar}{2m} (\Psi^* \vec{\nabla} \Psi - \vec{\nabla} \Psi^* \Psi). \quad (2.151)$$

Hence invariance of the action implies the continuity equation (2.150) that holds on solutions to the Schrödinger equation. It is further used to construct the conserved charge $P$ integrating the quantity $J$

$$P = \int d^3 x J = \int d^3 x \Psi^* \Psi. \quad (2.152)$$

The total probability is preserved as a consequence of the continuity equation
\[
\frac{dP}{dt} = \int_{\mathbb{R}^3} d^3x \partial_t J = -\int_{\mathbb{R}^3} d^3x \partial_i J^i = \int_{\partial \mathbb{R}^3} \vec{J} dS = 0. \tag{2.153}
\]

The third equality is due to Gauss’s theorem while the last one follows from the standard supposition that \( \Psi \) vanishes in spatial infinity (a particle cannot escape to infinity during a finite time interval).

**Exercises**

1. Confirm the preservation of probability, \( \frac{dP}{dt} = 0 \), by direct computation with use of the Schrödinger equation.
2. Obtain the charge \( P \) using the Hamiltonian action functional (2.133).

### 2.11 Hamiltonization Procedure in Terms of First-Order Action Functional

Here we describe a very elegant Hamiltonization recipe [10, 28] based on manipulations with the Lagrangian action. Let

\[
S = \int d\tau L(q^a, \dot{q}^a), \tag{2.154}
\]

be a Lagrangian action of a non-singular system. Let us introduce an extended phase space parameterized by independent coordinates \( q^a, p_a, v^a \). Starting from the Lagrangian given in Eq. (2.154), we can construct the following first-order action on the extended space

\[
S_1 = \int d\tau \left[ L(q^a, v^a) + p_a (\dot{q}^a - v^a) \right]. \tag{2.155}
\]

This implies the equations of motion

\[
\dot{q}^a = v^a, \quad \dot{p}_a = \frac{\partial L(q, v)}{\partial q^a}, \quad p_a = \frac{\partial L(q, v)}{\partial v^a}. \tag{2.156}
\]

The last equation determines the conjugate momenta (see (2.21)), while the first two equations coincide with the first-order equations of motion for the initial action (2.154), see Eq. (2.26). So the action (2.155) represents an equivalent formulation for the theory (2.154). In this formulation, equations for canonical momenta (2.21) appear as part of the equations of motion. The remainder of the Hamiltonization recipe consists of using the third equation to expel \( v^a \) from the first two equations. The corresponding computations coincide with those made in Sect. 2.1.2, starting from Eq. (2.26), and give the Hamiltonian equations (2.32).
We finish this section with a comment on the formal relationship between the different actions. Let us take the first order action as a basic object. The Lagrangian action can be obtained from the first order one by using the first equation from (2.156).

Solving the last equation from (2.156), \( v = v(q, p) \), and substituting the result into \( S_1 \), we obtain the Hamiltonian action.

We can also substitute \( p_a \) of the last equation from (2.156) into \( S_1 \) obtaining the following action in \( v, p \) space

\[
S_v = \int d\tau \left[ L(q^a, v^a) + \frac{\partial L(q, v)}{\partial v^a}(\dot{q}^a - v^a) \right].
\]  

(2.157)

The corresponding equations of motion are

\[
\frac{\delta S_v}{\delta q^a} = \frac{\partial L}{\partial q^a} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^a} + \frac{\partial^2 L}{\partial q^a \partial v^b}(\dot{q}^b - v^b) = 0,
\]

\[
\frac{\delta S_v}{\delta v^a} = \frac{\partial^2 L}{\partial v^a \partial v^b}(\dot{q}^b - v^b) = 0.
\]

(2.158)

For non-degenerate theory, they imply the Lagrangian equations for \( q^a \). Hence the action \( S_v \) can also be used to analyze the system.

### 2.12 Hamiltonization of a Theory with Higher-Order Derivatives

Here we discuss a theory which involves the higher-order derivatives. Inclusion of the higher derivatives into equations of motion is one of the ways to treat with the problem of divergences in perturbative quantum gravity theory. If such terms are added to the Einstein gravity, then the resulting quantum theory is renormalizable [29]. Detailed discussion of the subject can be found in [30].

#### 2.12.1 First-Order Trick

We start from a particular example of the action

\[
S = \int d\tau L(q_1, \dot{q}_1, \ddot{q}_1),
\]

(2.159)

where we use the notation \( q_1 \equiv (q^1, q^2, \ldots, q^n) \) for the configuration-space vector. This leads to the equations of motion of the fourth order

\[
\frac{\partial L}{\partial q_1} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{q}_1} \right) = 0.
\]

(2.160)
We suppose that the theory is nondegenerate, that is \( \det \frac{\partial^2 L}{\partial \dot{q}_a \partial \dot{q}_b} \neq 0 \).

The simplest way to obtain a Hamiltonian formulation is to apply the first-order trick of previous section to the \( \ddot{q}_1 \). We introduce the extended configuration space with the coordinates \( q_1, s, q_2 \), and write the action

\[
S_1 = \int d\tau \left[ L(q_1, \dot{q}_1, s) + q_2(\ddot{q}_1 - s) \right]
\]

\[
= \int d\tau \left[ L(q_1, \dot{q}_1, s) - \dot{q}_2 \ddot{q}_1 - q_2 s \right].
\]

In contrast to (2.159), this leads to the second-order equations of motion

\[
\ddot{q}_1 = s, \quad q_2 = \frac{\partial L}{\partial s},
\]

\[
\frac{\partial L}{\partial q_1} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_1} \right) + \ddot{q}_2 = 0.
\]

Using Eqs. (2.163) in (2.164) we reproduce the initial higher-order Eqs. (2.159), hence the two actions are equivalent.

Equation (2.162) represents one more example of a singular action. So, its Hamiltonian formulation is obtained according to the formalism that will be discussed in Chap. 8. For the later use, we present the final result for Hamiltonian equations of the variables \( q_1, q_2, p_1, p_2 \)

\[
\dot{q}_1 = -p_2, \quad \dot{p}_1 = \frac{\partial L}{\partial q_1},
\]

\[
\dot{q}_2 = -p_1 - \frac{\partial L}{\partial p_2}, \quad \dot{p}_2 = -s,
\]

where \( L = L(q_1, -p_2, s) \). The reader can verify that they imply (2.163) and (2.164).

We exclude the variable \( s \) from these equations. According to the rank condition \( \det \frac{\partial^2 L}{\partial \dot{q}_a \partial \dot{q}_b} \neq 0 \), the second equation from (2.163), \( q_2 = \frac{\partial L(q_1, -p_2, s)}{\partial s} \), can be resolved with respect to \( s \), \( s = s(q_1, q_2, p_2) \). Substituting the function \( s(q_1, q_2, p_2) \) into Eqs. (2.165), they read

\[
\dot{q}_1 = -p_2, \quad \dot{p}_1 = \frac{\partial L}{\partial q_1} \bigg|_s = \frac{\partial L}{\partial q_1} - q_2 \frac{\partial s}{\partial q_1},
\]

\[
\dot{q}_2 = -p_1 - \frac{\partial L}{\partial p_2} \bigg|_s = -p_1 - \frac{\partial L}{\partial p_2} + q_2 \frac{\partial s}{\partial p_2}, \quad \dot{p}_2 = -s,
\]

where on the r.h.s. we have \( L \equiv L(q_1, -p_2, s(q_1, q_2, p_2)) \). They follow from the Hamiltonian:
\[ H = -p_1 p_2 - L(q_1, -p_2, s(q_1, q_2, p_2)) + q_2 s(q_1, q_2, p_2). \]  

### 2.12.2 Ostrogradsky Method

Now consider the Hamiltonization of a theory with an action that depends on time derivatives up to \( N \)-th order. The procedure has been developed by Ostrogradsky \[31\]. Consider the action

\[ S = \int d\tau L(q_1, \dot{q}_1, \ddot{q}_1, \ldots, ^{(N)}q_1), \quad q_1 \equiv (q^1, q^2, \ldots, q^n). \]  

Disregarding a total derivative, variation of the action reads

\[ \delta S = \int d\tau \sum_{i=0}^{N} \frac{\partial L}{\partial q_1^{(i)}} \delta q_1^{(i)} = \int d\tau \left( \sum_{i=0}^{N} (-1)^i \frac{d^i}{d\tau^i} \frac{\partial L}{\partial q_1^{(i)}} \right) \delta q_1, \]  

so the Lagrangian equations are

\[ \frac{\partial L}{\partial q_1} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{d\tau} \frac{\partial L}{\partial \ddot{q}_1} + \ldots + (-1)^{N-1} \frac{d^{N-1}}{d\tau^{N-1}} \frac{\partial L}{\partial ^{(N)}q_1} \right) = 0. \]  

Computing derivatives with respect to \( \tau \) we conclude that the equations have the following structure

\[ L_{ab}^{(2N)} q_1^b = K_a(q_1, \dot{q}_1, \ldots, ^{(2N-1)}q_1), \quad L_{ab} = \frac{\partial^2 L}{\partial q_1^a \partial q_1^b}. \]  

Hence the Lagrangian with \( N \)-th order derivatives implies \( 2N \)-th order equations. The theory is called non-degenerate if

\[ \det L_{ab} \neq 0. \]  

In this case the equations can be written in the normal form, with higher derivatives separated on l.h.s. of the equations. Specification of the position \( q_1 \) as well as its \( 2N - 1 \) derivatives at some instant implies unique solution to the Cauchy problem.

To present the system in the Hamiltonian form, we introduce \( 2N \times n \) dimensional phase space spanned by the coordinates \( q_i, p_i, i = 1, 2, \ldots, N \). Let us specify their dynamics as follows. The variable \( q_1 \) obeys the Eq. (2.170), while other variables accompany its evolution according the equations
\begin{equation}
q_i = q_1, \quad i = 2, 3, \ldots, N,
\end{equation}

\begin{equation}
p_i = \sum_{j=i}^{N} (-1)^{j-i} \frac{d^{j-i}}{d\tau^{j-i}} \frac{\partial L}{\partial q_1}, \quad i = 1, 2, \ldots, N.
\end{equation}

(2.173)

In particular, expression for the momenta \( p_N \) reads

\begin{equation}
p_N = \frac{\partial L(q_1, \dot{q}_1)}{\partial \dot{q}_1} = \frac{\partial L(q_1, q_2, \ldots, q_N, \dot{q}_N)}{\partial \dot{q}_N}.
\end{equation}

(2.174)

According to the condition (2.172), it can be resolved algebraically with respect to \( \dot{q}_N \). Let us denote the solution by \( s_N \)

\begin{equation}
\dot{q}_N = s_N(q_1, q_2, \ldots, q_N, p_N).
\end{equation}

(2.175)

Using the solution, we rewrite the system (2.170) and (2.173) in the first order normal form

\begin{align}
\dot{q}_{i-1} &= q_i, \\
\dot{q}_N &= s_N(q_1, q_2, \ldots, q_N, p_N), \\
\dot{p}_1 &= \frac{\partial L}{\partial q_1} \bigg|_{s_N} = -\frac{\partial}{\partial q_i} (p_{NSN} - L(q_i, s_N)), \\
\dot{p}_i &= -p_{i-1} + \frac{\partial L}{\partial q_i} \bigg|_{s_N} \\
&= -\frac{\partial}{\partial q_i} (p_{i-1} q_i + p_{NSN} - L(q_i, s_N)).
\end{align}

(2.176) \quad (2.177) \quad (2.178) \quad (2.179)

Here \( i = 2, 3, \ldots, N \). At last, introducing the Hamiltonian

\begin{equation}
H(q_i, p_j) = p_1 q_2 + p_2 q_3 + \ldots + p_{N-1} q_N + p_{NSN} - L(q_i, s_N),
\end{equation}

(2.180)

the system (2.176), (2.177), (2.178), and (2.179) acquires the Hamiltonian form

\begin{align}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \equiv \{q_i, H\}, \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} \equiv \{p_i, H\}.
\end{align}

(2.181)

The Poisson brackets are defined by

\begin{equation}
\{q_i^a, p_{jb}\} = \delta_i^b \delta_{ab}.
\end{equation}

(2.182)

Equations (2.181) follow from the Hamiltonian action functional
\[ S_H = \int d\tau (p_i \dot{q}_i^i - H). \]  

(2.183)

In resume, for an \( N \)-th order Lagrangian, the Hamiltonian formulation implies introducing \( 2N \times n \) dimensional phase space with the Poisson brackets (2.182). The working recipe for construction the corresponding Hamiltonian can be formulated as follows. Define the momenta \( p_N \) according to the Eq. (2.174) and resolve it with respect to \( q_1^{(N)} \). Then the Hamiltonian is

\[ H(q_i, p_j) = \sum_{i=1}^{N} p_i^{(i)} q_1^{(i)} - L(q_1, q_1), \quad i = 1, 2, \ldots, N, \]  

(2.184)

where one substitutes \( q_{i+1} \) instead of \( q_1^{(i)}, i = 1, 2, \ldots, N-1 \), and \( s_N \) of Eq. (2.175) instead of \( q_1 \).

In conclusion, we point out that Eqs. (2.176), (2.177) can not be resolved with respect to the momenta, that is the Ostrogradsky equations (2.181) can not be obtained from a Lagrangian (without higher derivatives). To avoid the difficulty, one needs to make an appropriate canonical transformation. For instance, for the case of the Lagrangian \( L(q_1, \dot{q}_1, \ddot{q}_1) \) it is sufficient to make the transformation \( q_2 \to -p_2, p_2 \to q_2 \). After that, the Hamiltonian (2.180) and the Ostrogradsky equations (2.181) turn out into Eqs. (2.167) and (2.166).
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