Chapter 2
Geometric Properties of Generalized Bessel Functions

Abstract The goal of the present chapter is to study some geometric properties (like univalence, starlikeness, convexity, close-to-convexity) of generalized Bessel functions of the first kind. In order to achieve our goal we use several methods: differential subordinations technique, Alexander transform, results of L. Fejér, W. Kaplan, S. Owa and H.M. Srivastava, S. Ozaki, S. Ponnusamy and M. Vuorinen, H. Silverman, and Jack’s lemma. Moreover, we present some immediate applications of univalence and convexity involving generalized Bessel functions associated with the Hardy space and a monotonicity property of generalized and normalized Bessel functions of the first kind.

2.1 Univalence of Generalized Bessel Functions

Our aim in this section is to give sufficient conditions for the function $u_p$, defined in (1.20), to be univalent. The results of this section were picked up from the papers of Á. Baricz [38, 42, 43, 52].

Before we state our main results of this section we recall some basic facts. A single-valued function $f : \mathbb{E} \subseteq \mathbb{C} \rightarrow \mathbb{C}$ is said to be univalent (or schlicht) in a domain $\mathbb{E}$ if it never takes the same value twice, that is, if $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{E}$ implies that $z_1 = z_2$. The function $f_1(z) = \bar{z}$ is nowhere analytic, but it is univalent in the complex plane $\mathbb{C}$. A necessary condition for an analytic function $f : \mathbb{E} \subseteq \mathbb{C} \rightarrow \mathbb{C}$ to be univalent in $\mathbb{E}$ is that $f'(z) \neq 0$ in $\mathbb{E}$, but this condition is not sufficient; for example $e^z$ is not univalent in $\mathbb{C}$ though its derivative never vanishes in $\mathbb{C}$.

A domain $\mathbb{E} \subseteq \mathbb{C}$ is said to be starlike domain with respect to $z_0 \in \mathbb{E}$ if the line segment joining $z_0$ to every other point $z \in \mathbb{E}$ lies entirely in $\mathbb{E}$. If a function $f$ maps $\mathbb{E}$ onto a domain that is starlike with respect to $z_0$, then $f$ is said to be starlike with respect to $z_0$. In particular, if $z_0$ is the origin, then we say that $f$ is a starlike function.

Further a domain $\mathbb{E} \subseteq \mathbb{C}$ is said to be convex if the line segment joining any two points of $\mathbb{E}$ lies entirely in $\mathbb{E}$. If a function $f$ maps $\mathbb{E}$ onto a convex domain, then we say that $f$ is a convex function in $\mathbb{E}$.

Due to the famous Riemann mapping theorem, any simply connected domain in the complex plane $\mathbb{C}$ which is not the whole plane, can be mapped by an analytic
function conformally onto the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. More precisely, the Riemann mapping theorem states (see the book of P.L. Duren [90, p. 11]): if $U$ is a simply connected domain which is a proper subset of the complex plane and $\zeta$ is a given point in $U$, then there is a unique function $f$ which maps $U$ conformally onto the unit disk $\mathbb{D}$ and has the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$. Thus, the investigation of the analytic functions which are univalent in a simply connected domain with more than one boundary point can be confined to the investigation of analytic functions which are univalent in $\mathbb{D}$. Hence without loss of generality we assume that $E$ is the unit disk $\mathbb{D}$. This is to simplify and to give short and elegant formulae. Note that for functions defined in a domain bordered by an ellipse (see the paper of N.N. Pascu et al. [169]), by a closed Lamé curve or a so-called spindle curve (see the paper of Á. Baricz [40]), the conditions (2.1) and (2.3) become more complicated, thus this also justifies why we need to work on the unit disk.

It is known that an analytic function $f : \mathbb{D} \to \mathbb{C}$, which satisfies $f(0) = 0$ and $f'(0) \neq 0$, is starlike (see the book of P.L. Duren [90]) if and only if

$$\Re \left[ \frac{zf'(z)}{f(z)} \right] > 0 \quad \text{for all } z \in \mathbb{D}. \tag{2.1}$$

Moreover, the function $f$ is said (see the papers of B.C. Carlson and D.B. Shaffer [80], I.S. Jack [124] and M.S. Robertson [200]) to be starlike of order $\alpha$, where $\alpha \in [0, 1)$, if

$$\Re \left[ \frac{zf'(z)}{f(z)} \right] > \alpha \quad \text{for all } z \in \mathbb{D}. \tag{2.2}$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all functions which are starlike of order $\alpha$. Throughout this book, it should be understood that functions such as $zf'/f$, which have removable singularities at $z = 0$, have had these singularities removed in statements like (2.1), (2.2). It is known that $g : \mathbb{D} \to \mathbb{C}$, which satisfies $g'(0) \neq 0$, is convex (see the book of P.L. Duren [90]) if and only if

$$\Re \left[ 1 + \frac{zg''(z)}{g'(z)} \right] > 0 \quad \text{for all } z \in \mathbb{D}. \tag{2.3}$$

If in addition

$$\Re \left[ 1 + \frac{zg''(z)}{g'(z)} \right] > \alpha \quad \text{for all } z \in \mathbb{D},$$

where $\alpha \in [0, 1)$, then $g$ is called convex of order $\alpha$. We denote the class of convex functions of order $\alpha$ by $\mathcal{C}(\alpha)$. We remark that for all $\alpha \in [0, 1)$ we have

$$\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*, \quad \mathcal{C}(\alpha) \subseteq \mathcal{C}(0) = \mathcal{C}$$

and

$$\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha).$$
The classes $S^\ast(\alpha)$ and $C(\alpha)$ were first introduced by M.S. Robertson [200], and were studied subsequently by I.S. Jack [124], T.H. MacGregor [142], B. Pinchuk [172], A. Schild [206], and others.

Let $A$ denote the class of normalized functions of the form

$$f(z) = z + \sum_{n \geq 2} a_n z^n,$$

where $a_n \in \mathbb{C}$ for all $n \geq 2$, which are analytic in the unit disk $D$. Further let $\mathcal{S}$ denote the class of all functions in $A$, which are univalent in the unit disk $D$. It is known that for normalized functions belonging to the class $A$ we have $C(\alpha) \subset S^\ast(\alpha) \subset S$.

An analytic function $f : D \to \mathbb{C}$ is said to be close-to-convex with respect to a convex function $\phi : D \to \mathbb{C}$ if

$$\Re\left[\frac{f'(z)}{\phi'(z)}\right] > 0 \quad \text{for all} \quad z \in D.$$

If there exists a convex function $\phi : D \to \mathbb{C}$ such that $f$ is close-to-convex with respect to $\phi$, then we say that $f$ is close-to-convex. We note that $f$ is not required a priori to be univalent (cf. Lemma 2.3 below), and the associated function $\phi$ need not be a function belonging to the class $A$. Moreover, every starlike (and hence convex) function is close-to-convex. However, if $f$ is starlike then it is not necessary that it will be close-to-convex with respect to a particular convex function. For instance, it is known that if $f \in A$ is convex, then $f$ need not be close-to-convex with respect to the identity function.

Given a number $\alpha \in [0, 1)$, we say that $f : D \to \mathbb{C}$ is close-to-convex of order $\alpha$ with respect to a convex function $\phi : D \to \mathbb{C}$ if

$$\Re\left[\frac{f'(z)}{\phi'(z)}\right] > \alpha \quad \text{for all} \quad z \in D.$$

If there exists a convex function $\phi : D \to \mathbb{C}$ such that $f$ is close-to-convex of order $\alpha$ with respect to $\phi$, then we say that $f$ is close-to-convex of order $\alpha$. For a general reference for the special classes of univalent functions we refer to the books of P.L. Duren [90], A.W. Goodman [101, 102] and C. Pommerenke [173, 174].

In order to prove our results the following preliminary results will be helpful. The first result is due to J.W. Alexander [5] and is called Alexander’s duality theorem, and the second one is due to I.S. Jack [124].

**Lemma 2.1.** (J.W. Alexander [5]) Let $f : D \to \mathbb{C}$ be an analytic function. Then $f$ is convex if and only if $zf'$ is starlike.

**Lemma 2.2.** (I.S. Jack [124]) Let $f : D \to \mathbb{C}$ be an analytic function. Further, consider the function $g : D \to \mathbb{C}$, defined by $g(z) = z[f'(z)]^{1/(1-\alpha)}$, where $\alpha \in [0, 1)$. Then $f \in C(\alpha)$ if and only if $g \in \mathcal{S}^\ast(\alpha)$. Moreover, $f \in C(\alpha)$ if and only if $zf' \in \mathcal{S}^\ast(\alpha)$. 

Now, we state the following known condition of univalence (see also Theorem 2.17 in the book of P.L. Duren [90]).

**Lemma 2.3.** (W. Kaplan [129] and S. Ozaki [167]) *If* \( f : \mathbb{D} \rightarrow \mathbb{C} \) *is a close-to-convex function, then it is univalent in* \( \mathbb{D} \). *

We note that there are other known univalence criteria in the literature, which may be useful to solve similar problems such as treated in this book. For example, the known Nehari’s criterion for univalence states (see the paper of Z. Nehari [158]): if \( f : \mathbb{D} \rightarrow \mathbb{C} \) is analytic and locally univalent in \( \mathbb{D} \) and its Schwarzian derivative \( S_f \), defined by

\[
S_f(z) = \left[ \frac{f''(z)}{f'(z)} \right]' - \frac{1}{2} \left[ \frac{f''(z)}{f'(z)} \right]^2,
\]

satisfies

\[
(1 - |z|^2)^2 |S_f(z)| \leq 2 \quad \text{for all} \quad z \in \mathbb{D},
\]

then \( f \) is univalent in \( \mathbb{D} \). The constant 2 in Nehari’s criterion is best possible and cannot be replaced by any larger number, as it was shown by E. Hille [115] as an addendum to Nehari’s paper [158]. In addition to the above condition, in his original paper Z. Nehari [158] showed that \( |S_f(z)| \leq \pi^2/2 \) for all \( z \in \mathbb{D} \) is also a sufficient condition for the function \( f \) to be univalent in \( \mathbb{D} \). Here the constant \( \pi^2/2 \) is sharp. However, Nehari’s conditions may be awkward to verify because it requires the computation of the Schwarzian derivative. Thus, it is often simpler to work directly with the logarithmic derivative \( f''/f' \) of \( f' \), called sometimes as the pre-Schwarzian derivative. Such univalence criterion, which involves the pre-Schwarzian derivative is due to J. Becker [66], who proved that the condition

\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1}{1 - |z|^2},
\]

where \( z \in \mathbb{D} \), is a sufficient condition for the univalence of \( f \) in \( \mathbb{D} \). The constant 1 is sharp, as was proved later by J. Becker and C. Pommerenke [67]. Various other conditions of Nehari and Becker type are discussed in the survey article of F.G. Avhadiev and L.A. Aksentev [31]. For more details we refer also to the books of P.L. Duren [90], A.W. Goodman [101, 102] and C. Pommerenke [173, 174].

### 2.1.1 Sufficient Conditions Involving Jack’s Lemma

S. Owa and H. M. Srivastava in [168] proved several interesting results concerning univalent, starlike of order \( \alpha \) and convex of order \( \alpha \) generalized hypergeometric functions. An important result which was used by S. Owa and H. M. Srivastava is the Jack lemma, i.e.
Lemma 2.4. (I.S. Jack [124]) Let \( f \) be a nonconstant analytic function in the unit disk \( \mathbb{D} \) with \( f(0) = 0 \). If \( |f(z)| \) attains its maximum value on the circle \(|z| = r \in [0, 1]\) at a point \( z_0 \), then there exists a real number \( m \geq 1 \) such that \( z_0 f'(z_0) = mf(z_0) \).

By using their technique mutatis mutandis we can obtain easily the following result.

Theorem 2.1. (Á. Baricz [54]) Let \( u_p(z) = \sum_{n \geq 0} b_n z^n \) be the generalized and normalized Bessel function, where \( b_n \) is defined by (1.18). Suppose that \( b_1 > 0 \) and that \( u_p \) satisfies the condition

\[
|u_p'(z) - b_1|^{1-\beta} \cdot \left| \frac{z u_p''(z)}{u_p'(z)} \right|^\beta < (b_1)^{1-\beta} \left( \frac{1}{2} \right)^\beta \tag{2.4}
\]

for all \( z \in \mathbb{D} \) and for some fixed \( \beta \geq 0 \). Then \( u_p \) is univalent in \( \mathbb{D} \).

Proof. For the function \( h : \mathbb{D} \to \mathbb{C} \), defined by \( h(z) = (u_p(z) - 1)/b_1 \), the condition (2.4) implies

\[
|h'(z) - 1|^{1-\beta} \cdot \left| \frac{zh''(z)}{h'(z)} \right|^\beta < \left( \frac{1}{2} \right)^\beta \tag{2.5}
\]

Further, it is clear that \( h \in \mathcal{A} \). Now we define the function \( f : \mathbb{D} \to \mathbb{C} \) by \( f(z) = h'(z) - 1 \). Then it follows that \( f \) is analytic in the unit disk \( \mathbb{D} \) with \( f(0) = 0 \). From (2.5) we get

\[
|f(z)|^{1-\beta} \cdot \left| \frac{zf'(z)}{1 + f(z)} \right|^\beta < \left( \frac{1}{2} \right)^\beta \tag{2.6}
\]

or equivalently

\[
|f(z)| \cdot \left| \frac{zf'(z)}{f(z)} \cdot \frac{1}{1 + f(z)} \right|^\beta < \left( \frac{1}{2} \right)^\beta
\]

for all \( z \in \mathbb{D} \), where the comment about removable singularities applies just as in (2.1) and (2.2). Now assume that there exists a point \( z_0 \in \mathbb{D} \) such that

\[
\max \{|f(z)| : |z| \leq |z_0|\} = |f(z_0)| = 1.
\]

Then we can put \( z_0 f'(z_0)/f(z_0) = m \geq 1 \) by means of Lemma 2.4. Therefore, we obtain

\[
|f(z_0)| \cdot \left| \frac{z_0 f'(z_0)}{f(z_0)} \cdot \frac{1}{1 + f(z_0)} \right|^\beta \geq \left( \frac{m}{2} \right)^\beta \geq \left( \frac{1}{2} \right)^\beta,
\]

which contradicts the condition (2.6), and so also (2.4).

This shows that \( |f(z)| = |h'(z) - 1| < 1 \), which implies \( \text{Re} h'(z) > 0 \) for all \( z \in \mathbb{D} \). But this means that \( \text{Re}[u_p'(z)/b_1] > 0 \) for all \( z \in \mathbb{D} \). Note that \( \varphi(z) = b_1 z \) is
convex in \( \mathbb{D} \). For this \( \varphi \), the function \( u_p \) satisfies \( \text{Re}[u_p'(z)/\varphi'(z)] > 0 \) for all \( z \in \mathbb{D} \). Consequently, by Lemma 2.3 we deduce that \( u_p \) is univalent\(^1\) in \( \mathbb{D} \).

By setting \( \beta = 0 \) and \( \beta = 1 \) in Theorem 2.1, we obtain the following corollaries:

**Corollary 2.1.** (Á. Baricz [54]) If \( u_p(z) = \sum_{n=0}^{\infty} b_n z^n \), where \( b_1 > 0 \), satisfies the condition \(|u_p'(z) - b_1| < b_1 \) for all \( z \in \mathbb{D} \), then it is univalent in \( \mathbb{D} \).

**Corollary 2.2.** (Á. Baricz [54]) If \( u_p(z) = \sum_{n=0}^{\infty} b_n z^n \), where \( b_1 > 0 \), satisfies the following condition \(|zu_p''(z)/u_p'(z)| < 1/2 \) for all \( z \in \mathbb{D} \), then it is univalent in \( \mathbb{D} \).

**Remark 2.1.** In fact the results of Theorem 2.1 and Corollaries 2.1, 2.2 hold in the disk \(|z| < 4/|c|\), which for \( 0 < |c| < 4 \) is larger than the unit disk. For this we remind the reader that the function \( qF_1(\kappa, \cdot) \) is a special case of the function \( qF_r(a_1, \ldots, a_q, b_1, \ldots, b_r, \cdot) \) (see Sect. 1.1). By applying Theorem 1, Corollaries 1, 2 due to S. Owa and H.M. Srivastava [168] to the function \( F(z) = qF_1(\kappa, z) \), we know that

\[
|F'(z) - b_1|^{1-\beta} \cdot \left| \frac{zF''(z)}{F'(z)} \right|^\beta < (b_1)^{1-\beta} \left( \frac{1}{2} \right)^{\beta} \quad \text{for all } z \in \mathbb{D} \tag{2.7}
\]

and some fixed \( \beta \geq 0 \), then \( F \) is univalent in \( \mathbb{D} \). Now using the fact that \( F(z) = u_p(-4z/c) \) and changing \( z \) with \( -cz/4 \) the inequality (2.7) becomes (2.4), which means that \( u_p \) is univalent in the disk \(|-cz/4| < 1\).

### 2.1.2 Sufficient Conditions Involving the Admissible Function Method

The next lemma will be used to prove several theorems.

**Lemma 2.5.** (S.S. Miller and P.T. Mocanu [153, 155]) Let \( \mathcal{E} \subseteq \mathbb{C} \) be a set in the complex plane \( \mathbb{C} \) and \( \psi : \mathbb{C} \times \mathbb{D} \to \mathbb{C} \) a function, that satisfies the admissibility condition \( \psi(\rho, \sigma, \mu + \nu; z) \notin \mathcal{E} \), where \( z \in \mathbb{D} \), \( \rho, \sigma, \mu, \nu \in \mathbb{R} \) with \( \mu + \sigma \leq 0 \) and \( \sigma \leq -(1 + \rho^2)/2 \). If \( h \) is analytic in the unit disk \( \mathbb{D} \), with \( h(0) = 1 \) and \( \psi(h(z), zh'(z), z^2h''(z); z) \in \mathcal{E} \) for all \( z \in \mathbb{D} \), then \( \text{Re} h(z) > 0 \) for all \( z \in \mathbb{D} \). In particular, if we only have \( \psi : \mathbb{C} \times \mathbb{D} \to \mathbb{C} \), the admissibility condition reduces to \( \psi(\rho i, \sigma; z) \notin \mathcal{E} \) for all \( z \in \mathbb{D} \) and \( \rho, \sigma \in \mathbb{R} \) with \( \sigma \leq -(1 + \rho^2)/2 \).

Using the above lemma we can prove the following theorem.\(^2\)

\(^1\)We note that since \( b_1 > 0 \) here can be applied another univalence criteria. Namely, due to K. Noshiro [166] and S.E. Warschawski [226] it is known that if \( f : \mathbb{D} \to \mathbb{C} \) is analytic and \( \text{Re} f'(z) > 0 \) for all \( z \in \mathbb{D} \), then \( f \) is univalent in \( \mathbb{D} \). Now, since \( \text{Re} u_p'(z) > 0 \) for all \( z \in \mathbb{D} \), we conclude that \( u_p \) is univalent in \( \mathbb{D} \).

\(^2\)Note that there is a mistake in the mentioned paper [38] of the author, namely the expression \( "u_p(z) = z^{-p/2}v_p(z^{1/2})" \) should be changed with \( "u_p(z) = 2^p \cdot \Gamma(\kappa)z^{-p/2}v_p(z^{1/2})" \).
Theorem 2.2. (Á. Baricz [38]) If \( b, p, c \in \mathbb{R} \) are such that \( \kappa \geq |c|/4 + 1 \), then \( \text{Re} u_p(z) > 0 \) for all \( z \in \mathbb{D} \). Further if \( \kappa \geq |c|/4 \) and \( c \neq 0 \), then \( u_p \) is univalent in \( \mathbb{D} \).

Proof. When \( c = 0 \), then we have \( u_p(z) \equiv 1 \), thus \( \text{Re} u_p(z) > 0 \) for all \( z \in \mathbb{D} \). Next suppose that \( c \neq 0 \). Put \( h = u_p \). Since \( h \) satisfies (1.21), we have

\[
4z^2 h''(z) + 4\kappa zh'(z) + czh(z) = 0. \tag{2.8}
\]

If we consider \( \psi(r, s, t; z) = 4t + 4ks + czr \) and \( E = \{0\} \), then the equation (2.8) implies \( \psi(h(z), zh'(z), z^2h''(z); z) \in E \) for all \( z \in \mathbb{D} \). Next we will use Lemma 2.5 to prove that \( \text{Re} h(z) > 0 \) for all \( z \in \mathbb{D} \). If we put \( z = x + iy \), where \( x, y \in \mathbb{R} \), then

\[
\text{Re} \psi(\rho i, \sigma, \mu + vi; x + iy) = 4(\mu + \sigma) + 4(\kappa - 1)\sigma - c\rho y \quad \text{for all} \quad \rho, \sigma, \mu, v \in \mathbb{R}.
\]

Let \( \rho, \sigma, \mu, v \in \mathbb{R} \) satisfy \( \mu + \sigma \leq 0 \) and \( \sigma \leq -(1 + \rho^2)/2 \). Since \( \kappa > 1 \), we have

\[
\text{Re} \psi(\rho i, \sigma, \mu + vi; x + iy) \leq -2(\kappa - 1)\rho^2 - c\rho y - 2(\kappa - 1).
\]

Set \( Q_1(\rho) = -2(\kappa - 1)\rho^2 - c\rho y - 2(\kappa - 1) \). This value will be strictly negative for all real \( \rho \), because the discriminant \( \Delta \) of \( Q_1(\rho) \) satisfies

\[
\Delta = c^2y^2 - 16(\kappa - 1)^2 < c^2 - 16(\kappa - 1)^2 \leq 0
\]

whenever \( y \in (-1, 1) \). Consequently \( \psi \) satisfies the admissibility condition of Lemma 2.5. Hence by Lemma 2.5 we conclude \( \text{Re} h(z) = \text{Re} u_p(z) > 0 \) for all \( z \in \mathbb{D} \).

When \( \kappa \geq |c|/4 \) and \( c \neq 0 \), then the above result implies \( \text{Re} u_{p+1}(z) > 0 \) for all \( z \in \mathbb{D} \). Using Lemma 1.2 we conclude that

\[
\text{Re} \left[ -\frac{4\kappa}{c} u_p'(z) \right] = \text{Re} u_{p+1}(z) > 0 \quad \text{for all} \quad z \in \mathbb{D}.
\]

This means that \( u_p \) is close-to-convex with respect to the convex function \( \varphi : \mathbb{D} \to \mathbb{C} \), defined by \( \varphi(z) = -(cz)/(4\kappa) \). By Lemma 2.3 it follows that \( u_p \) is univalent in \( \mathbb{D} \).

\(\square\)

Note that similar results as in Theorem 2.2 for Gaussian and confluent hypergeometric functions were obtained by S.S. Miller and P.T. Mocanu [154].

In fact Theorem 2.2 may be extended for complex parameters, as we can see in the following result.

Theorem 2.3. (Á. Baricz [42]) If \( b, p, c \in \mathbb{C} \) are such that \( \text{Re} \kappa \geq |c|/4 + 1 \), then \( \text{Re} u_p(z) > 0 \) for all \( z \in \mathbb{D} \). Further if \( \text{Re} \kappa \geq |c|/4 \) and \( c \neq 0 \), then \( u_p \) is univalent in \( \mathbb{D} \).

Proof. The proof of this theorem is similar to the proof of the previous theorem. If \( c = 0 \), then \( u_p(z) \equiv 1 \), and consequently \( \text{Re} u_p(z) > 0 \) for all \( z \in \mathbb{D} \). Suppose that \( \text{Re} \kappa \geq |c|/4 + 1 \) and \( c \neq 0 \). Again we denote \( h = u_p \), \( \psi(r, s, t; z) = 4t + 4ks + czr \)
and $\mathbb{E} = \{0\}$. Then we have $\psi(h(z), zh'(z), z^2h''(z); z) \in \mathbb{E}$ for all $z \in \mathbb{D}$. Now we will use Lemma 2.5 to prove that $\text{Re} h(z) > 0$. If we put $z = x + iy$ and $c = c_1 + ic_2$, where $x, y, c_1, c_2 \in \mathbb{R}$, then

$$\text{Re} \psi(\rho i, \sigma, \mu + vi; x + iy) = 4(\mu + \sigma) + 4(\text{Re} \kappa - 1)\sigma - (c_1y + c_2x)\rho$$

for all $\rho, \sigma, \mu, v \in \mathbb{R}$. Let $\rho, \sigma, \mu, v \in \mathbb{R}$ satisfy $\mu + \sigma \leq 0$ and $\sigma \leq -(1 + \rho^2)/2$. Since $\text{Re} \kappa > 1$, we have

$$\text{Re} \psi(\rho i, \sigma, \mu + vi; x + iy) \leq -2(\text{Re} \kappa - 1)\rho^2 - (c_1y + c_2x)\rho - 2(\text{Re} \kappa - 1).$$

Set $Q_1(\rho) = -2(\text{Re} \kappa - 1)\rho^2 - (c_1y + c_2x)\rho - 2(\text{Re} \kappa - 1)$. This value will be strictly negative for all real $\rho$, because the discriminant $\Delta$ of $Q_1(\rho)$ satisfies

$$\Delta = (c_1y + c_2x)^2 - 16(\text{Re} \kappa - 1)^2$$

$$\leq |c|^2|z|^2 - 16(\text{Re} \kappa - 1)^2$$

$$< |c|^2 - 16(\text{Re} \kappa - 1)^2 \leq 0$$

whenever $z \in \mathbb{D}$. Consequently $\psi$ satisfies the admissibility condition of Lemma 2.5. Hence by Lemma 2.5 we conclude $\text{Re} h(z) = \text{Re} u_p(z) > 0$ for all $z \in \mathbb{D}$. The second part of the theorem is proved just like in the case of Theorem 2.2. \qed

Note that results similar to those given in Theorem 2.3 were obtained for confluent hypergeometric functions by S. Kanas and J. Stankiewicz [128].

Another extension of Theorem 2.2 for real parameters is the following theorem.

**Theorem 2.4.** (Á. Baricz [52]) Let $\alpha \in [0, 1/2)$, and let $b, p, c$ be arbitrary real numbers. If $\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4 + 1$, then $\text{Re} u_p(z) > \alpha$ for all $z \in \mathbb{D}$. Moreover, if $c \neq 0$ and $\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4$, then $u_p$ is close-to-convex of order $\alpha$ in $\mathbb{D}$.

**Proof.** First assume that $c = 0$. Then $u_p(z) \equiv 1$, and consequently $\text{Re} u_p(z) > \alpha$ for all $z \in \mathbb{D}$. Now suppose that $\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4 + 1$ and $c \neq 0$. Define the function $h : \mathbb{D} \to \mathbb{C}$ by

$$h(z) = \frac{u_p(z) - \alpha}{1 - \alpha}.$$ 

Since $u_p$ satisfies (1.21), $h$ will satisfy the following differential equation:

$$4z^2h''(z) + 4\kappa zh'(z) + cz\left[h(z) + \alpha \frac{1}{1 - \alpha}\right] = 0. \quad (2.9)$$

Using $\psi(r, s, t; z) = 4t + 4\kappa s + cz[r + \alpha/(1 - \alpha)]$ and $\mathbb{E} = \{0\}$, we see that equation (2.9) implies $\psi(h(z), zh'(z), z^2h''(z); z) \in \mathbb{E}$ for all $z \in \mathbb{D}$. Next we use Lemma 2.5 to prove that $\text{Re} h(z) > 0$ for all $z \in \mathbb{D}$. For $z = x + iy$, where $x, y \in \mathbb{R}$, we have

$$\text{Re} \psi(\rho i, \sigma, \mu + vi; x + iy) = 4(\mu + \sigma) + 4(\kappa - 1)\sigma - cp y + \alpha cx/(1 - \alpha)$$
for all $\rho, \sigma, \mu, \nu \in \mathbb{R}$. Let $\rho, \sigma, \mu, \nu \in \mathbb{R}$ satisfy $\mu + \sigma \leq 0$ and $\sigma \leq -(1 + \rho^2)/2$. Since $\kappa - 1 \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4 > 0$, we obtain

$$\text{Re } \psi(\rho i, \sigma, \mu + \nu i; x + iy) \leq -2(\kappa - 1)\rho^2 - cy\rho - 2(\kappa - 1) + \alpha cx/(1 - \alpha).$$

Set $Q_1(\rho) = -2(\kappa - 1)\rho^2 - cy\rho - 2(\kappa - 1) + \alpha cx/(1 - \alpha)$. This value is strictly negative for all real $\rho$, because the discriminant $\Delta_1$ of $Q_1(\rho)$ satisfies

$$\Delta_1 = c^2y^2 - 16(\kappa - 1)^2 + 8\alpha cx(\kappa - 1)/(1 - \alpha) < c^2(1 - x^2) - 16(\kappa - 1)^2 + 8\alpha cx(\kappa - 1)/(1 - \alpha) = Q_2(x) \leq 0,$$

whenever $x^2 + y^2 < 1$ (because $z$ is in the unit disk) and the discriminant $\Delta_2$ of $Q_2(x)$ is negative. $\Delta_2$ has the form

$$\Delta_2 = 4c^2 \left[ -16 \frac{1 - 2\alpha}{(1 - \alpha)^2}(\kappa - 1)^2 + c^2 \right]$$

and this is negative if and only if we have $\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4 + 1$. Hence by Lemma 2.5 we conclude

$$\text{Re } h(z) = \text{Re } \left[ \frac{1}{1 - \alpha}(u_p(z) - \alpha) \right] > 0 \text{ for all } z \in \mathbb{D},$$

and this means that $\text{Re } u_p(z) > \alpha$ for all $z \in \mathbb{D}$.

Now suppose that $\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4$ and $c \neq 0$. Then the above result implies $\text{Re } u_{p+1}(z) > \alpha$ for all $z \in \mathbb{D}$. Using again Lemma 1.2 we conclude that

$$\text{Re } \left[ \frac{-4\kappa}{c} u_p'(z) \right] = \text{Re } u_{p+1}(z) > \alpha \text{ for all } z \in \mathbb{D}.$$

This means that $u_p$ is close-to-convex of order $\alpha$ in $\mathbb{D}$ with respect to the function $\varphi(z) = -(cz)/(4\kappa)$.

Note that results similar to those given in Theorem 2.4 were obtained by J.H. Choi et al. [84] and by S. Ponnusamy and M. Vuorinen [187, 188] for Gaussian and confluent hypergeometric functions.

### 2.1.3 Sufficient Conditions Involving the Alexander Transform

The following results of L. Fejér and S. Ozaki will be used to prove another sufficient condition for the univalence of the function $u_p$, defined by (1.20). As we can
Lemma 2.8. (S. Ozaki [167]) If the function \( f(z) = \sum_{n \geq 1} A_n z^n \), where \( A_1 = 1 \) and \( A_n \geq 0 \) for all \( n \geq 2 \), is analytic in \( \mathbb{D} \), and the sequences \( \{ n A_n - (n + 1) A_{n+1} \} \) and \( \{ n A_n \} \) both are decreasing, then \( f \) is starlike in \( \mathbb{D} \).

Lemma 2.7. (L. Fejér [98]) If the function \( g(z) = \sum_{n \geq 1} C_n z^{n-1} \), where \( C_1 = 1 \) and \( C_n \geq 0 \) for all \( n \geq 2 \), is analytic in \( \mathbb{D} \) and if \( \{ C_n \} \) is a convex decreasing sequence, i.e., \( C_n - 2C_{n+1} + C_{n+2} \geq 0 \) and \( C_n - C_{n+1} \geq 0 \) for all \( n \geq 1 \), then

\[
\text{Re} g(z) > 1/2 \text{ for all } z \in \mathbb{D}.
\]

Lemma 2.8. (S. Ozaki [167]) If the function \( f(z) = z + \sum_{n \geq 2} A_n z^n \) is analytic in \( \mathbb{D} \) and if \( 1 \geq 2A_2 \geq \ldots \geq n A_n \geq \ldots \geq 0 \) or \( 1 \leq 2A_2 \leq \ldots \leq n A_n \leq \ldots \leq 2 \), then \( f \) is close-to-convex with respect to \(-\log(1-z)\).

We note that some generalizations of the coefficient conditions for close-to-convex functions from Lemma 2.8 were obtained recently by S. Ponnusamy [176]. Moreover, it is worth to mention here that the coefficient conditions from Lemma 2.8 does not imply necessarily that the function \( f \) is starlike in \( \mathbb{D} \). For example, the coefficients of the function \( f(z) = z + z^2/2 + z^3/3 \) satisfies \( 1 \geq 2A_2 \geq \ldots \geq n A_n \geq \ldots \geq 0 \), but \( f \) is not starlike in \( \mathbb{D} \). Indeed, for \( z = e^{i\theta} \) we have

\[
\text{Re} \left[ \frac{f(z)}{zf'(z)} \right] = \frac{3 + 8 \cos \theta}{6(1 + 2 \cos \theta)},
\]

which is negative for some values of \( \theta \), and thus \( f \) is not starlike in \( \mathbb{D} \). For more details see the paper of I.R. Nezhmetdinov and S. Ponnusamy [162].

It is also important to note here that I.R. Nezhmetdinov and S. Ponnusamy [162] using the duality technique have obtained other sufficient conditions over the MacLaurin coefficients of an analytic and normalized function \( f \) that imply its star-likeness. More precisely, I.R. Nezhmetdinov and S. Ponnusamy [162] in particular proved that if the function \( f(z) = z + \sum_{n \geq 2} a_n z^n \) is analytic in \( \mathbb{D} \) and in addition \( 2 \leq 3a_2 \leq \ldots \leq (n+1)a_n \leq \ldots \) and \( n a_n \leq 2 \) for all \( n \geq 2 \), or \( 2/3 \geq a_2 \geq 2a_3 \geq \ldots \geq (n-1)a_n \geq \ldots \geq 0 \) and \( na_n \geq a_2 \) for all \( n \geq 3 \), then \( f \) is starlike in \( \mathbb{D} \).

For convenience, as in the paper of S. Ponnusamy and M. Vuorinen [187], we denote by \( \mathcal{C} \mathcal{S}^* \) the family of all functions in \( \mathcal{A} \), which are close-to-convex with respect to \(-\log(1-z)\) and also starlike in \( \mathbb{D} \).

Now we state one of our main results which gives a sufficient condition for the Alexander transform \( A_f \) of \( f(z) = zu_p(z) \) to be in the family of starlike functions. The Alexander transform of \( f \) is defined on \( \mathbb{D} \) by

\[
A_f(z) = \int_0^z \frac{(f \ast l)(t)}{t} \, dt = \int_0^z \frac{f(t)}{t} \, dt = z + \sum_{n \geq 2} \frac{a_n}{n} z^n = (f \ast h)(z), \quad (2.10)
\]
where the functions \( f, l \) and \( h \) are defined by
\[
 f(z) = z + \sum_{n \geq 2} \alpha_n z^n, \quad l(z) = \frac{z}{1-z} = \sum_{n \geq 1} \zeta^n
\]
and
\[
 h(z) = -\log(1-z) = \sum_{n \geq 1} \frac{z^n}{n}. \]

We note that the convolution \( f \ast g \), or Hadamard product (see the book of P.L. Duren [90]), of two power series
\[
f(z) = z + \sum_{n \geq 2} \alpha_n z^n \quad \text{and} \quad g(z) = z + \sum_{n \geq 2} \beta_n z^n
\]
is defined as the power series
\[(f \ast g)(z) = z + \sum_{n \geq 2} \alpha_n \beta_n z^n.\]

**Theorem 2.5.** (Á. Baricz [52]) Let \( b, p \) be arbitrary real numbers and let \( c < 0 \). If \( N(c) = [-(c+2) + \sqrt{c^2/2 - 4c + 4}]/2, \kappa \geq N(c)/2 \) and \( f(z) = zu_p(z) \), then \( \Lambda_f \in \mathcal{C} \mathcal{P}^* \). Moreover, we have \( \text{Re} u_p(z) > 1/2 \) for all \( z \in \mathbb{D} \).

**Proof.** From (1.20) we have
\[
f(z) = z + \sum_{n \geq 2} b_{n-1} z^n = \sum_{n \geq 1} \frac{(-c/4)^{n-1} z^n}{(\kappa)_{n-1}(n-1)!}.
\]

So in this case the corresponding Alexander transform takes the form \( \Lambda_f(z) = \sum_{n \geq 1} A_n z^n \), where \( A_n = b_{n-1}/n \) for all \( n \geq 1 \), i.e.
\[
 A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1} n!}.
\]

Obviously we have \( A_1 = 1 \). Because \( c < 0 \) and \( 4\kappa \geq 2N(c) > -c > 0 \), we also have \( A_n > 0 \) for all \( n \geq 2 \). Next we prove that the sequence \( \{ nA_n \}_{n \geq 1} \) is decreasing. Fix any \( n \geq 1 \). From the definition of the Pochhammer symbol it follows
\[
 (n+1)A_{n+1} = -\frac{c}{4(\kappa+n-1)} \cdot A_n. \quad (2.11)
\]

Using (2.11) we have
\[
 nA_n - (n+1)A_{n+1} = \frac{U_1(n) \cdot A_n}{4(\kappa+n-1)}, \quad (2.12)
\]
where \( U_1(n) = 4n^2 + 4(\kappa - 1)n + c. \) Since \( n^2 \geq 2n - 1 \) and \( 4\kappa > -c \), we have

\[
U_1(n) \geq 4(\kappa + 1)n + c - 4 \geq U_1(1) = 4\kappa + c > 0.
\]

Consequently, (2.12) yields \( nA_n > (n + 1)A_{n+1} \). This shows that the sequence \( \{nA_n\}_{n \geq 1} \) is strictly decreasing.

Next, we show that the sequence \( \{nA_n - (n + 1)A_{n+1}\}_{n \geq 1} \) is also decreasing. For convenience we denote \( B_n = nA_n - (n + 1)A_{n+1} \) for each \( n \geq 1 \). Fix any \( n \geq 1 \). Using (2.12), we find that

\[
B_n - B_{n+1} = \frac{U_2(n) \cdot A_n}{2(n + 1)(\kappa + n)(\kappa + n - 1)},
\]

where

\[
U_2(n) = 2n^4 + 4\kappa n^3 + D_1n^2 + D_2n + D_3; \\
D_1 = 2\kappa^2 + 2\kappa + c - 2; \\
D_2 = 2\kappa^2 + (c - 2)\kappa + c; \\
D_3 = (c + 8\kappa)c/8.
\]

Our aim is to show that \( U_2(n) > 0 \). First we observe that \( n^4 \geq 4n^3 - 6n^2 + 4n - 1 \) holds. By using this inequality we obtain \( U_2(n) \geq V(n) \), where

\[
V(n) = 4(\kappa + 2)n^3 + (D_1 - 12)n^2 + (D_2 + 8)n + D_3 - 2.
\]

Clearly, the coefficient of \( n^3 \) in the above expression is nonnegative, since \( \kappa > 0 \). Therefore using that \( n^3 \geq 3n^2 - 3n + 1 \), we obtain \( V(n) \geq W(n) \), where

\[
W(n) = D_4n^2 + D_5n + D_6; \\
D_4 = 2\kappa^2 + 14\kappa + c + 10; \\
D_5 = 2\kappa^2 + (c - 14)\kappa + c - 16; \\
D_6 = c^2/8 + (c + 4)\kappa + 6.
\]

Now, we observe that \( D_4 \) is also nonnegative, because

\[
\kappa \geq [N(c)]/2 > -c/4 > [-7 + \sqrt{29 - 2c}]/2
\]

(the value \([-7 + \sqrt{29 - 2c}]/2 \) is the greatest root of the equation \( D_4 = 0 \)). Similarly \( n^2 \geq 2n - 1 \), therefore \( W(n) \geq X(n) \), where \( X(n) = D_7n + D_8 \). \( D_7 = 2D_4 + D_5 \) and \( D_8 = D_6 - D_4 \). Analogously, by the hypothesis, we can deduce easily that

\[
D_7 = 6\kappa^2 + (c + 14)\kappa + 3c + 4 > 0.
\]
Indeed, the relation
\[ \kappa \geq \frac{N(c)}{2} - c/4 > -\frac{(c + 14) + \sqrt{c^2 - 44c + 100}}{12} = \kappa \]
(here \(\kappa\) is the greatest root of the equation \(D_7 = 0\)) implies that \(D_7\) is nonnegative, and leads to \(X(n) \geq X(1)\). In this case
\[ X(1) = D_4 + D_5 + D_6 = 4\kappa^2 + 2(c + 2)\kappa + c^2/8 + 2c \]
is also positive, because \(\kappa \geq \frac{N(c)}{2} > -c/4 > 0\). Thus, we have proved a chain of inequalities
\[ U_2(n) \geq V(n) \geq W(n) \geq X(n) \geq X(1) > 0, \]
which implies \(B_n - B_{n+1} > 0\). Thus the sequence \(\{nA_n - (n+1)A_{n+1}\}_{n \geq 1}\) is strictly decreasing. By Lemma 2.6 we deduce that \(\Lambda_f\) is starlike in \(\mathbb{D}\).

The sequence \(\{nA_n\}_{n \geq 1}\) is strictly decreasing and \(2A_2 = b_1 = -c/(4\kappa) < 1\). Thus it follows by Lemma 2.8 that \(\Lambda_f\) is close-to-convex with respect to \(-\log(1-z)\). Consequently we have \(\Lambda_f \in \mathcal{CJ}^*\).

Now, we apply Lemma 2.7 to prove that \(\Re u_p(z) > 1/2\) for all \(z \in \mathbb{D}\). For this consider \(g = u_p\). Therefore we have \(C_n = b_{n-1} = nA_n\) for all \(n \geq 1\) and thus the sequence \(\{C_n\}_{n \geq 1}\) is strictly decreasing. In addition we have \(C_n - 2C_{n+1} + C_{n+2} = B_n - B_{n+1} > 0\) for all \(n \geq 1\). Now Lemma 2.7 yields the asserted property and this completes the proof. \(\square\)

Taking \(c = -1\) and \(b = 1\) in Theorem 2.5, we obtain:

**Corollary 2.3.** (Á. Baricz [52]) If \(p \geq -5 + \sqrt{17}/2\)/4 \(\simeq -0.5211310131\ldots\), then we have that \(\int_0^z r^{-p/2}J_p(i\sqrt{t})dt \) is in \(\mathcal{CJ}^*\). Moreover, \(\Re \mathcal{J}_p(\sqrt{z}) > 1/2\) for all \(z \in \mathbb{D}\).

**Proof.** We have \(N(-1) = -1 + \sqrt{17}/2)/2 \simeq 0.9577379740\ldots\). Therefore by Theorem 2.5 we have that \(\Lambda_f\) is in \(\mathcal{CJ}^*\), where \(f(z) = zu_p(z) = \mathcal{J}_p(z^{1/2})\). But from (1.24) it follows that \(\mathcal{J}_p(z^{1/2}) = 2^p\Gamma(p+1)z^{-p/2}I_p(z^{1/2})\). On the other hand, (1.5) and (1.7) imply \(I_p(z) = i^{-p}J_p(iz)\). Thus
\[ \Lambda_f(z) = \int_0^z \mathcal{J}_p(t^{1/2}) dt = (-2i)^p\Gamma(p+1) \int_0^z t^{-p/2}J_p(i\sqrt{t}) dt \]
and this completes the proof. \(\square\)

Another consequence of Theorem 2.5 is obtained by using the recursive relation \(4\kappa u'_p(z) = -cu_{p+1}(z)\) (see Lemma 1.2), namely

**Corollary 2.4.** (Á. Baricz [52]) Let \(b\) and \(p\) be arbitrary real numbers. If \(c < 0\), \(\kappa \geq -c/4 - 1\) and \(\kappa \neq 0\), then \(u_p\) is univalent in \(\mathbb{D}\).
Proof. By the proof of Theorem 2.5 the Alexander transform \( \int_0^z u_{p+1}(t) \, dt \) is close-to-convex with respect to the convex function \(- \log(1 - z)\) if \(\kappa + 1 \geq -c/4\), and therefore, in particular, it is univalent. Using the relation \(4\kappa u'_p(z) = -cu_{p+1}(z)\), we have

\[
\int_0^z u_{p+1}(t) \, dt = -\frac{4\kappa}{c} \int_0^z u'_p(t) \, dt = -\frac{4\kappa}{c} [u_p(z) - 1].
\]

Consequently \(-4\kappa[u_p(z) - 1]/c\) is univalent in \(D\). Since the addition of a constant and the multiplication by a nonzero quantity do not disturb the univalence, we immediately deduce that \(u_p\) is univalent in \(D\). This completes the proof.  

Remark 2.2. By Theorem 2.2 \(u_p\) is univalent in \(D\) when \(\kappa \geq |c|/4\) and \(c \neq 0\). If we consider \(c < 0\), then the above conditions become \(\kappa \geq -c/4\). Since \(-c/4 > -c/4 - 1\), it follows that the result of Corollary 2.4 is better than the result of Theorem 2.2.

Using the idea of the proof of Corollary 2.4, we deduce the following corollary, which is also an immediate consequence of Theorem 2.5.

Corollary 2.5. (Á. Baricz [52]) If for \(b, p \in \mathbb{R}, c < 0\) we have \(\kappa + 1 \geq N(c)/2\), where \(N(c)\) is defined in Theorem 2.5, then the function \((-4\kappa/c)[u_p(z) - 1]\) is in \(\mathcal{C}_M^*\).

2.1.4 Sufficient Conditions Involving Results of L. Fejér

As we can see in this section a direct application of Lemma 2.7 combined with Lemma 2.3 gives in some particular cases a sharper result than Corollary 2.4.

Lemma 2.9. (Á. Baricz [43]) Let \(b, p \in \mathbb{R}, c < 0\) and \(\alpha \in [0, 1)\) be fixed numbers and let

\[
c_0 = \frac{c}{8(\alpha - 1)}; \quad c_1 = \frac{-13 + \sqrt{77 - 2c}}{2}; \quad c_2 = \frac{-(c/2 + 27) + \sqrt{c^2/4 - 23c + 169}}{10}.
\]

If \(\kappa \geq \max\{c_0, c_1, c_2\}\), then the sequence \(\{A_n\}_{n \geq 1}\), defined by

\[
A_1 = 1 \quad \text{and} \quad A_n = \frac{b_{n-1}}{2(1 - \alpha)} \quad \text{for all} \ n \geq 2,
\]

is a nonnegative convex decreasing sequence, where \(b_n\) is defined by (1.18).

Proof. First we prove that the sequence \(\{A_n\}_{n \geq 1}\) is decreasing. The inequality \(A_2 \leq A_1\) is equivalent to \(\kappa \geq c_0 = c/[8(\alpha - 1)]\). Now we prove that \(A_n - A_{n+1} \geq 0\)
for all \( n \geq 2 \). Let \( n \geq 2 \) be fixed. By definition we have \( A_n - A_{n+1} = [b_{n-1} - b_n]/[2(1 - \alpha)] \). Therefore using the recursive relation \( 4n(\kappa + n - 1)b_n = -c b_{n-1} \), we obtain
\[
A_n - A_{n+1} = \frac{[4n^2 + 4(\kappa - 1)n + c] \cdot b_{n-1}}{8(1 - \alpha)n(\kappa + n - 1)}.
\]

Let us denote \( M_1(n) = 4n^2 + 4(\kappa - 1)n + c \). From \( (n - 2)^2 \geq 0 \), we obtain \( n^2 \geq 4n - 4 \). Therefore \( M_1(n) \geq 4(\kappa + 3)n + c - 16 \). By the hypothesis \( \kappa > 0 \), and therefore \( M_1(n) \geq 8\kappa + c + 8 \) and this value is nonnegative, because \( 8\kappa \geq c/(\alpha - 1) \geq -c - 8 \). Consequently we have shown that \( A_n - A_{n+1} \geq 0 \).

The assumptions of the lemma imply that the sequence \( \{A_n\}_{n \geq 1} \) is nonnegative. Therefore, we need only to show that this sequence is convex, i.e.
\[
A_n - 2A_{n+1} + A_{n+2} \geq 0 \text{ for all } n \geq 1. \tag{2.13}
\]

Observe that \( A_1 - 2A_2 + A_3 \geq 0 \) is equivalent to the inequality \( 16\kappa + c + 16 \geq 0 \). But \( 16\kappa + c + 16 \geq 8(\kappa + 1) > 0 \) and therefore the inequality (2.13) holds for \( n = 1 \). Next we verify inequality (2.13) for \( n \geq 2 \). Let \( n \geq 2 \) be fixed. From the definition of \( A_n \), we find that
\[
A_n - 2A_{n+1} + A_{n+2} = \frac{b_{n-1}M_2(n)}{4(1 - \alpha)n(n+1)(\kappa + n - 1)(\kappa + n)},
\]
where
\[
M_2(n) = 2n^4 + 4\kappa n^3 + D_1n^2 + D_2n + D_3;
\]
\[
D_1 = 2\kappa^2 + 2\kappa + c - 2;
\]
\[
D_2 = 2\kappa^2 + (c - 2)\kappa + c;
\]
\[
D_3 = (8\kappa + c)c/8.
\]

With some computation we get
\[
A_n - 2A_{n+1} + A_{n+2} = \frac{b_{n-1}M_3(n)}{2(1 - \alpha)n(n+1)(\kappa + n - 1)(\kappa + n)},
\]
where
\[
M_3(n) = (n - 2)^4 + 2(\kappa + 4)(n - 2)^3 + E_1(n - 2)^2 + E_2(n - 2) + E_3;
\]
\[
E_1 = \kappa^2 + 13\kappa + c/2 + 23;
\]
\[
E_2 = 5\kappa^2 + (c/2 + 27)\kappa + 5c/2 + 28;
\]
\[
E_3 = 9\kappa^2 + (3c/2 + 57)\kappa + c^2/16 + 9c/2 + 81.
\]

If \( c \in [-37, 0) \), then the greatest real root \( c_1 = [-13 + \sqrt{77 - 2c}]/2 \) of the equation \( E_1 = 0 \) is negative. From the condition \( c_1 < c/\lfloor 8(\alpha - 1) \rfloor \leq \kappa \) we deduce that \( E_1 \) is
nonnegative. Analogously \( c_2 = [-(c/2 + 27) + \sqrt{c^2/4 - 23c + 169}] / 10 \) is the greatest real root of the equation \( E_2 = 0 \) and \( c_2 \) is also negative (when \( c \in [-37, 0) \)). This and the condition \( c_2 < \kappa \), imply that \( E_2 \) is nonnegative. In this case \( (c \in [-37, 0)) \) the greatest root of the equation \( E_3 = 0 \) is \( c_3 = [-(c/2 + 19) + \sqrt{c + 37}] / 6 \) and this root is nonnegative. Moreover \( c_3 \leq c/[8(\alpha - 1)] \), therefore from the condition \( c_0 \leq \kappa \) we deduce that \( E_3 \) is also nonnegative.

For \( c < -37 \) the expression \( E_3 \) is nonnegative and because of \( \kappa \geq \max\{c_0, c_1, c_2\} \) it follows that \( E_1 \) and \( E_2 \) are also positive.

Therefore all the numbers \( E_i \) \((i \in \{1, 2, 3\})\) are nonnegative. From this observation we deduce that \( M_3(n) \geq 0 \). Thus \( \{A_n\}_{n \geq 1} \) is a convex sequence.

Using Lemma 2.9 we obtain the next result.

**Theorem 2.6.** (Á. Baricz [43]) Let \( \alpha \in [0, 1), b, p \in \mathbb{R}, c < 0 \). If \( \kappa \geq \max\{c_0, c_1, c_2\} \) (where \( c_0, c_1, c_2 \) are defined in Lemma 2.9), then \( \Re u_p(z) > \alpha \) for all \( z \in \mathbb{D} \).

**Proof.** For all \( n \geq 1 \) let \( A_n \) be defined as in Lemma 2.9. Then the conclusion \( \Re u_p(z) > \alpha \) for all \( z \in \mathbb{D} \) is equivalent to

\[
\Re \left[ 1 + \sum_{n \geq 2} A_n z^{n-1} \right] > 1/2 \text{ for all } z \in \mathbb{D}.
\]

By Lemma 2.9 and the hypotheses, we observe that the sequence \( \{A_n\}_{n \geq 1} \) is nonnegative, convex and decreasing. Therefore the conclusion follows from Lemma 2.7. ⊓⊔

In the proof of Lemma 2.9 we have seen that \( \max\{c_0, c_1, c_2\} = c_0 \) for \( c \in [-37, 0) \). Therefore by taking \( \alpha = 0 \) in Theorem 2.6 we immediately obtain the following result:

**Theorem 2.7.** (Á. Baricz [43]) If \( \kappa \geq -c/8 - 1 \) for \( b, p \in \mathbb{R}, c \in [-37, 0) \), then \( u_p \) is univalent in \( \mathbb{D} \).

**Proof.** We apply Theorem 2.6 and conclude that \( \Re u_{p+1}(z) > 0 \) for all \( z \in \mathbb{D} \). Using Lemma 1.2, it follows that

\[
\Re \left[ -\frac{4\kappa}{c} u'_p(z) \right] = \Re u_{p+1}(z) > 0 \text{ for all } z \in \mathbb{D}.
\]

Thus, \( u_p \) is close-to-convex with respect to the function \( \varphi(z) = -(cz)/(4\kappa) \). By Lemma 2.3 it follows that \( u_p \) is univalent in \( \mathbb{D} \). ⊓⊔

**Remark 2.3.** In Corollary 2.4 we have seen that \( u_p \) is univalent in \( \mathbb{D} \) for arbitrary \( b \in \mathbb{R}, c < 0, \kappa \geq -c/4 - 1 \). Clearly we have \( -c/4 - 1 > -c/8 - 1 \). Hence for \( c \in [-37, 0) \) the result of Theorem 2.7 is better than the result of Corollary 2.4.

Now we end this section with the following consequence of Theorem 2.6.
Corollary 2.6. (Á. Baricz [43]) If \( \alpha \in [0, 1), c \in [-37, 0), b \in \mathbb{R} \) and in addition \( 8(1-\alpha)(\kappa+1)+c \geq 0 \), then \( \Re\{(-4\kappa/c)u'_p(z)\} > \alpha \) for all \( z \in \mathbb{D} \).

Proof. By Lemma 1.2 we know that \( 4\kappa u'_p(z) = -cu_{p+1}(z) \) for all \( z \in \mathbb{D} \). Therefore we have \( u_{p+1}(z) = (-4\kappa/c)u'_p(z) \) for all \( z \in \mathbb{D} \). Using Theorem 2.6 for \( p + 1 \) the inequality follows.

\[ \Box \]

2.2 Starlikeness and Convexity Properties of Generalized Bessel Functions

Our aim in this section is to find sufficient conditions for the function \( u_p \), i.e. for the hypergeometric function \( {_0F_1}(\kappa, -cz/4) \), to be starlike, convex, starlike of order \( \alpha \), convex of order \( \alpha \) and close-to-convex with respect to some known functions. Moreover, in the third part of this section we present some immediate applications of convexity and univalence involving Bessel functions associated with the Hardy space of analytic functions.

Note that the results included in this section were proved by the author in the papers [38, 42, 43, 52]. As in the previous section we begin with an application of Jack’s lemma.

2.2.1 Sufficient Conditions Involving Jack’s Lemma

The following lemma proved by S. Owa and H.M. Srivastava [168] will be useful in this section to find sufficient conditions for the functions \( u_p, zu_p \) to be starlike, and for the function \( u_p \) to be convex of order \( \alpha \).

Lemma 2.10. (S. Owa and H.M. Srivastava [168]) If \( f \in \mathcal{A} \) and

\[
\left| \frac{zf''(z)}{f(z)} - 1 \right|^{1-\beta} \left| \frac{zf'''(z)}{f'(z)} \right|^{\beta} < (1-\alpha)^{1-2\beta}(1-3\alpha/2+\alpha^2)^{\beta}
\]

for some fixed \( \alpha \in [0, 1/2] \) and \( \beta \geq 0 \), and for all \( z \in \mathbb{D} \), then \( f \) is in the class \( \mathcal{S}^*(\alpha) \).

By applying Lemma 2.10, we prove the following theorems concerning the starlikeness of the generalized Bessel functions of order \( \alpha \).

Theorem 2.8. (Á. Baricz [54]) If the function \( u_p \), defined by (1.20), satisfies the condition

\[
\left| \frac{zu'_p(z)}{u_p(z)} \right| < 1 - \alpha,
\]

where \( \alpha \in [0, 1/2] \) and \( z \in \mathbb{D} \), then \( zu_p \in \mathcal{S}^*(\alpha) \).
Proof. Define the function $g$ by $g(z) = zu_p(z)$ for all $z \in \mathbb{D}$. The given condition becomes
\[
\frac{zg'(z)}{g(z)} - 1 < 1 - \alpha,
\]
where $z \in \mathbb{D}$. By taking $\beta = 0$ in Lemma 2.10, we thus conclude from the previous inequality that $g \in \mathcal{S}^*(\alpha)$, which proves Theorem 2.8. □

Remark 2.4. Evidently, since $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^* \subset \mathcal{S}$, the function $zu_p$ is univalent in $\mathbb{D}$ under the hypothesis of Theorem 2.8.

Theorem 2.9. (Á. Baricz [54]) If the function $u_p$, defined by (1.20), satisfies the condition
\[
\frac{|zu''_p(z)|}{|u'_p(z)|} < 1 - \frac{3\alpha/2 + \alpha^2}{1 - \alpha},
\]
where $\alpha \in [0, 1/2]$ and $z \in \mathbb{D}$, then it is starlike of order $\alpha$ with respect to 1.

Proof. We define the function $h : \mathbb{D} \to \mathbb{C}$ by $h(z) = [u_p(z) - b_0]/b_1$. Then $h \in \mathcal{S}$ and
\[
\frac{|zh''(z)|}{|h'(z)|} = \frac{|zu''_p(z)|}{|u'_p(z)|} < 1 - \frac{3\alpha/2 + \alpha^2}{1 - \alpha},
\]
where $\alpha \in [0, 1/2]$ and $z \in \mathbb{D}$. Consequently, it follows by Lemma 2.10 (choose $\beta = 1$) that $h \in \mathcal{S}^*(\alpha)$, i.e. $h$ is starlike of order $\alpha$ with respect to the origin for $\alpha \in [0, 1/2]$. Now Theorem 2.9 follows from the definition of the function $h$, because $b_0 = 1$. □

Remark 2.5. Since $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^* \subset \mathcal{S}$, the function $z \mapsto [u_p(z) - b_0]/b_1$ is univalent in $\mathbb{D}$ under the hypothesis of Theorem 2.9.

Corresponding to Theorem 2.8, we have the following result on the convexity of the generalized Bessel functions.

Theorem 2.10. (Á. Baricz [54]) If for $\alpha \in [0, 1/2]$ and $c \neq 0$ we have
\[
\frac{|zu'_{p+1}(z)|}{|u_{p+1}(z)|} < 1 - \alpha
\]
for all $z \in \mathbb{D}$, then $u_p \in \mathcal{C}(\alpha)$.

Proof. Theorem 2.8 implies that $zu_{p+1} \in \mathcal{S}^*(\alpha)$. On the other hand, Lemma 1.2 yields $b_1zu_{p+1}(z) = u'_{p}(z)$, where $b_1 = -c/(4\kappa) \neq 0$. Consequently $zu'_p \in \mathcal{S}^*(\alpha)$ and therefore $u_p \in \mathcal{C}(\alpha)$. □

Remark 2.6. The results of the Theorems 2.8, 2.9 and 2.10 hold in the disk $|z| < 4/|c|$, which for $0 < |c| < 4$ is larger than the unit disk. This can be proved by using the same argument as in Remark 2.1. By applying the Theorems 2, 3, and 5 by S. Owa and H.M. Srivastava [168] to the function $F(z) = _0F_1(\kappa, z)$, using the identity $F(z) = u_p(-4z/c)$ and changing $z$ with $-cz/4$, we obtain that the Theorems 2.8, 2.9 and 2.10 hold in the disk $|z| < 4/|c|$. 

2 Geometric Properties of Generalized Bessel Functions
2.2.2 Sufficient Conditions Involving the Admissible Function Method

In this section we use the admissible function method (or, with other words the technique of differential subordinations), i.e. Lemma 2.5, to give sufficient conditions for the function \( z^{up} \) to be starlike and for the function \( u_p \) to be convex.

**Theorem 2.11.** (Á. Baricz [38]) Let \( b, c, p \in \mathbb{R} \), of which \( c \neq 0 \). The functions \( w_p \) and \( u_p \), defined by (1.15) and (1.20), respectively, have the following properties:

(a) If \( \kappa \geq |c|/4 + 1/2 \), then \( u_p \in \mathcal{C} \).
(b) If \( \kappa \geq |c|/4 + 3/2 \), then \( z^{up} \in \mathcal{S}^* \).
(c) If \( \kappa \geq |c|/2 + 1 \), then \( z^{1-p}w_p \in \mathcal{S}^* \).

**Proof.**

(a) Since \( \kappa > |c|/4 \) and \( c \neq 0 \), Theorem 2.2 implies \( \text{Re} u_{p+1}(z) > 0 \) for all \( z \in \mathbb{D} \). According to Lemma 1.2 it follows that \( u_p'(z) \neq 0 \) for all \( z \in \mathbb{D} \). Define \( q : \mathbb{D} \rightarrow \mathbb{C} \) by

\[
q(z) = 1 + \frac{zu''(z)}{u'(z)}.
\]

The function \( q \) is analytic in \( \mathbb{D} \) and \( q(0) = 1 \). Suppose that \( z \neq 0 \). Since \( u_p \) satisfies the differential equation (1.21), we have

\[
4zu''(z) + 4\kappa u'(z) + cu(z) = 0.
\]

If we differentiate this equation, we obtain

\[
4zu'''(z) + 4(\kappa + 1)u''(z) + cu'(z) = 0.
\]

We know that \( u_p'(z) \neq 0 \), therefore if we divide both sides of this equation with \( u_p'(z) \), and multiply with \( z \), we obtain

\[
4 \left[ \frac{zu''(z)}{u'(z)} \right] + 4(\kappa + 1) \left[ \frac{zu''(z)}{u'(z)} \right] + cz = 0. \tag{2.14}
\]

Now we differentiate logarithmically and multiply with \( z \) on both sides of the equation \( q(z) - 1 = zu''/u'(z) \). Thus we obtain

\[
\frac{zq'(z)}{q(z) - 1} = 1 + \frac{zu''(z)}{u'(z)} - [q(z) - 1],
\]
According to (a) the function

\[ \frac{z u_p''(z)}{u_p'(z)} = \frac{z q'(z) + q^2(z) - 3q(z) + 2}{q(z) - 1}. \]

In view of (2.14) this result reveals that \( q \) satisfies the following differential equation:

\[ 4z q'(z) + 4q^2(z) + 4(\kappa - 2)q(z) + cz - 4(\kappa - 1) = 0. \] (2.15)

Obviously, this equation is also valid when \( z = 0 \).

If we use \( \psi(r,s;z) = 4s + 4r^2 + 4(\kappa - 2)r + cz - 4(\kappa - 1) \) and \( E = \{0\} \), then (2.15) implies \( \psi(q(z), z q'(z); z) \in E \) for all \( z \in \mathbb{D} \). Now we use Lemma 2.5 to prove that \( \text{Re} q(z) > 0 \) for all \( z \in \mathbb{D} \). For \( z = x + iy \in \mathbb{D} \) (with \( x,y \in \mathbb{R} \)) and \( \rho, \sigma \in \mathbb{R} \) satisfying \( \sigma \leq -(1 + \rho^2)/2 \), we obtain

\[ \text{Re} \psi(\rho i, \sigma; x + iy) = 4\sigma - 4\rho^2 + cx - 4(\kappa - 1) \]
\[ \leq -6\rho^2 + cx - 2(2\kappa - 1) \]
\[ < |c| - 2(2\kappa - 1) \leq 0. \]

By Lemma 2.5 we conclude that \( \text{Re} q(z) > 0 \) for all \( z \in \mathbb{D} \), which shows that \( u_p \) is convex in \( \mathbb{D} \).

(b) According to (a) the function \( u_{p-1} \) is convex. By Alexander’s duality theorem, i.e. Lemma 2.1 it follows that \( zu_p'_{p-1} \in \mathcal{S}^* \). But, on the other hand, Lemma 1.2 yields

\[ cz u_p(z) = -4(\kappa - 1)zu_{p-1}'(z). \]

Consequently, it results that \( zu_p \in \mathcal{S}^* \).

(c) According to Theorem 2.2 we have \( \text{Re} u_p(z) > 0 \) for all \( z \in \mathbb{D} \), hence \( u_p(z) \neq 0 \) for all \( z \in \mathbb{D} \). Define \( q : \mathbb{D} \rightarrow \mathbb{C} \) by

\[ q(z) = 1 + 2 \frac{zu_p'(z)}{u_p(z)}. \]

The function \( q \) is analytic in \( \mathbb{D} \) and \( q(0) = 1 \). Assume that \( z \neq 0 \). Because \( u_p \) satisfies the equation (1.21), it satisfies the following equation too:

\[ 4 \left[ \frac{zu_p''(z)}{u_p'(z)} \right] \left[ \frac{zu_p'(z)}{u_p(z)} \right] + 4\kappa \left[ \frac{zu_p'(z)}{u_p(z)} \right] + cz = 0. \] (2.16)

We proceed as in part (a), we differentiate logarithmically and multiply with \( z \) the expression \([q(z) - 1]/2 = [zu_p'(z)]/u_p(z)\). We obtain

\[ \frac{zu_p''(z)}{u_p'(z)} = \frac{2z q'(z) + q^2(z) - 4q(z) + 3}{2(q(z) - 1)}. \] (2.17)
In view of (2.16) this result reveals that $q$ satisfies the differential equation

$$2zq'(z) + q^2(z) + 2(\kappa - 2)q(z) + cz - 2(\kappa - 3/2) = 0, \quad (2.18)$$

which is also valid when $z = 0$.

If $\psi(r,s;z) = 2s + r^2 + 2(\kappa - 2)r + cz - 2(\kappa - 3/2)$ and $E = \{0\}$, then (2.18) implies $\psi(q(z), zq'(z); z) \in E$ for all $z \in \mathbb{D}$. We use Lemma 2.5 to prove that $\text{Re} \ q(z) > 0$ for all $z \in \mathbb{D}$. For $z = x + iy \in \mathbb{D}$ with $x, y \in \mathbb{R}$, and $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \leq -\left(1 + \rho^2\right)/2$, we obtain

$$\text{Re} \ \psi(\rho i, \sigma; x + iy) = 2\sigma - \rho^2 + cx - 2(\kappa - 3/2)$$

$$\leq -2\rho^2 + cx - 2(\kappa - 1)$$

$$< |c| - 2(\kappa - 1) \leq 0.$$

By Lemma 2.5 we conclude that $\text{Re} \ q(z) > 0$ for all $z \in \mathbb{D}$. Now consider the function $g_p : \mathbb{D} \to \mathbb{C}$, defined by $g_p(z) = z u_p(z)$. Since

$$\frac{zg'_p(z)}{g_p(z)} = \frac{1}{2} + \frac{1}{2} q(z),$$

it follows that

$$\text{Re} \left[ \frac{zg'_p(z)}{g_p(z)} \right] > \frac{1}{2} \quad \text{for all} \ z \in \mathbb{D},$$

which shows that $g_p$ is starlike of order $1/2$.

(d) Define the function $h_p : \mathbb{D} \to \mathbb{C}$ by $h_p(z) = z^{1-p} w_p(z)$. Since $h_p(z) = a_0(p) z u_p(z^2)$, where $a_0(p) = \left[2^p \Gamma(p + 1)\right]^{-1}$ (see (1.19)), it follows that

$$\frac{zh'_p(z)}{h_p(z)} = 2 \left[ \frac{z^2 g'_p(z^2)}{g_p(z^2)} - \frac{1}{2} \right].$$

But from part (c) we know that the function $g_p : \mathbb{D} \to \mathbb{C}$, defined by $g_p(z) = z u_p(z)$, is starlike of order $1/2$. Thus we conclude that

$$\text{Re} \left[ \frac{zh'_p(z)}{h_p(z)} \right] > 0 \quad \text{for all} \ z \in \mathbb{D},$$

and hence $h_p$ is starlike.

Taking in Theorems 2.2 and 2.11 the values $b = c = 1$, we obtain the following corollary.$^3$

---

$^3$ Note that in the papers of Á. Baricz [38] and V. Selinger [208] the expression “$f_p(z) = z^{-p/2} J_p(z^{1/2})$” should be replaced with “$f_p(z) = 2^p \Gamma(p + 1) z^{-p/2} J_p(z^{1/2})$.”
Corollary 2.7. (V. Selinger [208]) Let $p \in \mathbb{R}$. For the function

$$z \mapsto \mathcal{J}_p(z^{1/2}) = 2^p \Gamma(p + 1)z^{-p/2}J_p(z^{1/2}),$$

where $J_p$ is defined by (1.5), the following properties are true:

(a) If $p \geq 1/4$, then $\text{Re} \mathcal{J}_p(z^{1/2}) > 0$ for all $z \in \mathbb{D}$.

(b) If $p \geq -3/4$, then $z \mapsto \mathcal{J}_p(z^{1/2})$ is univalent in $\mathbb{D}$.

(c) If $p \geq -1/4$, then $z \mapsto \mathcal{J}_p(z^{1/2})$ is convex in $\mathbb{D}$.

(d) If $p \geq 3/4$, then $z \mapsto z \mathcal{J}_p(z^{1/2})$ is starlike in $\mathbb{D}$.

(e) If $p \geq 1/2$, then $z \mapsto z \mathcal{J}_p(z^{1/2})$ is starlike of order $1/2$ in $\mathbb{D}$.

(f) If $p \geq 1/2$, then $z \mapsto z^{1-p}J_p(z)$ is starlike in $\mathbb{D}$.

Remark 2.7. We note that R.K. Brown [78, Theorem 3] proved that if $p = p_1 + ip_2$ ($p_1, p_2 \in \mathbb{R}$) is a complex number satisfying one of the following conditions $p_2 \leq p_1 \in [0, 1)$ or $p_1 \geq 1, 2p_1 - 1 > p_2^2$, then the normalized Bessel function $z \mapsto z^{1-p}J_p(z)$ is regular, univalent, and spirallike in every circle $|z| < r = \rho_\mu^*$, where $\mu^2 = \text{Re}[p^2], \mu > 0$, and $\rho_\mu^*$ is the smallest positive zero of the function

$$r \mapsto rJ_\mu(r) + \text{Re}[1 - p]J_\mu(r).$$

In the particular case when $p$ is real the function $z \mapsto z^{1-p}J_p(z)$ is starlike in $|z| < \rho_\mu^*$, but it is not univalent in any larger circle. The method used by R.K. Brown is completely different from the method of differential subordinations, but if $p$ is real the inequality $2p_1 - 1 > p_2^2$ becomes $p_1 > 1/2$, which appears in part (f) of Corollary 2.7. Nevertheless the method of differential subordinations gives only fairly weak estimates, because for real values of $p$ we obtain that $\rho_\mu^*$ is precisely $\rho_\mu$ obtained by E. Kreyszig and J. Todd [138], and the circle $|z| < \rho_\mu$ is larger than the unit disk (we know the inequalities $2\sqrt{(p + 1)/3} < \rho_\mu < \sqrt{12(p + 2)/5}$ for all $p > -1$). In fact $\rho_\mu$ is the smallest positive zero of the function $z \mapsto z^{1-p}J_p(z)$. The authors proved in [138] that for $p > -1$ the radius of univalence $\rho_\mu$ of the function $z \mapsto z^{1-p}J_p(z)$ increases steadily with $p$. For more details the interested reader is referred also to the paper of H.S. Wilf [229], where the author proved that

$$\rho_\mu = \sqrt{2p_1}\left[1 + 1/(4p) + O(p^{-2})\right], \quad p \to \infty.$$ 

It is also worth mentioning that R.K. Brown [79] proved that if $p \in (-1/2, 0)$, then the normalized Bessel function $z \mapsto z^{1-p}J_p(z)$ is starlike in $\{z \in \mathbb{C} : |z| < \rho_\mu^*\}$, where $\rho_\mu^*$ is the smallest positive zero of the function $pJ_p'(\rho) + (1 - p)J_p(\rho)$. This result is sharp and extends the author’s previous result [78] obtained for $p \geq 0$.

Remark 2.8. Observe that if we choose $c = -1$ and $b = 1$ in Theorems 2.2 and 2.11, then for the function

$$z \mapsto \mathcal{J}_p(z^{1/2}) = 2^p \Gamma(p + 1)z^{-p/2}J_p(z^{1/2}),$$

the following properties hold:

(a) If $p \geq 1/4$, then $\text{Re} \mathcal{J}_p(z^{1/2}) > 0$ for all $z \in \mathbb{D}$.

(b) If $p \geq -3/4$, then $z \mapsto \mathcal{J}_p(z^{1/2})$ is univalent in $\mathbb{D}$.

(c) If $p \geq -1/4$, then $z \mapsto \mathcal{J}_p(z^{1/2})$ is convex in $\mathbb{D}$.

(d) If $p \geq 3/4$, then $z \mapsto z \mathcal{J}_p(z^{1/2})$ is starlike in $\mathbb{D}$.

(e) If $p \geq 1/2$, then $z \mapsto z \mathcal{J}_p(z^{1/2})$ is starlike of order $1/2$ in $\mathbb{D}$.

(f) If $p \geq 1/2$, then $z \mapsto z^{1-p}J_p(z)$ is starlike in $\mathbb{D}$.
2.2 Starlikeness and Convexity Properties of Generalized Bessel Functions

where $I_p$ is the modified Bessel function of the first kind, defined by (1.7), the properties are the same as for the function $z \mapsto J_p(z^{1/2})$, because in this case $|c| = 1$.

If we take $b = 2$ and $c = 1$, then the Theorems 2.2 and 2.11 yield.

**Corollary 2.8.** (Á. Baricz [38]) Let $p \in \mathbb{R}$. For the function

$$z \mapsto J_{p+\frac{1}{2}}(z^{1/2}) = 2^{p+\frac{1}{2}} \frac{\Gamma\left(p+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} z^{-\frac{1}{2}(p+\frac{1}{2})} J_{p+\frac{1}{2}}(z^{1/2}),$$

where $J_p$ is defined by (1.5), the following assertions are true:

(a) If $p \geq -1/4$, then $\text{Re} \ J_{p+1/2}(z^{1/2}) > 0$ for all $z \in \mathbb{D}$.
(b) If $p \geq -5/4$, then $z \mapsto J_{p+1/2}(z^{1/2})$ is univalent in $\mathbb{D}$.
(c) If $p \geq -3/4$, then $z \mapsto J_{p+1/2}(z^{1/2})$ is convex in $\mathbb{D}$.
(d) If $p \geq 1/4$, then $z \mapsto z J_{p+1/2}(z^{1/2})$ is starlike in $\mathbb{D}$.
(e) If $p \geq 0$, then $z \mapsto z J_{p+1/2}(z^{1/2})$ is starlike of order $1/2$ in $\mathbb{D}$.
(f) If $p \geq 0$, then $z \mapsto z^{-1} J_p(z)$ is starlike in $\mathbb{D}$.

**Remark 2.9.** If we consider the Bessel functions of the first kind of order $p$, which can be expressed with elementary functions as $\cos$, $\sin$, $\cosh$ and $\sinh$, we may obtain some interesting examples of univalent, starlike and convex functions, using the definitions of the functions $J_p$ and $J_p$.

(a) The functions $\cos \sqrt{z}$, $\cosh \sqrt{z}$ and $(\sin \sqrt{z})/\sqrt{z}$ are univalent in $\mathbb{D}$, because the functions

$$J_{-1/2}(z^{1/2}) = \sqrt{\pi/2} \cdot z^{1/4} J_{-1/2}(\sqrt{z}) = \cos \sqrt{z},$$

$$J_{-1/2}(z^{1/2}) = \sqrt{\pi/2} \cdot z^{1/4} I_{-1/2}(\sqrt{z}) = \cosh \sqrt{z},$$

$$J_{1/2}(z^{1/2}) = \sqrt{\pi/2} \cdot z^{-1/4} J_{1/2}(\sqrt{z}) = \frac{\sin \sqrt{z}}{\sqrt{z}}$$

satisfy the conditions of Theorem 2.2.

(b) The functions $(\sin \sqrt{z} - \sqrt{z} \cos \sqrt{z})/(\sqrt{z})$, $(\sinh \sqrt{z})/\sqrt{z}$ and $(\sin \sqrt{z})/\sqrt{z}$ are convex, because the functions

$$J_{3/2}(z^{1/2}) = \frac{3}{z} \left( \frac{\sin \sqrt{z}}{\sqrt{z}} - \cos \sqrt{z} \right),$$

$$J_{1/2}(z^{1/2}) = \sqrt{\pi/2} \cdot z^{-1/4} J_{1/2}(\sqrt{z}) = \frac{\sinh \sqrt{z}}{\sqrt{z}}$$

and $J_{1/2}(z^{1/2}) = (\sin \sqrt{z})/\sqrt{z}$ satisfy the conditions of Theorem 2.11 (first part).
(c) The functions $\sqrt{z}\sin\sqrt{z}$ and $\sqrt{z}\sinh\sqrt{z}$ are starlike functions of order 1/2, because the functions

$$zJ_{1/2}(z^{1/2}) = \sqrt{\frac{\pi}{2}} \cdot z^{3/4}J_{1/2}(\sqrt{z}) = \sqrt{z} \sin\sqrt{z}$$

and analogously

$$zJ_{1/2}(z^{1/2}) = \sqrt{\frac{\pi}{2}} \cdot z^{3/4}I_{1/2}(\sqrt{z}) = \sqrt{z} \sinh\sqrt{z}$$

satisfy the conditions of Theorem 2.11 (third part).

(d) The functions $\sin z$, $\sinh z$ and $(\sin\sqrt{z})/\sqrt{z} - \cos\sqrt{z}$ are starlike functions in $\mathbb{D}$, because the functions $z^{1/2}J_{1/2}(z) = \sqrt{2/\pi} \cdot \sin z$, $z^{1/2}I_{1/2}(z) = \sqrt{2/\pi} \cdot \sinh z$ and

$$zJ_{3/2}(z^{1/2}) = 3 \left( \frac{\sin\sqrt{z}}{\sqrt{z}} - \cos\sqrt{z} \right)$$

satisfy the conditions of Theorem 2.11 (second and fourth part).

Note that similar results as those given in Theorem 2.11 were obtained by S.S. Miller and P.T. Mocanu [154] for Gaussian and confluent hypergeometric functions.

Theorem 2.11 can be extended for complex parameters, as the following result reveals.

**Theorem 2.12.** (Á. Baricz [42]) Let $b, p, c \in \mathbb{C}$, of which $c \neq 0$. Then $w_p$ and $u_p$, defined by (1.15) and (1.20), respectively, have the following properties:

(a) If $\text{Re} \kappa \geq |c|/4 + (\text{Im} \kappa)^2/6 + 1/2$, then $u_p \in C$.

(b) If $\text{Re} \kappa \geq |c|/4 + (\text{Im} \kappa)^2/6 + 3/2$, then $zu_p \in S^*$.

(c) If $\text{Re} \kappa \geq |c|/2 + (\text{Im} \kappa)^2/4 + 1$, then $zu_p \in S^*(1/2)$.

(d) If $\text{Re} \kappa \geq |c|/2 + (\text{Im} \kappa)^2/4 + 1$, then $z^{1-p}w_p \in S^*$.

**Proof.** The proof of this theorem is very similar to the proof of Theorem 2.11. For convenience we just sketch the proof.

(a) Since $\text{Re}(\kappa + 1) \geq |c|/4 + (\text{Im} \kappa)^2/6 + 3/2 > |c|/4 + 1$ and $c \neq 0$, Theorem 2.3 implies $\text{Re} u_{p+1}(z) > 0$ for all $z \in \mathbb{D}$. According to Lemma 1.2 it follows that $u'_p(z) \neq 0$ for all $z \in \mathbb{D}$. Define $h : \mathbb{D} \to \mathbb{C}$ by

$$h(z) = 1 + \frac{zu'_p(z)}{u'_p(z)}.$$

The function $h$ is analytic in $\mathbb{D}$ and $h(0) = 1$. Just like in the proof of part (a) of Theorem 2.11 it is shown that $h$ satisfies the differential equation

$$4zh'(z) + 4h^2(z) + 4(\kappa - 2)h(z) + cz - 4(\kappa - 1) = 0.$$  \hspace{1cm} (2.19)
If
\[ \psi(r,s;z) = 4s + 4r^2 + 4(\kappa - 2)r + cz - 4(\kappa - 1) \]
and \( E = \{0\} \), then \((2.19)\) implies \( \psi(h(z), zh'(z); z) \in E \) for all \( z \in \mathbb{D} \). We use Lemma 2.5 to prove that \( \text{Re}h(z) > 0 \) for all \( z \in \mathbb{D} \). Let \( z = x + iy \in \mathbb{D} \) and \( c = c_1 + ic_2 \) (with \( x, y, c_1, c_2 \in \mathbb{R} \)). For all \( \rho, \sigma \in \mathbb{R} \) satisfying \( \sigma \leq -\frac{1}{2} - (\rho^2)/2 \) we obtain
\[
\text{Re} \psi(\rho i, \sigma; x + iy) = 4\sigma - 4\rho^2 - 4\rho \text{Im} \kappa + c_1x - c_2y - 4(\text{Re} \kappa - 1) \\
\leq -6\rho^2 - 4(\text{Im} \kappa)\rho + c_1x - c_2y - 2(2\text{Re} \kappa - 1) = Q_1(\rho).
\]
The discriminant \( \Delta_1 \) of the quadratic form \( Q_1(\rho) \) is
\[
\Delta_1 = 4[4(\text{Im} \kappa)^2 + 6(c_1x - c_2y) - 12(2\text{Re} \kappa - 1)].
\]
By the Cauchy–Bunyakovsky–Schwarz inequality (1.35) we have
\[
c_1x - c_2y \leq |c_1x - c_2y| \leq \sqrt{c_1^2 + c_2^2} \sqrt{x^2 + y^2} < |c|.
\]
Therefore we have
\[
\frac{\Delta_1}{4} < 4(\text{Im} \kappa)^2 + 6|c| - 12(2\text{Re} \kappa - 1) \leq 0.
\]
Thus, the quadratic form \( Q_1(\rho) \) is strictly negative, and consequently we have
\( \text{Re} \psi(\rho i, \sigma; x + iy) < 0 \). By Lemma 2.5 we conclude that \( \text{Re}h(z) > 0 \) for all \( z \in \mathbb{D} \), which shows that \( u_p \) is convex in \( \mathbb{D} \).

(b) Since \( \text{Re}(\kappa - 1) \geq |c|/4 + (\text{Im} \kappa)^2/6 + 1/2 = |c|/4 + (\text{Im}(\kappa - 1))^2/6 + 1/2 \), it follows from (a) that \( u_{p-1} \) is convex. By applying Alexander’s duality theorem, i.e. Lemma 2.1 we conclude that \( z u'_{p-1} \in \mathcal{J}^* \). But, on the other hand, Lemma 1.2 yields \( czu_p(z) = -4(\kappa - 1)zu'_{p-1}(z) \). Consequently, it results that \( z u_p \in \mathcal{J}^* \).

(c) Since the condition of the first part of Theorem 2.3 holds, i.e. we have \( \text{Re} \kappa > |c|/4 + 1 \), we deduce that \( u_p(z) \neq 0 \) for all \( z \in \mathbb{D} \). Define \( h : \mathbb{D} \rightarrow \mathbb{C} \) by
\[
h(z) = 1 + 2 \frac{zu_p'(z)}{u_p(z)}.
\]
The function \( h \) is analytic in \( \mathbb{D} \) and \( h(0) = 1 \). Just like in the proof of part (c) of Theorem 2.11 it is shown that \( h \) satisfies the differential equation
\[
2zh'(z) + h^2(z) + 2(\kappa - 2)h(z) + cz - (2\kappa - 3) = 0. \quad (2.20)
\]
If
\[
\psi(r,s;z) = 2s + r^2 + 2(\kappa - 2)r + cz - (2\kappa - 3)
\]
and $E = \{0\}$, then (2.20) implies $\psi(h(z), zh'(z); z) \in E$ for all $z \in D$. We use Lemma 2.5 to prove that $\Re h(z) > 0$ for all $z \in D$. If $z = x + iy \in D$ and $c = c_1 + ic_2$ (with $x, y, c_1, c_2 \in \mathbb{R}$), we obtain for all $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -(1 + \rho^2)/2$ that

$$\Re \psi(\rho i, \sigma; x + iy) = 2\sigma - \rho^2 - 2(\Im \kappa)\rho + c_1 x - c_2 y - (2\Re \kappa - 3)$$

$$\leq -2\rho^2 - 2(\Im \kappa)\rho + c_1 x - c_2 y - (2\Re \kappa - 2) = Q_2(\rho).$$

The discriminant $\Delta_2$ of the quadratic form $Q_2(\rho)$ is

$$\Delta_2 = 4(\Im \kappa)^2 + 8(c_1 x - c_2 y) - 8(2\Re \kappa - 2).$$

By the Cauchy–Bunyakovsky–Schwarz inequality (1.35), we know that $c_1 x - c_2 y < |c|$. Therefore we have $\Delta_2 < 4(\Im \kappa)^2 + 8|c| - 8(2\Re \kappa - 2) \leq 0$. Thus $Q_2(\rho) < 0$, and consequently $\Re \psi(\rho i, \sigma; x + iy) < 0$. By Lemma 2.5 we conclude that $\Re h(z) > 0$ for all $z \in D$, and this inequality shows that $g_\rho : D \to \mathbb{C}$, defined by $g_\rho(z) = z u_\rho(z)$, is starlike of order $1/2$.

(d) The proof of this assertion coincides with the proof of (d) in Theorem 2.11. $\square$

Note that similar results as those given in Theorem 2.12 were obtained by S. Kanas and J. Stankiewicz [128] for confluent hypergeometric functions. Moreover, it is worth mentioning here that part (d) of Theorem 2.12 slightly improves the result of R.K. Brown [78, Theorem 3]. More precisely, recall that R.K. Brown proved (see Remark 2.7) that if $p = \Re p + i\Im p$ satisfies one of the following conditions $\Im p \leq \Re p \in [0, 1)$ or $\Re p \geq 1$, $2\Re p - 1 > (\Im p)^2$, then the normalized Bessel function $z \mapsto z^{1-p} J_p(z)$ is univalent in every disk $D_r = \{z \in \mathbb{C} : |z| < \rho_{\mu}^*\}$, where $\mu^2 = \Re|p^2|$, $\mu > 0$, and $\rho_{\mu}^*$ is the smallest positive zero of the function

$$r \mapsto r J_\mu'(r) + \Re[1 - p] J_\mu(r).$$

Now, in the case of the unit disk, i.e. for $z \in D \cap D_r$ part (d) of Theorem 2.12 states that if $\Re p \geq 1/2 + (\Im p)^2/4$, then the normalized Bessel function $z \mapsto z^{1-p} J_p(z)$ is still starlike and hence univalent in $D$. And this slightly improves the above result of R.K. Brown because we have that $\Re p \geq 1/2 + (\Im p)^2/2 \geq 1/2 + (\Im p)^2/4$.

Another extension of Theorem 2.11 for real parameters is included in the next theorem.

**Theorem 2.13.** (Á. Baricz [52]) If $\alpha \in [0, 1)$ and $b, p, c$ are real numbers such that $c \neq 0$ and $4\alpha^2 + (|c| - 6)\alpha + 2 \geq 0$, then the functions $w_p$ and $u_p$, defined by (1.15) and (1.20), respectively, have the following properties:

(a) If $\kappa \geq \frac{|c| + 2(1 - \alpha)(1 - 2\alpha)}{4(1 - \alpha)}$, then $u_p \in \mathcal{C}(\alpha)$. Moreover, we have that

$$\Re [u_p'(z)]^{\frac{1}{2(1 - \alpha)}} > \frac{1}{2} \text{ for all } z \in D.$$
(b) If \( \kappa \geq \frac{|c| + 2(1 - \alpha)(3 - 2\alpha)}{4(1 - \alpha)} \), then \( z u_p \in S^*(\alpha) \) and
\[
z \left[ -\frac{c}{4(\kappa - 1)} u_p(z) \right]^{1/\alpha} \in S^*(\alpha).
\]

(c) If \( \kappa \geq \frac{|c| + 2(1 - \alpha)(3 - 2\alpha)}{4(1 - \alpha)} \) and \( \alpha \neq 0 \), then \( z^{\frac{2(1-\alpha)-p}{-2\alpha}} w_p \left( z^\frac{1}{\alpha} \right) \in S^* \).

**Proof.** (a) The equality
\[
\frac{|c| + 2(1 - \alpha)(1 - 2\alpha)}{4(1 - \alpha)} = \frac{|c|}{4} + \frac{4\alpha^2 + (|c| - 6\alpha + 2}{4(1 - \alpha)}
\]
implies \( \kappa \geq |c|/4 \). By applying Theorem 2.2 we conclude that \( \Re u_{p+1}(z) > 0 \) for all \( z \in \mathbb{D} \). According to Lemma 1.2 it follows that \( u'_p(z) \neq 0 \) for all \( z \in \mathbb{D} \).

Define \( q : \mathbb{D} \to \mathbb{C} \) by
\[
q(z) = 1 + \frac{zu''_p(z)}{(1 - \alpha)u'_p(z)}.
\]
The function \( q \) is analytic in \( \mathbb{D} \) and \( q(0) = 1 \). Since \( u_p \) satisfies the differential equation (1.21) it can be shown, as in the proof of Theorem 2.11, that \( q \) satisfies the following differential equation:
\[
4(1 - \alpha)zq'(z) + 4(1 - \alpha)^2q^2(z) + 2(1 - \alpha)e_1q(z) + cz - 2(1 - \alpha)e_2 = 0,
\]
where \( e_1 = 2\kappa + 4(\alpha - 1) \) and \( e_2 = 2\kappa + 2(\alpha - 1) \).

If
\[
\psi(r, s; z) = 4(1 - \alpha)s + 4(1 - \alpha)^2r^2 + 2(1 - \alpha)e_1r + cz - 2(1 - \alpha)e_2
\]
and \( \mathbb{E} = \{0\} \), then (2.21) implies \( \psi(q(z), zq'(z); z) \in \mathbb{E} \) for all \( z \in \mathbb{D} \). We use Lemma 2.5 to prove that \( \Re q(z) > 0 \) for all \( z \in \mathbb{D} \). For \( z = x + iy \in \mathbb{D} \) (with \( x, y \in \mathbb{R} \)) and \( \rho, \sigma \in \mathbb{R} \) satisfying \( \sigma \leq -(1 + \rho^2)/2 \), we obtain
\[
\Re \psi(\rho i, \sigma; x + iy) = 4(1 - \alpha)\sigma - 4(1 - \alpha)^2\rho^2 + cx - 2(1 - \alpha)e_2 \\
\leq -2(1 - \alpha)(3 - 2\alpha)\rho^2 + cx - 2(1 - \alpha)(1 + e_2) \\
< |c| + 2(1 - \alpha)(1 - 2\alpha) - 4(1 - \alpha)\kappa \leq 0.
\]

By Lemma 2.5 we conclude that \( \Re q(z) > 0 \) for all \( z \in \mathbb{D} \). This result implies
\[
\Re \left[ 1 + \frac{zu''_p(z)}{u'_p(z)} \right] = (1 - \alpha)\Re q(z) + \alpha > \alpha \quad \text{for all} \quad z \in \mathbb{D},
\]
which shows that $u_p$ is convex of order $\alpha$ in $\mathbb{D}$. On the other hand, it is known (see the paper of I.S. Jack [124, p. 473]) that if $f \in \mathcal{C}(\alpha)$, then

$$\text{Re} \left[ f'(z) \right] \frac{1}{2(1-\alpha)} > \frac{1}{2} \text{ for all } z \in \mathbb{D}.$$ 

Now, using the fact that $u_p$ is convex of order $\alpha$ in $\mathbb{D}$, the asserted inequality follows.

(b) Since

$$\kappa - 1 \geq \frac{|c| + 2(1 - \alpha)(3 - 2\alpha)}{4(1 - \alpha)} - 1 = \frac{|c| + 2(1 - \alpha)(1 - 2\alpha)}{4(1 - \alpha)},$$

it follows from (a) that $u_{p-1}$ is convex of order $\alpha$. By applying the general version of Alexander’s duality theorem, i.e. Lemma 2.2 we conclude that $zu'_{p-1} \in \mathcal{S}^*(\alpha)$ and $z[u'_{p-1}]^{1/(1-\alpha)} \in \mathcal{S}^*(\alpha)$. According to Lemma 1.2 these imply that

$$-\frac{c}{4(\kappa - 1)} zu_p(z) \in \mathcal{S}^*(\alpha)$$

and

$$z \left[ -\frac{c}{4(\kappa - 1)} u_p(z) \right]^{\frac{1}{1-\alpha}} \in \mathcal{S}^*(\alpha).$$

(c) Define the functions $g_p : \mathbb{D} \to \mathbb{C}$ and $h_p : \mathbb{D} \to \mathbb{C}$ by

$$g_p(z) = zu_p(z) \text{ and } h_p(z) = z^{[2(1-\alpha)-p]/(2\alpha)} w_p \left( z^{1/(2\alpha)} \right),$$

respectively. Since $h_p(z) = a_0(p) z^{(1-\alpha)/\alpha} u_p \left( z^{1/\alpha} \right)$, where $a_0(p) = [2^p \Gamma(p)]^{-1}$ (see (1.19)), it follows that

$$\frac{zh'_p(z)}{h_p(z)} = \frac{1}{\alpha} \left[ \frac{z^{1/\alpha} \frac{d}{dz} p(z)^{1/\alpha}}{g_p(z)^{1/\alpha}} - \alpha \right].$$

Finally because $g_p \in \mathcal{S}^*(\alpha)$, we deduce that $h_p \in \mathcal{S}^*$. \qed

### 2.2.3 Sufficient Conditions Involving Results of H. Silverman

In this section we place conditions on $b$, $p$ and $c$ to guarantee that $zu_p$ is in various subclasses of starlike and convex functions, where $u_p$ is defined by (1.20). Further constraints lead to coefficient characterizations of the families. An integral operator related to the generalized Bessel function is also examined. For similar results involving Gaussian hypergeometric functions we refer to the paper of H. Silverman [212], which is the starting point of this section.
Denote by \( \mathcal{S}_1^*(\alpha) \), where \( \alpha \in [0,1) \), the subclass of \( \mathcal{S}^*(\alpha) \) consisting of functions \( f \) for which
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad \text{for all } z \in D.
\]

A function \( f \) is said to be in \( \mathcal{C}_1(\alpha) \) if \( zf' \in \mathcal{S}_1^*(\alpha) \). Coefficient bounds and other extremal properties for \( \mathcal{S}_1^*(\alpha) \) were found in papers of P.J. Eenigenburg [92] and H. Silverman [211].

We will make use of the following lemmata.

**Lemma 2.11.** (H. Silverman [210]) Let \( \alpha \in [0,1) \). A sufficient condition for \( f(z) = z + \sum_{n \geq 2} a_n z^n \) to be in \( \mathcal{S}_1^*(\alpha) \) and \( \mathcal{C}_1(\alpha) \), respectively, is that
\[
\sum_{n \geq 2} (n - \alpha) |a_n| \leq 1 - \alpha,
\]
\[
\sum_{n \geq 2} n(n - \alpha) |a_n| \leq 1 - \alpha
\]
respectively.

**Lemma 2.12.** (E.P. Merkes, M.S. Robertson and W.T. Scott [150], H. Silverman [210]) Let \( \alpha \in [0,1) \). Suppose that \( f(z) = z - \sum_{n \geq 2} a_n z^n \), \( a_n \geq 0 \). Then a necessary and sufficient condition for \( f \) to be in \( \mathcal{S}_1^*(\alpha) \) and \( \mathcal{C}_1(\alpha) \), respectively, is that
\[
\sum_{n \geq 2} (n - \alpha) a_n \leq 1 - \alpha,
\]
\[
\sum_{n \geq 2} n(n - \alpha) a_n \leq 1 - \alpha
\]
respectively. In addition, \( f \in \mathcal{S}_1^*(\alpha) \iff f \in \mathcal{S}^*(\alpha) \), \( f \in \mathcal{C}_1(\alpha) \iff f \in \mathcal{C}(\alpha) \), and \( f \in \mathcal{S}^* \iff f \in \mathcal{F} \).

Our main results of this section are the following theorems.

**Theorem 2.14.** (Á. Baricz [54]) If \( \alpha \in [0,1) \), \( c < 0 \) and \( \kappa > 0 \), then a sufficient condition for \( zu_p \) to be in \( \mathcal{S}_1^*(\alpha) \) is
\[
u_p(1) + u'_p(1)/(1 - \alpha) \leq 2. \tag{2.22}
\]
Moreover, (2.22) is necessary and sufficient for \( \psi(z) = z[2 - u_p(z)] \) to be in \( \mathcal{S}_1^*(\alpha) \).

**Proof.** Since \( zu_p(z) = z + \sum_{n \geq 2} b_{n-1} z^n \), according to Lemma 2.11 we need only show that
\[
\sum_{n \geq 2} (n - \alpha) b_{n-1} \leq 1 - \alpha.
\]
We notice that
\[
\sum_{n \geq 2} (n - \alpha) b_{n-1} = \sum_{n \geq 2} (n-1) b_{n-1} + \sum_{n \geq 2} (1 - \alpha) b_{n-1}
\]
\[
= \sum_{n \geq 2} \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (1 - \alpha)[u_p(1) - 1].
\]

Taking into consideration that \((\kappa)_{n-1} = \kappa(\kappa + 1)_{n-2}\) we may write the above expression as follows
\[
\sum_{n \geq 2} (n - \alpha) b_{n-1} = -\frac{c}{4\kappa} \sum_{n \geq 2} \frac{(-c/4)^{n-2}}{(\kappa+1)_{n-2}(n-2)!} + (1 - \alpha)[u_p(1) - 1]
\]
\[
= (1 - \alpha)[u_p(1) - 1] + \left(-\frac{c}{4\kappa}\right) u_{p+1}(1).
\]

Using Lemma 1.2 we obtain
\[
\sum_{n \geq 2} (n - \alpha) b_{n-1} = (1 - \alpha)[u_p(1) - 1] + u_p'(1).
\]

This sum is bounded above by \(1 - \alpha\) if and only if (2.22) holds. Since
\[
z[2 - u_p(z)] = z - \sum_{n \geq 2} b_{n-1} z^n,
\]
the necessity of (2.22) for \(\psi\) to be in \(\mathcal{S}^*(\alpha)\) follows from Lemma 2.12.

\[\square\]

Remark 2.10. Condition (2.22) with \(\alpha = 0\) is both necessary and sufficient for \(\psi\) to be in \(\mathcal{S}_1(\alpha)\). For the convenience throughout this section we denote
\[
\varsigma_p(t) = \frac{2(1 - t^2)^{\kappa-3/2}}{B(\kappa - 1/2, 1/2)} \cosh(t \sqrt{-c}). \tag{2.23}
\]

By Lemma 1.3 the condition (2.22) may be written as
\[
(1 - \alpha) \int_0^1 \varsigma_p(t) \, dt - \frac{c}{4\kappa} \int_0^1 \varsigma_{p+1}(t) \, dt \leq 2(1 - \alpha).
\]

In particular when \(c = -1\) and \(b = 1\) (in this case \(u_p(z)\) becomes \(\mathcal{I}_p(z^{1/2})\), defined by (1.24)) we obtain that condition (2.22) simplifies to
\[
2^{p-2}[I_{p+1}(1) + 2(1 - \alpha)I_p(1)] \cdot \Gamma(p + 1) \leq 1 - \alpha,
\]
which guarantees that
\[ z \mathcal{I}_p(z^{1/2}) = 2^p \Gamma(p + 1)z^{1-p/2}I_p(z^{1/2}) \in \mathcal{K}^*(\alpha). \]

Moreover, the above condition is necessary and sufficient for \( z[2 - \mathcal{I}_p(z^{1/2})] \) to be in \( \mathcal{K}^*(\alpha). \)

Our next theorem is similar to Theorem 2.14, but it deals with the convex case.

**Theorem 2.15.** (Á. Baricz [54]) If \( \alpha \in [0, 1), c < 0 \) and \( \kappa > 0 \), then a sufficient condition for \( uz_p \) to be in \( \mathcal{C}_1(\alpha) \) is
\[ u''_p(1) + (3 - \alpha)u'_p(1) + (1 + \alpha)u_p(1) \leq 2. \quad (2.24) \]

Moreover, this condition is necessary and sufficient for \( \psi(z) = z[2 - u_p(z)] \) to be in \( \mathcal{C}_1(\alpha) \).

**Proof.** In view of Lemma 2.11, we need only show that
\[ \sum_{n \geq 2} n(n - \alpha)b_{n-1} \leq 1 - \alpha. \]

We notice that
\[ \sum_{n \geq 2} n(n - \alpha)b_{n-1} = \sum_{n \geq 0} (n + 2)(n + 2 - \alpha)b_{n+1} = A - \alpha B, \]

where
\[ A = \sum_{n \geq 0} (n + 2)b_{n+1} \quad \text{and} \quad B = \sum_{n \geq 0} (n + 2)b_{n+1}. \]

Thus, we have to prove that \( A - \alpha B \leq 1 - \alpha \). Computing \( A \) and \( B \), we obtain:

\[
A = \sum_{n \geq 0} \frac{(n+1)(-c/4)^{n+1}}{(n+1)!} + 2 \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(n+1)!} + \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(n+1)!} \\
= \sum_{n \geq 1} \frac{(-c/4)^{n+1}}{(n+1)!} + 3 \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(n+1)!} + \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(n+1)!} \\
= \frac{(-c/4)^2}{\kappa(\kappa + 1)} \sum_{n \geq 1} \frac{(-c/4)^{n-1}}{(n+2)(n-1)!} + 3 \frac{(-c/4)^n}{\kappa} \sum_{n \geq 0} \frac{(-c/4)^n}{(\kappa + 1)n!} + \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(\kappa + 1)n!} \\
= \frac{(-c/4)^2}{\kappa(\kappa + 1)} u_{p+2}(1) - 3 \frac{c}{4\kappa} u_{p+1}(1) + u_p(1) - 1 \\
= u''_p(1) + 3u'_p(1) + u_p(1) - 1,
\]
where we used that \((a)_n = a(a + 1)_{n-1}\) and Lemma 1.2, i.e. the relations

\[ u'_p(z) = -\frac{c}{4\kappa} u_{p+1}(z), \quad u''_p(z) = \frac{(-c/4)^2}{\kappa(\kappa + 1)} u_{p+2}(z). \]

Analogously, we obtain that

\[ B = \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(\kappa)_{n+1} n!} + \sum_{n \geq 0} \frac{(-c/4)^{n+1}}{(\kappa)_{n+1} (n + 1)!} = u'_p(1) + u_p(1) - 1. \]

Therefore it is clear that the expression

\[ A - \alpha B = u''_p(1) + (3 - \alpha) u'_p(1) + (1 + \alpha) u_p(1) - (1 + \alpha) \]

is bounded above by \(1 - \alpha\) if and only if (2.24) holds. Lemma 2.12 implies that (2.24) is also necessary for \(\psi\) to be in \(C_1(\alpha)\). \(\square\)

**Remark 2.11.** Using (2.23) and Lemma 1.3 we see that condition (2.24) may be written as follows:

\[ \frac{(-c/4)^2}{\kappa(\kappa + 1)} \int_0^1 \varsigma_{p+2}(t) \, dt + (3 - \alpha) \left( -\frac{c}{4\kappa} \right) \int_0^1 \varsigma_{p+1}(t) \, dt + (1 + \alpha) \int_0^1 \varsigma_p(t) \, dt \leq 2. \]

In particular, when \(c = -1\) and \(b = 1\), the condition (2.24) becomes

\[ 2^{p-3} [I_{p+2}(1) + (3 - \alpha) I_{p+1}(1) + 4(1 + \alpha) I_p(1)] \cdot \Gamma(p+1) \leq 2, \]

which guarantees that

\[ zJ_p(z^{1/2}) = 2^p \Gamma(p+1) z^{1-p/2} I_p(z^{1/2}) \in C_1(\alpha). \]

Moreover, the above condition is necessary and sufficient for \(z[2 - J_p(z^{1/2})]\) to be in \(C_1(\alpha)\).

As in the paper of H. Silverman [212] one can look at other linear operators acting on \(u_p\) to obtain similar results. We illustrate this idea in the case of a particular integral operator.

**Theorem 2.16.** (Á. Baricz [54]) If \(c < 0, \kappa > 0\) and \(u_p(1) \leq 2\), then \(\int_0^z u_p(t) \, dt \in \mathcal{F}^s\).

**Proof.** Since

\[ \int_0^z u_p(t) \, dt = \sum_{n \geq 0} \frac{b_n}{n+1} z^{n+1} = z + \sum_{n \geq 2} \frac{b_{n-1}}{n} z^n, \]
we note that
\[ \sum_{n \geq 2} n \cdot \frac{b_{n-1}}{n} = \sum_{n \geq 2} b_{n-1} = u_p(1) - 1 \leq 1 \]
if and only if \( u_p(1) \leq 2 \).

**Remark 2.12.** Comparable bounds to Theorem 2.16 may be obtained for positive order of starlikeness too. Denoting \( f(z) = z u_p(z) \) and \( g(z) = \int_0^z u_p(t) \, dt \), it is clear that \( g \in \mathcal{C}_1(\alpha) \) if and only if \( f \in \mathcal{S}_*(\alpha) \). This follows observing that \( g'(z) = u_p(z), \ g''(z) = u'_p(z), \) and so \( 1 + zg''(z)/g'(z) = 1 + zu_p(z)/u_p(z) = zf'(z)/f(z) \). Thus any starlikeness type result concerning \( z u_p \) leads to a convexity result concerning \( g \), for example condition (2.22) guarantees that \( g \in \mathcal{C}_1(\alpha) \).

### 2.2.4 Close-to-Convexity with Respect to Certain Functions

Motivated by the papers of S. Ponnusamy and M. Vuorinen [187, 188], we discuss in this section a few conditions concerning the parameters of \( u_p \), which guarantee the close-to-convexity with respect to the functions

\[ - \log(1 - z) \quad \text{and} \quad \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right). \]

For this we need the following result of S. Ozaki [167].

**Lemma 2.13.** (S. Ozaki [167]) If \( f(z) = z + \sum_{n \geq 2} B_{2n-1} z^{2n-1} \) is analytic in \( \mathbb{D} \) and if

\[ 1 \geq 3B_3 \geq \ldots \geq (2n-1)B_{2n-1} \geq \ldots \geq 0 \quad \text{or} \quad 1 \leq 3B_3 \leq \ldots \leq (2n-1)B_{2n-1} \leq \ldots \leq 2, \]

then \( f \) is univalent in \( \mathbb{D} \).

We note that, as S. Ponnusamy and M. Vuorinen pointed out in [187], proceeding exactly as in the proof of Lemma 2.8 (also due to S. Ozaki [167]), one can verify directly that if a function \( f : \mathbb{D} \to \mathbb{C} \) satisfies the hypothesis of Lemma 2.13, then it is close-to-convex with respect to the convex function

\[ z \mapsto \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right). \]

**Theorem 2.17.** (Á. Baricz [52]) If \( c < 0 \) and \( b, p \in \mathbb{R} \), then \( u_p \), defined by (1.20), has the following properties:

(a) If \( \kappa \geq -c/2 \), then \( z \mapsto zu_p(z) \) is close-to-convex with respect to the function \( - \log(1 - z) \), and consequently it is univalent in \( \mathbb{D} \).

(b) If \( \kappa \geq -3c/4 \), then \( z \mapsto zu_p(z^2) \) is close-to-convex with respect to the function \( \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right) \), and consequently it is univalent in \( \mathbb{D} \).
Proof. (a) Set

\[ f(z) = zu_\rho(z) = z + \sum_{n \geq 2} b_{n-1}z^n. \]

We have \(b_{n-1} > 0\) for all \(n \geq 2\) and \(2b_1 = -c/(2\kappa) \leq 1\). From the definition of the ascending factorial notation (we use the formula \((\kappa)_n = (\kappa + n - 1)(\kappa)_{n-1}\)) we observe that

\[ b_n = -\frac{c}{4n(\kappa + n - 1)}b_{n-1} \quad \text{for all} \quad n \geq 2. \]

We use Lemma 2.8 to prove that \(f\) is close-to-convex with respect to the function \(-\log(1 - z)\). Therefore, we show that \(\{nb_{n-1}\}_{n \geq 2}\) is a decreasing sequence. By a short computation we obtain

\[ nb_{n-1} - (n+1)b_n = b_{n-1}\left[ n + \frac{c(n+1)}{4n(\kappa + n - 1)} \right] = \frac{b_{n-1} \cdot U_1(n)}{4n(\kappa + n - 1)}, \]

where \(U_1(n) = 4n^3 + 4(\kappa - 1)n^2 + cn + c\). Using the inequalities \(n^3 \geq 3n^2 - 3n + 1\), \(n^2 \geq 2n - 1\) and \(8\kappa + c + 4 > 0\), we obtain

\[ U_1(n) \geq 4(\kappa + 2)n^2 + (c - 12)n + c + 4 \geq (8\kappa + c + 4)n - 4(\kappa + 2) + c + 4 \geq 2(2\kappa + c) \geq 0. \]

This implies that \(nb_{n-1} - (n+1)b_n \geq 0\) for all \(n \geq 2\), thus, \(\{nb_{n-1}\}_{n \geq 2}\) is a decreasing sequence. By Lemma 2.8 it follows that \(f\) is close-to-convex with respect to the convex function \(-\log(1 - z)\). Consequently, \(f\) is univalent in \(\mathbb{D}\) (see Lemma 2.3).

(b) Set

\[ f(z) = zu_\rho(z^2) = z + \sum_{n \geq 2} B_{2n-1}z^{2n-1}, \]

where \(B_{2n-1} = b_{n-1}\) for all \(n \geq 2\). Therefore we have \(3B_3 = 3b_1 = -(3c)/(4\kappa) \leq 1\) and \(B_{2n-1} > 0\) for all \(n \geq 2\). We want to show that \(\{(2n-1)B_{2n-1}\}_{n \geq 2}\) is a decreasing sequence. Fix \(n \geq 2\). Then we have

\[ (2n-1)B_{2n-1} - (2n+1)B_{2n+1} = \frac{B_{2n-1} \cdot U_2(n)}{4n(\kappa + n - 1)}, \]

where \(U_2(n) = 8n^3 + 8(\kappa - 3/2)n^2 - 4(\kappa - c/2 - 1)n + c\). Using the inequalities \(n^3 \geq 3n^2 - 3n + 1\) and \(n^2 \geq 2n - 1\), we obtain
2.3 Applications Involving Bessel Functions with Hardy Space of Analytic Functions

\[ U_2(n) \geq 8(\kappa + 3/2)n^2 - 4(\kappa - c/2 + 5)n + c + 8 \]
\[ \geq 12(\kappa + c/6 + 1/3)n - 8(\kappa + 3/2) + c + 8 \]
\[ \geq 4(\kappa + 3c/4) \geq 0. \]

Hence \( \{ (2n - 1)B_{2n-1} \}_{n \geq 2} \) is a decreasing sequence. But \( z \mapsto \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \) is convex in \( \mathbb{D} \), so by applying Lemma 2.3 the desired conclusion follows. \( \square \)

In particular if we choose \( c = -1 \) and \( b = 1 \), then we obtain the following sufficient condition for the close-to-convexity of the normalized and modified Bessel functions of the first kind of order \( p \).

**Corollary 2.9.** (Á. Baricz [52]) If \( p \geq -1/2 \), then \( z \mapsto zJ_p(\sqrt{z}) \) is close-to-convex with respect to the function \( -\log(1-z) \), and consequently it is univalent in \( \mathbb{D} \). Moreover, if \( p \geq -1/4 \), then \( z \mapsto zJ_p(z) \) is close-to-convex with respect to the function \( \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \), and consequently it is univalent in \( \mathbb{D} \).

**Remark 2.13.** If we consider the Bessel functions which can be expressed with elementary functions as \( \cosh \) and \( \sinh \) (see Remark 2.9), we may obtain the following examples of close-to-convex functions.

(a) The functions \( z\cosh \sqrt{z} \) and \( \sqrt{z} \sinh \sqrt{z} \) are close-to-convex functions with respect to the convex function \( -\log(1-z) \), because \( zJ_{-1/2}(z^{1/2}) = \sqrt{\pi/2} \cdot z^{5/4}I_{-1/2}(\sqrt{z}) = z\cosh \sqrt{z} \) and \( zJ_{1/2}(z^{1/2}) = \sqrt{\pi/2} \cdot z^{1/4}I_{1/2}(\sqrt{z}) = \sqrt{z} \sinh \sqrt{z} \) satisfy the condition of Theorem 2.17 (first part).

(b) The function \( zJ_0(iz) \) is close-to-convex with respect to \( z \mapsto \frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \), because \( zJ_0(z) = zJ_0(z) = zJ_0(iz) \) satisfies the condition of Theorem 2.17 (second part). We note that this property of the Bessel function of the first kind of zero order was established also by S. Ponnusamy and M. Vuorinen [187, p. 83].

## 2.3 Applications Involving Bessel Functions Associated with Hardy Space of Analytic Functions

Our aim in this section is to find conditions on the parameters which guarantee that the generalized and normalized Bessel function belongs to a certain Hardy space. Moreover, we present a monotonicity property of the generalized and normalized Bessel functions. In addition, we find the radius of the smallest disk centered at 1, which contains the image region \( u_p(\mathbb{D}) \).
2.3.1 Bessel Transforms and Hardy Space of Generalized Bessel Functions

In this section we present some immediate applications of convexity and univalence involving Bessel functions associated with the Hardy space of analytic functions, i.e. we obtain conditions for the function \( u_\nu \), defined by (1.20), to belong to the Hardy space \( \mathcal{H}^\infty \).

Let \( \mathcal{H} \) be the set of all analytic functions on \( \mathbb{D} \). For any \( \mu \in (0, \infty) \), any function \( f \in \mathcal{H} \) and any \( r \in [0, 1) \) set

\[
M_\mu (r, f) = \left\{ \begin{array}{ll}
\left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\mu \, d\theta \right)^{1/\mu}, & \text{if } 0 < \mu < \infty \\
\max \left\{|f(z)| : |z| \leq r \right\}, & \text{if } \mu = \infty.
\end{array} \right.
\]

By definition, the function \( f \in \mathcal{H} \) is said to belong to the Hardy space \( \mathcal{H}_\mu \), where \( 0 < \mu \leq \infty \), if the set \( \{ M_\mu (r, f) : r \in [0, 1) \} \) is bounded. We note that for \( 1 \leq \mu \leq \infty \), \( \mathcal{H}_\mu \) is actually a Banach space with the norm defined by \( \|f\|_\mu = \lim_{r \to 1^-} M_\mu (r, f) \).

Furthermore, \( \mathcal{H}^\infty \) is the class of bounded analytic functions in \( \mathcal{H} \). We note that for \( 0 < \mu \leq \nu \leq \infty \), it can be shown that \( \mathcal{H}_\nu \) is a subset of \( \mathcal{H}_\mu \) (see the book of P.L. Duren [89]).

For \( \alpha < 1 \) we introduce the class

\[
\mathcal{P}(\alpha) = \{ f \in \mathcal{H} : f(0) = 1, \exists \, \eta \in \mathbb{R} \text{ such that } \Re \{ e^{i\eta} f(z) \} > \alpha \text{ for all } z \in \mathbb{D} \}
\]

and define \( \mathcal{R}(\alpha) = \{ f \in \mathcal{A} : f' \in \mathcal{P}(\alpha) \} \), i.e.

\[
\mathcal{R}(\alpha) = \{ f \in \mathcal{A} : \exists \, \eta \in \mathbb{R} \text{ such that } \Re \{ e^{i\eta} f'(z) \} > \alpha \text{ for all } z \in \mathbb{D} \},
\]

where we used the fact that \( f'(0) = 1 \) for every \( f \in \mathcal{A} \). When \( \eta = 0 \) we denote \( \mathcal{P}(\alpha) \) and \( \mathcal{R}(\alpha) \) simply by \( \mathcal{P}_0(\alpha) \) and \( \mathcal{R}_0(\alpha) \), respectively; for \( \alpha = 0 \) we denote \( \mathcal{P}_0(\alpha) \) and \( \mathcal{R}_0(\alpha) \) simply by \( \mathcal{P} \) and \( \mathcal{R} \), respectively.

The next lemmata will be used to prove several theorems.

**Lemma 2.14.** (S. Ponnusamy [180]) For \( \alpha, \beta < 1 \) and \( \gamma = 1 - 2(1 - \alpha)(1 - \beta) \), we have \( \mathcal{R}(\alpha) * \mathcal{R}_0(\beta) \subset \mathcal{P}(\gamma) \), or equivalently \( \mathcal{P}(\alpha) \ast \mathcal{P}_0(\beta) \subset \mathcal{P}(\gamma) \).

**Lemma 2.15.** (P.J. Eenigenburg and F.R. Keogh [93]) Let \( \alpha \in [0, 1) \). If the function \( f \in \mathcal{C}(\alpha) \) is not of the form

\[
\begin{align*}
\begin{cases}
f(z) = \mu + \nu z(1 - ze^{i\gamma})^{2\alpha - 1}, & \alpha \neq 1/2 \\
f(z) = \mu + \nu \log(1 - ze^{i\gamma}), & \alpha = 1/2
\end{cases}
\end{align*}
\]
for some $\mu, \nu \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, then the following statements hold:

(a) There exists $\delta = \delta(f) > 0$ such that $f' \in \mathcal{H}^{\delta+1/2(1-\alpha)}$.
(b) If $\alpha \in [0, 1/2)$, then there exists $\tau = \tau(f) > 0$ such that $f \in \mathcal{H}^{\tau+1/(1-2\alpha)}$.
(c) If $\alpha \geq 1/2$, then $f \in \mathcal{H}^\infty$.

Our main results read as follows.

**Theorem 2.18.** (Á. Baricz [43]) Let $\alpha \in [0, 1)$ and $b, p, c$ be real numbers such that $c \neq 0$ and $4\alpha^2 + (|c| - 6)\alpha + 2 \geq 0$. If $4(1-\alpha)\kappa \geq |c| + 2(1-\alpha)(1-2\alpha)$, then the following assertions are true:

(a) If $\alpha \in [0, 1/2)$, then $u_p \in \mathcal{H}^{1/(1-2\alpha)}$.
(b) If $\alpha \geq 1/2$, then $u_p \in \mathcal{H}^\infty$.

**Proof.** First we observe that

$$
\mu + \frac{\nu z}{(1 - ze^{i\gamma})^{1-2\alpha}} = \mu + \nu z F(1, 1 - 2\alpha, 1, ze^{i\gamma})
$$

$$
= \mu + \nu \sum_{n \geq 0} \frac{(1 - 2\alpha)_n}{n!} e^{imz n + 1}
$$

for $\mu, \nu \in \mathbb{C}$, $\alpha \neq 1/2$ and for real $\gamma$, where $F(a, b; c; z)$ is the Gaussian hypergeometric series defined in (1.1). On the other hand

$$
\mu + \nu \log(1 - ze^{i\gamma}) = \mu - \nu z F(1, 1, 2, ze^{i\gamma})
$$

$$
= \mu - \nu \sum_{n \geq 0} \frac{1}{n + 1} e^{imz n + 1}.
$$

Therefore, since $\gamma$ is real, the function $u_p$ cannot be of the form $\mu + \nu z (1 - ze^{i\gamma})^{2\alpha-1}$ (for $\alpha \neq 1/2$) or $\mu + \nu \log(1 - ze^{i\gamma})$ (for $\alpha = 1/2$). We know that $u_p$ is convex of order $\alpha$ (property (a) in Theorem 2.13). Hence by Lemma 2.15 the proof is completed.

**Theorem 2.19.** (Á. Baricz [43]) If $b, p, c \in \mathbb{R}$ such that $4\kappa \geq |c| - 2$ and $c \neq 0$, then $u_p \in \mathcal{H}^\infty$.

**Proof.** Since $\kappa + 1 \geq |c|/4 + 1/2$, it follows by Theorem 2.11 that $u_{p+1} \in \mathcal{C} = \mathcal{C}(0)$. In addition, by the hypergeometric series representation we observe that $u_{p+1}$ is not of the form

$$
\mu + \nu z (1 - ze^{i\gamma})^{-1} = \mu + \nu z F(1, 1, 1, ze^{i\gamma})
$$

$$
= \mu + \nu \sum_{n \geq 0} e^{imz n + 1},
$$
where \( \mu, \nu \in \mathbb{C} \) and \( \gamma \in \mathbb{R} \). Taking into consideration that \( 4k\mu'p(z) = -c\nu_{p+1}(z) \) (see Lemma 1.2), we conclude that \( u'_p \) is convex in \( \mathbb{D} \) and is not of the form \( \mu + \nu z(1-ze^{i\gamma})^{-1} \). Hence by Lemma 2.15 we have \( u'_p \in \mathcal{H}^1 \). On the other hand, it is known that an analytic function \( f : \mathbb{C} \to \mathbb{C} \) is continuous in \( \overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and absolutely continuous on \( \partial \mathbb{D} = \{ z \in \mathbb{C} : |z| = 1 \} \) if and only if \( f' \in \mathcal{H}^1 \) (see the book of P.L. Duren [89, Theorem 3.11]). From this we deduce that \( u_p \) is continuous in \( \overline{\mathbb{D}} \), so \( u_p \) is a bounded analytic function in \( \mathbb{D} \). This completes the proof of Theorem 2.19.

We note that \( u_p \) is in fact an entire function, i.e. it is holomorphic everywhere on the whole complex plane. Thus, since every continuous function on a compact set is bounded, we deduce that \( u_p \in \mathcal{H}^\infty \) for all admissible values of the parameters \( b, c \) and \( p \).

The main aim in the sequel is to find conditions on \( \alpha_1 \) and \( \alpha_2 \) and the parameters \( b, c \) and \( p \) such that \( Zu_p * f \) maps \( \mathcal{R}(\alpha_1) \) into \( \mathcal{R}(\alpha_2) \).

**Theorem 2.20.** (Á. Baricz [43]) Let \( \alpha \in [0, 1/2) \), and let \( b, p, c \) be real numbers such that \( \kappa \geq (1-\alpha)(1-2\alpha)^{-1/2} |c|/4 + 1 \). If \( f \in \mathcal{R}(\alpha_1) \), where \( \alpha_1 < 1 \), then \( Zu_p * f \in \mathcal{R}(\gamma) \), where \( \gamma = 1 - 2(1 - \alpha_1)(1 - \alpha) \).

**Proof.** Set \( g(z) = Zu_p(z) * f(z) \). Then we have \( g'(z) = u_p(z) * f'(z) \). By the hypotheses and Theorem 2.4, we have \( u_p \in \mathcal{P}_0(\alpha) \). Using Lemma 2.14 and the fact that \( f' \in \mathcal{P}(\alpha_1) \), we immediately obtain that the function \( g' \) belongs to \( \mathcal{P}(\gamma) \), where \( \gamma = 1 - 2(1 - \alpha_1)(1 - \alpha) \). But \( g' \in \mathcal{P}(\gamma) \) implies \( g \in \mathcal{R}(\gamma) \), therefore the proof is complete. \( \square \)

As an immediate consequence of Theorem 2.20 we have the following corollary.

**Corollary 2.10.** (Á. Baricz [43]) Let \( \alpha, b, p, c \) satisfy the hypotheses of Theorem 2.20. If \( f \in \mathcal{R}(\alpha_1) \), where \( \alpha_1 = (1 - 2\alpha)/(2 - 2\alpha) \), then \( Zu_p * f \in \mathcal{R}(0) \).

Taking \( \alpha = 0 \) in the above Corollary we obtain the next result.

**Corollary 2.11.** (Á. Baricz [43]) Let \( b, p, c \in \mathbb{R} \) such that \( \kappa \geq |c|/4 + 1 \). If \( f \in \mathcal{R}(1/2) \), then \( Zu_p * f \in \mathcal{R}(0) \).

Now we present the analogue of Theorem 2.20, using Lemma 2.9 and Theorem 2.6.

**Theorem 2.21.** (Á. Baricz [43]) Let \( \alpha_1 < 1, \alpha \in [0, 1) \), \( b \in \mathbb{R}, c \in [-3\gamma, 0) \) such that \( 8(1-\alpha)\kappa + c \geq 0 \). If \( f \in \mathcal{R}(\alpha_1) \), then \( Zu_p * f \in \mathcal{R}(\gamma) \), with \( \gamma = 1 - 2(1 - \alpha_1)(1 - \alpha) \).

**Proof.** Using Theorem 2.6, we have that \( u_p \in \mathcal{P}_0(\alpha) \). Therefore the proof is the same as the proof of Theorem 2.20. \( \square \)

As an immediate consequence of Theorem 2.21, we have for \( \gamma = 0 \) and \( \alpha = 0 \), respectively.
Corollary 2.12. (Á. Baricz [43]) Let $\alpha \in [0, 1)$, $b \in \mathbb{R}$ and $c \in [-37, 0)$ be such that $8(1 - \alpha)\kappa + c \geq 0$. If $f \in \mathcal{R}(\alpha_1)$, where $\alpha_1 = (1 - 2\alpha)/(2 - 2\alpha)$, then $zu_p \ast f \in \mathcal{R}(0)$. In particular, let $b \in \mathbb{R}$, $c \in [-37, 0)$ be such that $\kappa \geq -c/8$. If $f \in \mathcal{R}(1/2)$, then $zu_p \ast f \in \mathcal{R}(0)$.

Note that it is easy to verify that the results of Corollary 2.12 for $c \in [-37, 0)$ are better than the results of Corollaries 2.10 and 2.11. The situation is the same for Theorems 2.21 and 2.20. So in certain cases the “method of sequences” is better than the method of differential subordinations.

For Gaussian and confluent hypergeometric functions (and as well for generalized hypergeometric functions) may be found similar results with those given in this section in the works published by J.H. Choi et al. [84] and S. Ponnusamy [175, 176].

We end this section by applying the above presented theorems and corollaries for some particular cases. By applying the Theorems 2.18, 2.19, 2.20, 2.21 and Corollaries 2.10, 2.11, 2.12 to the function

$$z \mapsto \mathcal{I}_p(z^{1/2}) = 2^p \Gamma(p + 1)z^{-p/2}I_p(z^{1/2})$$

we obtain the following result.

Corollary 2.13. (Á. Baricz [43]) Let $p \in \mathbb{R}$. For the function $z \mapsto \mathcal{I}_p(z^{1/2})$ the following assertions are true:

(a) Let $\alpha \in [0, 1)$ satisfy $4(1 - \alpha)p \geq 4\alpha^2 - 2\alpha - 1$. If $\alpha \geq 1/2$, then $\mathcal{I}_p(z^{1/2}) \in \mathcal{H}_\infty$; if $\alpha \in [0, 1/2)$ we have $\mathcal{I}_p(z^{1/2}) \in \mathcal{H}_{1/(1 - 2\alpha)}$.

(b) If $p \geq -5/4$, then $\mathcal{I}_p(z^{1/2}) \in \mathcal{H}_\infty$.

(c) Let $\alpha_1 < 1$, $4p\sqrt{1 - 2\alpha} \geq 1 - \alpha$, where $\alpha \in [0, 1/2)$. If $f \in \mathcal{R}(\alpha_1)$, then $z\mathcal{I}_p(z^{1/2}) \ast f(z) \in \mathcal{R}(\gamma)$, with $\gamma = 1 - 2(1 - \alpha_1)/(1 - \alpha)$. Moreover, if $\alpha_1 = (1 - 2\alpha)/(2 - 2\alpha)$ and $f \in \mathcal{R}(\alpha_1)$, then $z\mathcal{I}_p(z^{1/2}) \ast f(z) \in \mathcal{R}(0)$. In particular, if $p \geq 1/4$ and $f \in \mathcal{R}(1/2)$, then $z\mathcal{I}_p(z^{1/2}) \ast f(z) \in \mathcal{R}(0)$.

(d) Let $\alpha_1 < 1$, $8(1 - \alpha)(p + 1) \geq 1$, where $\alpha \in [0, 1)$. If $f \in \mathcal{R}(\alpha_1)$, then $z\mathcal{I}_p(z^{1/2}) \ast f(z) \in \mathcal{R}(\gamma)$, with $\gamma = 1 - 2(1 - \alpha_1)/(1 - \alpha)$. If $\alpha_1 = (1 - 2\alpha)/(2 - 2\alpha)$ and $f \in \mathcal{R}(\alpha_1)$, then $z\mathcal{I}_p(z^{1/2}) \ast f(z) \in \mathcal{R}(0)$. In particular, if $p \geq -7/8$ and $f \in \mathcal{R}(1/2)$, then $z\mathcal{I}_p(z^{1/2}) \ast f(z) \in \mathcal{R}(0)$.

Remark 2.14. For a normalized function $f \in \mathcal{A}$ define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} \, dt, \quad (2.25)$$

where $\mu$ is a real-valued, nonnegative weight function normalized so that

$$\int_0^1 \mu(t) \, dt = 1.$$ 

Now, for $\alpha < 1$ and $0 \leq \beta \leq 1$ we define the class of functions

$$\mathcal{P}_\beta(\alpha) = \{ f \in \mathcal{A} : \exists \eta \in \mathbb{R} \text{ such that } \text{Re} \Delta_\beta(z) > \alpha \text{ for all } z \in \mathbb{D} \},$$
where
\[ \Delta_\beta(z) = e^{i\eta} \left( (1 - \beta) \frac{f(z)}{z} + \beta f'(z) \right). \]

In the last two decades there has been a vivid interest concerning the integral transform (2.25). By using the duality technique among others many authors have found conditions for \( f \in \mathcal{P}_\beta(\alpha) \) such that \( V_\mu(f) \in \mathcal{S}^* \) or \( V_\mu(f) \in \mathcal{P}_\beta(\delta) \). The reason for the considerable interest in finding such conditions is that for appropriate choices of \( \mu \) the integral transform \( V_\mu(f) \) reduces to some known operators like Alexander, Bernardi, Libera and Komatu transforms, respectively. For more details and for related results we refer to the papers of R.M. Ali and V. Singh [6], R. Fournier and S. Ruscheweyh [99], S. Ponnusamy and F. Rønning [181–184], Y.C. Kim and F. Rønning [134], and to references therein. In the study of the above conditions the Gaussian hypergeometric function plays an important role. More precisely, the convolution operator \( V_{a,b,c}(f)(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}f(tz) \frac{dt}{t} \) is exactly an integral operator of the form (2.25), which reduces for \( a = 1 \) to the Carlson-Shaffer operator
\[ V_{1,b,c}(f)(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}f(tz) \frac{dt}{t}. \]

The hypergeometric transform \( V_{a,b,c}(f) \) has been studied intensively in the last two decades. For example, S. Ponnusamy [178] has found conditions on \( \alpha_1, \alpha_2 \) and on the parameters \( a, b \) and \( c \) such that the operator \( V_{a,b,c}(f) \) maps \( \mathcal{R}(\alpha_1) \) into \( \mathcal{R}(\alpha_2) \), Y.C. Kim and F. Rønning [134] have found the sharp value of \( \alpha \) such that \( V_{a,b,c}(f) \in \mathcal{S}^* \), where \( f \in \mathcal{P}_\beta(\alpha) \). For other important geometric properties of the integral transform (2.25) and of the hypergeometric transform \( V_{a,b,c}(f) \) we refer to the papers of R. Balasubramanian et al. [33–35, 37], R.W. Barnard et al. [65], J.H. Choi et al. [83] and to the references therein.

### 2.3.2 A Monotonicity Property of Generalized Bessel Functions

Let \( z \mapsto F(a,b,c,z) \) be the Gaussian hypergeometric function, defined by (1.1). Based on numerical experiments S. Ponnusamy and M. Vuorinen [188, p. 351] enunciated the following interesting open problems (which to our knowledge are still open):

1. There exist positive numbers \( \alpha_1, \alpha_2 \) such that for \( a \in (0, \alpha_1) \) and \( b \in (0, \alpha_2) \) the normalized function \( z \mapsto zF(a,b,a+b,z) \) (\( z \mapsto zF(a,b,a+b,z^2) \), respectively) maps the \( \mathbb{D} \) into a strip domain. For example, the functions
\[ -\log(1-z) = zF(1,1,2,z) \]
and
\[
\frac{1}{2} \log \left( \frac{1+z}{1-z} \right) = zF \left( 1, \frac{1}{2}, \frac{3}{2}; z^2 \right)
\]
map \( \mathbb{D} \) into a strip. Therefore, the problem here is to find the exact range of the constants \( \alpha_1, \alpha_2 \) and conditions on \( a \) and \( b \) such that this property holds.

2. We recall that the Koebe function
\[
\frac{z}{(1-z)^2} = zF(1,2,1,z)
\]
maps \( \mathbb{D} \) into the complement of the ray \( \{ z = x + iy \in \mathbb{C} : x = 0, y \leq -1/4 \} \).

This function raises the following question: Suppose that \( a, b, c > 0 \) with \( c < a + b \). Do there exist \( \alpha_3, \alpha_4 > 0 \) such that for \( a \in (0, \alpha_3) \) and \( b \in (0, \alpha_4) \) the function \( z \mapsto zF(a,b,c,z) \) has the property that the image domain is completely contained in a sector type domain where the “angle” depends on \( a + b - c \)?

Let \( u_p \) be the generalized and normalized Bessel function of the first kind of order \( p \) depending on the parameters \( b, c, p \) and let \( \lambda_p(z) = u_p(z^2), z \in \mathbb{D} \). Motivated by the above open problems, we prove in this section that the image region \( \lambda_p(\mathbb{D}) \) is completely contained in a disk type domain (or egg-shaped domain). We also find the radius of the smallest disk centered at 1 which contains the image region \( \lambda_p(\mathbb{D}) \). Further, we aim to prove that if \( p > q \), then \( \lambda_p(\mathbb{D}) \subset \lambda_q(\mathbb{D}) \), under certain conditions on the parameters \( b, c, p, q \). The original results of this section were obtained by S. András and Á. Baricz [24].

Due to Lommel’s expression of \( J_p(z) \) as an integral of Poisson type (see the book of G.N. Watson [227, p. 48]) we know that if \( \text{Re } p > -1/2 \) and \( z \in \mathbb{C} \) then
\[
J_p(z) = \frac{(z/2)^p}{\Gamma(p + \frac{1}{2})} \int_0^\pi e^{iz\cos \theta} \sin^{2p} \theta d\theta.
\] (2.26)

If \( p \) is real and greater than \(-1/2\), then it follows from (2.26) (see G.N. Watson [227, p. 49]) that
\[
|J_p(z)| \leq \frac{|(z/2)^p|}{\Gamma(p + \frac{1}{2})} \int_0^\pi e^{i|\text{Im } z|} \sin^{2p} \theta d\theta = \frac{|(z/2)^p|}{\Gamma(p + 1)} e^{i|\text{Im } z|}.
\] (2.27)

By using the expression \( \sqrt{2/(\pi z)} \cos z \) for \( J_{-1/2}(z) \) it may be shown that (2.27) is also valid when \( p = -1/2 \).

Let us consider now the function \( J_p \) defined by (1.23). Recall that in particular we have
\[
J_{-1/2}(z) = \sqrt{\pi/2} \cdot z^{1/2}J_{-1/2}(z) = \cos z,
\] (2.28)
\[
J_{1/2}(z) = \sqrt{\pi/2} \cdot z^{-1/2}J_{1/2}(z) = \frac{\sin z}{z},
\] (2.29)
\[ J_{3/2}(z) = 3\sqrt{\pi/2} \cdot z^{-3/2} J_{3/2}(z) = 3 \left( \frac{\sin z}{z^3} - \frac{\cos z}{z^2} \right). \] (2.30)

From (2.27) it is clear that for \( p \geq -1/2 \) and \( z \in \mathbb{C} \) we have the inequality \( |J_p(z)| \leq e^{|\text{Im}z|} \). Thus if \( z \) lies in \( \mathbb{D} \) and \( p \geq -1/2 \), then \( |J_p(z)| \leq e \). Comparing this inequality with the following inequalities (see the books of M. Abramowitz and I.A. Stegun [1, p. 75], D.S. Mitrinović [156, p. 323])

\[ |J_{-1/2}(z)| = |\cos z| < 2, \quad |J_{1/2}(z)| = \left| \frac{\sin z}{z} \right| < \frac{6}{5}, \text{ whenever } z \in \mathbb{D}, \quad (2.31) \]

it is natural to seek the radius of the smallest disk centered at the origin which contains the image region \( J_p(\mathbb{D}) \). In Corollary 2.14 we give the radius of this disk.

Moreover, using a disk with center at 1, we obtain a much sharper result.

**Remark 2.15.** When \( \text{Re} p > -1/2 \), then the function \( J_p \) under discussion admits the integral representation (see the handbook of M. Abramowitz and I.A. Stegun [1, p. 360] or the relation (1.26) for \( c = 1 \) and \( b = 1 \))

\[ J_p(z) = \int_0^1 \cos(tz) d\mu_p(t), \quad (2.32) \]

where \( d\mu_p(t) = \mu_p(t) dt \) with

\[ \mu_p(t) = \frac{2(1 - t^2)^{p-1/2}}{B\left(p + \frac{1}{2}, \frac{1}{2}\right)}, \]

\( \mu_p \) being the probability measure on \([0,1]\). From the first inequality in (2.31) we easily get

\[ |J_p(z)| = \left| \int_0^1 \cos(tz) d\mu_p(t) \right| \leq \int_0^1 |\cos(tz)| d\mu_p(t) < \int_0^1 2d\mu_p(t) = 2. \]

In other words, the above particular inequality \( |\cos z| < 2 \) implies \( |J_p(z)| < 2 \) for all \( z \in \mathbb{D} \), but this upper bound is far from being the best possible.

**Remark 2.16.** Numerical computations and graphics in Derive6, Maple6 guarantee the following properties of the functions \( J_{-1/2}, J_{1/2} \) and \( J_{3/2} \) defined above by the relations (2.28), (2.29) and (2.30), respectively:

1. \( J_{-1/2}(\mathbb{D}) \) is contained in the disk centered at 1, with radius 0.543080635..., i.e. we have \( |J_{-1/2}(z) - 1| < 0.543080635... \) for all \( z \in \mathbb{D} \) (see Fig. 2.1).
2. \( J_{1/2}(\mathbb{D}) \) is contained in the disk centered at 1, with radius 0.175201194..., i.e. we have \( |J_{1/2}(z) - 1| < 0.175201194... \) for all \( z \in \mathbb{D} \) (see Fig. 2.2).
3. \( J_{3/2}(\mathbb{D}) \) is contained in the disk centered at 1, with radius 0.103638323, i.e. we have \( |J_{3/2}(z) - 1| < 0.103638323... \) for all \( z \in \mathbb{D} \).
2.3 Applications Involving Bessel Functions with Hardy Space of Analytic Functions

Fig. 2.1 The graph of the function $f : \mathbb{D} \rightarrow \mathbb{C}$, defined by $f(z) = J_{-1/2}(z)$

Fig. 2.2 The graph of the function $f : \mathbb{D} \rightarrow \mathbb{C}$, defined by $f(z) = J_{1/2}(z)$
Taking into account the first property we may conclude that in fact \( J_{-1/2}(D) \) is contained in the disk centered at origin with radius \( \cosh 1 = 1.543080635\ldots \), and clearly this is the smallest disk (centered at origin) which contains \( J_{-1/2}(D) \), because

\[
|\cos z| = |\cos(x + iy)| = \sqrt{\cos^2 x + \sinh^2 y} \leq \cosh y < \cosh 1.
\]

Thus using the same argument as in Remark 2.15 we can conclude that for \( \Re p \geq -1/2 \), the inequality \( |J_p(z)| < 1.543080635\ldots \) holds for all \( z \in \mathbb{D} \), and this upper bound is the best possible for \( p = -1/2 \).

In view of Remark 2.16 and Figs. 2.3 and 2.4 we may ask the following questions:

(a) Is it true that if \( p > -1 \) increases, then the image region \( J_p(D) \) decreases, more precisely if \( p > q > -1 \), then \( J_p(D) \subset J_q(D) \)?

(b) Find the radius \( r_p \) (depending on \( p \)) of the smallest disk centered at 1 which contains the image region \( J_p(D) \) when \( p > -1 \).

Our aim in this section is to answer the above questions for generalized and normalized Bessel functions, Theorem 2.22 is the main result of this section. In particular we prove that \( r_p = J_p(1) - 1 \), where \( J_p \) is defined by (1.24). Moreover we prove that if \( p > q \geq -1/4 \), then \( J_p(D) \subset J_q(D) \) holds (see Corollary 2.14 below). Part (a) of the above questions for \( p > q \) and \( q \in (-1,-1/4) \) remains open.

The following result contains conditions on the parameters \( b,c,p,q \) such that \( \lambda_p(D) \subset \lambda_q(D) \) holds.

**Fig. 2.3** The graph of the functions \( f,g : D \to \mathbb{C} \), defined by \( f(z) = J_{-1/2}(z) \) and \( g(z) = J_{1/2}(z) \)
2.3 Applications Involving Bessel Functions with Hardy Space of Analytic Functions

The graph of the functions $f, g, h : \mathbb{D} \rightarrow \mathbb{C}$, defined by $f(z) = J_{-1/2}(z)$, $g(z) = J_{1/2}(z)$ and $h(z) = J_{3/2}(z)$

**Theorem 2.22.** (S. András and Á. Baricz [24]) Let $p, b, c \in \mathbb{C}$ be such that $2\kappa_p = 2p + b + 1 \neq 0, -2, -4, \ldots$. The radius of the smallest disk centered at 1 which contains the domain $\lambda_p(D)$ is

$$\sum_{n \geq 1} \frac{|c/4|^n}{n! \cdot (\kappa_p)_n}.$$  

Moreover, if $q \in \mathbb{C}$ such that $\text{Re} \kappa_p > \text{Re} \kappa_q$ and

$$12 \text{Re} \kappa_q \geq 3|c| + 2(\text{Im} \kappa_q)^2 + 6,$$

where $2\kappa_q = 2q + b + 1$, then $\lambda_p(D) \subset \lambda_q(D)$.

**Proof.** First observe that

$$\left| \sum_{m=1}^{n} \frac{(-c/4)^m z^{2m}}{(\kappa_p)_m m!} \right| \leq \sum_{m=1}^{n} \frac{|-c/4|^m z^{2m}}{(\kappa_p)_m m!} < \sum_{m=1}^{n} \frac{|-c/4|^m}{(\kappa_p)_m m!} = \frac{1}{m!}$$  

holds for all $z \in \mathbb{D}$. For $z \to \pm i$ we obtain equality, so this is the best upper bound for the whole disk. From the D’Alembert ratio test the radius of convergence for the power series $\lambda_p(z) - \lambda_p(0)$ is infinity, so if in (2.34) $n$ tends to infinity, we get the required result.
By the definition of the function \( \lambda_p \), it is enough to show that under the above assumptions \( u_p(\mathbb{D}) \subset u_q(\mathbb{D}) \) holds. Due to condition (2.33) from part (a) of Theorem 2.12 we know that \( u_q \) is a convex function in \( \mathbb{D} \), i.e. \( u_q(\mathbb{D}) \) is convex. From Lemma 1.4, equation (1.31) we conclude, by using a known integral mean-value formula, that \( u_p(\mathbb{D}) \) is contained in the convex hull of \( \{u_q(t\mathbb{D}) : t \in [0,1]\} \). Since \( u_q(\mathbb{D}) \) is convex we obtain that \( u_p(\mathbb{D}) \subset u_q(\mathbb{D}) \). Thus the proof is complete. \( \square \)

Choosing \( c = 1, b = 1 \) in Theorem 2.22 we obtain the following result:

**Corollary 2.14.** (S. András and Á. Baricz [24]) Let \( p > -1 \). The radius \( r_p \) of the smallest disk centered at 1 which contains the domain \( \mathcal{I}_p(\mathbb{D}) \) is \( r_p(1) - 1 \). Moreover, if \( p, q \in \mathbb{C} \) such that \( \text{Re } p > \text{Re } q \geq -1/4 + (\text{Im } q)^2 / 6 \), then \( \mathcal{I}_p(\mathbb{D}) \subset \mathcal{I}_q(\mathbb{D}) \) holds. In particular, if \( p, q \) are real numbers such that \( p > q \geq -1/4 \), then \( \mathcal{I}_p(\mathbb{D}) \subset \mathcal{I}_q(\mathbb{D}) \).

**Remark 2.17.** Let \( \mathcal{I}_p \) defined by (1.24). Analogously with the relations (2.28), (2.29) and (2.30) we have

\[
\mathcal{I}_{-1/2}(z) = \sqrt{\pi/2} z^{1/2} I_{-1/2}(z) = \cosh z,
\]

\[
\mathcal{I}_{1/2}(z) = \sqrt{\pi/2} z^{-1/2} I_{1/2}(z) = \frac{\sinh z}{z},
\]

\[
\mathcal{I}_{3/2}(z) = 3 \sqrt{\pi/2} z^{-3/2} I_{3/2}(z) = -3 \left( \frac{\sinh z}{z^3} - \frac{\cosh z}{z^2} \right).
\]

Thus, we have

\[
r_{-1/2} = \mathcal{I}_{-1/2}(1) - 1 = \cosh 1 - 1 \simeq 0.543080635 \ldots,
\]

\[
r_{1/2} = \mathcal{I}_{1/2}(1) - 1 = \sinh 1 - 1 \simeq 0.175201194 \ldots
\]

and

\[
r_{3/2} = \mathcal{I}_{3/2}(1) - 1 = 3(\cosh 1 - \sinh 1) - 1 \simeq 0.103638323 \ldots.
\]

**Remark 2.18.** Note that under the condition (2.33) the image region \( u_q(\mathbb{D}) \) (and consequently \( \lambda_q(\mathbb{D}) \)) is clearly starlike with respect to 1, i.e. \( u_q \) is a starlike function with respect to 1. This is justified by convexity of \( u_q(\mathbb{D}) \) and by the relation \( u_q(0) = 1 \). On the other hand, \( \lambda_p(\mathbb{D}) \) is symmetric with respect to the real axis, since according to (1.26) we have

\[
\lambda_p(z) = \int_0^1 \cos(rz\sqrt{c})d\mu_{p,b}(t) = \int_0^1 \cos(tz\sqrt{c})d\mu_{p,b}(t) = \lambda_p(z) \text{ for all } z \in \mathbb{D}.
\]
Moreover, using results of S. Owa and H.M. Srivastava [168, Lemma 3, p. 1062] we can find easily other sufficient conditions for the function $u_q$ to be starlike with respect to 1. For example, if $u_q$ satisfies the condition $|zu''_q(z)/u'_q(z)| < 1$, where $\kappa_q \neq 0, -1, -2, \ldots$ and $z \in \mathbb{D}$, then it is starlike with respect to 1 (see Theorem 2.9).

**Remark 2.19.** The Schwarz lemma (see the book of P.L. Duren [90, p. 3]) states: if $f : \mathbb{D} \to \mathbb{D}$ is analytic with $f(0) = 0$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. On the other hand, an analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be subordinate (see the book of P.L. Duren [90]) to an analytic function $g : \mathbb{D} \to \mathbb{C}$ (written $f \prec g$) if $f = g \circ \varphi$ for some analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$. The superordinate function $g$ need not be univalent. From Schwarz’s lemma it is clear that if $f \prec g$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Moreover if the function $g$ is univalent, then the inverse implication also holds. Thus, taking into consideration that $u_p(0) = u_q(0) = 1$ and that the function $u_q$ is univalent when $\text{Re} \kappa_q \geq |c|/4$ (by Theorem 2.3), Theorem 2.22 can be written in terms of subordination, i.e. if the hypotheses of Theorem 2.22 hold, then $u_p \prec u_q$. Finally, note that similar results for confluent and Gaussian hypergeometric functions were obtained by S.S. Miller and P.T. Mocanu [154].
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