In this chapter we introduce the notion of a cellular automaton. We fix a group and an arbitrary set which will be called the alphabet. A configuration is defined as being a map from the group into the alphabet. Thus, a configuration is a way of attaching an element of the alphabet to each element of the group. There is a natural action of the group on the set of configurations which is called the shift action (see Sect. 1.1). A cellular automaton is a self-mapping of the set of configurations defined from a system of local rules commuting with the shift (see Definition 1.4.1). We equip the configuration set with the prodiscrete topology; that is, the topology of pointwise convergence associated with the discrete topology on the alphabet (see Sect. 1.2). It turns out that every cellular automaton is continuous with respect to the prodiscrete topology (Proposition 1.4.8) and commutes with the shift (Proposition 1.4.4). Conversely, when the alphabet is finite, every continuous self-mapping of the configuration space which commutes with the shift is a cellular automaton (Theorem 1.8.1). Another important fact in the finite alphabet case is that every bijective cellular automaton is invertible, in the sense that its inverse map is also a cellular automaton (Theorem 1.10.2). We give examples showing that, when the alphabet is infinite, a continuous self-mapping of the configuration space which commutes with the shift may fail to be a cellular automaton and a bijective cellular automaton may fail to be invertible. In Sect. 1.9, we introduce the prodiscrete uniform structure on the configuration space. We show that a self-mapping of the configuration space is a cellular automaton if and only if it is uniformly continuous and commutes with the shift (Theorem 1.9.1).

1.1 The Configuration Set and the Shift Action

Let $G$ be a group. For $g \in G$, denote by $L_g$ the left multiplication by $g$ in $G$, that is, the map $L_g : G \to G$ given by
\[ L_g(g') = gg' \] for all \( g' \in G \).

Observe that for all \( g_1, g_2, g' \in G \) one has
\[
(L_{g_1} \circ L_{g_2})(g') = L_{g_1}(L_{g_2}(g')) = L_{g_1}(g_2g') = g_1g_2g' = L_{g_1g_2}(g')
\]
which shows that
\[ L_{g_1} \circ L_{g_2} = L_{g_1g_2}. \] (1.1)

Let \( A \) be a set. Consider the set \( A^G \) consisting of all maps from \( G \) to \( A \):
\[
A^G = \prod_{g \in G} A = \{ x : G \to A \}.
\]
The set \( A \) is called the alphabet. The elements of \( A \) are called the letters, or the states, or the symbols, or the colors. The group \( G \) is called the universe. The set \( A^G \) is called the set of configurations.

Given an element \( g \in G \) and a configuration \( x \in A^G \), we define the configuration \( gx \in A^G \) by
\[
(gx)(g') = x(g^{-1}g') \quad \text{for all } g' \in G.
\]
Thus one has
\[
gx(g') = x(g^{-1}g') \quad \text{for all } g' \in G.
\]
The map
\[
G \times A^G \to A^G \\
(g, x) \mapsto gx
\]
is a left action of \( G \) on \( A^G \). Indeed, for all \( g_1, g_2 \in G \) and \( x \in A^G \), one has
\[
g_1(g_2x) = g_1(x \circ L_{g_2^{-1}}) = x \circ L_{g_2^{-1}} \circ L_{g_1^{-1}} = x \circ L_{g_2^{-1}g_1^{-1}} = x \circ L_{(g_1g_2)^{-1}} = (g_1g_2)x,
\]
where the third equality follows from (1.1). Also, denoting by \( 1_G \) the identity element of \( G \) and by \( \text{Id}_G : G \to G \) the identity map, one has
\[
1_Gx = x \circ L_{1_G} = x \circ \text{Id}_G = x.
\]
This left action of \( G \) on \( A^G \) is called the \( G \)-shift on \( A^G \).

A pattern over the group \( G \) and the alphabet \( A \) is a map \( p : \Omega \to G \) defined on some finite subset \( \Omega \) of \( G \). The set \( \Omega \) is then called the support of \( p \).
1.2 The Prodiscrete Topology

Let $G$ be a group and let $A$ be a set.

We equip each factor $A_i$ of $A^G$ with the discrete topology (all subsets of $A_i$ are open) and $A^G$ with the associated product topology (see Sect. A.4). This topology is called the prodiscrete topology on $A^G$. This is the smallest topology on $A^G$ for which the projection map $\pi_g : A^G \to A$, given by $\pi_g(x) = x(g)$, is continuous for every $g \in G$ (cf. Sect. A.4). The elementary cylinders

$$C(g,a) = \pi_g^{-1}(\{a\}) = \{x \in A^G : x(g) = a\} \quad (g \in G, a \in A)$$

are both open and closed in $A^G$. A subset $U \subset A^G$ is open if and only if $U$ can be expressed as a (finite or infinite) union of finite intersections of elementary cylinders.

For a subset $\Omega \subset G$ and a configuration $x \in A^G$ let $x|_{\Omega} \in A^\Omega$ denote the restriction of $x$ to $\Omega$, that is, the map $x|_{\Omega} : \Omega \to A$ defined by $x|_{\Omega}(g) = x(g)$ for all $g \in \Omega$.

If $x \in A^G$, a neighborhood base of $x$ is given by the sets

$$V(x, \Omega) = \{y \in A^G : x|_{\Omega} = y|_{\Omega}\} = \bigcap_{g \in \Omega} C(g, x(g)),$$

where $\Omega$ runs over all finite subsets of $G$.

**Proposition 1.2.1.** The space $A^G$ is Hausdorff and totally disconnected.

**Proof.** The discrete topology on $A$ is Hausdorff and totally disconnected, and, by Proposition A.4.1 and Proposition A.4.2, a product of Hausdorff (resp. totally disconnected) topological spaces is Hausdorff (resp. totally disconnected). \hfill $\square$

Recall that an action of a group $G$ on a topological space $X$ is said to be continuous if the map $\varphi_g : X \to X$ given by $\varphi_g(x) = gx$ is continuous on $X$ for each $g \in G$.

**Proposition 1.2.2.** The action of $G$ on $A^G$ is continuous.

**Proof.** Let $g \in G$ and consider the map $\varphi_g : A^G \to A^G$ defined by $\varphi_g(x) = gx$. The map $\pi_h \circ \varphi_g$ is equal to $\pi_g \circ (\pi_h^{-1} \circ \varphi_g)$ and is therefore continuous on $A^G$ for every $h \in G$. Consequently, $\varphi_g$ is continuous (cf. Sect. A.4). \hfill $\square$

1.3 Periodic Configurations

Let $G$ be a group and let $A$ be a set. Let $H$ be a subgroup of $G$. A configuration $x \in A^G$ is called $H$-periodic if $x$ is fixed by $H$, that is, if one has
Let $\text{Fix}(H)$ denote the subset of $A^G$ consisting of all $H$-periodic configurations.

**Examples 1.3.1.** (a) One has $\text{Fix}(\{1_G\}) = A^G$.

(b) The set $\text{Fix}(G)$ consists of all constant configurations and may be therefore identified with $A$.

(c) For $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, $n \geq 1$, the set $\text{Fix}(H)$ is the set of sequences $x: \mathbb{Z} \to A$ which admit $n$ as a (not necessarily minimal) period, that is, such that $x(i + n) = x(i)$ for all $i \in \mathbb{Z}$.

**Proposition 1.3.2.** Let $H$ be a subgroup of $G$. Then the set $\text{Fix}(H)$ is closed in $A^G$ for the prodiscrete topology.

**Proof.** We have 

$$\text{Fix}(H) = \bigcap_{h \in H} \{ x \in A^G : hx = x \}. \quad (1.4)$$

The space $A^G$ is Hausdorff by Proposition 1.2.1 and the action of $G$ on $A^G$ is continuous by Proposition 1.2.2. Thus the set of fixed points of the map $x \mapsto gx$ is closed in $A^G$ for each $g \in G$. Therefore $\text{Fix}(H)$ is closed in $A^G$ by (1.4). \qed

Consider the set $H \backslash G = \{ Hg : g \in G \}$ consisting of all right cosets of $H$ in $G$ and the canonical surjective map

$$\rho: G \to H \backslash G$$

$$g \mapsto Hg.$$ 

Given an element $y \in A^{H \backslash G}$, i.e., a map $y: H \backslash G \to A$, we can form the composite map $y \circ \rho: G \to A$ which is an element of $A^G$. In fact, we have $y \circ \rho \in \text{Fix}(H)$ since 

$$(h(y \circ \rho))(g) = y \circ \rho(h^{-1}g) = y(\rho(h^{-1}g)) = y(\rho(g)) = y \circ \rho(g)$$

for all $g \in G$ and $h \in H$.

**Proposition 1.3.3.** Let $H$ be a subgroup of $G$ and let denote by $\rho: G \to H \backslash G$ the canonical surjection. Then the map $\rho^*: A^{H \backslash G} \to \text{Fix}(H)$ defined by $\rho^*(y) = y \circ \rho$ for all $y \in A^{H \backslash G}$ is bijective.

**Proof.** If $y_1, y_2 \in A^{H \backslash G}$ satisfy $y_1 \circ \rho = y_2 \circ \rho$, then $y_1 = y_2$ since $\rho$ is surjective. Thus $\rho^*$ is injective.

If $x \in \text{Fix}(H)$, then $hx = x$ for all $h \in H$, that is, 

$$x(h^{-1}g) = x(g) \quad \text{for all } h \in H, g \in G.$$ 

Thus, the configuration $x$ is constant on each right coset of $G$ modulo $H$, that is, $x$ is in the image of $\rho^*$. This shows that $\rho^*$ is surjective. \qed
Corollary 1.3.4. If the set $A$ is finite and $H$ is a subgroup of finite index of $G$, then the set $\text{Fix}(H)$ is finite and one has $|\text{Fix}(H)| = |A| [G:H]$, where $[G:H]$ denotes the index of $H$ in $G$. \hfill \Box$

Example 1.3.5. Let $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, where $n \geq 1$. If $A$ is finite of cardinality $k$, then $|\text{Fix}(H)| = kn$.

Suppose now that $N$ is a normal subgroup of $G$, that is, $gN = Ng$ for all $g \in G$. Then, there is a natural group structure on $G/N = N \setminus G$ for which the canonical surjection $\rho: G \to G/N$ is a homomorphism.

Proposition 1.3.6. Let $N$ be a normal subgroup of $G$. Then $\text{Fix}(N)$ is a $G$-invariant subset of $A^G$.

Proof. Let $x \in \text{Fix}(N)$ and $g \in G$. Given $h \in N$, then there exists $h' \in N$ such that $hg = gh'$, since $N$ is normal in $G$. Thus, we have

$$h(gx) = g(h'x) = gx$$

which shows that $gx \in \text{Fix}(N)$. \hfill \Box

Since every element of $\text{Fix}(N)$ is fixed by $N$, the action of $G$ on $\text{Fix}(N)$ induces an action of $G/N$ on $\text{Fix}(N)$ which satisfies $\rho(g)x = gx$ for all $g \in G$ and $x \in \text{Fix}(N)$.

Suppose that a group $\Gamma$ acts on two sets $X$ and $Y$. A map $\varphi: X \to Y$ is called $\Gamma$-equivariant if one has $\varphi(\gamma x) = \gamma \varphi(x)$ for all $\gamma \in \Gamma$ and $x \in X$.

Proposition 1.3.7. Let $N$ be a normal subgroup of $G$ and let $\rho: G \to G/N$ denote the canonical epimorphism. Then the map $\rho^*: A^{G/N} \to \text{Fix}(N)$ defined by $\rho^*(y) = y \circ \rho$ for all $y \in A^{G/N}$ is a $G/N$-equivariant bijection.

Proof. We already know that $\rho^*$ is bijective (Proposition 1.3.3).

Let $g \in G$ and $y \in A^{G/N}$. For all $g' \in G$, we have

$$\rho(g)\rho^*(y)(g') = g\rho^*(y)(g')$$

$$= \rho^*(y)(g^{-1}g')$$

$$= (y \circ \rho)(g^{-1}g')$$

$$= y(\rho(g^{-1}g'))$$

$$= y((\rho(g))^{-1}\rho(g'))$$

$$= \rho(g)y(\rho(g'))$$

$$= \rho^*(\rho(g)y)(g').$$

Thus $\rho(g)\rho^*(y) = \rho^*(\rho(g)y)$. This shows that $\rho^*$ is $G/N$-equivariant. \hfill \Box
1.4 Cellular Automata

Let $G$ be a group and let $A$ be a set.

**Definition 1.4.1.** A *cellular automaton* over the group $G$ and the alphabet $A$ is a map $\tau: A^G \to A^G$ satisfying the following property: there exist a finite subset $S \subset G$ and a map $\mu: A^S \to A$ such that

$$\tau(x)(g) = \mu((g^{-1}x)|_S) \quad (1.5)$$

for all $x \in A^G$ and $g \in G$, where $(g^{-1}x)|_S$ denotes the restriction of the configuration $g^{-1}x$ to $S$.

Such a set $S$ is called a *memory set* and $\mu$ is called a *local defining map* for $\tau$.

Observe that formula (1.5) says that the value of the configuration $\tau(x)$ at an element $g \in G$ is the value taken by the local defining map $\mu$ at the pattern obtained by restricting to the memory set $S$ the shifted configuration $g^{-1}x$.

**Remark 1.4.2.** (a) Equality (1.5) may also be written

$$\tau(x)(g) = \mu((x \circ L_g)|_S) \quad (1.6)$$

by (1.2).

(b) For $g = 1_G$, formula (1.5) gives us

$$\tau(x)(1_G) = \mu(x|_S). \quad (1.7)$$

As the restriction map $A^G \to A^S$, $x \mapsto x|_S$, is surjective, this shows that if $S$ is a memory set for the cellular automaton $\tau$, then there is a unique map $\mu: A^S \to A$ which satisfies (1.5). Thus one says that this unique $\mu$ is the *local defining map* for $\tau$ *associated with* the memory set $S$.

**Examples 1.4.3.** (a) **The cellular automaton associated with the Game of Life.** Consider an infinite two-dimensional orthogonal grid of square cells, each of which is in one of two possible states, live or dead. Every cell $c$ interacts with its eight neighboring cells, namely the North, North-East, East, South-East, South, South-West, West and North-West cells (see Fig. 1.1).

At each step in time, the following rules for the evolution of the states of the cells are applied (in Figs. 1.2–1.5 we label with a “•” a live cell and with a “◦” a dead cell):

- **(birth):** a cell that is dead at time $t$ becomes alive at time $t + 1$ if and only if three of its neighbors are alive at time $t$ (cf. Fig. 1.2);
- **(survival):** a cell that is alive at time $t$ will remain alive at time $t + 1$ if and only if it has exactly two or three live neighbors at time $t$ (cf. Fig. 1.3);
1.4 Cellular Automata

Fig. 1.1 The cell $c$ and its eight neighboring cells

\[
\begin{array}{|c|c|c|}
\hline
\text{c(NW)} & \text{c(N)} & \text{c(NE)} \\
\hline
\text{c(W)} & c & \text{c(E)} \\
\hline
\text{c(SW)} & \text{c(S)} & \text{c(SE)} \\
\hline
\end{array}
\]

Fig. 1.2 A cell that is dead at time $t$ becomes alive at time $t+1$ if and only if three of its neighbors are alive at time $t$

Fig. 1.3 A cell that is alive at time $t$ will remain alive at time $t+1$ if and only if it has exactly two or three live neighbors at time $t$
Fig. 1.4 A live cell that has at most one live neighbor at time $t$ will be dead at time $t + 1$

- **(death by loneliness):** a live cell that has at most one live neighbor at time $t$ will be dead at time $t + 1$ (cf. Fig. 1.4);
- **(death by overcrowding):** a cell that is alive at time $t$ and has four or more live neighbors at time $t$, will be dead at time $t + 1$ (cf. Fig. 1.5).

Let us show that the map which transforms a configuration of cells at time $t$ into the configuration at time $t + 1$ according to the above rules is indeed a cellular automaton.

Consider the group $G = \mathbb{Z}^2$ and the finite set $S = \{-1, 0, 1\}^2 \subset G$. Then there is a one-to-one correspondence between the cells in the grid and the elements in $G$ in such a way that the following holds. If $c$ is a given cell, then $c + (0, 1)$ is the neighboring North cell, $c + (1, 1)$ is the neighboring North-East cell, and so on; in other words, $c$ and its eight neighboring cells correspond to the group elements $c + s$ with $s \in S$ (see Fig. 1.6).

Consider the alphabet $A = \{0, 1\}$. The state 0 (resp. 1) corresponds to absence (resp. presence) of life. With each configuration of the states of the cells in the grid we associate a map $x \in A^G$ defined as follows. Given a cell $c$ we set $x(c) = 1$ (resp. 0) if the cell $c$ is alive (resp. dead).
Consider the map $\mu : A^S \to A$ given by

$$
\mu(y) = \begin{cases} 
1 & \text{if } \sum_{s \in S} y(s) = 3 \\
0 & \text{otherwise}
\end{cases}
\quad \text{or}
\begin{cases} 
\sum_{s \in S} y(s) = 4 \text{ and } y((0,0)) = 1,
\end{cases}
$$

for all $y \in A^S$.

A moment of thought tells us that $\mu$ just expresses the rules for the Game of Life.

The cellular automaton $\tau : A^G \to A^G$ with memory set $S$ and local defining map $\mu$ is called the cellular automaton associated with the Game of Life.

(b) The Discrete Laplacian. Let $G = \mathbb{Z}$ and $A = \mathbb{R}$. Consider the map $\Delta : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}$ defined by

$$
\Delta(x)(n) = 2x(n) - x(n-1) - x(n+1).
$$

Then $\Delta$ is the cellular automaton over $\mathbb{Z}$ with memory set $S = \{-1,0,1\}$ and local defining map $\mu : \mathbb{R}^S \to \mathbb{R}$ given by

$$
\mu(y) = 2y(0) - y(-1) - y(1) \quad \text{for all } y \in \mathbb{R}^S.
$$

This may be generalized in the following way. Let $G$ be an arbitrary group and let $S$ be a nonempty finite subset of $G$. Let $\mathbb{K}$ be a field. Consider the map $\Delta_S = \Delta_S^G : \mathbb{K}^G \to \mathbb{K}^G$ defined by

$$
\Delta_S(x)(g) = |S|x(g) - \sum_{s \in S} x(gs).
$$
Then $\Delta_S$ is a cellular automaton over $G$ with memory set $S \cup \{1_G\}$ and local defining map $\mu: \mathbb{K}^{S \cup \{1_G\}} \rightarrow \mathbb{K}$ given by

$$\mu(y) = |S|y(1_G) - \sum_{s \in S} y(s) \text{ for all } y \in \mathbb{K}^{S \cup \{1_G\}}.$$ 

This cellular automaton is called the \textit{discrete Laplacian} over $\mathbb{K}$ associated with $G$ and $S$.

(c) \textbf{The Majority action cellular automaton}. Let $G$ be a group and let $S$ be a finite subset of $G$. Take $A = \{0, 1\}$ and consider the map $\tau: A^G \rightarrow A^G$ defined by

$$\tau(x)(g) = \begin{cases} 1 & \text{if } \sum_{s \in S} x(gs) > \frac{|S|}{2} \\ 0 & \text{if } \sum_{s \in S} x(gs) < \frac{|S|}{2} \\ x(g) & \text{if } \sum_{s \in S} x(gs) = \frac{|S|}{2} \end{cases}$$

for all $x \in A^G$. Then $\tau$ is a cellular automaton over $G$ with memory set $S \cup \{1_G\}$ and local defining map $\mu: A^{S \cup \{1_G\}} \rightarrow A$ given by

$$\mu(y) = \begin{cases} 1 & \text{if } \sum_{s \in S} y(s) > \frac{|S|}{2} \\ 0 & \text{if } \sum_{s \in S} y(s) < \frac{|S|}{2} \\ y(1_G) & \text{if } \sum_{s \in S} y(s) = \frac{|S|}{2} \end{cases}$$

for all $y \in A^{S \cup \{1_G\}}$.

The cellular automaton $\tau$ is called the \textit{majority action} cellular automaton associated with $G$ and $S$ (see Figs. 1.7–1.8).

The terminology comes from the fact that given $x \in A^G$ and $g \in G$, the value $\tau(x)(g)$ is equal to $a \in \{0, 1\}$ if there is a strict majority of elements of $gS$ at which the configuration $x$ takes the value $a$, or to $x(g)$ if no such majority exists.

(d) Let $G$ be a group, $A$ a set, and $f: A \rightarrow A$ a map from $A$ into itself. Then the map $\tau: A^G \rightarrow A^G$ defined by $\tau(x) = f \circ x$ is a cellular automaton with memory set $S = \{1_G\}$ and local defining map $\mu: A^S \rightarrow A$ given by $\mu(y) = f(y(1_G))$. Note that, if $f$ is the identity map $\text{Id}_A$ on $A$, then $\tau$ equals the identity map $\text{Id}_{A^G}$ on $A^G$.

(e) Let $G$ be a group, $A$ a set, and $s_0$ an element of $G$. Let $R_{s_0}: G \rightarrow G$ denote the right multiplication by $s_0$ in $G$, that is, the map $R_{s_0}: G \rightarrow G$ defined by $R_{s_0}(g) = gs_0$. Then the map $\tau: A^G \rightarrow A^G$ defined by $\tau(x) = x \circ R_{s_0}$ is a cellular automaton with memory set $S = \{s_0\}$ and local defining map $\mu: A^S \rightarrow A$ given by $\mu(y) = y(s_0)$.

\textbf{Proposition 1.4.4.} Let $G$ be a group and let $A$ be a set. Then every cellular automaton $\tau: A^G \rightarrow A^G$ is $G$-equivariant.
Proof. Let $S$ be a memory set for $\tau$ and let $\mu: A^S \to A$ be the associated local defining map. For all $g, h \in G$ and $x \in A^G$, we have
\[
\tau(gx)(h) = \mu((h^{-1}gx)|S) = \mu(((g^{-1}h)^{-1}x)|S) = \tau(x)(g^{-1}h) = g\tau(x)(h).
\]
Thus $\tau(gx) = g\tau(x)$. \hfill $\Box$

**Corollary 1.4.5.** Let $\tau: A^G \to A^G$ be a cellular automaton and let $H$ be a subgroup of $G$. Then one has $\tau(\text{Fix}(H)) \subset \text{Fix}(H)$.

Proof. Let $x \in \text{Fix}(H)$. By the previous Proposition, we have, for every $h \in H$,
\[
h\tau(x) = \tau(hx) = \tau(x).
\]
Thus $\tau(x) \in \text{Fix}(H)$. \hfill $\Box$

The following characterization of cellular automata will be useful in the sequel.
Proposition 1.4.6. Let $G$ be a group and let $A$ be a set. Consider a map $\tau: A^G \to A^G$. Let $S$ be a finite subset of $G$ and let $\mu: A^S \to A$. Then the following conditions are equivalent:

(a) $\tau$ is a cellular automaton admitting $S$ as a memory set and $\mu$ as the associated local defining map;
(b) $\tau$ is $G$-equivariant and one has $\tau(x)(1_G) = \mu(x|S)$ for every $x \in A^G$.

Proof. The fact that (a) implies (b) follows from Proposition 1.4.4 and formula (1.7)

Conversely, suppose (b). Then, by using the $G$-equivariance of $\tau$, we get

$$\tau(x)(g) = \tau(g^{-1}x)(1_G) = \mu((g^{-1}x)|S)$$

for all $x \in A^G$ and $g \in G$. Consequently, $\tau$ satisfies (a). $\Box$

An important feature of cellular automata is their continuity (with respect to the prodiscrete topology). In the proof of this property, we shall use the following.

Lemma 1.4.7. Let $G$ be a group and let $A$ be a set. Let $\tau: A^G \to A^G$ be a cellular automaton with memory set $S$ and let $g \in G$. Then $\tau(x)(g)$ depends only on the restriction of $x$ to $gS$.

Proof. This is an immediate consequence of (1.5) since $(g^{-1}x)(s) = x(gs)$ for all $s \in S$. $\Box$

Proposition 1.4.8. Let $G$ be a group and let $A$ be a set. Then every cellular automaton $\tau: A^G \to A^G$ is continuous.

Proof. Let $S$ be a memory set for $\tau$. Let $x \in A^G$ and let $W$ be a neighborhood of $\tau(x)$ in $A^G$. Then we can find a finite subset $\Omega \subset G$ such that (cf. equation (1.3))

$$V(\tau(x), \Omega) \subset W.$$

Consider the finite set $\Omega S = \{gs : g \in \Omega, s \in S\}$. If $y \in A^G$ coincide with $x$ on $\Omega S$, then $\tau(y)$ and $\tau(x)$ coincide on $\Omega$ by Lemma 1.4.7. Thus, we have

$$\tau(V(x, \Omega S)) \subset V(\tau(x), \Omega) \subset W.$$

This shows that $\tau$ is continuous. $\Box$

Proposition 1.4.9. Let $G$ be a group and let $A$ be a set. Let $\sigma: A^G \to A^G$ and $\tau: A^G \to A^G$ be cellular automata. Then the composite map $\sigma \circ \tau: A^G \to A^G$ is a cellular automaton. Moreover, if $S$ (resp. $T$) is a memory set for $\sigma$ (resp. $\tau$), then $ST = \{st : s \in S, t \in T\}$ is a memory set for $\sigma \circ \tau$.

Proof. It is clear that the map $\sigma \circ \tau$ is $G$-equivariant since $\sigma$ and $\tau$ are $G$-equivariant (by Proposition 1.4.4). Let $S$ (resp. $T$) be a memory set for
σ (resp. τ). For every \( x \in A^G \), we have \( \sigma \circ \tau(x)(1_G) = \sigma(\tau(x))(1_G) \). By Lemma 1.4.7, \( \sigma(\tau(x))(1_G) \) depends only on the restriction of \( \tau(x) \) to \( S \). By using Lemma 1.4.7 again, we deduce that, for every \( s \in S \), the element \( \tau(x)(s) \) depends only on the restriction of \( x \) to \( ST \). Therefore, \( \sigma \circ \tau(x)(1_G) \) depends only on the restriction of \( x \) to \( ST \). By applying Proposition 1.4.6, we conclude that \( \sigma \circ \tau \) is a cellular automaton admitting \( ST \) as a memory set.

\[ \square \]

Remark 1.4.10. With the hypotheses and notation of the previous proposition, denote by \( \mu: A^S \to A \) and \( \nu: A^T \to A \) the local defining maps for \( \sigma \) and \( \tau \), respectively. Then, the local defining map \( \kappa: A^{ST} \to A \) for \( \sigma \circ \tau \) may be described in the following way.

For \( y \in A^{ST} \) and \( s \in S \) define \( y_s \in A^T \) by setting \( y_s(t) = y(st) \) for all \( t \in T \). Also, denote by \( \overline{y} \in A^S \) the map defined by \( \overline{y}(s) = \nu(y_s) \) for all \( s \in S \). We finally define the map \( \kappa: A^{ST} \to A \) by setting

\[ \kappa(y) = \mu(\overline{y}) \quad (1.9) \]

for all \( y \in A^{ST} \).

Let \( x \in A^G \), \( g \in G \), \( s \in S \), and \( t \in T \). We then have

\[
(s^{-1}g^{-1}x)|_T(t) = s^{-1}g^{-1}x(t) \\
= g^{-1}x(st) \\
= (g^{-1}x)|_{ST}(st) \\
= ((g^{-1}x)|_{ST})_s(t).
\]

This shows that

\[
(s^{-1}g^{-1}x)|_T = ((g^{-1}x)|_{ST})_s
\]

and therefore

\[
\tau(g^{-1}x)(s) = \nu \left( (s^{-1}g^{-1}x)|_T \right) = \nu \left( ((g^{-1}x)|_{ST})_s \right) = (g^{-1}x)|_{ST}(s).
\]

As a consequence,

\[
\tau(g^{-1}x)|_S = (g^{-1}x)|_{ST}. \quad (1.10)
\]

Finally, one has

\[
(\sigma \circ \tau)(x)(g) = \sigma(\tau(x))(g) \\
= \mu \left( (g^{-1}\tau(x))|_S \right) \\
= \mu(\tau(g^{-1}x)|_S) \quad (1.11)
\]

(by (1.10))

\[
\quad (by \ (1.9)) \quad = \kappa \left( (g^{-1}x)|_{ST} \right).
\]

Recall that a monoid is a set equipped with an associative binary operation admitting an identity element. Denote by \( \text{CA}(G; A) \) the set consisting of all
cellular automata $\tau : A^G \to A^G$. In Example 1.4.3(d) we have seen that the identity map $\text{Id}_{A^G} : A^G \to A^G$ is a cellular automaton. Thus we have:

**Corollary 1.4.11.** The set $\text{CA}(G; A)$ is a monoid for the composition of maps.

### 1.5 Minimal Memory

Let $G$ be a group and let $A$ be a set. Let $\tau : A^G \to A^G$ be a cellular automaton. Let $S$ be a memory set for $\tau$ and let $\mu : A^S \to A$ be the associated defining map. If $S'$ is a finite subset of $G$ such that $S \subset S'$, then $S'$ is also a memory set for $\tau$ and the local defining map associated with $S'$ is the map $\mu' : A^{S'} \to A$ given by $\mu' = \mu \circ p$, where $p : A^{S'} \to A^S$ is the canonical projection (restriction map). This shows that the memory set of a cellular automaton is not unique in general. However, we shall see that every cellular automaton admits a unique memory set of minimal cardinality. Let us first establish the following result.

**Lemma 1.5.1.** Let $\tau : A^G \to A^G$ be a cellular automaton. Let $S_1$ and $S_2$ be memory sets for $\tau$. Then $S_1 \cap S_2$ is also a memory set for $\tau$.

**Proof.** Let $x \in A^G$. Let us show that $\tau(x)(1_G)$ depends only on the restriction of $x$ to $S_1 \cap S_2$. To see this, consider an element $y \in A^G$ such that $x|_{S_1 \cap S_2} = y|_{S_1 \cap S_2}$. Let us choose an element $z \in A^G$ such that $z|_{S_1} = x|_{S_1}$ and $z|_{S_2} = y|_{S_2}$ (we may take for instance the configuration $z \in A^G$ which coincides with $x$ on $S_1$ and with $y$ on $G \setminus S_1$). We have $\tau(x)(1_G) = \tau(z)(1_G)$ since $x$ and $z$ coincide on $S_1$, which is a memory set for $\tau$. On the other hand, we have $\tau(y)(1_G) = \tau(z)(1_G)$ since $y$ and $z$ coincide on $S_2$, which is also a memory set for $\tau$. It follows that $\tau(x)(1_G) = \tau(y)(1_G)$.

Thus there exists a map $\mu : A^{S_1 \cap S_2} \to A$ such that

$$\tau(x)(1_G) = \mu(x|_{S_1 \cap S_2})$$

for all $x \in A^G$.

As $\tau$ is $G$-equivariant (Proposition 1.4.4), we deduce that $S_1 \cap S_2$ is a memory set for $\tau$ by using Proposition 1.4.6. \qed 

**Proposition 1.5.2.** Let $\tau : A^G \to A^G$ be a cellular automaton. Then there exists a unique memory set $S_0 \subset G$ for $\tau$ of minimal cardinality. Moreover, if $S$ is a finite subset of $G$, then $S$ is a memory set for $\tau$ if and only if $S_0 \subset S$.

**Proof.** Let $S_0$ be a memory set for $\tau$ of minimal cardinality. As we have seen at the beginning of this section, every finite subset of $G$ containing $S_0$ is also a memory set for $\tau$. Conversely, let $S$ be a memory set for $\tau$. As $S \cap S_0$ is a memory set for $\tau$ by Lemma 1.5.1, we have $|S \cap S_0| \geq |S_0|$. This implies
1.6 Cellular Automata over Quotient Groups

\(S \cap S_0 = S_0\), that is, \(S_0 \subset S\). In particular, \(S_0\) is the unique memory set of minimal cardinality. ∎

The memory set of minimal cardinality of a cellular automaton is called its \textit{minimal} memory set.

\textbf{Remark 1.5.3.} A map \(F: A^G \to A^G\) is constant if there exists a configuration \(x_0 \in A^G\) such that \(F(x) = x_0\) for all \(x \in A^G\). By \(G\)-equivariance, a cellular automaton \(\tau: A^G \to A^G\) is constant if and only if there exists \(a \in A\) such that \(\tau(x)(g) = a\) for all \(x \in A^G\) and \(g \in G\). Observe that a cellular automaton \(\tau: A^G \to A^G\) is constant if and only if its minimal memory set is the empty set.

\section*{1.6 Cellular Automata over Quotient Groups}

Let \(G\) be a group and let \(A\) be a set. Let \(\tau: A^G \to A^G\) be a cellular automaton. Suppose that \(N\) is a normal subgroup of \(G\) and let \(\rho: G \to G/N\) denote the canonical epimorphism. It follows from Proposition 1.3.7 that the map \(\rho^*: A^{G/N} \to \text{Fix}(N)\), defined by \(\rho^*(y) = y \circ \rho\) for all \(y \in A^{G/N}\), is a bijection from the set \(A^{G/N}\) of configurations over the group \(G/N\) onto the set \(\text{Fix}(N) \subset A^G\) of \(N\)-periodic configurations over \(G\). On the other hand, the set \(\text{Fix}(N)\) satisfies \(\tau(\text{Fix}(N)) \subset \text{Fix}(N)\) by Corollary 1.4.5. Thus, we can define a map \(\overline{\tau}: A^{G/N} \to A^{G/N}\) by setting

\[\overline{\tau} = (\rho^*)^{-1} \circ \tau|_{\text{Fix}(N)} \circ \rho^*.\] (1.12)

In other words, the map \(\overline{\tau}\) is obtained by conjugating by \(\rho^*\) the restriction of \(\tau\) to \(\text{Fix}(N)\), so that the diagram

\[\begin{array}{ccc}
A^{G/N} & \xrightarrow{\rho^*} & \text{Fix}(N) \subset A^G \\
\tau| & & \downarrow \tau|_{\text{Fix}(N)} \\
A^{G/N} & \xrightarrow{\rho^*} & \text{Fix}(N)
\end{array}\]

is commutative.

Suppose that \(S \subset G\) is a memory set for \(\tau\) and that \(\mu: A^S \to A\) is the associated local defining map. Consider the finite subset \(\overline{S} = \rho(S) \subset G/N\) and the map \(\overline{\mu}: A^{\overline{S}} \to A\) defined by \(\overline{\mu} = \mu \circ \pi\), where \(\pi: A^{\overline{S}} \to A^S\) is the injective map induced by \(\rho\).

\textbf{Proposition 1.6.1.} The map \(\overline{\tau}: A^{G/N} \to A^{G/N}\) is a cellular automaton over the group \(G/N\) admitting \(\overline{S}\) as a memory set and \(\overline{\mu}: A^{\overline{S}} \to A\) as the associated local defining map.
Proof. Let \( y \in A^{G/N} \), \( g \in G \), and \( \overline{g} = \rho(g) \). We have
\[
\overline{\tau}(y)(\overline{g}) = \tau(y \circ \rho)(g) \\
= \mu((g^{-1}(y \circ \rho))|_S) \\
= \overline{\mu}((\overline{g}^{-1}y)|_{\overline{S}}).
\]
Thus \( \tau \) is a cellular automaton with memory set \( \overline{S} \) and local defining map \( \overline{\mu}: A^{\overline{S}} \rightarrow A \).

Consider now the map \( \Phi: \text{CA}(G; A) \rightarrow \text{CA}(G/N; A) \) given by \( \Phi(\tau) = \tau \), where \( \tau \) is defined by (1.12). We have the following:

**Proposition 1.6.2.** The map \( \Phi: \text{CA}(G; A) \rightarrow \text{CA}(G/N; A) \) is a monoid epimorphism.

**Proof.** Let \( \sigma: A^{G/N} \rightarrow A^{G/N} \) be a cellular automaton over \( G/N \) with memory set \( T \subset G/N \) and local defining map \( \nu: A^T \rightarrow A \). Let \( S \subset G \) be a finite set such that \( \rho \) induces a bijection \( \phi: S \rightarrow T \). Consider the map \( \mu: A^S \rightarrow A \) defined by \( \mu(y) = \nu(y \circ \phi^{-1}) \) for all \( y \in A^S \). Let \( \tau: A^G \rightarrow A^G \) be the cellular automaton over \( G \) with memory set \( S \) and local defining map \( \mu \). We have
\[
\overline{\mu}(z) = (\mu \circ \pi)(z) = \nu(\pi(z) \circ \phi^{-1}) = \nu(z)
\]
for all \( z \in A^{\overline{S}} \). It follows that \( \overline{\mu} = \nu \) and \( \overline{\tau} = \sigma \). This shows that \( \Phi \) is surjective.

The fact that \( \Phi \) is a monoid morphism immediately follows from (1.12).

**Examples 1.6.3.** Let \( G \) be a group, \( S \subset G \) a finite subset, and \( N \) a normal subgroup of \( G \). Denote by \( \rho: G \rightarrow G/N \) the canonical epimorphism and suppose that \( \rho \) induces a bijection between \( S \) and \( \overline{S} = \rho(S) \subset G/N \).

(a) Consider the discrete laplacian \( \Delta_S: \mathbb{R}^G \rightarrow \mathbb{R}^G \) associated with \( G \) and \( S \) (cf. Example 1.4.3(b)). Then \( \Phi(\Delta_S): \mathbb{R}^{G/N} \rightarrow \mathbb{R}^{G/N} \) is the discrete laplacian associated with \( G/N \) and \( \overline{S} \).

(b) Consider the majority action cellular automaton \( \tau: \{0, 1\}^G \rightarrow \{0, 1\}^G \) associated with \( G \) and \( S \) (cf. Example 1.4.3(c)). Then \( \Phi(\tau): \{0, 1\}^{G/N} \rightarrow \{0, 1\}^{G/N} \) is the majority action cellular automaton associated with \( G/N \) and \( \overline{S} \).

### 1.7 Induction and Restriction of Cellular Automata

Let \( G \) be a group and let \( A \) be a set. Let \( H \) be a subgroup of \( G \).

Let \( \text{CA}(G, H; A) \) denote the set consisting of all cellular automata \( \tau: A^G \rightarrow A^G \) admitting a memory set \( S \) such that \( S \subset H \). Thus, \( \text{CA}(G, H; A) \) is the
subset of $\text{CA}(G; A)$ consisting of the cellular automata whose minimal memory set is contained in $H$.

Recall that a subset $N$ of a monoid $M$ is called a submonoid if the identity element $1_M$ is in $N$ and $N$ is stable under the monoid operation (that is, $xy \in N$ for all $x, y \in N$). If $N$ is a submonoid of a monoid $M$, then the monoid operation induces by restriction a monoid structure on $N$.

**Proposition 1.7.1.** The set $\text{CA}(G, H; A)$ is a submonoid of $\text{CA}(G; A)$.

**Proof.** The identity element of $\text{CA}(G; A)$ is the identity map $\text{Id}_{A^G}$. We have $\text{Id}_{A^G} \in \text{CA}(G, H; A)$ since $\{1_G\}$ is a memory set for $\text{Id}_{A^G}$ and $\{1_G\} \subset H$.

Let $\sigma, \tau \in \text{CA}(G, H; A)$. Let $S$ (resp. $T$) be a memory set for $\sigma$ (resp. $\tau$) such that $S \subset H$ (resp. $T \subset H$). It follows from Proposition 1.4.9 that $ST$ is a memory set for $\sigma \circ \tau$. Since $ST \subset H$, this implies that $\sigma \circ \tau \in \text{CA}(G, H; A)$. This shows that $\text{CA}(G, H; A)$ is a submonoid of $\text{CA}(G; A)$. \qed

Let $\tau \in \text{CA}(G, H; A)$. Let $S$ be a memory set for $\tau$ such that $S \subset H$ and let $\mu : A^S \to A$ denote the associated local defining map. Then, the map $\tau_H : A^H \to A^H$ defined by

$$
\tau_H(x)(h) = \mu((h^{-1}x)|_S) \quad \text{for all } x \in A^H, h \in H,
$$

is a cellular automaton over the group $H$ with memory set $S$ and local defining map $\mu$. Observe that if $\tilde{x} \in A^G$ is such that $\tilde{x}|_H = x$, then

$$
\tau_H(x)(h) = \tau(\tilde{x})(h) \quad \text{for all } h \in H. \quad (1.13)
$$

This shows in particular that $\tau_H$ does not depend on the choice of the memory set $S \subset H$. One says that $\tau_H$ is the restriction of the cellular automaton $\tau$ to $H$.

Conversely, let $\sigma : A^H \to A^H$ be a cellular automaton with memory set $S$ and local defining map $\mu : A^S \to A$. Then the map $\sigma^G : A^G \to A^G$ defined by

$$
\sigma^G(\tilde{x})(g) = \mu((g^{-1}\tilde{x})|_S) \quad \text{for all } \tilde{x} \in A^G, g \in G,
$$

is a cellular automaton over $G$ with memory set $S$ and local defining map $\mu$. If $S_0$ is the minimal memory set of $\sigma$ and $\mu_0 : A^{S_0} \to A$ is the associated local defining map then $\mu = \mu_0 \circ \pi$, where $\pi : A^S \to A^{S_0}$ is the restriction map (see Sect. 1.5). Thus, one has

$$
\sigma^G(\tilde{x})(g) = \mu((g^{-1}\tilde{x})|_S) = \mu_0 \circ \pi((g^{-1}\tilde{x})|_S) = \mu_0((g^{-1}\tilde{x})|_{S_0})
$$

for all $\tilde{x} \in A^G$ and $g \in G$. This shows in particular that $\sigma^G$ does not depend on the choice of the memory set $S \subset H$. One says that $\sigma^G \in \text{CA}(G, H; A)$ is the cellular automaton induced by $\sigma \in \text{CA}(H; A)$.

**Proposition 1.7.2.** The map $\tau \mapsto \tau_H$ is a monoid isomorphism from $\text{CA}(G, H; A)$ onto $\text{CA}(H; A)$ whose inverse is the map $\sigma \mapsto \sigma^G$. 
Proof. To simplify notation, denote by $\alpha \colon CA(G, H; A) \to CA(H; A)$ and $\beta \colon CA(H; A) \to CA(G, H; A)$ the maps defined by $\alpha(\tau) = \tau_H$ and $\beta(\sigma) = \sigma^G$ respectively. It is clear from the definitions given above that $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity maps. Therefore, $\alpha$ is bijective with inverse $\beta$.

It remains to show that $\alpha$ is a monoid homomorphism.

Let $x \in A^H$ and let $\bar{x} \in A^G$ extending $x$. By applying (1.13), we get

$$\alpha(\text{Id}_{A^G})(x)(h) = \text{Id}_{A^G}(\bar{x})(h) = \bar{x}(h) = x(h)$$

for all $h \in H$. This shows that $\alpha(\text{Id}_{A^G})(x) = x$ for all $x \in A^H$, that is, $\alpha(\text{Id}_{A^G}) = \text{Id}_{A^H}$.

Let $\sigma, \tau \in CA(G, H; A)$. Let $x \in A^H$ and let $\bar{x} \in A^G$ extending $x$. By applying (1.13) again, we have

$$\alpha(\sigma \circ \tau)(x)(h) = (\sigma \circ \tau)(\bar{x})(h) = \sigma(\tau(\bar{x}))(h)$$

for all $h \in H$. On the other hand, since $\tau(\bar{x})$ extends $\alpha(\tau)(x)$, we have

$$\alpha(\sigma)(\alpha(\tau)(x))(h) = \sigma(\tau(\bar{x}))(h)$$

that is,

$$\alpha(\sigma) \circ \alpha(\tau))(x)(h) = \sigma(\tau(\bar{x}))(h)$$

for all $h \in H$. From (1.14) and (1.15), we deduce that $\alpha(\sigma \circ \tau)(x) = (\alpha(\sigma) \circ \alpha(\tau))(x)$ for all $x \in A^H$, that is, $\alpha(\sigma \circ \tau) = \alpha(\sigma) \circ \alpha(\tau)$. \qed

Let $\tau \in CA(G, H; A)$. In order to analyze the way $\tau$ transforms a configuration $\bar{x} \in A^G$, we now introduce the set $G/H = \{gh : g \in G\}$ consisting of all left cosets of $H$ in $G$. Since the cosets $c \in G/H$ form a partition of $G$, we have a natural identification $A^G = \prod_{c \in G/H} A^c$. With this identification, we have

$$\bar{x} = (\bar{x}|_c)_{c \in G/H}$$

for each $\bar{x} \in A^G$, where $\bar{x}|_c \in A^c$ denotes the restriction of $\bar{x}$ to $c$. Observe now that if $c \in G/H$ and $g \in c$, then $\tau(\bar{x})(g)$ depends only on $\bar{x}|_c$ (this directly follows from Lemma 1.4.7 since if $S$ is a memory set for $\tau$ with $S \subset H$, then $gS \subset c$). This implies that $\tau$ may be written as a product

$$\tau = \prod_{c \in G/H} \tau_c,$$

where $\tau_c \colon A^c \to A^c$ is the unique map which satisfies $\tau_c(\bar{x}|_c) = (\tau(\bar{x}))|_c$ for all $\bar{x} \in A^G$. Note that the notation is coherent when $c = H$, since, in this case, $\tau_c = \tau_H \colon A^H \to A^H$ is the cellular automaton obtained by restriction of $\tau$ to $H$.

Given a coset $c \in G/H$ and an element $g \in c$, denote by $\phi_g \colon H \to c$ the bijective map defined by $\phi_g(h) = gh$ for all $h \in H$. Then $\phi_g$ induces a bijective map $\phi_g^* \colon A^c \to A^H$ given by
\[ \phi^*_g(x) = x \circ \phi_g \]  \hspace{1cm} (1.17)

for all \( x \in A^c \). It turns out that the maps \( \tau_c \) and \( \tau_H \) are conjugate by \( \phi^*_g \):

**Proposition 1.7.3.** With the above notation, we have,

\[ \tau_c = (\phi^*_g)^{-1} \circ \tau_H \circ \phi^*_g. \]  \hspace{1cm} (1.18)

In other words, the following diagram

\[
\begin{array}{ccc}
A^c & \xrightarrow{\tau_c} & A^c \\
\downarrow{\phi^*_g} & & \downarrow{\phi^*_g} \\
A^H & \xrightarrow{\tau_H} & A^H
\end{array}
\]

is commutative.

**Proof.** Let \( x \in A^c \) and let \( \tilde{x} \in A^G \) extending \( x \). For all \( h \in H \), we have

\[
(\phi^*_g \circ \tau_c)(x)(h) = \phi^*_g(\tau_c(x))(h) = (\tau_c(x) \circ \phi_g)(h) = \tau_c(x)(gh) = \tau(\tilde{x})(gh) = g^{-1}\tau(\tilde{x})(h) = \tau(g^{-1}\tilde{x})(h),
\]

where the last equality follows from the \( G \)-equivariance of \( \tau \) (Proposition 1.4.4). Now observe that the configuration \( g^{-1}\tilde{x} \in A^G \) extends \( x \circ \phi_g \in A^H \). Thus, we have

\[
(\phi^*_g \circ \tau_c)(x)(h) = \tau_H(x \circ \phi_g)(h) = \tau_H(\phi^*_g(x))(h) = (\tau_H \circ \phi^*_g)(x)(h).
\]

This shows that \( \phi^*_g \circ \tau_c = \tau_H \circ \phi^*_g \), which gives (1.18) since \( \phi^*_g \) is bijective. \( \square \)

The following statement will be used in the proof of Proposition 3.2.1:

**Proposition 1.7.4.** Let \( G \) be a group and let \( A \) be a set. Let \( H \) be a subgroup of \( G \) and let \( \tau \in \text{CA}(G, H; A) \). Let \( \tau_H : A^H \to A^H \) denote the cellular automaton obtained by restriction of \( \tau \) to \( H \). Then the following hold:

(i) \( \tau \) is injective if and only if \( \tau_H \) is injective;
(ii) \( \tau \) is surjective if and only if \( \tau_H \) is surjective;
(iii) \( \tau \) is bijective if and only if \( \tau_H \) is bijective.
Proof. It immediately follows from (1.16) that $\tau$ is injective (resp. surjective, resp. bijective) if and only if $\tau_c$ is injective (resp. surjective, resp. bijective) for all $c \in G/H$.

Now, (1.18) says that, given $c \in G/H$ and $g \in G$, the map $\tau_c$ and $\tau_H$ are conjugate by the bijection $\phi_g$. We deduce that $\tau_c$ is injective (resp. surjective, resp. bijective) if and only if $\tau_H$ is injective (resp. surjective, resp. bijective).

Thus, $\tau$ is injective (resp. surjective, resp. bijective) if and only if $\tau_H$ is injective (resp. surjective, resp. bijective). \[\square\]

1.8 Cellular Automata with Finite Alphabets

Let $G$ be a group and let $A$ be a finite alphabet. As a product of finite spaces is compact by Tychonoff theorem (see Corollary A.5.3), it follows that $A^G$ is compact. This topological property is very useful in the study of cellular automata over finite alphabets. In particular, it may be used to prove the following:

**Theorem 1.8.1 (Curtis-Hedlund theorem).** Let $G$ be a group and let $A$ be a finite set. Let $\tau: A^G \to A^G$ be a map and equip $A^G$ with its prodiscrete topology. Then the following conditions are equivalent:

(a) the map $\tau$ is a cellular automaton;
(b) the map $\tau$ is continuous and $G$-equivariant.

Proof. The fact that (a) implies (b) directly follows from Proposition 1.4.4 and Proposition 1.4.8 (this implication does not require the finiteness assumption on the alphabet $A$).

Conversely, suppose (b). Let us show that $\tau$ is a cellular automaton. As the map $\varphi: A^G \to A$ defined by $\varphi(x) = \tau(x)(1_G)$ is continuous, we can find, for each $x \in A^G$, a finite subset $\Omega_x \subset G$ such that if $y \in A^G$ coincide with $x$ on $\Omega_x$, that is, if $y \in V(x, \Omega_x)$, then $\tau(y)(1_G) = \tau(x)(1_G)$. The sets $V(x, \Omega_x)$ form an open cover of $A^G$. As $A^G$ is compact, there is a finite subset $F \subset A^G$ such that the sets $V(x, \Omega_x)$, $x \in F$, cover $A^G$. Let us set $S = \bigcup_{x \in F} \Omega_x$ and suppose that two configurations $y, z \in A^G$ coincide on $S$. Let $x_0 \in F$ be such that $y \in V(x_0, \Omega_{x_0})$, that is, $y|_{\Omega_{x_0}} = x_0|_{\Omega_{x_0}}$. As $S \supset \Omega_{x_0}$ we have $y|_{\Omega_{x_0}} = z|_{\Omega_{x_0}}$ and therefore $\tau(y)(1_G) = \tau(x_0)(1_G) = \tau(z)(1_G)$. Thus there is a map $\mu: A^S \to A$ such that $\tau(x)(1_G) = \mu(x|_S)$ for all $x \in A^G$. As $\tau$ is $G$-equivariant, it follows from Proposition 1.4.6 that $\tau$ is a cellular automaton with memory set $S$ and local defining map $\mu$. \[\square\]

When the alphabet $A$ is infinite, a continuous and $G$-equivariant map $\tau: A^G \to A^G$ may fail to be a cellular automaton. In other words, the implication (b) $\Rightarrow$ (a) in Theorem 1.8.1 becomes false if we suppress the finiteness hypothesis on $A$. This is shown by the following example.
Example 1.8.2. Let $G$ be an arbitrary infinite group and take $A = G$ as the alphabet set. To avoid confusion, we denote by $g \cdot h$ the product of two elements $g$ and $h$ in $G$. Consider the map $\tau: A^G \to A^G$ defined by

$$\tau(x)(g) = x(g \cdot x(g))$$

for all $x \in A^G$ and $g \in G$.

Given $x \in A^G$ and $g, h \in G$ we have

$$g(\tau(x))(h) = \tau(x)(g^{-1} \cdot h)$$

$$= x(g^{-1} \cdot h \cdot (g^{-1} \cdot h))$$

$$= x(g^{-1} \cdot h \cdot [gx](h))$$

$$= gx(h \cdot [gx](h))$$

$$= \tau(gx)(h).$$

This shows that $g(\tau(x)) = \tau(gx)$ for all $x \in A^G$ and $g \in G$. Therefore, $\tau$ is $G$-equivariant.

Moreover, $\tau$ is continuous. Indeed, given $x \in A^G$ and a finite set $K \subset G$, let us show that there exists a finite set $F \subset G$ such that, if $y \in A^G$ and $y \in V(x, F)$, then $\tau(y) \in V(\tau(x), K)$. Set $F = K \cup \{k \cdot x(k): k \in K\}$. Then, if $y \in V(x, F)$, then, for all $k \in K$ one has

$$\tau(x)(k) = x(k \cdot x(k)) = y(k \cdot x(k)) = y(k \cdot y(k)) = \tau(y)(k).$$

This shows that $\tau(y) \in V(\tau(x), K)$. Thus, $\tau$ is continuous.

However, $\tau$ is not a cellular automaton. Indeed, fix $g_0 \in G \setminus \{1_G\}$ and, for all $g \in G$, consider the configurations $x_g$ and $y_g$ in $A^G$ defined by

$$x_g(h) = \begin{cases} 
  g & \text{if } h = 1_G \\
  g_0 & \text{if } h = g \\
  1_G & \text{otherwise}
\end{cases}$$

and

$$y_g(h) = \begin{cases} 
  g & \text{if } h = 1_G \\
  1_G & \text{otherwise}
\end{cases}$$

for all $h \in G$.

Note that $x_g|_{G \setminus \{g\}} = y_g|_{G \setminus \{g\}}$. Let $F \subset G$ be a finite set and choose $g \in G \setminus F$ (this is possible because $G$ is infinite). Then one has $x_g|_F = y_g|_F$ while

$$\tau(x_g)(1_G) = x_g(x_g(1_G)) = x_g(g) = g_0$$

and

$$\tau(y_g)(1_G) = y_g(y_g(1_G)) = y_g(g) = 1_G,$$
so that \( \tau(x_g)(1_G) \neq \tau(y_g)(1_G) \). It follows that there is no finite set \( F \subset G \) such that, for all \( x \in A^G \), the value of \( \tau(x) \) at \( 1_G \) only depends on the values of \( x |_F \). This shows that \( \tau \) is not a cellular automaton (cf. Remark 1.4.2(b)).

### 1.9 The Prodiscrete Uniform Structure

Let \( G \) be a group and let \( A \) be a set. The prodiscrete uniform structure on \( A^G \) is the product uniform structure obtained by taking the discrete uniform structure on each factor \( A \) of \( A^G = \prod_{g \in G} A \) (see Appendix B for definition and basic facts about uniform structures).

A base of entourages for the prodiscrete uniform structure on \( A^G \) is given by the sets \( W_\Omega \subset A^G \times A^G \), where

\[
W_\Omega = \{(x, y) \in A^G \times A^G : x|_\Omega = y|_\Omega\}
\]

and \( \Omega \) runs over all finite subsets of \( G \).

Observe that, using the notation introduced in (1.3), we have

\[
V(x, \Omega) = \{y \in A^G : (x, y) \in W_\Omega\}
\]

for all \( x \in A^G \).

The following statement gives a global characterization of cellular automata in terms of the prodiscrete uniform structure and the \( G \)-shift on \( A^G \).

**Theorem 1.9.1.** Let \( A \) be a set and let \( G \) be a group. Let \( \tau: A^G \to A^G \) be a map and equip \( A^G \) with its prodiscrete uniform structure. Then the following conditions are equivalent:

(a) \( \tau \) is a cellular automaton;

(b) \( \tau \) is uniformly continuous and \( G \)-equivariant.

**Proof.** Suppose that \( \tau: A^G \to A^G \) is a cellular automaton. We already know that \( \tau \) is \( G \)-equivariant by Proposition 1.4.4. Let us show that \( \tau \) is uniformly continuous. Let \( S \) be a memory set for \( \tau \). It follows from Lemma 1.4.7 that if two configurations \( x, y \in A^G \) coincide on \( gS \) for some \( g \in G \), then \( \tau(x)(g) = \tau(y)(g) \). Consequently, if the configurations \( x \) and \( y \) coincide on \( \Omega S = \{gs : g \in \Omega, s \in S\} \) for some subset \( \Omega \subset G \), then \( \tau(x) \) and \( \tau(y) \) coincide on \( \Omega \).

Observe that \( \Omega S \) is finite whenever \( \Omega \) is finite. Using the notation introduced in (1.19), we deduce that

\[
(\tau \times \tau)(W_\Omega S) \subset W_\Omega
\]

for every finite subset \( \Omega \) of \( G \). As the sets \( W_\Omega \), where \( \Omega \) runs over all finite subsets of \( G \), form a base of entourages for the prodiscrete uniform structure
on $A^G$, it follows that $\tau$ is uniformly continuous. This shows that (a) implies (b).

Conversely, suppose that $\tau$ is uniformly continuous and $G$-equivariant. Let us show that $\tau$ is a cellular automaton. Consider the subset $\Omega = \{1_G\} \subset G$. Since $\tau$ is uniformly continuous, there exists a finite subset $S \subset G$ such that $(\tau \times \tau)(W_S) \subset W_\Omega$. This means that $\tau(x)(1_G)$ only depends on the restriction of $x$ to $S$. Thus, there is a map $\mu: A^S \to A$ such that

$$\tau(x)(1_G) = \mu(x|_S)$$

for all $x \in A^G$. Using the $G$-equivariance of $\tau$, we get

$$\tau(x)(g) = [g^{-1}\tau(x)](1_G) = \tau(g^{-1}x)(1_G) = \mu((g^{-1}x)|_S)$$

for all $x \in A^G$ and $g \in G$. This shows that $\tau$ is a cellular automaton with memory set $S$ and local defining map $\mu$. Consequently, (b) implies (a). \(\square\)

Every uniformly continuous map between uniform spaces is continuous with respect to the associated topologies, and the converse is true when the source space is compact (Theorem B.2.3). The topology defined by the prodiscrete uniform structure on $A^G$ is the prodiscrete topology (see Example (1) in Sect. B.3). In the case when $A$ is finite, the prodiscrete topology on $A^G$ is compact by Tychonoff theorem (Theorem A.5.2). Thus Theorem 1.9.1 reduces to the Curtis-Hedlund theorem (Theorem 1.8.1) in this case.

Remark 1.9.2. Suppose that $G$ is countable and $A$ is an arbitrary set. Then the prodiscrete uniform structure (and hence the prodiscrete topology) on $A^G$ is metrizable. To see this, choose an increasing sequence

$$\emptyset = E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots$$

of finite subsets of $G$ such that $\bigcup_{n \geq 0} E_n = G$. Then the sets $W_{E_n}$, $n \geq 0$, form a base of entourages for the prodiscrete uniform structure on $A^G$. Consider now the metric $d$ on $A^G$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-\max_{n \geq 0} \{n \geq 0: x|_{E_n} = y|_{E_n}\}} & \text{if } x \neq y. \end{cases}$$

for all $x, y \in A^G$. Then we have

$$W_{E_n} = \{(x, y) \in A^G \times A^G : d(x, y) < 2^{-n+1}\}$$

for every $n \geq 0$. Consequently, $d$ defines the prodiscrete uniform structure on $A^G$.

Let $G$ be a group and let $A$ be a set. Let $H$ be a subgroup of $G$. Let us equip $A^{H \backslash G}$ with its prodiscrete uniform structure and $\text{Fix}(H) \subset A^G$ with the
uniform structure induced by the prodiscrete uniform structure on $A^G$. Recall from Proposition 1.3.7 that there is a natural bijection $\rho^* : A^{H\setminus G} \to A^G$ defined by $\rho^*(y) = y \circ \rho$, where $\rho : G \to H\setminus G$ is the canonical surjection.

**Proposition 1.9.3.** The map $\rho^* : A^{H\setminus G} \to \text{Fix}(H)$ is a uniform isomorphism.

**Proof.** For $g \in G$, let $\pi_g : A^G \to A$ and $\pi'_g : A^{H\setminus G} \to A$ denote the projection maps given by $x \mapsto x(g)$ and $y \mapsto y(\rho(g))$ respectively. Observe that $\pi_g \circ \rho^* = \pi'_g$ is uniformly continuous for all $g \in G$. This shows that $\rho^*$ is uniformly continuous. Similarly, the uniform continuity of $(\rho^*)^{-1}$ follows from the fact that $\pi'_g \circ (\rho^*)^{-1} = \pi_g|_{\text{Fix}(H)}$ is uniformly continuous for each $g \in G$. Consequently, $\rho^*$ is a uniform isomorphism. \(\Box\)

### 1.10 Invertible Cellular Automata

Let $G$ be a group and let $A$ be a set. One says that a cellular automaton $\tau : A^G \to A^G$ is invertible (or reversible) if $\tau$ is bijective and the inverse map $\tau^{-1} : A^G \to A^G$ is also a cellular automaton. This is equivalent to the existence of a cellular automaton $\sigma : A^G \to A^G$ such that $\tau \circ \sigma = \sigma \circ \tau = \text{Id}_{A^G}$. Thus, the set of invertible cellular automata over the group $G$ and the alphabet $A$ is exactly the group $\text{ICA}(G; A)$ consisting of all invertible elements of the monoid $\text{CA}(G; A)$.

**Theorem 1.10.1.** Let $A$ be a set and let $G$ be a group. Let $\tau : A^G \to A^G$ be a map and equip $A^G$ with its prodiscrete uniform structure. Then the following conditions are equivalent:

(a) $\tau$ is an invertible cellular automaton;

(b) $\tau$ is a $G$-equivariant uniform automorphism of $A^G$.

**Proof.** It is clear that the inverse map of a bijective $G$-equivariant map from $A^G$ onto itself is also $G$-equivariant. Therefore, the equivalence of conditions (a) and (b) follows from the characterization of cellular automata given in Theorem 1.9.1. \(\Box\)

Bijective cellular automata over finite alphabets are always invertible:

**Theorem 1.10.2.** Let $G$ be a group and let $A$ be a finite set. Then every bijective cellular automaton $\tau : A^G \to A^G$ is invertible.

**Proof.** Let $\tau : A^G \to A^G$ be a bijective cellular automaton. The map $\tau^{-1}$ is $G$-equivariant since $\tau$ is $G$-equivariant. On the other hand, $\tau^{-1}$ is continuous with respect to the prodiscrete topology by compactness of $A^G$. Consequently, $\tau^{-1}$ is a cellular automaton by Theorem 1.8.1. \(\Box\)
Example 1.10.3. Let $\mathbb{K}$ be a field. Let us take as the alphabet set the ring $A = \mathbb{K}[[t]]$ of all formal power series in one indeterminate $t$ with coefficients in $\mathbb{K}$. Thus, an element of $A$ is just a sequence $a = (k_i)_{i \in \mathbb{N}}$ of elements of $\mathbb{K}$ written in the form

$$a = k_0 + k_1 t + k_2 t^2 + k_3 t^3 + \cdots = \sum_{i \in \mathbb{N}} k_i t^i,$$

and the addition and multiplication of two elements $a = \sum_{i \in \mathbb{N}} k_i t^i$ and $b = \sum_{i \in \mathbb{N}} k_i' t^i$ are respectively given by $a + b = \sum_{i \in \mathbb{N}} (k_i + k_i') t^i$ and $ab = \sum_{i \in \mathbb{N}} k_i'' t^i$ with $k_i'' = \sum_{i_1 + i_2 = i} k_{i_1} k_{i_2}'$ for all $i \in \mathbb{N}$. We take $G = \mathbb{Z}$. Thus, a configuration $x \in A^G$ is a map $x: \mathbb{Z} \to \mathbb{K}[[t]]$. Consider the map $\tau: A^G \to A^G$ defined by

$$\tau(x)(n) = x(n) - tx(n + 1)$$

for all $x \in A^G$ and $n \in \mathbb{Z}$. Clearly $\tau$ is a cellular automaton admitting $S = \{0, 1\}$ as a memory set (the local defining map associated with $S$ is the map $\mu: A^S \to A$ defined by $\mu(x_0, x_1) = x_0 - tx_1$ for all $x_0, x_1 \in A$).

Let us show that $\tau$ is bijective. Consider the map $\sigma: A^G \to A^G$ given by

$$\sigma(x)(n) = x(n) + tx(n + 1) + t^2 x(n + 2) + t^3 x(n + 3) + \cdots$$

for all $x \in A^G$ and $n \in \mathbb{Z}$. Observe that $\sigma(x)(n) \in \mathbb{K}[[t]]$ is well defined by the preceding formula. In fact, if we develop $x(n) \in \mathbb{K}[[t]]$ in the form

$$x(n) = \sum_{i \in \mathbb{N}} x_{n,i} t^i \quad (n \in \mathbb{Z}, x_{n,i} \in \mathbb{K}),$$

then

$$\sigma(x)(n) = \sum_{i \in \mathbb{N}} \left( \sum_{j=0}^i x_{n+j,i-j} \right) t^i.$$

One immediately checks that $\sigma \circ \tau = \tau \circ \sigma = \text{Id}_{A^G}$. Therefore, $\tau$ is bijective with inverse map $\tau^{-1} = \sigma$.

Let us show that the map $\sigma: A^G \to A^G$ is not a cellular automaton.

Let $F$ be a finite subset of $\mathbb{Z}$ and choose an integer $M \geq 0$ such that $F \subset (-\infty, M]$. Consider the configuration $y$ defined by $y(n) = 0$ if $n \leq M$ and $y(n) = 1$ if $n \geq M + 1$, and the configuration $z$ defined by $z(n) = 0$ for all $n \in \mathbb{Z}$. Then $y$ and $z$ coincide on $F$. However, the value at 0 of $\sigma(y)$ is

$$\sigma(y)(0) = t^{M+1} + t^{M+2} + t^{M+3} + \cdots$$

while the value of $\sigma(z)$ at 0 is $\sigma(z)(0) = 0$. It follows that there is no finite subset $F \subset \mathbb{Z}$ such that $\sigma(x)(0)$ only depends on the restriction of $x \in A^G$. The following example shows that Theorem 1.10.2 becomes false if we omit the finiteness hypothesis on the alphabet set $A$. 
to $F$. This shows that $\sigma$ is not a cellular automaton. Consequently, $\tau$ is a bijective cellular automaton which is not invertible.

In the next proposition we show that invertibility is preserved under the operations of induction and restriction.

**Proposition 1.10.4.** Let $G$ be a group and let $A$ be a set. Let $H$ be a subgroup of $G$ and let $\tau \in \text{CA}(G, H; A)$. Let $\tau_H \in \text{CA}(H; A)$ denote the cellular automaton obtained by restriction of $\tau$ to $H$. Then the following conditions are equivalent:

(a) $\tau$ is invertible;
(b) $\tau_H$ is invertible.

Moreover, if $\tau$ is invertible, then $\tau^{-1} \in \text{CA}(G, H; A)$ and one has

$$\tau^{-1}_H = (\tau_H)^{-1}. \tag{1.20}$$

**Proof.** First recall from (1.16) the factorizations

$$A^G = \prod_{c \in G/H} A^c \quad \text{and} \quad \tau = \prod_{c \in G/H} \tau_c, \tag{1.21}$$

where $\tau_c: A^c \to A^c$ satisfies $\tau_c(\tilde{x}|_c) = (\tau(\tilde{x}))|_c$ for all $\tilde{x} \in A^G$.

Suppose that $\tau$ is invertible. Denote by $\sigma \in \text{CA}(G; A)$ the inverse cellular automaton $\tau^{-1}$. It follows from (1.21) that the map $\tau_c: A^c \to A^c$ is bijective for each $c \in G/H$ and that

$$\sigma = \prod_{c \in G/H} (\tau_c)^{-1}, \tag{1.22}$$

where $(\tau_c)^{-1}: A^c \to A^c$ is the inverse map of $\tau_c$. Let us show that $\sigma \in \text{CA}(G, H; A)$. Let $S \subset G$ be a memory set for $\sigma$. Let $\tilde{x} \in A^G$. It follows from (1.22) that $(\sigma(\tilde{x}))|_H = (\tau_H)^{-1}(\tilde{x}|_H)$. Thus, we have

$$\sigma(\tilde{x})(1_G) = (\sigma(\tilde{x}))|_H(1_G) = (\tau_H)^{-1}(\tilde{x}|_H)(1_G).$$

This shows that $\sigma(\tilde{x})(1_G)$ only depends on $\tilde{x}|_H$. Arguing as in the proof of Lemma 1.5.1, we deduce that $S \cap H$ is a memory set for $\sigma$. Indeed, suppose that two configurations $\tilde{x}, \tilde{y} \in A^G$ coincide on $S \cap H$. Consider the configuration $\tilde{z} \in A^G$ which coincide with $\tilde{x}$ on $S$ and with $\tilde{y}$ on $G \setminus S$. We have $\sigma(\tilde{x})(1_G) = \sigma(\tilde{z})(1_G)$ since $\tilde{x}$ and $\tilde{z}$ coincide on $S$. On the other hand, we have $\sigma(\tilde{y})(1_G) = \sigma(\tilde{z})(1_G)$ since $\tilde{y}$ and $\tilde{z}$ coincide on $H$. This implies $\sigma(\tilde{x})(1_G) = \sigma(\tilde{y})(1_G)$. Thus, there is a map $\mu: A^{S \cap H} \to A$ such that

$$\sigma(\tilde{x})(1_G) = \mu(\tilde{x}|_{S \cap H})$$

for all $\tilde{x} \in A^G$. By applying Proposition 1.4.6, it follows that $S \cap H$ is a memory set for $\sigma$. Since $S \cap H \subset H$, this shows that $\tau^{-1} = \sigma \in \text{CA}(G, H; A)$. 

Moreover, it follows from (1.22) that

\[(\tau^{-1})_H = \sigma_H = (\tau_H)^{-1}\]

which gives us (1.20).

The equivalence (a) ⇔ (b) is then an immediate consequence of Proposition 1.7.2 which tells us that the restriction map $CA(G, H; A) \to CA(H; A)$ is a monoid isomorphism. \(\square\)

### Notes

Cellular automata were introduced by J. von Neumann (see [vNeu2]) who used them to describe theoretical models of self-reproducing machines. He first attempted to get such models by means of partial differential equations in $\mathbb{R}^3$. Later he changed the perspective and tried to use ideas and methods coming from robotics and electrical engineering. Eventually, in 1952, following a suggestion of S. Ulam, his former colleague at the Los Alamos Laboratories, he constructed a cellular automaton over the group $\mathbb{Z}^2$ with an alphabet consisting of 29 states. He then outlined the construction of a pattern, containing approximately 200,000 cells, which would reproduce itself. The details were later filled in by A.W. Burks in the 1960s [Bur].

The branch of mathematics which is concerned with the study of the dynamical properties of the shift action is known as *symbolic dynamics*. Many authors trace the birth of symbolic dynamics back to a paper published in 1898 by J. Hadamard [Had] in which words on two letters were used to code geodesics on certain surfaces with negative curvature. However, as it was pointed out by E.M. Coven and Z.W. Nitecki [CovN], Hadamard’s symbolic description of geodesics is purely static and involves only finite words. According to the authors of [CovN], the beginning of symbolic dynamics should be placed in a paper by G.A. Hedlund [Hed-1] published in 1944. Symbolic dynamics has important applications in dynamical systems, especially in the study of hyperbolic dynamical systems for which symbolic codings may be obtained from Markov partitions. One of the first examples of such an application was the use of the properties of the Thue-Morse sequence (see Exercise 3.41) by M. Morse [Mors] in 1921 to prove the existence of non-periodic recurrent geodesics on surfaces with negative curvature. A detailed exposition of symbolic dynamics over $\mathbb{Z}$ may be found for example in the books by B. Kitchens [Kit], by P. Kůrka [Kur], and by D. Lind and B. Marcus [LiM].

In the mid-1950s, Hedlund studied the so-called shift-commuting block maps which turn out to be exactly cellular automata over the group $\mathbb{Z}$. The Curtis-Hedlund theorem (Theorem 1.8.1), also called Curtis-Hedlund-Lyndon’s theorem or Hedlund’s theorem, is named after Hedlund [Hed-3].
who proved it in 1969. Its generalization to infinite alphabets, as stated as in Theorem 1.9.1, was proved by the authors in [CeC7].

Cellular automata were intensely studied from the 1960s, both by pure and applied mathematicians, under different names such as tessellation automata, parallel maps, cellular spaces, iterative automata, homogeneous structures, universal spaces, and sliding block codes (cf. [LiM, Section 1.1]). In most cases, these researches focused on cellular automata with finite alphabet over the groups $\mathbb{Z}$ or $\mathbb{Z}^2$ (see the surveys [BanMS], [BKM], [Kar3], [Wolfr3]).

The Game of Life was invented by the British mathematician J.H. Conway. This cellular automaton was described for the first time by M. Gardner [Gar-1] in the October 1970 issue of the Scientific American. From a theoretical computer science point of view, it is important because it has the power of a universal Turing machine, that is, anything that can be computed algorithmically can be computed by using the Game of Life.

In the 1980s, S. Wolfram [Wolfr1], [Wolfr2] started a systematic study and empirical classification of elementary cellular automata, that is, of cellular automata over $\mathbb{Z}$ with alphabet $A = \{0, 1\}$ and memory set $S = \{-1, 0, 1\}$. There are $2^{2^3} = 256$ such elementary cellular automata. Wolfram introduced a naming scheme for them which is nowadays widely used. Each elementary cellular automaton $\tau: A^\mathbb{Z} \to A^\mathbb{Z}$ is uniquely determined by the eight bit sequence

$$\mu(111)\mu(110)\mu(101)\mu(100)\mu(011)\mu(010)\mu(001)\mu(000) \in A^8,$$

where $\mu: A^S \to A$ is the associated local defining map. This bit sequence is the binary expansion of an integer in the interval $[0, 255]$, called the Wolfram number of $\tau$. For example, the majority action $\tau: A^\mathbb{Z} \to A^\mathbb{Z}$ associated with $S = \{-1, 0, 1\}$ (cf. Example 1.4.3(c)) is an elementary cellular automaton. Its local defining map $\mu$ gives

$$\mu(111)\mu(110)\mu(101)\mu(100)\mu(011)\mu(010)\mu(001)\mu(000) = 11101000$$

(cf. Fig. 1.7). It follows that the Wolfram number of $\tau$ is 232. Let us also mention that the elementary cellular automaton with Wolfram number 110 was recently proved computationally universal by M. Cook.

Wolfram introduced an empirical classification of elementary cellular automata into four classes according to the behavior of random initial configurations under iterations. These are known as Wolfram classes and are defined as follows:

(W1) Almost all initial configurations lead to the same uniform fixed-point configuration,

(W2) Almost all initial configurations lead to a periodic configuration,

(W3) Almost all initial configurations lead to chaos,

(W4) Localized structures with complex interactions emerge.

The survey paper [BKM] contains other dynamical classifications of cellular automata over $\mathbb{Z}$ with finite alphabet due to R. Gilman and to M. Hurley and P. Kůrka.
Invertible cellular automata are used to model time-reversible processes occurring in physics and biology. A group $G$ is called periodic if every element $g \in G$ has finite order. In [CeC11] it is shown that if $G$ is a non-periodic group, then for every infinite set $A$ there exists a bijective cellular automaton $\tau: A^G \to A^G$ which is not invertible (cf. Theorem 1.10.2 and Example 1.10.3). It was shown by Amoroso and Patt in 1972 [Amo] that it is decidable whether a given cellular automaton with finite alphabet over $\mathbb{Z}$ is invertible. This means that there exists an algorithm which establishes, after a finite number of steps, whether the cellular automaton corresponding to a given local defining map is invertible or not. On the other hand, J. Kari [Kar1], [Kar2], [Kar3] proved that the similar problem for cellular automata with finite alphabet over $\mathbb{Z}^d$, $d \geq 2$, is undecidable. Its proof is based on R. Berger’s undecidability result for the tiling problem of Wang tiles.

Exercises

1.1. An action of a group $\Gamma$ on a topological space $X$ is said to be topologically mixing if for each pair of nonempty subsets $U$ and $V$ of $X$ there exists a finite subset $F \subset \Gamma$ such that $U \cap \gamma V \neq \emptyset$ for all $\gamma \in \Gamma \setminus F$. Show that if $G$ is a group and $A$ is a set then the $G$-shift on $A^G$ is topologically mixing for the prodiscrete topology on $A^G$.

1.2. Let $G$ be a group and let $A$ be a set. Let $x \in A^G$ and let $\Omega_1$ and $\Omega_2$ be two subsets of $G$. Show that $V(x, \Omega_1 \cup \Omega_2) = V(x, \Omega_1) \cap V(x, \Omega_2)$ and $W_{\Omega_1 \cup \Omega_2} = W_{\Omega_1} \cap W_{\Omega_2}$ (see (1.3) and (1.19) for the definition of $V(x, \Omega)$ and $W_{\Omega}$).

1.3. Let $G$ be a countable group and let $A$ be a set. Show that the metric $d$ on $A^G$ introduced in Remark 1.9.2 is complete.

1.4. Let $G$ be an uncountable group and let $A$ be a set having at least two elements. Prove that the prodiscrete topology on $A^G$ is not metrizable. Hint: Prove that this topology does not satisfy the first axiom of countability.

1.5. Let $G$ be a group. Let $A$ and $B$ be two sets. Let $\tau_A: A^G \to A^G$ and $\tau_B: B^G \to B^G$ be cellular automata. For $x \in (A \times B)^G$, let $x_A \in A^G$ and $x_B \in B^G$ be the configurations defined by $x(g) = (x_A(g), x_B(g))$ for all $g \in G$. Show that the map $\tau: (A \times B)^G \to (A \times B)^G$ given by $\tau(x)(g) = (\tau_A(x_A)(g), \tau_B(x_B)(g))$ for all $g \in G$ is a cellular automaton.

1.6. Let $G$ be a group and let $S$ be a finite subset of $G$ of cardinality $k$. Let $A$ be a finite set of cardinality $n$. Show that there are exactly $n^k$ cellular automata $\tau: A^G \to A^G$ admitting $S$ as a memory set.
1.17. Let \( G = \mathbb{Z}^2 \) and \( A = \{0, 1\} \). Let \( \tau: A^G \to A^G \) denote the cellular automaton associated with the Game of Life. Let \( y \in A^G \) be the constant configuration defined by \( y(g) = 1 \) for all \( g \in G \) (all cells are alive). Find a configuration \( x \in A^G \) such that \( y = \tau(x) \).

1.18. Let \( A \) be a set and suppose that \( G \) is a trivial group. Show that the monoid \( CA(G; A) \) is canonically isomorphic to the monoid consisting of all maps from \( A \) to \( A \) (with composition of maps as the monoid operation). Also show that the group \( ICA(G; A) \) is canonically isomorphic to the symmetric group of \( A \).

1.19. Let \( G \) be a group and let \( A \) be a set. Let \( \tau: A^G \to A^G \) be a cellular automaton. Show that \( \tau \) admits a memory set which is reduced to a single element if and only if there exist an element \( s \in G \) and a map \( f: A \to A \) such that one has \( \tau(x)(g) = f(x(gs)) \) for all \( x \in A^G \) and \( g \in G \).

1.10. Prove that there are exactly 218 cellular automata \( \tau: \{0, 1\}^Z \to \{0, 1\}^Z \) whose minimal memory set is \( \{−1, 0, 1\} \).

1.11. Let \( \tau \in CA(\mathbb{Z}^2; \{0, 1\}) \) denote the cellular automaton associated with the Game of Life. Show that the minimal memory set of \( \tau \) is the set \( \{−1, 0, 1\}^2 \).

1.12. Let \( G \) be a group and let \( A \) be a set. Let \( \sigma, \tau \in CA(G; A) \). Let \( S_0 \) (resp. \( T_0 \), resp. \( C_0 \)) denote the minimal memory set of \( \sigma \) (resp. \( \tau \), resp. \( \sigma \circ \tau \)). Prove that \( C_0 \subset S_0 T_0 \). Give an example showing that this inclusion may be strict.

1.13. Let \( G \) be a group and let \( A \) be a set. Let \( H \) be a subgroup of \( G \) and let \( \tau \in CA(G, H; A) \). Show that \( \tau \) and \( \tau_H \) have the same minimal memory set.

1.14. Prove Proposition 1.4.9 by applying Theorem 1.9.1. Hint: Observe that the composite of two uniformly continuous maps is a uniformly continuous map.

1.15. Let \( G \) be a group and let \( A \) be a set. Let \( F \) be a nonempty finite subset of \( G \) and set \( B = A^F \). The sets \( A^G \) and \( B^G \) are equipped with their prodiscrete uniform structures and with the \( G \)-shift action. Show that the map \( \Phi_F: A^G \to B^G \) defined by \( \Phi_F(x)(g) = (g^{-1}x)|_F \) for all \( x \in A^G \) and \( g \in G \) is a \( G \)-equivariant uniform embedding.

1.16. Let \( G \) be a group, \( H \subset G \) a subgroup of \( G \), and let \( A \) be a set. Let \( H \setminus G = \{Hg : g \in G\} \) be the set of all right cosets of \( H \) in \( G \) and set \( B = A^{H \setminus G} \). The set \( A^G \) (resp. \( B^H \)) is equipped with its prodiscrete uniform structure and with the \( G \)-shift (resp. \( H \)-shift) action. Let \( T \subset G \) be a complete set of representatives for the right cosets of \( H \) in \( G \) so that \( G = \bigsqcup_{t \in T} Ht \). Show that the map \( \Psi = \Psi(H, T): A^G \to B^H \) defined by \( \Psi(x)(h)(Ht) = x(ht) \) for all \( x \in A^G \), \( h \in H \) and \( t \in T \) is an \( H \)-equivariant uniform isomorphism.
1.17. Let $G$ be a group and let $A$ be a set. For each $s \in G$, let $\tau_s: A^G \to A^G$ be the cellular automaton defined by $\tau_s(x)(g) = x(gs)$ for all $x \in A^G$, $g \in G$ (cf. Example 1.4.3(e)).

(a) Show that $\tau_s \in ICA(G; A)$ for every $s \in G$.

(b) Prove that the map $\Phi: G \to ICA(G; A)$ defined by $\phi(s) = \tau_s$ for all $s \in G$ is a group homomorphism.

(c) Prove that if $A$ has at least two elements, then $\Phi$ is injective but not surjective.

1.18. Let $G$ be a group and let $A$ be a set. Prove that the set consisting of all invertible cellular automata $\tau: A^G \to A^G$ admitting a memory set which is reduced to a single element is a subgroup of $ICA(G; A)$ isomorphic to the direct product $G \times \mathrm{Sym}(A)$.

1.19. Let $G = \mathbb{Z}$ and $A = \{0, 1\}$. Fix an integer $n \geq 3$ and let $S = \{-1, 0, 1, \ldots, n\}$. Consider the element $\alpha \in A^S$ (resp. $\beta \in A^S$) defined by $\alpha(-1) = \alpha(n) = 0$ and $\alpha(k) = 1$ for $0 \leq k \leq n - 1$ (resp. $\beta(-1) = \beta(0) = \beta(n) = 0$ and $\beta(k) = 1$ for $1 \leq k \leq n - 1$) and the map $\mu: A^S \to A$ defined by $\mu(\alpha) = 0$, $\mu(\beta) = 1$ and $\mu(y) = y(0)$ for $y \in A^S \setminus \{\alpha, \beta\}$. Let $\tau: A^G \to A^G$ be the cellular automaton with memory set $S$ and local defining map $\mu$.

(a) Show that $S$ is the minimal memory set of $\tau$.

(b) Show that $\tau$ is an invertible cellular automaton and that $\tau^{-1} = \tau$.

1.20. Show that the inverse map of the bijective cellular automaton $\tau: A^Z \to A^Z$ studied in Example 1.10.3 is discontinuous, with respect to the prodiscrete topology on $A^Z$, at every configuration $x \in A^Z$.

1.21. Let $G$ be a group and let $A$ be a set. A subshift of the configuration space $A^G$ is a subset $X \subset A^G$ which is $G$-invariant (i.e., such that $gx \in X$ for all $x \in X$ and $g \in G$) and closed in $A^G$ with respect to the prodiscrete topology.

(a) Show that $\varnothing$ and $A^G$ are subshifts of $A^G$.

(b) Show that if $x \in A^G$ then its orbit closure $\overline{Gx} \subset A^G$ is a subshift.

(c) Show that if $(X_i)_{i \in I}$ is a family of subshifts of $A^G$ then $\bigcap_{i \in I} X_i$ is a subshift of $A^G$.

(d) Show that if $(X_i)_{i \in I}$ is a finite family of subshifts of $A^G$ then $\bigcup_{i \in I} X_i$ is a subshift of $A^G$.

(e) Suppose that $A$ is finite. Show that if $X \subset A^G$ is a subshift and $\tau: A^G \to A^G$ is a cellular automaton then $\tau(X)$ is a subshift of $A^G$. Note: This last statement becomes false when $A$ is infinite (see Example 3.3.3).

1.22. Let $G$ be a group and let $A$ and $B$ be two sets. Let $f: A \to B$ be a map and consider the map $f_*: A^G \to B^G$ defined by $f_*(x) = f \circ x$ for all $x \in A^G$.

(a) Show that if $A$ is finite and $X$ is a subshift of $A^G$ then $f_*(X)$ is a subshift of $B^G$. Hint: Use the compactness of the configuration space $A^G$. 
Let \( f \) be the configurations \( x \) where (for forbidden patterns)

(b) Let \( G = \mathbb{Z}, A = \mathbb{Z}, B = \{0, 1\} \) and let \( f: A \to B \) be defined by \( f(n) = 0 \) if \( n < 0 \) and \( f(n) = 1 \) otherwise. Let \( X = \{ x_n : n \in \mathbb{Z} \} \subset A^\mathbb{Z} \)

where \( x_n(m) = n + m \) for all \( n, m \in \mathbb{Z} \). In other words, \( X \) is the \( \mathbb{Z} \)-orbit \( \mathbb{Z} x_0 \) of the configurations \( x_0 \). Show that \( X \) is a subshift of \( A^\mathbb{Z} \) but \( f^*(X) \) is not a subshift of \( B^\mathbb{Z} \).

1.23. Let \( G \) be a group and let \( A \) be a set. Given a set of patterns \( P \subset \bigcup A^\Omega \), where the union runs over all finite subsets \( \Omega \) of \( G \), we set

\[
X_P = \{ x \in A^G : (gx)|_\Omega \notin P \text{ for all } g \in G \text{ and all finite subsets } \Omega \subset G \},
\]

where \( (gx)|_\Omega \) denotes the restriction of the configuration \( gx \) to \( \Omega \).

(a) Let \( P \subset \bigcup A^\Omega \) be a set of patterns. Show that \( X_P \) is a subshift of \( A^G \).

(b) Conversely, show that if \( X \subset A^G \) is a subshift, then there exists a subset \( P \subset \bigcup A^\Omega \) such that \( X = X_P \). Such a set \( P \) is called a defining set of forbidden patterns for \( X \).

1.24. Let \( G \) be a group and let \( A \) be a set. Given a finite subset \( \Omega \subset G \) and a finite subset \( A \subset A^\Omega \) we set

\[
X(\Omega, A) = \{ x \in A^G : (gx)|_\Omega \in A \text{ for all } g \in G \}.
\]

(a) Let \( \Omega \subset G \) and \( A \subset A^\Omega \) be finite subsets. Show that \( X(\Omega, A) \) is a subshift of \( A^G \). A subshift \( X \subset A^G \) is said to be of finite type if there exists a finite subset \( \Omega \subset G \) and a finite subset \( A \subset A^\Omega \) such that \( X = X(\Omega, A) \). Such a set \( A \) is then called a defining set of admissible patterns for \( X \) and the subset \( \Omega \) is called a memory set for \( X \).

(b) Suppose that \( A \) is finite. Show that a subshift \( X \subset A^G \) is of finite type if and only if it admits a finite defining set of forbidden patterns.

1.25. Let \( G \) be a countable group and let \( A \) be a finite set. Show that there are at most countably many distinct subshifts \( X \subset A^G \) of finite type.

1.26. Let \( G \) be a group and let \( A \) be a finite set.

(a) Let \( \tau_1, \tau_2: A^G \to A^G \) be two cellular automata. Show that the set \( \{ x \in A^G : \tau_1(x) = \tau_2(x) \} \subset A^G \) is a subshift of finite type.

(b) Deduce from (a) that if \( \tau: A^G \to A^G \) is a cellular automaton then the set \( \text{Fix}(\tau) = \{ x \in A^G : \tau(x) = x \} \subset A^G \) is a subshift of finite type.

(c) Conversely, show that if \( X \subset A^G \) is a subshift of finite type then there exists a cellular automaton \( \tau: A^G \to A^G \) such that \( X = \text{Fix}(\tau) \).

1.27. Suppose that a group \( \Gamma \) acts continuously on a topological space \( Z \). One says that the action of \( \Gamma \) on \( Z \) is topologically transitive if for any pair of nonempty open subsets \( U \) and \( V \) of \( Z \) there exists an element \( \gamma \in \Gamma \) such that \( U \cap \gamma V \neq \emptyset \).

Let \( G \) be a group and let \( A \) be a set. A subshift \( X \subset A^G \) is said to be irreducible if for any finite subset \( \Omega \) of \( G \) and any two elements \( x_1, x_2 \in X \),
there exist a configuration \( x \in X \) and an element \( g \in G \) such that \( x|_\Omega = x_1|_\Omega \) and \( (gx)|_\Omega = x_2|_\Omega \).

Suppose that \( X \subset A^G \) is a subshift. Show that the action of \( G \) on \( X \) induced by the \( G \)-shift is topologically transitive if and only if \( X \) is irreducible.

1.28. Let \( G \) be a group and let \( A \) be a set. Let \( B \subset A \) and consider the subsets \( X,Y \subset A^G \) defined by \( X = \{ \bar{b} : b \in B \} \subset A^G \), where \( \bar{b} \) denotes the constant configuration given by \( \bar{b}(g) = b \) for all \( g \in G \), and \( Y = \{ y \in A^G : y(g) \in B \text{ for all } g \in G \} \).

(a) Show that \( X \) and \( Y \) are subshifts of \( A^G \).
(b) Show that if \( G \) is infinite then \( Y \) is irreducible.
(c) Suppose that \( B \) has at least two distinct elements. Show that \( X \) is not irreducible.

1.29. Let \( G \) be a group acting continuously on a nonempty complete metric space \( X \) whose topology satisfies the second axiom of countability (i.e., admitting a countable base of open subsets). Show that the following conditions are equivalent:

(i) the action of \( G \) on \( X \) is topologically transitive;
(ii) there is a point \( x \in X \) whose \( G \)-orbit is dense in \( X \);
(iii) there is a dense subset \( D \subset X \) such that the \( G \)-orbit of each point \( x \in D \) is dense in \( X \).

Hint: The implications (iii) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (i) are straightforward. To prove (i) \( \Rightarrow \) (iii), consider a sequence \( (U_n)_{n \in \mathbb{N}} \) of nonempty open subsets of \( X \) which form a base of the topology and denote by \( \Omega_n \) the set of points \( x \in X \) whose \( G \)-orbit meets \( U_n \). Then observe that each \( \Omega_n \) is open and dense subset of \( X \) if (i) is satisfied and apply Baire’s theorem (Theorem I.1.1, also cf. Remark I.1.2(ii)).

1.30. Let \( G \) be a countable group and let \( A \) be a countable (e.g. finite) set. Let \( X \subset A^G \) be a nonempty subshift. Show that the following conditions are equivalent:

(i) the subshift \( X \) is irreducible;
(ii) there is a configuration \( x \in X \) whose \( G \)-orbit is dense in \( X \);
(iii) there is a dense subset \( D \subset X \) such that the \( G \)-orbit of each configuration \( x \in D \) is dense in \( X \).

Hint: Use the results of Exercises 1.3 and 1.29.

1.31. Let \( G \) be a group and let \( A \) be a set. One says that a subshift \( X \subset A^G \) is topologically mixing if the action of \( G \) on \( X \) induced by the \( G \)-shift is topologically mixing (cf. Exercise 1.1).

(a) Let \( X \subset A^G \) be a subshift. Show that \( X \) is topologically mixing if and only if for any finite subset \( \Omega \) of \( G \) and any two configurations \( x_1,x_2 \in X \), there exists a finite subset \( F \subset G \) such that, for all \( g \in G \setminus F \), there exists a configuration \( x \in X \) satisfying \( x|_\Omega = x_1|_\Omega \) and \( (gx)|_\Omega = x_2|_\Omega \).
(b) Show that if \( G \) is infinite then every topologically mixing subshift \( X \subset A^G \) is irreducible.
1.32. Let $G$ be a group and let $A$ be a set. Let $\Delta \subset G$ be a finite subset. A subshift $X \subset A^G$ is said to be $\Delta$-irreducible if it satisfies the following condition: if $\Omega_1$ and $\Omega_2$ are two finite subsets of $G$ such that $\Omega_1$ and $\Omega_2\delta$ are disjoint for all $\delta \in \Delta$, then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ which satisfies $x|\Omega_1 = x_1|\Omega_1$ and $x|\Omega_2 = x_2|\Omega_2$. A subshift $X \subset A^G$ is said to be strongly irreducible if there exists a finite subset $\Delta \subset G$ such that $X$ is $\Delta$-irreducible.

(a) Show that the subshift $Y \subset A^G$ described in Exercise 1.28 is $\{1_G\}$-irreducible and therefore strongly irreducible.

(b) Show that every strongly irreducible subshift is topologically mixing.

1.33. Let $G$ be a group and let $A$ be a set. Let $H$ be a subgroup of $G$. For $x \in A^G$ and $g \in G$, denote by $x_g^H \in A^H$ the configuration defined by

$$x_g^H = (gx)|_H.$$

(a) Check that $hx_g^H = x_{hg}^H$ for all $x \in A^G$, $h \in H$ and $g \in G$.

(b) Let $X \subset A^H$ be a subshift. Show that the set $X^{(G)}$ defined by $X^{(G)} = \{x \in A^G : x_g^H \in X \text{ for all } g \in G\}$ is a subshift of $A^G$.

(c) Show that if $H \neq G$ then the subshift $X^{(G)} \subset A^G$ is irreducible for any subshift $X \subset A^H$.

(d) Let $X \subset A^H$ be a subshift. Show that $X^{(G)} \subset A^G$ is of finite type (resp. topologically mixing, resp. strongly irreducible) if and only if $X$ is of finite type (resp. topologically mixing, resp. strongly irreducible).

(e) Suppose that $\sigma : A^H \rightarrow A^H$ is a cellular automaton and let $\sigma^G : A^G \rightarrow A^G$ be the induced cellular automaton (cf. Sect. 1.7). Check that one has

$$\sigma^G(x)_g = \sigma(x_g^H)$$

for all $x \in A^G$ and $g \in G$.

(f) Show that if $X \subset A^H$ is a subshift such that $\sigma(X) \subset X$ then one has $\sigma^G(X^{(G)}) \subset X^{(G)}$.

1.34. Let $G$ be a group and let $A$ be a set. Let $F$ be a nonempty finite subset of $G$ and consider the map $\Phi_F : A^G \rightarrow B^G$ defined in Exercise 1.15, where $B = A^F$. Let $X \subset A^G$ be a subshift and set $X^{[F]} = \Phi_F(X)$.

(a) Show that $X^{[F]}$ is a subshift of $B^G$.

(b) Show that $X$ is irreducible (resp. topologically mixing, resp. strongly irreducible) if and only if $X^{[F]}$ is irreducible (resp. topologically mixing, resp. strongly irreducible).

(c) Show that if $X$ is of finite type then $X^{[F]}$ is of finite type.

1.35. Let $G$ be a group, $H \subset G$ a subgroup of $G$, and let $A$ be a set. Let also $T \subset G$ be a complete set of representatives for the right cosets of $H$ in $G$ and consider the map $\Psi : A^G \rightarrow B^H$ defined in Exercise 1.16, where $B = A^{H \setminus G}$. Let $X \subset A^G$ be a subshift and set $X^{(H,T)} = \Psi(X)$.

(a) Show that $X^{(H,T)}$ is a subshift of $B^H$.

(b) Show that if $X$ is of finite type then $X^{(H,T)}$ is of finite type.

1.36. Let $A$ be a set. Let $A^*$ denote the monoid consisting of all words in the alphabet $A$ (cf. Sect. D.1). Recall that any word $w \in A^*$ can be uniquely
Let and irreducible.

The subshift $X$ is strongly irreducible (and therefore topologically mixing and irreducible). Exercises 35

The integer $n$ is called the length of the word $w$ and it is denoted by $\ell(w)$. In the sequel, we shall identify the word $w = a_1a_2\cdots a_n$ with the pattern $p: \{1,2,\ldots,n\} \rightarrow A$ defined by $p(i) = a_i$ for $1 \leq i \leq n$. Given a subshift $X \subset A^\mathbb{Z}$ and an integer $n \geq 0$, we denote by $L_n(X) \subset A^n$ the set consisting of all words $w \in A^n$ for which there exists an element $x \in X$ such that $w = x(1)x(2)\cdots x(n)$. The set $L(X) = \cup_{n \in \mathbb{N}} L_n(X)$ is called the language of $X$. The elements $w \in L(X)$ are called the admissible words of $X$ (or, simply, the $X$-admissible words). The elements $w \in A^n \setminus L(X)$ are called the forbidden words of $X$.

(a) Let $X$ and $Y$ be two subshifts of $A^\mathbb{Z}$. Show that one has $X \subset Y$ (resp. $X = Y$) if and only if $L(X) \subset L(Y)$ (resp. $L(X) = L(Y)$).

(b) One says that a word $u \in A^*$ is a subword of a word $w \in A^*$ if there exist $v_1,v_2 \in A^*$ such that $w = v_1uv_2$. Let $X \subset A^\mathbb{Z}$ be a subshift and let $L = L(X)$. Show that $L$ satisfies the following conditions:

(i) if $w \in L$, then $u \in L$ for every subword $u$ of $w$;

(ii) if $w \in L$, then there exist $a,a' \in A$ such that $awa' \in L$.

(c) Conversely, show that if a subset $L \subset A^*$ satisfies conditions (i) and (ii), then there exists a unique subshift $X \subset A^\mathbb{Z}$ such that $L = L(X)$.

1.37. Let $A$ be a set and let $X \subset A^\mathbb{Z}$ be a subshift.

(a) Show that $X$ is of finite type if and only if the following holds: there exists an integer $n_0 \geq 0$ such that if the words $u,v,w \in A^*$ satisfy $\ell(v) \geq n_0$ and $uv,vw \in L(X)$, then one has $uvw \in L(X)$.

(b) Show that $X$ is irreducible if and only if for every pair of words $u$ and $v$ in $L(X)$, there exists a word $w \in A^*$ such that $uvw \in L(X)$.

(c) Show that $X$ is topologically mixing if and only if the following holds: for every pair of words $u$ and $v$ in $L(X)$, there exists an integer $n_0 \geq 0$ such that for every integer $n \geq n_0$ there exists a word $w \in A^n$ of length $\ell(w) = n$ satisfying $uvw \in L(X)$.

(d) Show that $X$ is strongly irreducible if and only if the following holds: there exists an integer $n_0 \geq 0$ such that, for every pair of words $u$ and $v$ in $L(X)$, and for every integer $n \geq n_0$, there exists a word $w \in A^n$ of length $\ell(w) = n$ such that $uvw \in L(X)$.

1.38. Let $A = \{0,1\}$ and let $X \subset A^\mathbb{Z}$ be the set of all $x \in A^\mathbb{Z}$ such that the following holds: if $x(n) = 1, x(n+1) = x(n+2) = \cdots = x(n+k) = 0, x(n+k+1) = 1$, for some $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, then $k$ is even.

(a) Show that $X$ is a subshift (it is called the even subshift).

(b) Show that $X$ is not of finite type.

(c) Show that $X$ is strongly irreducible (and therefore topologically mixing and irreducible).

1.39. Let $A = \{0,1\}$ and consider the subshift of finite type $X \subset A^\mathbb{Z}$ defined by $X = X_{\{11\}} = \{x \in A^\mathbb{Z} : (x(n), x(n+1)) \neq (1,1) \text{ for all } n \in \mathbb{Z}\}$. Show that $X$ is strongly irreducible (and therefore topologically mixing and irreducible). The subshift $X$ is called the golden mean subshift.
1.40. Let $A = \{0, 1\}$ and let $X \subset A^\mathbb{Z}$ be the set consisting of the two configurations $x, y \in A^\mathbb{Z}$ defined by

$$x(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad y(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{Z}$. Show that $X$ is an irreducible subshift of finite type which is not topologically mixing (and therefore not strongly irreducible either).

1.41. Let $A$ be a set. Let $X \subset A^\mathbb{Z}$ be a subshift of finite type. Show that $X$ is topologically mixing if and only if $X$ is strongly irreducible.

1.42. Let $A = \{0, 1\}$ and let $X \subset A^\mathbb{Z}$ be the subshift with defining set of forbidden words $\{01^k0^h1 : 1 \leq h \leq k, k = 1, 2, \ldots\}$. Show that $X$ is topologically mixing (and therefore irreducible) but not strongly irreducible.

1.43. Cellular automata between subshifts. Let $G$ be a group and let $A$ be a set. The set $A^G$ is equipped with its prodiscrete uniform structure and with the $G$-shift action. Let $X, Y \subset A^G$ be two subshifts and $\tau: X \to Y$ a map. Then the following are equivalent:

(i) there exists a cellular automaton $\tau: A^G \to A^G$ such that $\tau(x) = \tau(x)$ for all $x \in X$;

(ii) $\tau$ is $G$-equivariant and uniformly continuous.

One says that $\tau: X \to Y$ is a cellular automaton if the two equivalent conditions above are satisfied.

1.44. Let $G$ be a group and let $A$ be a finite set. Let $\tau: X \to Y$ be a map between subshifts $X, Y \subset A^G$. Show that the following conditions are equivalent:

(i) $\tau$ is a cellular automaton,

(ii) $\tau$ is $G$-equivariant and continuous (with respect to the topologies induced on $X$ and $Y$ by the prodiscrete topology on $A^G$).

1.45. Let $G$ be a group and let $A$ be a finite set. Let $\tau: A^G \to A^G$ be a cellular automaton and let $X \subset A^G$ be an irreducible (resp. topologically mixing, resp. strongly irreducible) subshift. Show that $\tau(X)$ is an irreducible (resp. topologically mixing, resp. strongly irreducible) subshift of $A^G$. 
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