Algorithms for Finding Optimal Flows in Dynamic Networks

Maria Fonoberova

Abstract This article presents an approach for solving some power systems problems by using optimal dynamic flow problems. The classical optimal flow problems on networks are extended and generalized for the cases of nonlinear cost functions on arcs, multicommodity flows, and time- and flow-dependent transactions on arcs of the network. All parameters of networks are assumed to be dependent on time. The algorithms for solving such kind of problems are developed by using special dynamic programming techniques based on the time-expanded network method together with classical optimization methods.

Keywords Dynamic networks · Minimum cost flow problem · Multicommodity flows · Network flows · Optimal flows

Introduction

This chapter is addressed to the elaboration of methods and algorithms for determining optimal single-commodity and multicommodity dynamic network flows, which are widely used for studying and solving a large class of practical problems, including power systems problems as well as some theoretical problems. Transportation, production and distribution, scheduling, telecommunication, management, and many other problems can be formulated and solved as optimal dynamic flow problems. Dynamic flows can be used to solve problems related to the power transmission or the optimal scheduling of generation resources in power systems as well as problems related to the coordination of operations. Power systems problems are very difficult to solve because of these systems dimensions, complexity, and their dependence on many factors. This article presents an approach to solve some power systems problems by using optimal dynamic flow problems.

M. Fonoberova
Aimdyn Inc, 1919 State St, Santa Barbara, CA 93101, USA
e-mail: mfonoberova@aimdyn.com

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In many practical optimization problems, the factor time is a key ingredient to the problem formulation. In this paper the time-varying flow models that capture the essential properties of flows arising in real-life applications are studied. We consider minimum cost flow problems on networks with demand-supply and capacity functions that depend on time, and cost functions that are nonlinear and depend on both time and flow. We also investigate the dynamic model with transit time functions that depend on the amount of flow and the entering time-moment of flow in the arc. These problems generalize the classical flow problems on static networks and extend some known dynamic optimization models on networks (Ahuja et al. 1993; Aronson 1989; Assad 1978; Cai et al. 2001; Carey and Subrahmanian 2000; Fleischer 2001a; Fleischer 2001b; Fleisher and Skutella 2002; Ford and Fulkerson 1958; Ford and Fulkerson 1962; Glockner and Nemhauser 2002; Hoppe and Tardos 2000; Klinz and Woeginger 1995, 1998; Ma et al. 2004; Pardalos and Guisewite 1991; Powell et al. 1995). To solve the considered dynamic flow problems, we elaborate the corresponding algorithms on the basis of the time-expanded network method.

1 Optimal Dynamic Network Flow Models and Power Industry

The optimal dynamic network flow models can be used for solving various problems in the power industry, including problems related to power generation, transmission, and distribution. To operate a power system, a lot of characteristic and parameters have to be taken into account, among which are equipment capabilities, time factor, optimal cost, and many others (Weber 2005; Wood and Wollenberg 1996). The proposed optimal dynamic flow models can provide an adequate framework for such problems.

A power system is an integrated system consisting of generating plants, transmission lines, distribution facilities, and other facilities that must operate simultaneously in real-time or in a synchronous manner to provide power to consumers (Chambers and Kerr 1996; Denny and Dismukes 2002; McDonald et al. 1997; Pansini 2005). The operation and control of such systems is very complex because of their huge dimensions as well as many interconnected factors that can influence the functionality of the systems (Contreras et al. 2002; Cook et al. 2000; Rajan 2006).

Optimal dynamic flows can be used for the optimal scheduling of generating resources to meet anticipated demand, what includes the scheduling of water, fossil fuels as well as equipment maintenance, and other factors (Batut and Renaud 1992; Feltenmark and Lindberg 1997). With the expansion of power systems and the increase of the number and size of generating units, the higher operability can be attained through properly established interconnections between different components of the system, which can be done by using dynamic network flows.

Optimal dynamic flow models can be a good choice for solving problems related to power transmission from generating stations or power plants to major load centers as well as further distribution of the power to final consumers (Kersting 2006; Kim and Baldick 1997; Short 2003; Weber et al. 2007; Willis 2004; Willis et al. 2000). The power transmission networks have multiple redundant lines between points so that power can be routed through a variety of routes from any power plant to any load center. The transmission problems occupy a very important place in the power industry, because in many cases the capital cost of electric power stations is so high and electric demand is so variable that it is cheaper to import some portion of the variable load than to generate it locally. Wide area transmission grids span across countries and even large portions of continents. The electricity cost can be reduced by allowing multiple generating plants to be interconnected over a large area. However, in this case various techniques have to be applied to provide the functionality of the whole system as well as the efficient and feasible transmission taking into account economic factors, network safety, and redundancy. Optimal dynamic flow problems can be used to find the best route or to minimize the total cost of the transmission. The dynamic flow problems with flow storage at nodes can be used to match the loads with the generation by using the buffering capability.

In general, problems of power generation, transmission, and distribution can be regarded as generalization of optimal multicommodity flow problems. Different producers negotiate with different consumers for the power. It is impossible to model each of such negotiation separately, because of the nature of the problem, and so we have to consider the picture in the whole. Multicommodity dynamic flows can also be used for solving problems in the interconnection between members of power pools. For example, many electric utilities in the contiguous United States and a large part of Canada operate as members of power pools. Each individual utility in such pools operates independently, but has contractual arrangements with other members.

2 Minimum Cost Dynamic Single: Commodity Flow Problems and Algorithms for Their Solving

In this section we formulate and study minimum cost dynamic flow problems on networks with nonlinear cost functions, which depend on flow and time, and demand-supply functions and capacity functions, which depend on time. To solve the considered problems we propose algorithms based on the reduction of dynamic problems to static ones on auxiliary networks. We analyze dynamic problems with transit time functions that depend on flow and time and elaborate algorithms for solving such problems.

2.1 The Minimum Cost Dynamic Flow Problem Formulation

A dynamic network $N = (V, E, \tau, d, u, \varphi)$ is determined by directed graph $G = (V, E)$ with set of vertices V, $|V| = n$, and set of arcs E, $|E| = m$, transit time function $\tau\colon E \to R_+$, demand-supply function $d\colon V \times \mathbb{T} \to R$, capacity function $u\colon E \times \mathbb{T} \to R_+$, and cost function $\varphi\colon E \times R_+ \times \mathbb{T} \to R_+$. It is considered the discrete time model, in which all times are integral and bounded by horizon T. Time horizon is the time until which the flow can travel in the network, and it defines the set $\mathbb{T} = \{0, 1, \ldots, T\}$ of the considered time moments. Time is measured in discrete steps, so that if one unit of flow leaves vertex z at time t on arc $e = (z, v)$, one unit of flow arrives at vertex v at time $t + \tau_e$, where τ_e is the transit time on arc e. The continuous flow model formulations can be found in Fleisher (2000); Fleischer (2001a); Fleisher and Skutella (2002).

In order for the flow to exist, it is required that $\sum_{t\in\mathbb{T}} \sum_{v\in V} d_v(t) = 0$. If for an arbitrary node $v \in V$ at a moment of time $t \in \mathbb{T}$, the condition $d_v(t) > 0$ holds, then this node v at the time-moment t is treated as a source. If at a moment of time $t \in \mathbb{T}$ the condition $d_v(t) < 0$ holds, then the node v at the time-moment t is regarded as a sink. In the case $d_v(t) = 0$ at a moment of time $t \in \mathbb{T}$, the node v at the time-moment t is considered as an intermediate node. In such a way, the same node $v \in V$ at different moments of time can serve as a source, a sink, or an intermediate node.

Without losing generality we consider that the set of vertices V is divided into three disjoint subsets V_+, V_-, V_*, such that we have the following:

V_+ consists of nodes $v \in V$, for which $d_v(t) \geq 0$ for $t \in \mathbb{T}$ and there exists at least one moment of time $t_0 \in \mathbb{T}$ such that $d_v(t_0) > 0$

V_- consists of nodes $v \in V$, for which $d_v(t) \leq 0$ for $t \in \mathbb{T}$ and there exists at least one moment of time $t_0 \in \mathbb{T}$ such that $d_v(t_0) < 0$

V_* consists of nodes $v \in V$, for which $d_v(t) = 0$ for every $t \in \mathbb{T}$

So, V_+ is a set of sources, V_- is a set of sinks, and V_* is a set of intermediate nodes of the network N.

A feasible dynamic flow in network N is a function $x\colon E \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{e\in E^-(v)} x_e(t) - \sum_{\substack{e\in E^+(v)\\ t-\tau_e\geq 0}} x_e(t-\tau_e) = d_v(t),\ \forall\, t \in \mathbb{T},\ \forall\, v \in V; \tag{1}$$

$$0 \leq x_e(t) \leq u_e(t),\quad \forall\, t \in \mathbb{T},\ \forall\, e \in E; \tag{2}$$

$$x_e(t) = 0,\ \forall\, e \in E,\ t = \overline{T - \tau_e + 1, T}, \tag{3}$$

where $E^-(v) = \{(v, z) \mid (v, z) \in E\},\quad E^+(v) = \{(z, v) \mid (z, v) \in E\}$.

Here the function x defines the value $x_e(t)$ of flow entering arc e at time t. The flow does not enter arc e at time t if it has to leave the arc after time T; this is ensured by condition (3). Restrictions (2) are capacity constraints. Conditions (1) represent flow conservation constraints.

To model transit costs, which may change over time, it is defined as the cost function $\varphi_e(x_e(t), t)$ with the meaning that flow of value $\rho = x_e(t)$ entering arc e at time t will incur a transit cost of $\varphi_e(\rho, t)$. It is assumed that $\varphi_e(0, t) = 0$ for all $e \in E$ and $t \in \mathbb{T}$.

The total cost of the dynamic flow x in the network N is defined as follows:

$$F(x) = \sum_{t \in \mathbb{T}} \sum_{e \in E} \varphi_e(x_e(t), t). \tag{4}$$

The minimum cost dynamic flow problem consists in finding a feasible dynamic flow that minimizes the objective function (4).

2.2 The Algorithm for Solving the Minimum Cost Dynamic Flow Problem

In the following, the approach based on the reduction of the dynamic problem to a corresponding static problem is proposed to solve the formulated above problem. It is shown that the minimum cost dynamic flow problem on network $N = (V, E, \tau, d, u, \varphi)$ can be reduced to a minimum cost static flow problem on an auxiliary time-expanded network $N^T = (V^T, E^T, d^T, u^T, \varphi^T)$. The advantage of such an approach is that it turns the problem of determining an optimal flow over time into a classical network flow problem.

The essence of the time-expanded network is that it contains a copy of the vertex set of the dynamic network for each moment of time $t \in \mathbb{T}$, and the transit times and flows are implicit in arcs linking those copies. The network N^T is defined as follows:

1. $V^T \colon = \{v(t) \,|\, v \in V,\ t \in \mathbb{T}\}$
2. $E^T \colon = \{e(t) = (v(t), z(t + \tau_e)) \,|\, e \in E,\ 0 \le t \le T - \tau_e\}$
3. $d_{v(t)}{}^T \colon = d_v(t)$ for $v(t) \in V^T$
4. $u_{e(t)}{}^T \colon = u_e(t)$ for $e(t) \in E^T$
5. $\varphi_{e(t)}{}^T (x_{e(t)}{}^T) \colon = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$

In the following we construct the time-expanded network N^T for the dynamic network N given in Fig. 1. The set of time moments is $\mathbb{T} = \{0, 1, 2, 3\}$. The transit times on each arc are as follows: $\tau_{e_1} = 1$, $\tau_{e_2} = 1$, and $\tau_{e_3} = 2$. The capacity, demand-supply, and cost functions are considered to be given. The constructed time-expanded network is presented in Fig. 2.

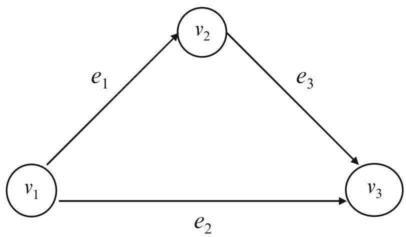

Fig. 1 The dynamic network N

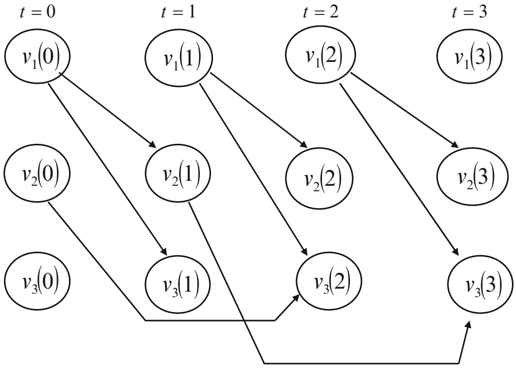

Fig. 2 The time-expanded network N^T

The correspondence between feasible flows in the dynamic network N and feasible flows in the time-expanded network N^T is stated in the following way. Let $x_e(t)$ be a flow in the dynamic network N, then the function x^T is defined as follows:

$$x_{e(t)}{}^T = x_e(t), \ \forall \, e(t) \in E^T, \tag{5}$$

which represents a flow in the time-expanded network N^T.

Lemma 1. *The correspondence (5) is a bijection from the set of feasible flows in the dynamic network N onto the set of feasible flows in the time-expanded network N^T.*

Proof. It is obvious that the correspondence (5) is a bijection from the set of T-horizon functions in the dynamic network N onto the set of functions in the time-expanded network N^T. In the following we have to show that each dynamic flow in the dynamic network N is put into the correspondence with a static flow in the time-expanded network N^T and vice-versa.

Let $x_e(t)$ be a dynamic flow in N, and let $x_{e(t)}{}^T$ be a corresponding function in N^T. Let us prove that $x_{e(t)}{}^T$ satisfies the conservation constraints in the static network N^T. Let $v \in V$ be an arbitrary vertex in N and t, $0 \le t \le T - \tau_e$, an arbitrary moment of time:

$$\begin{aligned} d_v(t) &\overset{(i)}{=} \sum_{e \in E^-(v)} x_e(t) - \sum_{\substack{e \in E^+(v) \\ t-\tau_e \ge 0}} x_e(t-\tau_e) = \\ &= \sum_{e(t) \in E^-(v(t))} x_{e(t)}{}^T - \sum_{e(t-\tau_e) \in E^+(v(t))} x_{e(t-\tau_e)}{}^T \overset{(ii)}{=} d_{v(t)}{}^T. \end{aligned} \tag{6}$$

Note that according to the definition of the time-expanded network, the set of arcs $\{e(t-\tau_e) | e(t-\tau_e) \in E^+(v(t))\}$ consists of all arcs that enter $v(t)$, while the set of arcs $\{e(t) | e(t) \in E^-(v(t))\}$ consists of all arcs that originate from $v(t)$. Therefore, all necessary conditions are satisfied for each vertex $v(t) \in V^T$. Hence, $x_{e(t)}{}^T$ is a flow in the time-expanded network N^T.

Let ${x_{e(t)}}^T$ be a static flow in the time-expanded network N^T and let $x_e(t)$ be a corresponding function in the dynamic network N. Let $v(t) \in V^T$ be an arbitrary vertex in N^T. The conservation constraints for this vertex in the static network are expressed by equality (ii) from (6), which holds for all $v(t) \in V^T$ at all times t, $0 \leq t \leq T - \tau_e$. Therefore, equality (i) holds for all $v \in V$ at all moments of time t, $0 \leq t \leq T - \tau_e$. In such a way $x_e(t)$ is a flow in the dynamic network N.

It is easy to verify that a feasible flow in the dynamic network N is a feasible flow in the time-expanded network N^T and vice-versa. Indeed,

$$0 \leq {x_{e(t)}}^T = x_e(t) \leq u_e(t) = {u_{e(t)}}^T.$$

The lemma is proved. □

Theorem 1. *If x is a flow in the dynamic network N and x^T is a corresponding flow in the time-expanded network N^T, then*

$$F(x) = F^T(x^T),$$

where

$$F^T(x^T) = \sum_{t \in \mathrm{T}} \sum_{e(t) \in E^T} {\varphi_{e(t)}}^T({x_{e(t)}}^T)$$

is the total cost of the static flow x^T in the time-expanded network N^T.

Moreover, for each minimum cost flow x^* in the dynamic network N, there is a corresponding minimum cost flow x^{*T} in the static network N^T such that

$$F(x^*) = F^T(x^{*T})$$

and vice-versa.

Proof. Let $x : E \times T \to R_+$ be an arbitrary dynamic flow in the dynamic network N. Then according to Lemma 1, the unique flow x^T in N^T corresponds to the flow x in N, and therefore we have

$$F(x) = \sum_{t \in \mathrm{T}} \sum_{e \in E} \varphi_e(x_e(t), t) = \sum_{t \in \mathrm{T}} \sum_{e(t) \in E^T} {\varphi_{e(t)}}^T({x_{e(t)}}^T) = F^T(x^T).$$

So, the first part of the theorem is proved.

To prove the second part of the theorem, we again use Lemma 1. Let $x^* : E \times T \to R_+$ be the optimal dynamic flow in N and x^{*T} be the corresponding optimal flow in N^T. Then

$$F(x^*) = \sum_{t \in \mathrm{T}} \sum_{e \in E} \varphi_e(x_e^*(t), t) = \sum_{t \in \mathrm{T}} \sum_{e(t) \in E^T} {\varphi_{e(t)}}^T({x_{e(t)}^*}^T) = F^T(x^{*T}).$$

The converse proposition is proved in an analogous way. □

The following algorithm for solving the minimum cost dynamic flow problem can be proposed:

1. To build the time-expanded network N^T for the dynamic network N
2. To solve the classical minimum cost flow problem on the static network N^T (Ahuja et al. 1993; Bland and Jensen 1985; Ermoliev and Melnic 1968; Goldberg and Tarjan 1987a,b; Hu 1970; Papadimitrou and Steiglitz 1982)
3. To reconstruct the solution of the static problem on the network N^T to the dynamic problem on the network N

Building the time-expanded network and reconstructing the solution of the minimum cost static flow problem to the dynamic one has complexity $O(nT + mT)$. The complexity of step 2 depends on the complexity of the algorithm used for the minimum cost flow problem on static networks. If such an algorithm has complexity $O(f(n', m'))$, where n' is a number of vertices and m' is a number of arcs in the network, then the complexity of solving the minimum cost flow problem on the time-expanded network employing the same algorithm is $O(f(nT, mT))$.

Some specific algorithms are proposed in Lozovanu and Fonoberova (2006) to minimize the size of the auxiliary static network. In the case of uncapacitated dynamic networks with cost functions that are concave with regard to flow value and do not change over time, the problem can be reduced to the minimum cost flow problem on a static network of equal size, not the time-expanded network.

2.3 The Dynamic Model with Flow Storage at Nodes

The previous mathematical model can be extended for the case with flow storage at nodes if we associate a transit time τ_v to each node $v \in V$, which means that the flow passage through this node takes τ_v units of time. If in addition we associate the capacity function $u_v(t)$ and the cost function $\varphi_v(x_v(t), t)$ to each node v, a more general model can be obtained. In this case the problem can be reduced to the previous one by simple transformation of the network where each node v is changed by a couple of vertices v' and v'' connected with directed arc $e_v = (v', v'')$. Here v' preserves all entering arcs and v'' preserves all leaving arcs of the previous network. The transit time $\tau_{e_v} = \tau_v$, the cost function $\varphi_{e_v}(x_{e_v}(t), t) = \varphi_v(x_v(t), t)$, and the capacity function $u_{e_v}(t) = u_v(t)$ are associated to arc e_v.

An important particular case of the minimum cost dynamic flow problem is the one when all amount of flow is dumped into the network from sources $v \in V_+$ at the time-moment $t = 0$ and it arrives at sinks $v \in V_-$ at the time-moment $t = T$. This means that the supply-demand function $d : V \times T \to R$ satisfies the conditions

(a) $d_v(0) > 0$, $d_v(t) = 0$, $t = 1, 2, \ldots, T$, for $v \in V_+$

(b) $d_v(T) < 0$, $d_v(t) = 0$, $t = 0, 1, 2, \ldots, T - 1$, for $v \in V_-$

So let us consider the minimum cost flow problem on the dynamic network with flow storage at nodes and integral constant demand-supply functions. Let

$N = (V, E, \tau, d, u, \varphi)$ be a given dynamic network, where the demand-supply function $d : V \to R$ does not depend on time. Without losing generality, we assume that no arcs enter sources or exit sinks. In order for a flow to exist, supply must equal demand: $\sum_{v \in V} d_v = 0$.

The mathematical model of the minimum cost flow problem on this dynamic network is the following:

$$\sum_{e \in E^-(v)} \sum_{t=0}^{T} x_e(t) - \sum_{e \in E^+(v)} \sum_{t=\tau_e}^{T} x_e(t - \tau_e) = d_v, \ \forall\, v \in V; \tag{7}$$

$$\sum_{e \in E^-(v)} \sum_{t=0}^{\theta} x_e(t) - \sum_{e \in E^+(v)} \sum_{t=\tau_e}^{\theta} x_e(t - \tau_e) \le 0, \ \forall v \in V_*, \ \forall \theta \in \mathbb{T}; \tag{8}$$

$$0 \le x_e(t) \le u_e(t), \ \ \forall\, t \in \mathbb{T}, \ \forall\, e \in E; \tag{9}$$

$$x_e(t) = 0, \ \forall\, e \in E, \ t = \overline{T - \tau_e + 1, T}. \tag{10}$$

Condition (10) ensures that there is no flow in the network after time horizon T. Conditions (9) are capacity constraints. As flow travels through the network, unlimited flow storage at the nodes is allowed, but any deficit is prohibited by constraint (8). Finally, all demands must be met, flow must not remain in the network after time T, and each source must not exceed its supply. This is ensured by constraint (7).

As earlier we seek for a feasible dynamic flow x that minimizes the total cost:

$$F(x) = \sum_{t \in \mathbb{T}} \sum_{e \in E} \varphi_e(x_e(t), t).$$

We would like to mention that the more general model can be obtained if we define the cost function as also dependent on the flow storage at nodes. In this case the problem can be solved by using the similar approach.

To solve the formulated above minimum cost dynamic flow problem, we use the modified time-expanded network method. The auxiliary static network N^T is constructed as follows:

1. $V^T := \{v(t) | v \in V, t \in \mathbb{T}\}$
2. $V_+^T := \{v(0) | v \in V_+\}$ and $V_-^T := \{v(T) | v \in V_-\}$
3. $E^T := \{(v(t), z(t + \tau_e)) \mid e = (v, z) \in E, \ 0 \le t \le T - \tau_e\} \cup$
   $\{v(t), v(t + 1) \mid v \in V, \ 0 \le t < T\}$
4. ${d_{v(t)}}^T := d_v$ for $v(t) \in V_+^T \cup V_-^T$; otherwise ${d_{v(t)}}^T := 0$
5. ${u_{(v(t),z(t+\tau_{(v,z)}))}}^T := u_{(v,z)}(t)$ for $(v(t), z(t + \tau_{(v,z)})) \in E^T$
   ${u_{(v(t),v(t+1))}}^T := \infty$ for $(v(t), v(t + 1)) \in E^T$
6. ${\varphi_{(v(t),z(t+\tau_{(v,z)}))}}^T({x_{(v(t),z(t+\tau_{(v,z)}))}}^T) := \varphi_{(v,z)}(x_{(v,z)}(t), t)$
   for $(v(t), z(t + \tau_{(v,z)})) \in E^T$
   ${\varphi_{(v(t),v(t+1))}}^T({x_{(v(t),v(t+1))}}^T) := 0$ for $(v(t), v(t + 1)) \in E^T$

If the flow correspondence is the following: ${x_{e(t)}}^T := x_e(t)$, where ${x_{(v(t),v(t+1))}}^T$ in N^T corresponds to the flow in N stored at node v at period of time from t to $t+1$, then the minimum cost flow problem on dynamic networks can be solved by solving the minimum cost static flow problem on the time-expanded network.

2.4 Determining the Minimum Cost Dynamic Flows in Networks with Transit Time Functions that Depend on Flow and Time

In the above dynamic models, the transit time functions are assumed to be constant on each arc of the network. In this setting, the time it takes to traverse an arc does not depend on the current flow situation on the arc and the moment of time. Intuitively, it is clear that in many applications the amount of time needed to traverse an arc of the network increases as the arc becomes more congested, and it also depends on the entering time-moment of flow in the arc. If these assumptions are taken into account, a more realistic model can be obtained. In this model we assume that the transit time function $\tau_e(x_e(t), t)$ is a nonnegative nondecreasing left-continuous step function with respect to the amount of flow $x_e(t)$ for every fixed time-moment $t \in \mathbb{T}$ and an arbitrary given arc $e \in E$. We also consider two-side restrictions on arc capacities $u'_e(t) \leq x_e(t) \leq u''_e(t), \forall\, t \in \mathbb{T}, \forall\, e \in E$, where $u', u'' \colon E \times \mathbb{T} \to R_+$ are lower and upper capacities, respectively.

It is shown (Fonoberova and Lozovanu 2007) that the minimum cost flow problem on dynamic network with transit time functions that depend on the amount of flow and the entering time-moment of flow in the arc can be reduced to a static problem on a special time-expanded network $N^T = (V^T, E^T, d^T, u'^T, u''^T, \varphi^T)$, which is defined as follows:

1. $\overline{V}^T \colon = \{v(t) \mid v \in V,\ t \in \mathbb{T}\}$
2. $\widetilde{V}^T \colon = \{e(v(t)) \mid v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathbb{T} \setminus \{T\}\}$
3. $V^T \colon = \overline{V}^T \cup \widetilde{V}^T$
4. $\widetilde{E}^T \colon = \{\widetilde{e}(t) = (v(t), e(v(t))) \mid v(t) \in \overline{V}^T$ and corresponding $e(v(t)) \in \widetilde{V}^T,\ t \in \mathbb{T} \setminus \{T\}\}$
5. $\overline{E}^T \colon = \{e^p(t) = (e(v(t)), z(t + \tau_e^p(x_e(t), t))) \mid e(v(t)) \in \widetilde{V}^T,\ z(t + \tau_e^p(x_e(t), t)) \in \overline{V}^T,\ e = (v, z) \in E,\ 0 \leq t \leq T - \tau_e^p(x_e(t), t),\ p \in P_{e,t}$ – set of numbers of steps of the transit time function $\tau_e^p(x_e(t), t)\}$
6. $E^T \colon = \overline{E}^T \cup \widetilde{E}^T$
7. ${d_{v(t)}}^T \colon = d_v(t)$ for $v(t) \in \overline{V}^T$
   ${d_{e(v(t))}}^T \colon = 0$ for $e(v(t)) \in \widetilde{V}^T$
8. ${u'_{\widetilde{e}(t)}}^T := u'_e(t)$ for $\widetilde{e}(t) \in \widetilde{E}^T$
   ${u''_{\widetilde{e}(t)}}^T \colon = u''_e(t)$ for $\widetilde{e}(t) \in \widetilde{E}^T$
   ${u'_{e^p(t)}}^T \colon = \overline{x_e^{p-1}}(t)$ for $e^p(t) \in \overline{E}^T$, where $\overline{x_e^0}(t) = u'_e(t)$
   ${u''_{e^p(t)}}^T \colon = \overline{x_e^p}(t)$ for $e^p(t) \in \overline{E}^T$

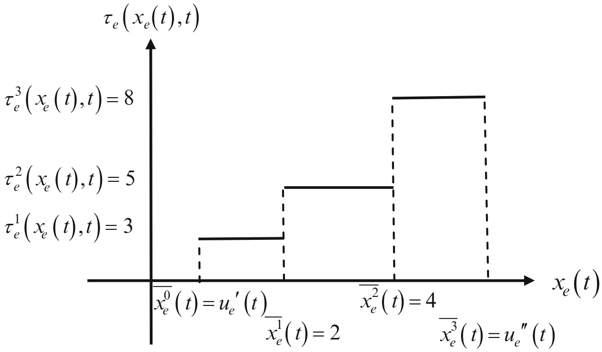

Fig. 3 The transit time function for the fixed moment of time t and the given arc $e = (v, z)$

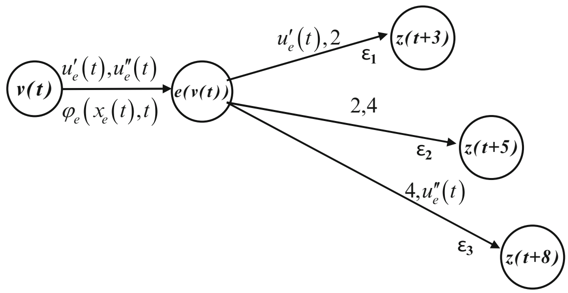

Fig. 4 The part of the constructed time-expanded network N^T for the fixed moment of time t for the arc $e = (v, z)$

9. $\varphi_{\widetilde{e}(t)}{}^T(x_{\widetilde{e}(t)}{}^T) := \varphi_e(x_e(t), t)$ for $\widetilde{e}(t) \in \widetilde{E}^T$
   $\varphi_{e^p(t)}{}^T(x_{e^p(t)}{}^T) := \varepsilon_p$ for $e^p(t) \in \overline{E}^T$, where $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{|P_{e,t}|}$ are small numbers

Let us consider, for example, the transit time function $\tau_e = \tau_e(x_e(t), t)$, graphic representation of which for the fixed moment of time t and the given arc e is presented in Fig. 3. Here $P_{e,t} = \{1, 2, 3\}$. So, for the fixed moment of time t on the given arc e, the transit time is equal to 3 if the value of flow belongs to interval $[u_e'(t), 2]$; the transit time is equal to 5 if the value of flow belongs to interval $(2, 4]$; the transit time is equal to 8 if the value of flow belongs to interval $(4, u_e''(t)]$.

In Fig. 4, a part of the obtained time-expanded network is presented for the fixed moment of time t for the given arc $e = (v, z)$, with the transit time function in Fig. 3. Lower and upper capacities of arcs are written above each arc and costs are written below each arc.

The solution of the dynamic problem can be found on the basis of the following results.

Lemma 2. *Let $x^T : E^T \to R_+$ be a flow in the static network N^T. Then the function $x : E \times \mathbb{T} \to R_+$ defined as follows:*

$$
\begin{aligned}
& x_e(t) = x_{\widetilde{e}(t)}{}^T = x_{e^p(t)}{}^T \\
& \textit{for } e = (v,z) \in E,\ \widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T, \\
& e^p(t) = (e(v(t)), z(t + \tau_e^p(x_e(t), t))) \in \overline{E}^T, \\
& p \in P_{e,t} \textit{ is such that } x_{\widetilde{e}(t)}{}^T \in (\overline{x_e^{p-1}}(t), \overline{x_e^p}(t)],\ t \in \mathbb{T},
\end{aligned}
$$

which represents a flow in the dynamic network N.

Let $x\colon E \times \mathbb{T} \to R_+$ *be a flow in the dynamic network* N. *Then the function* $x^T\colon E^T \to R_+$ *is defined as follows:*

$$
\begin{aligned}
& x_{\widetilde{e}(t)}{}^T = x_e(t) \textit{ for } \widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T,\ e = (v,z) \in E,\ t \in \mathbb{T}; \\
& x_{e^p(t)}{}^T = x_e(t) \textit{ for such } p \in P_{e,t} \textit{ that } x_e(t) \in (\overline{x_e^{p-1}}(t), \overline{x_e^p}(t)] \\
& \textit{and } x_{e^p(t)}{}^T = 0 \textit{ for all other } p \in P_{e,t} \\
& \textit{for } e^p(t) = (e(v(t)), z(t + \tau_e^p(x_e(t), t))) \in \overline{E}^T,\ e = (v,z) \in E,\ t \in \mathbb{T},
\end{aligned}
$$

which represents a flow in the static network N^T.

Theorem 2. *If* x^{*T} *is a static minimum cost flow in the static network* N^T, *then the corresponding one according to Lemma 2 dynamic flow* x^* *in the dynamic network* N *is also a minimum cost flow and vice-versa.*

The proofs of the above lemma and theorem can be obtained by using the arguments similar to the ones in the proofs of Lemma 1 and Theorem 1.

3 Minimum Cost Dynamic Multicommodity Flow Problems and Algorithms for Their Solving

In this section we formulate and investigate minimum cost multicommodity flow problems on dynamic networks. The multicommodity flow problem consists of shipping several different commodities from their respective sources to their sinks through a given network satisfying certain objectives in such a way that the total flow going through arcs does not exceed their capacities. No commodity ever transforms into another commodity, so that each one has its own flow conservation constraints, but they compete for the resources of the common network. In this section we consider the minimum cost multicommodity flow problems on dynamic networks with time-varying capacities of arcs and transit times on arcs that depend on sort of commodity entering them. We assume that cost functions, defined on arcs, are nonlinear and depend on time and flow, and demand-supply functions depend on time. For solving the considered problems, we propose algorithms based on the modification of the time-expanded network method. We also consider dynamic problems with transit time functions that depend on flow and time and propose algorithms for their solving.

3.1 The Minimum Cost Dynamic Multicommodity Flow Problem Formulation

It is considered a dynamic network $N = (V, E, K, \tau, d, u, w, \varphi)$, determined by directed graph $G = (V, E)$, where V is a set of vertices and E is a set of arcs, $K = \{1, 2, \ldots, q\}$ is set of commodities that must be routed through the same network, $\tau \colon E \times K \to R_+$ is transit time function, $d \colon V \times K \times \mathrm{T} \to R$ is demand-supply function, $u \colon E \times \mathrm{T} \to R_+$ is mutual capacity function, $w \colon E \times K \times \mathrm{T} \to R_+$ is individual capacity function, and $\varphi \colon E \times R_+ \times \mathrm{T} \to R_+$ is cost function. So, $\tau_e = (\tau_e^1, \tau_e^2, \ldots, \tau_e^q)$ is a vector, each component of which reflects the transit time on arc $e \in E$ for commodity $k \in K$. It is considered the discrete time model, where all times are integral and bounded by horizon T, which defines the set $\mathrm{T} = \{0, 1, \ldots, T\}$ of time moments.

In order for the flow to exist, it is required that $\sum_{t \in \mathrm{T}} \sum_{v \in V} d_v^k(t) = 0, \forall k \in K$. As earlier without losing generality we consider that, for every commodity $k \in K$, the set of vertices V is divided into three disjoint subsets V_+^k, V_-^k, V_*^k, such that we have the following:

V_+^k consists of nodes $v \in V$, for which $d_v^k(t) \geq 0$ for $t \in \mathrm{T}$, and there exists at least one moment of time $t_0 \in \mathrm{T}$ such that $d_v^k(t_0) > 0$

V_-^k consists of nodes $v \in V$, for which $d_v^k(t) \leq 0$ for $t \in \mathrm{T}$, and there exists at least one moment of time $t_0 \in \mathrm{T}$ such that $d_v^k(t_0) < 0$

V_*^k consists of nodes $v \in V$, for which $d_v^k(t) = 0$ for every $t \in \mathrm{T}$

So, V_+^k is a set of sources, V_-^k is a set of sinks, and V_*^k is a set of intermediate nodes for the commodity $k \in K$ in the network N

A feasible dynamic multicommodity flow in the network N is determined by a function $x \colon E \times K \times \mathrm{T} \to R_+$ that satisfies the following conditions:

$$\sum_{e \in E^-(v)} x_e^k(t) - \sum_{\substack{e \in E^+(v) \\ t - \tau_e^k \geq 0}} x_e^k(t - \tau_e^k) = d_v^k(t), \ \forall\, t \in \mathrm{T}, \ \forall\, v \in V, \ \forall k \in K; \tag{11}$$

$$\sum_{k \in K} x_e^k(t) \leq u_e(t), \ \forall\, t \in \mathrm{T}, \ \forall e \in E; \tag{12}$$

$$0 \leq x_e^k(t) \leq w_e^k(t), \ \ \forall\, t \in \mathrm{T}, \ \forall\, e \in E, \ \forall k \in K; \tag{13}$$

$$x_e^k(t) = 0, \ \forall\, e \in E, \ t = \overline{T - \tau_e^k + 1, T}, \ \forall k \in K. \tag{14}$$

Here the function x defines the value $x_e^k(t)$ of flow of commodity k entering arc e at moment of time t. Condition (14) ensures that the flow of commodity k does not enter arc e at time t if it has to leave the arc after time horizon T. Individual and mutual capacity constraints (13) and (12) are called weak and strong forcing constraints, respectively. Conditions (11) represent flow conservation constraints.

The total cost of the dynamic multicommodity flow x in the network N is defined as follows:

$$F(x) = \sum_{t \in \mathbb{T}} \sum_{e \in E} \varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^q(t), t). \tag{15}$$

The minimum cost dynamic multicommodity flow problem consists in finding a feasible dynamic multicommodity flow that minimizes the objective function (15).

3.2 The Algorithm for Solving the Minimum Cost Dynamic Multicommodity Flow Problem

To solve the formulated problem, we propose an approach based on the reduction of the dynamic problem to a static problem. It is shown that the minimum cost multicommodity flow problem on network N can be reduced to a static problem on a special auxiliary network N^T. In the case of the minimum cost multicommodity flow problem on dynamic network with different transit times on an arc for different commodities, the auxiliary time-expanded network $N^T = (V^T, E^T, K, d^T, u^T, w^T, \varphi^T)$ is defined in the following way:

1. $\overline{V}^T :\, = \{v(t) \mid v \in V,\ t \in \mathbb{T}\}$
2. $\widetilde{V}^T :\, = \{e(v(t)) \mid v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathbb{T} \setminus \{T\}\}$
3. $V^T :\, = \overline{V}^T \cup \widetilde{V}^T$
4. $\widetilde{E}^T :\, = \{\widetilde{e}(t) = (v(t), e(v(t))) \mid v(t) \in \overline{V}^T$ and corresponding $e(v(t)) \in \widetilde{V}^T$, $t \in \mathbb{T} \setminus \{T\}\}$
5. $\overline{E}^T :\, = \{e^k(t) = (e(v(t)), z(t + \tau_e^k)) \mid e(v(t)) \in \widetilde{V}^T,\ z(t + \tau_e^k) \in \overline{V}^T,$ $e = (v, z) \in E,\ 0 \le t \le T - \tau_e^k,\ k \in K\}$
6. $E^T :\, = \overline{E}^T \cup \widetilde{E}^T$
7. ${d^k_{v(t)}}^T :\, = d_v^k(t)$ for $v(t) \in \overline{V}^T,\ k \in K$

   ${d^k_{e(v(t))}}^T :\, = 0$ for $e(v(t)) \in \widetilde{V}^T,\ k \in K$
8. ${u_{\widetilde{e}(t)}}^T :\, = u_e(t)$ for $\widetilde{e}(t) \in \widetilde{E}^T$

   ${u_{e^k(t)}}^T :\, = \infty$ for $e^k(t) \in \overline{E}^T$
9. ${w^l_{e^k(t)}}^T :\, = \begin{cases} w_e^k(t), & \text{if } l = k \text{ for } e^k(t) \in \overline{E}^T,\ l \in K \\ 0, & \text{if } l \neq k \text{ for } e^k(t) \in \overline{E}^T,\ l \in K \end{cases}$

   ${w^l_{\widetilde{e}(t)}}^T = \infty$ for $\widetilde{e}(t) \in \widetilde{E}^T,\ l \in K$
10. ${\varphi_{\widetilde{e}(t)}}^T({x^1_{\widetilde{e}(t)}}^T, {x^2_{\widetilde{e}(t)}}^T, \dots, {x^q_{\widetilde{e}(t)}}^T) :\, = \varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^q(t), t)$ for $\widetilde{e}(t) \in \widetilde{E}^T$

    ${\varphi_{e^k(t)}}^T({x^1_{e^k(t)}}^T, {x^2_{e^k(t)}}^T, \dots, {x^q_{e^k(t)}}^T) :\, = 0$ for $e^k(t) \in \overline{E}^T$

In the following we construct the time-expanded network N^T for the dynamic network N given in Fig. 1 with set of two commodities $K = \{1, 2\}$, set of time moments $\mathbb{T} = \{0, 1, 2, 3\}$, and transit times $\tau_{e_1}^1 = 2$, $\tau_{e_1}^2 = 1$, $\tau_{e_2}^1 = 1$, $\tau_{e_2}^2 = 3$,

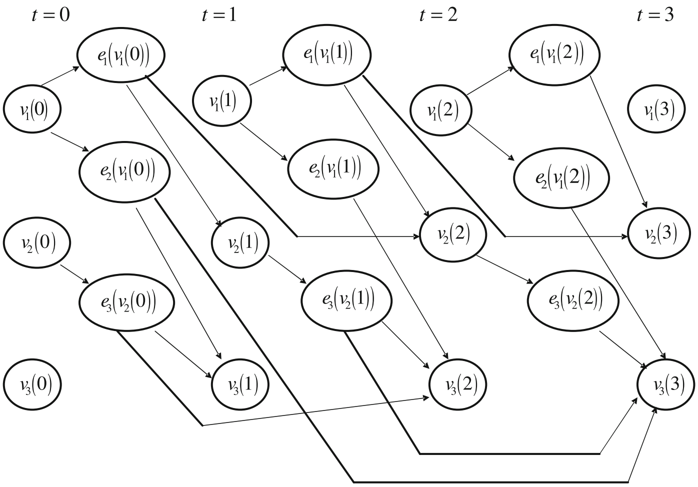

Fig. 5 The time-expanded network (case of different transit times on an arc for different commodities)

$\tau^1_{e_3} = 1$, $\tau^2_{e_3} = 2$. The mutual capacity, individual capacity, demand-supply, and cost functions are considered to be known. The constructed time-expanded network N^T is presented in Fig. 5.

Lemma 3. *Let $x^T\colon E^T \times K \to R_+$ be a multicommodity flow in the static network N^T. Then the function $x\colon E \times K \times \mathbb{T} \to R_+$ is defined in the following way:*

$$\begin{aligned}&x^k_e(t) = x^{k}_{e^k(t)}{}^T = x^{k}_{\widetilde{e}(t)}{}^T\\ &\text{for } e = (v,z) \in E,\ e^k(t) = (e(v(t)), z(t+\tau^k_e)) \in \overline{E}^T,\\ &\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T,\ k \in K,\ t \in \mathbb{T},\end{aligned}$$

which represents a multicommodity flow in the dynamic network N.

Let $x\colon E \times K \times \mathbb{T} \to R_+$ be a multicommodity flow in the dynamic network N. Then the function $x^T\colon E^T \times K \to R_+$ is defined in the following way:

$$\begin{aligned}&x^{k}_{\widetilde{e}(t)}{}^T = x^k_e(t) \text{ for } \widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T,\ e = (v,z) \in E,\ k \in K,\ t \in \mathbb{T};\\ &x^{k}_{e^k(t)}{}^T = x^k_e(t);\ x^{l}_{e^k(t)}{}^T = 0,\ l \neq k\\ &\text{for } e^k(t) = (e(v(t)), z(t+\tau^k_e)) \in \overline{E}^T,\ e = (v,z) \in E,\ l,k \in K,\ t \in \mathbb{T},\end{aligned}$$

which represents a multicommodity flow in the static network N^T.

Proof. To prove the first part of the lemma, we have to show that conditions (11)–(14) for the defined above x in the dynamic network N are true. These conditions evidently result from the following definition of multicommodity flows in the static network N^T:

$$\sum_{e(t)\in E^-(v(t))} {x_{e(t)}^k}^T - \sum_{e(t-\tau_e^k)\in E^+(v(t))} {x_{e(t-\tau_e^k)}^k}^T = {d_{v(t)}^k}^T, \quad (16)$$
$$\forall\, v(t) \in V^T,\ \forall k \in K;$$

$$\sum_{k\in K} {x_{e(t)}^k}^T \le {u_{e(t)}}^T,\ \forall e(t) \in E^T; \quad (17)$$

$$0 \le {x_{e(t)}^k}^T \le {w_{e(t)}^k}^T,\ \forall\, e(t) \in E^T,\ \forall k \in K; \quad (18)$$

$${x_{e(t)}^k}^T = 0,\ \forall\, e(t) \in E^T,\ t = \overline{T - \tau_e^k + 1, T},\ \forall k \in K, \quad (19)$$

where by $v(t)$ and $e(t)$ we denote $\overline{v}(t)$ or $\widetilde{v}(t)$ and $\overline{e}(t)$ or $\widetilde{e}(t)$, respectively, against context.

To prove the second part of the lemma, it is sufficient to show that conditions (16)–(19) hold for x^T defined above. Correctness of these conditions results from the procedure of constructing the time-expanded network, the correspondence between flows in static and dynamic networks, and the satisfied conditions (11)–(14).

The lemma is proved. □

Theorem 3. *If ${x^*}^T$ is a minimum cost multicommodity flow in the static network N^T, then the corresponding one according to Lemma 3 multicommodity flow x^* in the dynamic network N is also a minimum cost one and vice-versa.*

Proof. Taking into account the correspondence between static and dynamic multicommodity flows on the basis of Lemma 3, we obtain that costs of the static multicommodity flow in the time-expanded network N^T and the corresponding dynamic multicommodity flow in the dynamic network N are equal. To solve the minimum cost multicommodity flow problem on the static time-expanded network N^T, we have to solve the following problem:

$$F^T(x^T) = \sum_{t\in\mathbb{T}} \sum_{e(t)\in E^T} {\varphi_{e(t)}}^T({x_{e(t)}^1}^T, {x_{e(t)}^2}^T, \ldots, {x_{e(t)}^q}^T) \to \min$$
subject to (16)–(19).

□

In the case of the minimum cost multicommodity flow problem on dynamic network with common transit times on an arc for different commodities, the time-expanded network N^T can be constructed more simply:

1. $V^T :=\{v(t) \mid v \in V,\ t \in \mathbb{T}\}$
2. $E^T :=\{e(t) = (v(t), z(t+\tau_e)) \mid v(t) \in V^T,\ z(t+\tau_e) \in V^T,\ e = (v,z) \in E,$ $0 \le t \le T - \tau_e\}$

3. ${d^k_{v(t)}}^T := d^k_v(t)$ for $v(t) \in V^T,\ k \in K$
4. ${u_{e(t)}}^T := u_e(t)$ for $e(t) \in E^T$
5. ${w^k_{e(t)}}^T := w^k_e(t)$ for $e(t) \in E^T,\ k \in K$
6. ${\varphi_{e(t)}}^T({x^1_{e(t)}}^T, {x^2_{e(t)}}^T, \ldots, {x^q_{e(t)}}^T) := \varphi_e(x^1_e(t), x^2_e(t), \ldots, x^q_e(t), t)$ for $e(t) \in E^T$

The following lemma and theorem can be considered as particular cases of Lemma 3 and Theorem 3.

Lemma 4. *Let $x^T : E^T \times K \to R_+$ be a multicommodity flow in the static network N^T. Then the function $x: E \times K \times \mathbb{T} \to R_+$ is defined as follows:*

$$x^k_e(t) = {x^k_{e(t)}}^T \text{ for } e \in E,\ e(t) \in E^T,\ k \in K,\ t \in \mathbb{T},$$

which represents the multicommodity flow in the dynamic network N.

Let $x: E \times K \times \mathbb{T} \to R_+$ be a multicommodity flow in the dynamic network N. Then the function $x^T : E^T \times K \to R_+$ is defined as follows:

$${x^k_{e(t)}}^T = x^k_e(t) \text{ for } e(t) \in E^T,\ e \in E,\ k \in K,\ t \in \mathbb{T},$$

which represents the multicommodity flow in the static network N^T.

Theorem 4. *If ${x^*}^T$ is a minimum cost multicommodity flow in the static network N^T, then the corresponding one according to Lemma 4 multicommodity flow x^* in the dynamic network N is also a minimum cost one and vice-versa.*

In such a way, to solve the minimum cost multicommodity flow problem on dynamic networks we have the following:

1. To build the time-expanded network N^T for the given dynamic network N
2. To solve the classical minimum cost multicommodity flow problem on the static network N^T (Assad 1978; McBride 1998; Castro and Nabona 1996; Castro 2000, 2003; Ermoliev and Melnic 1968; Fleisher 2000).
3. To reconstruct the solution of the static problem on N^T to the dynamic problem on N.

The complexity of this algorithm depends on the complexity of the algorithm used for the minimum cost multicommodity flow problem on the static network. If such an algorithm has complexity $O(f(n', m'))$, where n' is a number of vertices and m' is a number of arcs in the network, then the complexity of solving the minimum cost multicommodity flow problem on the time-expanded network employing the same algorithm is $O(f((n+m)T, m(k+1)T))$, where n is the number of vertices in the dynamic network, m is the number of arcs in the dynamic network, and k is the number of commodities.

3.3 The Reduced Time-Expanded Network for Acyclic Graphs

In this subsection we consider the minimum cost multicommodity flow problem on acyclic dynamic network $N = (V, E, K, \tau, d, u, w, \varphi)$ with time horizon $T = +\infty$ and common transit times on an arc for different commodities. Without losing generality, we assume that no arcs enter sources or exit sinks. Let $T^* = \max\{|L|\} = \max\{\sum_{e \in L} \tau_e\}$, where L is a directed path in the graph $G = (V, E)$. It is not difficult to show that $x_e^k(t) = 0$ for $\forall e \in E$, $\forall k \in K$, $\forall t \geq T^*$. This fact allows us to replace the infinite time horizon with the finite one, by substituting T^* for the positive infinity.

In many cases a big number of nodes is not connected with a directed path both to a sink and a source. Removing such nodes from the considered network does not influence the set of flows in this network. These nodes are called irrelevant to the flow problem. Nodes that are not irrelevant are relevant. The static network obtained by eliminating the irrelevant nodes and all arcs adjacent to them from the time-expanded network is called the reduced time-expanded network.

The following algorithm is proposed for constructing the reduced network $N^{rT^*} = (V^{rT^*}, E^{rT^*}, d^{rT^*}, u^{rT^*}, w^{rT^*}, \varphi^{rT^*})$, which is based on the process of elimination of irrelevant nodes from the time-expanded network:

1. To build the time-expanded network N^{T^*} for the given dynamic network N.
2. To perform a breadth-first parse of the nodes for each source from the time expanded-network. The result of this step is the set $V_-(V_-^{T^*})$ of the nodes that can be reached from at least a source in V^{T^*}.
3. To perform a breadth-first parse of the nodes beginning with the sink for each sink and parsing the arcs in the direction opposite to their normal orientation. The result of this step is the set $V_+(V_+^{T^*})$ of nodes from which at least a sink in V^{T^*} can be reached.
4. The reduced network will consist of a subset of nodes V^{T^*} and arcs from E^{T^*} determined in the following way:

$$V^{rT^*} = V^{T^*} \cap V_-(V_-^{T^*}) \cap V_+(V_+^{T^*}),$$

$$E^{rT^*} = E^{T^*} \cap (V^{rT^*} \times V^{rT^*}).$$

5. ${d^{rk}_{v(t)}}^{T^*} := d_v^k(t)$ for $v(t) \in V^{rT^*}$, $k \in K$.
6. ${u^r{}_{e(t)}}^{T^*} := u_e(t)$ for $e(t) \in E^{rT^*}$.
7. ${w^{rk}_{e(t)}}^{T^*} := w_e^k(t)$ for $e(t) \in E^{rT^*}$, $k \in K$.
8. ${\varphi^r{}_{e(t)}}^{T^*}({x^1_{e(t)}}^{T*}, {x^2_{e(t)}}^{T*}, \ldots, {x^q_{e(t)}}^{T*}) := \varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^q(t), t)$ for $e(t) \in E^{rT^*}$.

The complexity of this algorithm can be estimated to be the same as the complexity of constructing the time-expanded network. It can be proven by using the similar approach as in Lozovanu and Stratila (2001) that the reduced network can be used in place of the time-expanded network.

We would like to mention that the proposed above approach with some modifications can be used for constructing the reduced time-expanded network for the optimal single-commodity dynamic flow problems and the optimal multicommodity dynamic flow problems with different transit times on an arc for different commodities.

3.4 The Minimum Cost Multicommodity Dynamic Flow Problem with Transit Time Functions that Depend on Flow and Time

In this subsection an approach for solving the minimum cost multicommodity dynamic flow problem with transit time functions that depend on flow and time is proposed. This problem is considered on dynamic networks with time-varying lower and upper capacity functions, time-varying mutual capacity function, and time-varying demand-supply function. It is assumed that cost functions, defined on arcs, are nonlinear and depend on flow and time. The transit time function $\tau_e^k(x_e^k(t), t)$ is considered to be a nonnegative nondecreasing left-continuous step function for each commodity $k \in K$.

The method for solving the minimum cost multicommodity dynamic flow problem with transit time functions that depend on flows and time is based on the reduction of the dynamic problem to a static problem on an auxiliary time-expanded network $N^T = (V^T, E^T, d^T, u^T, w'^T, w''^T, \varphi^T)$, which is defined as follows:

1. $\overline{V}^T := \{v(t) \mid v \in V,\ t \in \mathbb{T}\}$
2. $\widetilde{V}^T := \{e(v(t)) \mid v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathbb{T} \setminus \{T\}\}$
3. $V^T := \overline{V}^T \cup \widetilde{V}^T$
4. $\widetilde{E}^T := \{\widetilde{e}(t) = (v(t), e(v(t))) \mid v(t) \in \overline{V}^T$ and corresponding $e(v(t)) \in \widetilde{V}^T,\ t \in \mathbb{T} \setminus \{T\}\}$
5. $\overline{E}^T := \{e^{k,p}(t) = (e(v(t)), z(t + \tau_e^{k,p}(x_e^k(t), t))) \mid e(v(t)) \in \widetilde{V}^T,\ z(t + \tau_e^{k,p}(x_e^k(t), t)) \in \overline{V}^T, e = (v, z) \in E,\ 0 \le t \le T - \tau_e^{k,p}(x_e^k(t), t),\ p \in P_{e,t}^k -$ set of numbers of steps of the transit time function $\tau_e^k(x_e^k(t), t), k \in K\}$
6. $E^T := \overline{E}^T \cup \widetilde{E}^T$
7. ${d_{v(t)}^k}^T := d_v^k(t)$ for $v(t) \in \overline{V}^T,\ k \in K$
   ${d_{e(v(t))}^k}^T := 0$ for $e(v(t)) \in \widetilde{V}^T,\ k \in K$
8. ${u_{\widetilde{e}(t)}}^T := u_e(t)$ for $\widetilde{e}(t) \in \widetilde{E}^T$
   ${u_{e^{k,p}(t)}}^T := \infty$ for $e^{k,p}(t) \in \overline{E}^T$

9. $w'^{l}_{e^{k,p}(t)}{}^{T} := \begin{cases} \overline{x_e^{k,p-1}}(t), \text{ if } l=k \text{ for } e^{k,p}(t)\in\overline{E}^{T},\ l\in K, \text{ where } \overline{x_e^{k,0}}(t)=w'^{k}_{e}(t) \\ 0, \quad \text{if } l \neq k \ \text{ for } e^{k,p}(t) \in \overline{E}^{T},\ l \in K \end{cases}$

$w''^{l}_{e^{k,p}(t)}{}^{T} := \begin{cases} \overline{x_e^{k,p}}(t), \text{ if } l = k \text{ for } e^{k,p}(t) \in \overline{E}^{T},\ l \in K \\ 0, \quad \text{if } l \neq k \ \text{ for } e^{k,p}(t) \in \overline{E}^{T},\ l \in K \end{cases}$

$w'^{l}_{\widetilde{e}(t)}{}^{T} = -\infty;\ w''^{l}_{\widetilde{e}(t)}{}^{T} = +\infty \text{ for } \widetilde{e}(t) \in \widetilde{E}^{T},\ l \in K$

10. $\varphi_{\widetilde{e}(t)}{}^{T}(x^{1}_{\widetilde{e}(t)}{}^{T}, x^{2}_{\widetilde{e}(t)}{}^{T}, \ldots, x^{q}_{\widetilde{e}(t)}{}^{T}) := \varphi_e(x_e^1(t), x_e^2(t), \ldots, x_e^q(t), t)$ for $\widetilde{e}(t) \in \widetilde{E}^{T}$

$\varphi_{e^{k,p}(t)}{}^{T}(x^{1}_{e^{k,p}(t)}{}^{T}, x^{2}_{e^{k,p}(t)}{}^{T}, \ldots, x^{q}_{e^{k,p}(t)}{}^{T}) := \varepsilon^{k,p}$ for $e^{k,p}(t) \in \overline{E}^{T}$,
where $\varepsilon^{k,1} < \varepsilon^{k,2} < \cdots < \varepsilon^{k,|P^{k}_{e,t}|}$ are small numbers

For example, let us consider the transit time functions for an arc $e = (v, z)$ at the moment of time t presented in Figs. 6 and 7, which correspond to commodities 1 and 2, respectively.

The constructed part of the time-expanded network for the fixed moment of time t for the arc $e = (v, z)$ is presented in Fig. 8.

The following lemma and theorem give us the relationship between flows in network N and flows in network N^T.

Lemma 5. *Let $x^T: E^T \times K \to R_+$ be a multicommodity flow in the static network N^T. Then the function $x: E \times K \times \mathbb{T} \to R_+$ is defined in the following way:*

$$x_e^k(t) = x^{k}_{\widetilde{e}(t)}{}^{T} = x^{k}_{e^{k,p}(t)}{}^{T}$$
$$\textit{for } e = (v,z) \in E,\ \widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^{T},$$
$$e^{k,p}(t) = (e(v(t)), z(t + \tau_e^{k,p}(x_e^k(t), t))) \in \overline{E}^{T},$$
$$p \in P^k_{e,t} \textit{ is such that } x^{k}_{\widetilde{e}(t)}{}^{T} \in (\overline{x_e^{k,p-1}}(t), \overline{x_e^{k,p}}(t)],\ t \in \mathbb{T},\ k \in K,$$

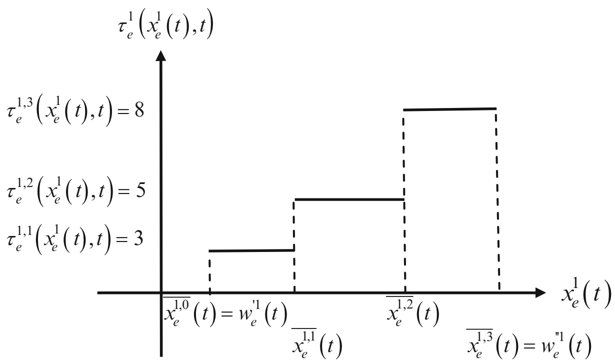

Fig. 6 The transit time function for commodity 1 for the fixed moment of time t and the given arc $e = (v, z)$

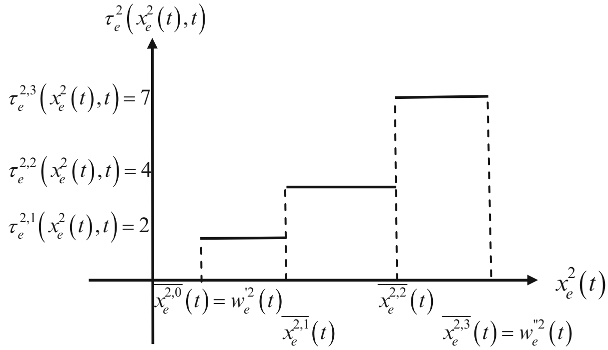

Fig. 7 The transit time function for commodity 2 for the fixed moment of time t and the given arc $e = (v, z)$

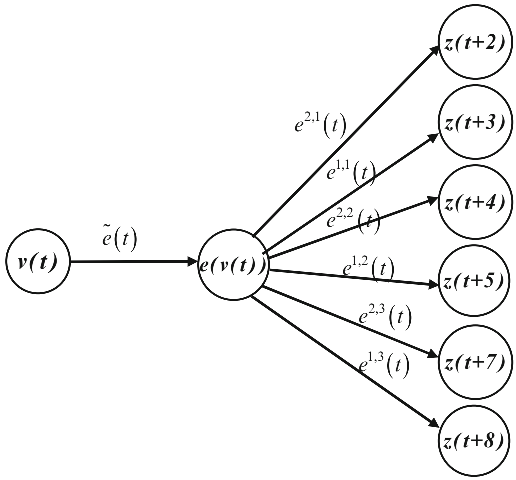

Fig. 8 The part of the constructed time-expanded network N^T for the fixed moment of time t for the arc $e = (v, z)$

which represents a multicommodity flow in the dynamic network N.

Let x: $E \times K \times \mathbb{T} \to R_+$ be a multicommodity flow in the dynamic network N. Then the function x^T: $E^T \times K \to R_+$ is defined in the following way:

$$x^{k}_{\widetilde{e}(t)}{}^{T} = x_e^k(t)$$
$$for\, \widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T, \; e = (v, z) \in E, \; k \in K, \; t \in \mathbb{T};$$

$$x^{l}_{e^{k,p}(t)}{}^{T} = 0, \; l \neq k;$$

$$x_{e^{k,p}(t)}^{k}{}^{T} = x_e^k(t) \textit{ for such } p \in P_{e,t}^k \textit{ that } x_e^k(t) \in (\overline{x_e^{k,p-1}}(t), \overline{x_e^{k,p}}(t)],$$
$$x_{e^{k,p}(t)}^{k}{}^{T} = 0 \textit{ for all other } p \in P_{e,t}^k$$
$$\textit{for } e^{k,p}(t) = (e(v(t)), z(t + \tau_e^{k,p}(x_e^k(t), t))) \in \overline{E}^T,\ e = (v,z) \in E,\ l,k \in K,$$
$$t \in \mathbb{T},$$

which represents a multicommodity flow in the static network N^T.

Theorem 5. *If x^{*T} is a minimum cost multicommodity flow in the static network N^T, then the corresponding one according to Lemma 5 multicommodity flow x^* in the dynamic network N is also a minimum cost one and vice-versa.*

Conclusions

In this chapter we extended and generalized the classical optimal flow problems on networks, which can be used for solving problems related to power generation, transmission, and distribution. We considered optimal single-commodity and multi-commodity flow problems on networks when all parameters are time-dependent. To solve the considered problems, we elaborated the corresponding algorithms on the basis of the time-expanded network method.

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