Chapter 9
Some Tools from the Semiclassical Calculus

In this chapter we shall describe some tools from Semiclassical Analysis, that will be used in the subsequent chapters to obtain localization properties of the large eigenvalues of an elliptic positive NCHO $Q^{\mathrm{w}}_{(\alpha,\beta)}(x,D)$, and more generally of an elliptic positive NCHO (with no sub-principal term), relating them to properties of the *periods of the bicharacteristics of the (principal) symbol*.

9.1 The Semiclassical Calculus

We recall here some basic properties of semiclassical the Weyl-quantization and the semiclassical calculus. (In general, for Semiclassical Analysis, we address the reader to Dimassi-Sjöstrand [7], Evans-Zworski [15], Ivrii [34], Martinez [40], Robert [65], Shubin [67] and Voros [71].)

Definition 9.1.1 (Semiclassical symbols). We shall say that a function $a(X;h) = a(\cdot;h) \in C^\infty(\mathbb{R}^{2n}_X)$, possibly depending on a parameter $h \in (0,h_0]$, $h_0 \in (0,1]$, belongs to the symbol class $S^k_\delta(m^\mu,g)$, $k,\mu \in \mathbb{R}$ and $\delta \in [0,1/2]$, if for all $\alpha \in \mathbb{Z}^{2n}_+$ there exists $C_\alpha > 0$ such that

$$|\partial^\alpha_X a(X;h)| \le C_\alpha m(X)^{\mu-|\alpha|} h^{-k-|\alpha|\delta},\ \forall X \in \mathbb{R}^{2n},\ \forall h \in (0,h_0]. \tag{9.1}$$

As before, $S^k_\delta(m^\mu,g;\mathsf{M}_N) = S^k_\delta(m^\mu,g)\otimes \mathsf{M}_N$, and $S^k_\delta(m^\mu,g;\mathsf{V}) = S^k_\delta(m^\mu,g)\otimes \mathsf{V}$, for any given finite-dimensional complex vector space V.

Given $a \in S^k_\delta(m^\mu,g;\mathsf{V})$, we shall consider its h-Weyl quantization

$$a^{\mathrm{w}}(x,hD)u(x) = (2\pi h)^{-n}\iint e^{ih^{-1}\langle x-y,\xi\rangle} a(\frac{x+y}{2},\xi;h)u(y)dyd\xi,\ u \in \mathscr{S}.$$

It is easy to see that $a^{\mathrm{w}}(x,hD)$ is in fact the Weyl-quantization of the symbol $a(x,h\xi;h)$.

We shall also need more general classes, defined as follows (see Dimassi-Sjöstrand [7]). We say that a smooth function $a(\cdot;h) \in C^\infty(\mathbb{R}^{2n}_X)$, possibly depending

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on a parameter $h \in (0,h_0]$, $h_0 \in (0,1]$, belongs to the symbol class $S^k_\delta(m^\mu)$, $k,\mu \in \mathbb{R}$ and $\delta \in [0,1/2]$, if for all $\alpha \in \mathbb{Z}^n_+$ there exists $C_\alpha > 0$ such that

$$|\partial^\alpha_X a(X;h)| \le C_\alpha m(X)^\mu h^{-k-|\alpha|\delta},\ \forall X \in \mathbb{R}^{2n},\ \forall h \in (0,h_0]. \tag{9.2}$$

Notice that, compared to (9.1), no further decay in m is obtained by taking derivatives of the symbol. The h-Weyl quantization of a symbol $a \in S^k_\delta(m^\mu)$ is defined as before. The vector and matrix-valued symbol classes are defined as before.

Notice that
$$S^k_\delta(m^\mu) = S^k_\delta(m^\mu, |dX|^2),$$
where $|dX|^2$ is the Euclidean metric in $\mathbb{R}^{2n}_X$, and that

$$S(m^\mu, g) \subset S^0_0(m^\mu, g) \subset S^k_\delta(m^\mu, g) \subset S^k_\delta(m^\mu). \tag{9.3}$$

Remark 9.1.2. Let $E > 0$. Define the L^2-isometry (also automorphism of $\mathscr{S}'$ and $\mathscr{S}$),
$$U_E\colon u(x) \longmapsto E^{-n/4} u(x/\sqrt{E}). \tag{9.4}$$
Then, given any symbol $a \in S^k_0(m^\mu; \mathsf{M}_N)$ one has from Theorem 3.1.12

$$U_E^{-1} a^{\mathrm{w}}(x,hD) U_E = a^{\mathrm{w}}(\sqrt{E}x, \sqrt{E}\tilde{h}D),\ \ \textit{where}\ \tilde{h} = \frac{h}{E}. \tag{9.5}$$

In particular
$$U_h^{-1} a^{\mathrm{w}}(x,hD) U_h = a^{\mathrm{w}}(\sqrt{h}x, \sqrt{h}D). \tag{9.6}$$
Notice that since U_h corresponds to the symplectic transformation

$$\kappa_h\colon (x,\xi) \longmapsto (h^{1/2}x, h^{-1/2}\xi),$$

by using $h^{1/2}m(X) \le m(h^{1/2}X) \le m(X)$, we have that if $a \in S(m^\mu, g)$, then

$$(a \circ \kappa_h)(\cdot, h\cdot) = a(\sqrt{h}\cdot, \sqrt{h}\cdot) \in S_0^{-\min\{\mu,0\}/2}(m^\mu, g).$$

△

By Proposition 3.2.15 we have the following result, that we shall frequently use when dealing with "classical semiclassical symbols" in the class $S^k_{0,\mathrm{cl}}(m^\mu, g)$ (see Definition 9.1.9 below).

Proposition 9.1.3. *Let $\mu \in \mathbb{R}$. Let $a_j \in S(m^{\mu-2j}, g)$, $j \in \mathbb{Z}_+$ (in particular, the a_j are* **independent** *of h). Then there exists $a \in S^0_0(m^\mu, g)$ such that*

$$a \sim \sum_{j\ge 0} h^j a_j,$$

that is, for all $r \in \mathbb{Z}_+$ we have

$$a - \sum_{j=0}^{r} h^j a_j \in S_0^{-(r+1)}(m^{\mu-2(r+1)}, g).$$

If another symbol a' has the same property, then

$$a - a' \in S^{-\infty}(m^{-\infty}, g) := \bigcap_{k \in \mathbb{Z}_+} S_0^{-k}(m^{\mu-2k}, g). \tag{9.7}$$

Proof. One takes an excision function $\chi \in C^\infty(\mathbb{R}_t; [0,1])$, such that $\chi(t) = 0$ for $|t| \leq 2$ and $\chi(t) = 1$ for $|t| \geq 4$, and defines

$$a(X;h) := \sum_{j=0}^{\infty} h^j \chi\left(\frac{m(X)}{hR_j}\right) a_j(X),$$

where $\{R_j\}_{j \in \mathbb{Z}_+} \subset [1, +\infty)$ is an increasing sequence with $R_j \nearrow +\infty$ as $j \to +\infty$ sufficiently fast. Following the steps of the proof of Proposition 3.2.15, using the fact that $h \in (0,1]$ and that on the support of $1 - \chi(m(X)/hR_j)$ we have $|X| \leq 4R_j$ and $1 \leq h^{-1} \leq 4R_j$ (so that we may divide and multiply by h^r, for any given $r \in \mathbb{N}$), one sees that the sequence R_j can indeed be so chosen that $a(X;h)$ has the required properties. □

Using a Borel-summation argument one also has the following proposition (see Evans-Zworski [15]).

Proposition 9.1.4. *Let $k_j \searrow -\infty$, $k_j > k_{j+1}$, $j \in \mathbb{Z}_+$, be a monotone decreasing sequence of real numbers. Let $a_j \in S_\delta^{k_j}(m^\mu)$. Then there exists $a \in S_\delta^{k_0}(m^\mu)$ such that $a \sim \sum_{j \geq 0} a_j$, that is, for all $r \in \mathbb{Z}_+$*

$$a - \sum_{j=0}^{r} a_j \in S_\delta^{k_{r+1}}(m^\mu).$$

If another symbol a' has the same property, then

$$a - a' \in S^{-\infty}(m^\mu) := \bigcap_{k \in \mathbb{R}} S_\delta^k(m^\mu). \tag{9.8}$$

Proof. One chooses $\chi \in C^\infty([0, +\infty); \mathbb{R})$ such that

$$0 \leq \chi \leq 1,\ \chi\big|_{[0,1]} = 1,\ \chi\big|_{[2,+\infty)} \equiv 0,$$

and then defines

$$a(X) := \sum_{j \geq 0} \chi(\lambda_j h) a_j(X),\ \ X \in \mathbb{R}^{2n},$$

where the sequence $1 \le \lambda_1 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots \to +\infty$ must be picked. The choice of the λ_j is therefore made as follows: for each multiindex α with $|\alpha| \le j$, we have

$$\begin{aligned}
|\partial_X^\alpha \big(\chi(\lambda_j h) a_j(X)\big)| &= |\chi(\lambda_j h)\partial_X^\alpha a_j(X)| \le C_{j,\alpha} h^{-k_j-\delta|\alpha|}\chi(\lambda_j h) m(X)^\mu \\
&= C_{j,\alpha} h^{-k_j-\delta|\alpha|}\frac{\lambda_j h}{\lambda_j h}\chi(\lambda_j h) m(X)^\mu \\
&\qquad (\text{since } \lambda_j h \le 2 \text{ in the support of } \chi) \\
&\le 2C_{j,\alpha}\frac{1}{\lambda_j} h^{-k_j-1-\delta|\alpha|} m(X)^\mu \\
&\le h^{-k_j-1-\delta|\alpha|} 2^{-j} m(X)^\mu,\ \forall X \in \mathbb{R}^{2n},
\end{aligned}$$

if λ_j is picked sufficiently large. Since we can accomplish this for all j and all α with $|\alpha| \le j$, and trivially make it possible to have $\lambda_{j+1} \ge \lambda_j$, the sequence $\{\lambda_j\}_j$ is therefore determined. One then concludes in a way similar to that used in the proof of Proposition 3.2.15. □

We leave it as an exercise for the reader to fill in the details of the proofs of Proposition 9.1.3 and of Proposition 9.1.4.

From Evans-Zworski [15] (see also Dimassi-Sjöstrand [7]) we have the following important result.

Theorem 9.1.5. *We have that for $a \in S^0_\delta(m^\mu; \mathsf{M}_N)$, the operator $a^{\mathrm{w}}(x, hD)$ as a linear map*

$$a^{\mathrm{w}}(x,hD)\colon \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$$

and as a linear map

$$a^{\mathrm{w}}(x,hD)\colon \mathscr{S}'(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathscr{S}'(\mathbb{R}^n;\mathbb{C}^N)$$

is **continuous**. *In particular, by virtue of (9.3), this is the case also for $a \in S^0_\delta(m^\mu, g; \mathsf{M}_N)$.*

As regards the L^2-continuity we have the following theorem.

Theorem 9.1.6. *Let $a \in S^0_\delta(1; \mathsf{M}_N)$, $0 \le \delta \le 1/2$. Then*

$$a^{\mathrm{w}}(x,hD)\colon L^2(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow L^2(\mathbb{R}^n;\mathbb{C}^N)$$

is bounded and there is a constant $C > 0$, **independent of** *h, such that*

$$\|a^{\mathrm{w}}(x,hD)\|_{L^2\to L^2} \le C,\ \ \forall h \in (0,1].$$

As regards the composition one has the following theorems.

For the classes $S^k_\delta(m^\mu; \mathsf{M}_N)$ we have the following result (see Dimassi-Sjöstrand [7] and Evans-Zworski [15]).

Theorem 9.1.7. *Given $a \in S^0_\delta(m^{\mu_1};\mathsf{M}_N)$ and $b \in S^0_\delta(m^{\mu_2};\mathsf{M}_N)$, one has*

$$a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = (a\sharp_h b)^{\mathrm{w}}(x,hD),$$

where

$$a\sharp_h b = e^{ih\sigma(D_X;D_Y)/2}(a(X)b(Y))\big|_{X=Y} \in S^0_\delta(m^{\mu_1+\mu_2};\mathsf{M}_N).$$

Here, recall, $\sigma(D_X;D_Y) = \sigma(D_x,D_\xi;D_y,D_\eta)$. Furthermore, when $\delta \in [0,1/2)$ one has

$$a\sharp_h b \sim \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{ih}{2}\sigma(D_X;D_Y)\Big)^k a(X)b(Y)\big|_{X=Y}. \tag{9.9}$$

When $\delta = 1/2$ one has (see [15]) that if $a,b \in S^0_{1/2}(1;\mathsf{M}_N)$ with

$$\operatorname{supp} a \subset K, \;\; \textit{and} \;\; \operatorname{dist}(\operatorname{supp} a, \operatorname{supp} b) \geq \gamma > 0,$$

where the **compact** *K and the constant γ are* **independent of** *h, then*

$$\|a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD)\|_{L^2\to L^2} = O(h^\infty). \tag{9.10}$$

Recall that $f(h) = O(h^{N_0})$ for some $N_0 \in \mathbb{Z}_+$ if there exists $C_{N_0} > 0$ such that $|f(h)| \leq C_{N_0}h^{N_0}$. We say that $f(h) = O(h^\infty)$ if for any given $N_0 \in \mathbb{Z}_+$ one has $f(h) = O(h^{N_0})$.

For the classes $S^k_\delta(m^\mu, g;\mathsf{M}_N)$ we have the following result (see Shubin [67, p. 245]).

Theorem 9.1.8. *Let $\delta \in [0,1/2)$. For any given symbols $a \in S^{k_1}_\delta(m^{\mu_1},g;\mathsf{M}_N)$ and $b \in S^{k_2}_\delta(m^{\mu_2},g;\mathsf{M}_N)$ we have*

$$a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = (a\sharp_h b)^{\mathrm{w}}(x,hD)$$

where

$$a\sharp_h b = e^{ih\sigma(D_X;D_Y)/2}(a(X)b(Y))\big|_{X=Y} \in S^{k_1+k_2}_\delta(m^{\mu_1+\mu_2},g;\mathsf{M}_N),$$

and for all $N_0 \in \mathbb{Z}_+$

$$(a\sharp_h b) = \sum_{k=0}^{N_0} \frac{1}{k!}\Big(\frac{ih}{2}\sigma(D_X;D_Y)\Big)^k a(X)b(Y)\big|_{X=Y} + h^{N_0+1} r_{N_0+1}, \tag{9.11}$$

where $r_{N_0+1} \in S^{k_1+k_2+2(N_0+1)\delta}_\delta(m^{\mu_1+\mu_2-2(N_0+1)},g;\mathsf{M}_N)$.

We next define the classical symbols in the semiclassical setting.

Definition 9.1.9 (Classical semiclassical symbols).

1. We shall say that a semiclassical symbol $a \in S_0^k(m^\mu;\mathsf{M}_N)$ is **classical** and write $a \in S_{\mathrm{cl}}^k(m^\mu;\mathsf{M}_N)$ if
$$a(X;h) \sim h^{-k}\sum_{j\geq 0} h^j a_j(X) \;\text{ in }\; S_0^k(m^\mu;\mathsf{M}_N),$$
where the $a_j \in S_0^0(m^\mu;\mathsf{M}_N)$ are **independent of** h, $j \geq 0$.
2. We shall say that a semiclassical symbol $a \in S_0^k(m^\mu,g;\mathsf{M}_N)$ is **classical** and write $a \in S_{0,\mathrm{cl}}^k(m^\mu,g;\mathsf{M}_N)$ if there exists a sequence $\{a_{\mu-2j}\}_{j\geq 0}$ of symbols $a_{\mu-2j} \in S(m^{\mu-2j},g;\mathsf{M}_N)$, $j \geq 0$, with the $a_{\mu-2j}$ **independent of** h, such that for any given $N_0 \in \mathbb{Z}_+$
$$a(X;h) - h^{-k}\sum_{j=0}^{N_0} h^j a_{\mu-2j}(X) \in S_0^{k-(N_0+1)}(m^{\mu-2(N_0+1)},g;\mathsf{M}_N).$$
We shall write
$$a(X;h) \sim h^{-k}\sum_{j\geq 0} h^j a_{\mu-2j}(X) \;\text{ in }\; S_0^k(m^\mu,g;\mathsf{M}_N).$$
For a classical semiclassical symbol $a \sim a_\mu + h a_{\mu-2} + \ldots$, one calls a_μ the **principal** symbol of a, and $a_{\mu-2}$ the **subprincipal** symbol of a.
3. We shall say that a classical semiclassical symbol $a \in S_{0,\mathrm{cl}}^0(m^\mu,g;\mathsf{M}_N)$ is **elliptic** if its principal symbol a_μ belongs to $S(m^\mu,g;\mathsf{M}_N)$ and $a_\mu(X)^{-1}$ exists for all $X \in \mathbb{R}^{2n}$ and belongs to $S(m^{-\mu},g;\mathsf{M}_N)$. Equivalently, one requires that $\det a_\mu \in S(m^{N\mu},g)$ be such that $|\det a_\mu(X)| \gtrsim m(X)^{N\mu}$ for all $X \in \mathbb{R}^{2n}$.
4. We shall say that a classical semiclassical symbol $a = a^* \in S_{0,\mathrm{cl}}^0(m^\mu,g;\mathsf{M}_N)$ is **positive elliptic** if its principal symbol $a_\mu = a_\mu^*$ belongs to $S(m^\mu,g;\mathsf{M}_N)$ and there are $0 < c_1 < c_2$ such that
$$c_1 m(X)^\mu |v|^2_{\mathbb{C}^N} \leq \langle a_\mu(X)v,v\rangle_{\mathbb{C}^N} \leq c_2 m(X)^\mu |v|^2_{\mathbb{C}^N},$$
for all $v \in \mathbb{C}^N$ and all $X \in \mathbb{R}^{2n}$.

Remark 9.1.10. From Theorem 9.1.7 and Theorem 9.1.8, respectively, we have that also the classes $S_{\mathrm{cl}}^k(m^\mu;\mathsf{M}_N)$ and $S_{0,\mathrm{cl}}^k(m^\mu,g;\mathsf{M}_N)$, respectively, are well-behaved under composition. △

We now wish to consider *inverses* in the semiclassical calculus.

We need the following result, due to R. Beals, which characterizes pseudodifferential operators in the semiclassical setting (see Dimassi-Sjöstrand [7] or Evans-Zworski [15]; we give a statement in the scalar case for simplicity).

Theorem 9.1.11. *Let $A = A_h \colon \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ be a continuous linear operator, $0 < h \le 1$. Then the following statements are equivalent:*

1. *$A = a^{\mathrm{w}}(x, hD)$ for some $a(x,\xi;h) = a \in S_0^0(1)$;*
2. *For every $N \in \mathbb{N}$ and for every family $\ell_1(x,\xi), \dots, \ell_N(x,\xi)$ of linear forms on $\mathbb{R}^{2n}$, the operator $\mathrm{ad}_{\ell_1(x,hD)} \circ \dots \circ \mathrm{ad}_{\ell_N(x,hD)} A_h$ is* **continuous** *in $L^2(\mathbb{R}^n)$, i.e. it belongs to $\mathscr{L}(L^2, L^2)$, with norm $O(h^N)$ in that space. (Recall that $\mathrm{ad}_A B = [A, B]$.)*

One may then prove the following proposition about "elliptic" elements in $S_0^0(m^\mu; \mathsf{M}_N)$.

Theorem 9.1.12. *Let $a(X;h) = a \in S_0^0(m^\mu; \mathsf{M}_N)$ be elliptic, that is, by definition, $a(X;h)^{-1} = a^{-1} \in S_0^0(m^{-\mu}; \mathsf{M}_N)$, for $0 < h \le h_0$. Then, by possibly shrinking h_0, there exists a symbol $b \in S_0^0(m^{-\mu}; \mathsf{M}_N)$ such that*

$$b^{\mathrm{w}}(x,hD) a^{\mathrm{w}}(x,hD) = I = a^{\mathrm{w}}(x,hD) b^{\mathrm{w}}(x,hD), \quad 0 < h \le h_0,$$

and, furthermore, $b^{\mathrm{w}}(x,hD)$ possesses an asymptotic expansion, that is,

$$b \sim a^{-1} + h(a^{-1} \sharp_h r) + h^2 (a^{-1} \sharp_h r \sharp_h r) + \dots,$$

where $r \in S_0^0(1; \mathsf{M}_N)$ is such that $(a^{-1})^{\mathrm{w}}(x,hD) a^{\mathrm{w}}(x,hD) = I - h r^{\mathrm{w}}(x,hD)$.

Proof. Let $\tilde{b} = a^{-1} \in S_0^0(m^{-\mu}; \mathsf{M}_N)$. Then

$$\tilde{b}^{\mathrm{w}}(x,hD) a^{\mathrm{w}}(x,hD) = I - h r_L^{\mathrm{w}}(x,hD), \;\; r_L \in S_0^0(1; \mathsf{M}_N),$$

and

$$a^{\mathrm{w}}(x,hD) \tilde{b}^{\mathrm{w}}(x,hD) = I - h r_R^{\mathrm{w}}(x,hD), \;\; r_R \in S_0^0(1; \mathsf{M}_N).$$

Then by Theorem 9.1.6

$$I - h r_L^{\mathrm{w}}(x,hD) \colon L^2 \longrightarrow L^2 \;\text{ and }\; I - h r_R^{\mathrm{w}}(x,hD) \colon L^2 \longrightarrow L^2$$

are both invertible, provided $h \in (0, h_0]$, if $h_0 \in (0,1]$ is taken sufficiently small. By the Beals Theorem 9.1.11 it follows that there exists $c_L, c_R \in S_0^0(1; \mathsf{M}_N)$ such that

$$\Big(I - h r_L^{\mathrm{w}}(x,hD)\Big)^{-1} = c_L(x,hD), \;\; \Big(I - h r_R^{\mathrm{w}}(x,hD)\Big)^{-1} = c_R(x,hD),$$

so that

$$\Big(I - h r_L^{\mathrm{w}}(x,hD)\Big)^{-1} \tilde{b}^{\mathrm{w}}(x,hD) =: b_L^{\mathrm{w}}(x,hD), \;\; b_L \in S_0^0(m^{-\mu}; \mathsf{M}_N),$$

and

$$\tilde{b}^{\mathrm{w}}(x,hD) \Big(I - h r_R^{\mathrm{w}}(x,hD)\Big)^{-1} =: b_R^{\mathrm{w}}(x,hD), \;\; b_R \in S_0^0(m^{-\mu}; \mathsf{M}_N),$$

whence we get the existence of a left-inverse $b_L(x,\xi;h) = b_L$ and of a right-inverse $b_R(x,\xi;h) = b_R$, both belonging to the symbol class $S_0^0(m^{-\mu};\mathsf{M}_N)$, such that

$$b_L^{\mathrm{w}}(x,hD)a^{\mathrm{w}}(x,hD) = I = a^{\mathrm{w}}(x,hD)b_R^{\mathrm{w}}(x,hD), \quad 0 < h \le h_0,$$

and, finally,

$$b_L^{\mathrm{w}}(x,hD) = b_L^{\mathrm{w}}(x,hD)a^{\mathrm{w}}(x,hD)b_R^{\mathrm{w}}(x,hD) = b_R^{\mathrm{w}}(x,hD).$$

Put therefore $b = b_L = b_R$.

In addition, we have that b possesses an asymptotic expansion. In fact, for any given $N_0 \in \mathbb{Z}_+$, let

$$\begin{aligned} & b_{N_0}^{\mathrm{w}}(x,hD) \\ & \quad := \tilde{b}^{\mathrm{w}}(x,hD)\Big(I + hr_R^{\mathrm{w}}(x,hD) + h^2 r_R^{\mathrm{w}}(x,hD)^2 + \ldots + h^{N_0} r_R^{\mathrm{w}}(x,hD)^{N_0}\Big). \end{aligned}$$

Then

$$a^{\mathrm{w}}(x,hD)b_{N_0}^{\mathrm{w}}(x,hD) = I - h^{N_0+1} r_R(x,hD)^{N_0+1},$$

and it follows that

$$\begin{aligned} b_{N_0}^{\mathrm{w}}(x,hD) &= b^{\mathrm{w}}(x,hD)a^{\mathrm{w}}(x,hD)b_{N_0}^{\mathrm{w}}(x,hD) \\ &= b^{\mathrm{w}}(x,hD) - h^{N_0+1} b^{\mathrm{w}}(x,hD) r_R^{\mathrm{w}}(x,hD)^{N_0+1} \\ &= b^{\mathrm{w}}(x,hD) + h^{N_0+1} r_{N_0+1}^{\mathrm{w}}(x,hD), \end{aligned}$$

where we have put $r_{N_0+1}^{\mathrm{w}}(x,hD) := -b^{\mathrm{w}}(x,hD)r_R^{\mathrm{w}}(x,hD)^{N_0+1}$. By Theorem 9.1.7 we have $r_{N_0+1} \in S_0^0(m^{-\mu};\mathsf{M}_N)$, and this concludes the proof. □

It will be also useful to have the following variation of Theorem 9.1.12 for elliptic classical semiclassical symbols $a \in S_{0,\mathrm{cl}}^0(m^{\mu},g;\mathsf{M}_N)$. The usual parametrix construction (using h as "parameter of homogeneity") gives the following theorem.

Theorem 9.1.13. *Let $a \in S_{0,\mathrm{cl}}^0(m^{\mu},g;\mathsf{M}_N)$ be elliptic, that is, with $a \sim \sum_{j\ge 0} h^j a_{\mu-2j}$, let $a_{\mu}^{-1} \in S(m^{-\mu},g;\mathsf{M}_N)$. Then there exists a classical semiclassical symbol $b \in S_{0,\mathrm{cl}}^0(m^{-\mu},g;\mathsf{M}_N)$ such that*

$$b^{\mathrm{w}}(x,hD)a^{\mathrm{w}}(x,hD) = I + r_L^{\mathrm{w}}(x,hD),$$

$$a^{\mathrm{w}}(x,hD)b^{\mathrm{w}}(x,hD) = I + r_R^{\mathrm{w}}(x,hD),$$

where $r_L, r_R \in S^{-\infty}(m^{-\infty},g;\mathsf{M}_N)$ (see (9.7)).

9.2 Decoupling a System

In this section, using an adaptation taken from Parenti-Parmeggiani [49] of the classical decoupling argument of Taylor [69], we prove the following result. (See also Helffer-Sjöstrand [21].)

Theorem 9.2.1. *Let* $\mu > 0$ *and let* $a = a^* \sim \sum_{j\geq 0} h^j a_{\mu-2j} \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$*. Suppose there exists* $e_0 \in S(1,g;\mathsf{M}_N)$ *such that* $e_0^* e_0 = e_0 e_0^* = I$ *and*

$$e_0^* a_\mu e_0 = b_\mu = \begin{bmatrix} \lambda_{1,\mu} & 0 \\ 0 & \lambda_{2,\mu} \end{bmatrix}, \tag{9.12}$$

where $\lambda_{j,\mu} = \lambda_{j,\mu}^* \in S(m^\mu, g; \mathsf{M}_{N_j})$, $j = 1,2$, $N_1 + N_2 = N$, *and*

$$d_{\lambda_1,\lambda_2}(X) \gtrsim m(X)^\mu, \quad \forall X \in \mathbb{R}^{2n}, \tag{9.13}$$

where, for each $X \in \mathbb{R}^{2n}$,

$$d_{\lambda_1,\lambda_2}(X) = \inf\Big\{ |\mu_1 - \mu_2|;\ \mu_1 \in \mathrm{Spec}(\lambda_{1,\mu}(X)), \mu_2 \in \mathrm{Spec}(\lambda_{2,\mu}(X)) \Big\}. \tag{9.14}$$

Then there exists $e \in S^0_{0,\mathrm{cl}}(1,g;\mathsf{M}_N)$ *with principal symbol* e_0 *such that:*

1. *One has*

$$e^{\mathrm{w}}(x,hD)^* e^{\mathrm{w}}(x,hD) - I,\ e^{\mathrm{w}}(x,hD) e^{\mathrm{w}}(x,hD)^* - I \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N);$$

2. $e^{\mathrm{w}}(x,hD)^* a^{\mathrm{w}}(x,hD) e^{\mathrm{w}}(x,hD) - b^{\mathrm{w}}(x,hD) \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N)$, *where the symbol* $b \sim \sum_{j\geq 0} h^j b_{\mu-2j} \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ *is* **blockwise diagonal***, with*

$$b_{\mu-2j}(X) = \begin{bmatrix} b_{1,\mu-2j}(X) & 0 \\ 0 & b_{2,\mu-2j}(X) \end{bmatrix},\ \forall X \in \mathbb{R}^{2n},\ \forall j \geq 0,$$

with blocks $b_{j,\mu}$ *of sizes* N_j, $j = 1,2$, *respectively, and with principal symbol*

$$b_\mu(X) = \begin{bmatrix} \lambda_{1,\mu}(X) & 0 \\ 0 & \lambda_{2,\mu}(X) \end{bmatrix},\ \forall X \in \mathbb{R}^{2n}.$$

We shall call b **an** h^∞**-(block)-diagonalization** *of* a*. Notice that* b *depends on* a *and* e_0.

Proof. We immediately observe that once $e^{\mathrm{w}}(x,hD)$ has been found with the property that its principal symbol is e_0 and

$$e^{\mathrm{w}}(x,hD) e^{\mathrm{w}}(x,hD)^* = I + r^{\mathrm{w}}(x,hD), \text{ with } r \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N),$$

then by the ellipticity (using Theorem 9.1.13) we also get

$$e^{\mathrm{w}}(x,hD)^* e^{\mathrm{w}}(x,hD) = I + s^{\mathrm{w}}(x,hD), \text{ with } s \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N).$$

Hence it suffices to prove the existence of $e^{\mathrm{w}}(x,hD)$ and b with the required properties. We show that for every integer $k \in \mathbb{Z}_+$ there exist

$$(i) \qquad e_{-2k} \in S(m^{-2k}, g; \mathsf{M}_N),$$

and

$$(ii) \qquad b_{j,\mu-2k} \in S(m^{\mu-2k}, g; \mathsf{M}_{N_j}), \quad j = 1,2,$$

such that, with $E_{N_0}(X) := \sum_{k=0}^{N_0} h^k e_{-2k}(X)$,

$$E_{N_0} \sharp_h E_{N_0}^* = I + h^{N_0+1} S^0_{0,\mathrm{cl}}(m^{-2(N_0+1)}, g; \mathsf{M}_N),$$

and

$$E_{N_0}^* \sharp_h a \sharp_h E_{N_0} = \sum_{k=0}^{N_0} b_{\mu-2k} + h^{N_0+1} S^0_{0,\mathrm{cl}}(m^{\mu-2(N_0+1)}, g; \mathsf{M}_N),$$

where the $b_{\mu-2k} = \begin{bmatrix} b_{1,\mu-2k} & 0 \\ 0 & b_{2,\mu-2k} \end{bmatrix}$ are in block-diagonal form. We shall then take $e \sim \sum_{k\geq 0} h^k e_{-2k}$.

Hence we proceed by induction. So, suppose we have already constructed symbols $e_0, e_{-2}, \ldots, e_{-2N_0}$, and $b_\mu, b_{\mu-2}, \ldots, b_{\mu-2N_0}$, *independent of* h, with the required properties. Put hence

$$S^{\mathrm{w}}_{N_0}(x,hD) := E^{\mathrm{w}}_{N_0}(x,hD) E^{\mathrm{w}}_{N_0}(x,hD)^* - I, \tag{9.15}$$

where $S_{N_0} \in h^{N_0+1} S^0_{0,\mathrm{cl}}(m^{-2(N_0+1)}, g; \mathsf{M}_N)$. We shall write, for short, $E^{\mathrm{w}}_{N_0}$, $S^{\mathrm{w}}_{N_0}$ and e^{w}_{-2k} in place of $E^{\mathrm{w}}_{N_0}(x,hD)$ etc.

We look for a matrix-symbol $e_{-2(N_0+1)} \in S(m^{-2(N_0+1)}, g; \mathsf{M}_N)$ such that

$$\Big(E^{\mathrm{w}}_{N_0} + h^{N_0+1} e^{\mathrm{w}}_{-2(N_0+1)}\Big)\Big((E^{\mathrm{w}}_{N_0})^* + h^{N_0+1}(e^{\mathrm{w}}_{-2(N_0+1)})^*\Big) - I = h^{N_0+2} r^{\mathrm{w}}(x,hD),$$

where $r \in S^0_{0,\mathrm{cl}}(m^{-2(N_0+2)}, g; \mathsf{M}_N)$, that is, we look for $e_{-2(N_0+1)}$ such that

$$S^{\mathrm{w}}_{N_0} + h^{N_0+1}\Big(e^{\mathrm{w}}_0 (e^{\mathrm{w}}_{-2(N_0+1)})^* + e^{\mathrm{w}}_{-2(N_0+1)}(e^{\mathrm{w}}_0)^*\Big) = h^{N_0+2}\tilde{r}^{\mathrm{w}}(x,hD),$$

with $\tilde{r} \in S^0_{0,\mathrm{cl}}(m^{-2(N_0+2)}, g; \mathsf{M}_N)$. Using the composition formula (9.11) we thus look at the coefficient of h^{N_0+1} and require that it be zero, obtaining the equation

$$s_{-2(N_0+1)} + e_0(e_{-2(N_0+1)})^* + e_{-2(N_0+1)} e_0^* = 0. \tag{9.16}$$

Notice that $s^*_{-2(N_0+1)} = s_{-2(N_0+1)}$ (this follows from (9.15)). Equation (9.16) has general solution

$$e_{-2(N_0+1)} = -\frac{1}{2}s_{-2(N_0+1)}e_0 + \phi_{-2(N_0+1)}, \tag{9.17}$$

where $\phi_{-2(N_0+1)}$ solves

$$e_0\phi^*_{-2(N_0+1)} + \phi_{-2(N_0+1)}e_0^* = 0,$$

which in turn gives that $\phi_{-2(N_0+1)}$ has the form

$$\phi_{-2(N_0+1)} = \alpha_{-2(N_0+1)}e_0, \tag{9.18}$$

where $\alpha_{-2(N_0+1)} \in S(m^{-2(N_0+1)}, g; \mathsf{M}_N)$ and

$$\alpha^*_{-2(N_0+1)} + \alpha_{-2(N_0+1)} = 0.$$

We next perform a choice of $\alpha_{-2(N_0+1)}$ for obtaining the blocks $b_{j,\mu-2(N_0+1)}$, $j = 1,2$. Since on the one hand

$$(E^{\mathrm{w}}_{N_0+1})a^{\mathrm{w}}E^{\mathrm{w}}_{N_0+1} = \sum_{k=0}^{N_0+1} h^k b^{\mathrm{w}}_{\mu-2k} + h^{N_0+2}r_1^{\mathrm{w}},$$

with $r_1 \in S^0_{0,\mathrm{cl}}(m^{\mu-2(N_0+2)}, g; \mathsf{M}_N)$, and on the other

$$\begin{aligned}(E^{\mathrm{w}}_{N_0+1})a^{\mathrm{w}}E^{\mathrm{w}}_{N_0+1} &= (E^{\mathrm{w}}_{N_0})^*a^{\mathrm{w}}E^{\mathrm{w}}_{N_0} + h^{N_0+1}\Big((e^{\mathrm{w}}_{-2(N_0+1)})^*a^{\mathrm{w}}e_0^{\mathrm{w}} \\ &\qquad + (e_0^{\mathrm{w}})^*a^{\mathrm{w}}e^{\mathrm{w}}_{-2(N_0+1)}\Big) + h^{N_0+2}r_2^{\mathrm{w}} \\ &=: T^{\mathrm{w}}_{N_0} + h^{N_0+2}r_2^{\mathrm{w}},\end{aligned} \tag{9.19}$$

with $r_2 \in S^0_{0,\mathrm{cl}}(m^{\mu-2(N_0+2)}, g; \mathsf{M}_N)$, the conditions for the blocks $b_{j,\mu-2k}$ are already satisfied for $0 \le k \le N_0$, *independently of* $e_{-2(N_0+1)}$. Let $q_{\mu-2(N_0+1)}$ be the coefficient of h^{N_0+1} in $E^*_{N_0}\sharp_h a \sharp_h E_{N_0}$. Then the coefficient of h^{N_0+1} in T_{N_0} is

$$\begin{aligned}&q_{\mu-2(N_0+1)} + e^*_{-2(N_0+1)}a_\mu e_0 + e_0^*a_\mu e_{-2(N_0+1)} \\ &\quad = q_{\mu-2(N_0+1)} + (e^*_{-2(N_0+1)}e_0)e_0^*a_\mu e_0 + e_0^*a_\mu e_0(e_0^*e_{-2(N_0+1)}).\end{aligned} \tag{9.20}$$

Using (9.17) and (9.18), we write

$$\begin{cases} e_0^*e_{-2(N_0+1)} = -\dfrac{1}{2}e_0^*s_{-2(N_0+1)}e_0 + e_0^*\alpha_{-2(N_0+1)}e_0 =: \tau + \beta, \\[2ex] e^*_{-2(N_0+1)}e_0 = \tau - \beta, \end{cases} \tag{9.21}$$

where

$$\tau = -\frac{1}{2} e_0^* s_{-2(N_0+1)} e_0 = \tau^*, \text{ and } \beta = e_0^* \alpha_{-2(N_0+1)} e_0 = -\beta^*.$$

By (9.21), the term (9.20) goes over to

$$q_{\mu-2(N_0+1)} + (e_0^* a_\mu e_0)\tau + \tau(e_0^* a_\mu e_0) + (e_0^* a_\mu e_0)\beta - \beta(e_0^* a_\mu e_0). \tag{9.22}$$

We now show that β, hence in turn

$$\alpha_{-2(N_0+1)} = e_0 \beta e_0^*, \tag{9.23}$$

can be so chosen as to kill the off-diagonal terms in (9.22). In fact, upon writing

$$q_{\mu-2(N_0+1)} + (e_0^* a_\mu e_0)\tau + \tau(e_0^* a_\mu e_0) = \begin{bmatrix} u_1 & \gamma \\ \gamma^* & u_2 \end{bmatrix},$$

where the $u_j = u_j^*$ are $N_j \times N_j$ blocks, $j = 1,2$, we look for β in the form

$$\beta = \begin{bmatrix} 0 & \delta \\ -\delta^* & 0 \end{bmatrix},$$

and, using (9.12), we are therefore led to the matrix equation

$$\lambda_{1,\mu}\delta - \delta\lambda_{2,\mu} = -\gamma. \tag{9.24}$$

By Lemma 9.2.2 below, hypothesis (9.13) yields that equation (9.24) has a unique smooth $N_1 \times N_2$ matrix-valued solution $\delta \in S(m^{-2(N_0+1)}, g; \mathrm{Mat}_{N_1 \times N_2}(\mathbb{C}))$. Since this fixes β, and hence $\alpha_{-2(N_0+1)}$, the terms $b_{j,\mu-2(N_0+1)}$ are then the block-diagonal terms in (9.22). This concludes the inductive step and the proof of the theorem. □

Lemma 9.2.2. *Let $E = E^* \in S(m^\mu, g; \mathsf{M}_{N_1})$ and $F = F^* \in S(m^\mu, g; \mathsf{M}_{N_2})$ be such that (recall (9.14))*

$$d_{E,F}(X) \geq c_0\, m(X)^\mu, \ \forall X \in \mathbb{R}^{2n}.$$

Then for each $X \in \mathbb{R}^{2n}$ the map

$$\begin{cases} \Phi_{E,F}(X) \colon \mathrm{Mat}_{N_1 \times N_2}(\mathbb{C}) \longrightarrow \mathrm{Mat}_{N_1 \times N_2}(\mathbb{C}), \\ \Phi_{E,F}(X)T = E(X)T - TF(X), \end{cases} \tag{9.25}$$

is an isomorphism. Moreover,

$$\|\Phi_{E,F}(X)^{-1}\| \leq \frac{C}{m(X)^\mu}, \ \forall X \in \mathbb{R}^{2n}, \tag{9.26}$$

for a **universal** *constant* $C > 0$. *Hence, if* $S \in S(m^{\mu-2k}, g; \mathrm{Mat}_{N_1\times N_2}(\mathbb{C}))$, *for some* $k \in \mathbb{Z}_+$, *we have that*

$$X \longmapsto T(X) := \Phi_{E,F}(X)^{-1}\big(S(X)\big) \in S(m^{-2k}, g; \mathrm{Mat}_{N_1\times N_2}(\mathbb{C})). \tag{9.27}$$

Proof. For each fixed $X \in \mathbb{R}^{2n}$, let $\mathrm{Spec}(E(X)) = \{e_1(X), \dots, e_{\nu_X}(X)\}$. Hence $e_j(X) \neq e_{j'}(X)$ if $j \neq j'$. Consider the contour in $\mathbb{C}$

$$\gamma(X) = \bigcup_{j=1}^{\nu_X} \gamma_j(X), \ \text{ with } \ \gamma_j(X) \cap \gamma_{j'}(X) = \emptyset \ \text{ for } \ j \neq j',$$

counter-clockwise oriented, where each $\gamma_j(X) = \partial D_j(X)$ is a small circle that encloses only the eigenvalue $e_j(X)$ of $E(X)$, $1 \le j \le \nu_X$, and

$$\mathrm{Spec}(F(X)) \subset \mathbb{C} \setminus \bigcup_{j=1}^{\nu_X} D_j(X).$$

Then, on considering the equation $E(X)T - TF(X) = S(X)$ we have that the solution can be written as

$$\begin{aligned} T = T(X) &= \frac{1}{2\pi i}\int_{\gamma(X)} (\zeta - E(X))^{-1} S(X) (\zeta - F(X))^{-1} d\zeta \\ &= \sum_{j=1}^{\nu_X} P_{E,j}(X) S(X) \big(e_j(X) - F(X)\big)^{-1}, \end{aligned} \tag{9.28}$$

where $P_{E,j}(X) \colon \mathbb{C}^{N_1} \to \mathbb{C}^{N_1}$ is the orthogonal projection associated with the eigenvalue $e_j(X)$. This shows that for each fixed X the map $\Phi_{E,F}(X)$ is an isomorphism (for it is injective and linear). Moreover, using the fact that the norm of the resolvent of a normal operator equals the spectral radius, we also obtain from (9.28) that

$$|T(X)| = |\Phi_{E,F}(X)^{-1}\big(S(X)\big)| \le \frac{C}{m(X)^{\mu}} |S(X)|, \ \ \forall X \in \mathbb{R}^{2n},$$

where $C = N_1/c_0$. This shows (9.26). Now, (9.26) gives that $T(X)$ is continuous and differentiable to all orders, for we have that for any given $X_1, X_2 \in \mathbb{R}^{2n}$

$$\begin{aligned} &\Phi_{E,F}(X_1)\Big(T(X_1) - T(X_2)\Big) \\ &\quad = \Big(E(X_2) - E(X_1)\Big)T(X_2) + T(X_2)\Big(F(X_1) - F(X_2)\Big) + S(X_1) - S(X_2), \end{aligned} \tag{9.29}$$

from which, since (9.26) yields $|T(X_2)| \lesssim |S(X_2)|$ for all X_2 (recall that $m(X) \ge 1$ for all X), we obtain the continuity of T by considering the map $\Phi_{E,F}(X_1)^{-1}$, taking the limit as $X_2 \to X_1$, and using the continuity of E, F and S. The differentiability claim follows from (9.29) by induction.

It remains to control the growth of $|\partial_X^\alpha T(X)|$. For any given $\alpha \in \mathbb{Z}_+^{2n}$, the matrix $\partial_X^\alpha T$ is a solution to

$$E\partial_X^\alpha T - \partial_X^\alpha T F + S_\alpha = \partial_X^\alpha S,$$

where, by the Leibniz rule,

$$S_\alpha := \sum_{\substack{\beta\le\alpha\\ \beta\neq\alpha}} \binom{\alpha}{\beta} \left(\partial_X^{\alpha-\beta} E \partial_X^\beta T - \partial_X^\beta T \partial_X^{\alpha-\beta} F\right).$$

Hence

$$\partial_X^\alpha T(X) = \Phi_{E,F}(X)^{-1}\Big(\partial_X^\alpha S(X) - S_\alpha(X)\Big),$$

so that assuming by induction on α that $|\partial_X^\beta T| \lesssim m^{-2k-|\beta|}$, gives by (9.26)

$$|\partial_X^\alpha T(X)| \le C_\alpha m(X)^{-2k-|\alpha|},\ \forall X \in \mathbb{R}^{2n},$$

which concludes the proof. □

As a consequence of the proof of Theorem 9.2.1 we have the following analogue for classical symbols (which is used in Parmeggiani [52]).

Theorem 9.2.3. *Let $\mu > 0$ and let $a = a^* \sim \sum_{j\ge 0} a_{\mu-2j} \in S_{\rm cl}(m^\mu, g; \mathsf{M}_N)$. Suppose there exists $e_0 \in C^\infty(\mathbb{R}^{2n}\setminus\{0\}; \mathsf{M}_N)$, positively homogeneous of degree 0, such that*

$$e_0^* e_0 = e_0 e_0^* = I,\ \ \text{and}\ \ e_0^* a_\mu e_0 = b_\mu = \begin{bmatrix} \lambda_{1,\mu} & 0 \\ 0 & \lambda_{2,\mu}\end{bmatrix},\ X \neq 0,$$

where $\lambda_{j,\mu} = \lambda_{j,\mu}^* \in C^\infty(\mathbb{R}^{2n}\setminus\{0\}; \mathsf{M}_{N_j})$ are **positively homogeneous** *of degree μ, $j = 1, 2$, and*

$$\mathrm{Spec}(\lambda_{1,\mu}(X)) \cap \mathrm{Spec}(\lambda_{2,\mu}(X)) = \emptyset,\ \ \forall X \in \mathbb{R}^{2n},\ |X| = 1. \tag{9.30}$$

Then there exists $e \in S_{\rm cl}(1, g; \mathsf{M}_N)$ with principal symbol e_0 such that

1. *$e^{\rm w}(x,D)^* e^{\rm w}(x,D) - I$, $e^{\rm w}(x,D) e^{\rm w}(x,D)^* - I \in S(m^{-\infty}, g; \mathsf{M}_N)$;*
2. *$e^{\rm w}(x,D)^* a^{\rm w}(x,D) e^{\rm w}(x,D) - b^{\rm w}(x,D) \in S(m^{-\infty}, g; \mathsf{M}_N)$, where the symbol $b \sim \sum_{j\ge 0} b_{\mu-2j} \in S_{\rm cl}(m^\mu, g; \mathsf{M}_N)$ is* **blockwise diagonal**, *with*

$$b_{\mu-2j}(X) = \begin{bmatrix} b_{1,\mu-2j}(X) & 0 \\ 0 & b_{2,\mu-2j}(X)\end{bmatrix},\ \ \forall X \in \mathbb{R}^{2n}\setminus\{0\},\ \forall j \ge 0,$$

with blocks $b_{j,\mu}$ of sizes N_j, $j=1,2$, respectively, and with principal symbol

$$b_\mu(X) = \begin{bmatrix} \lambda_{1,\mu}(X) & 0 \\ 0 & \lambda_{2,\mu}(X) \end{bmatrix}, \ \forall X \in \mathbb{R}^{2n} \setminus \{0\}.$$

We shall call such a symbol b **a (block)-diagonalization** *of a. Notice that b depends on a and e_0.*

Remark 9.2.4. Given $N \times N$ matrices a and b, a straightforward computation gives the following very useful formula:

$$\{a,b\}^* = -\{b^*,a^*\} \tag{9.31}$$

which yields in particular that

$$\frac{i}{2}\{a,a^*\} \ \textit{and} \ \frac{i}{2}\{a^*,a\} \ \textit{are } \textbf{Hermitian} \textit{ matrices.}$$

△

It will be useful to compute the subprincipal part $b_{\mu-2}$ (that is, the coefficient of the h term) of the block-diagonal system obtained in Theorem 9.2.1. The same formula holds for the version given in Theorem 9.2.3 for classical operators. We have the following proposition.

Proposition 9.2.5. *For the subprincipal part $b_{\mu-2}$ of the h^∞-diagonalization given in Theorem 9.2.1 one has, by (9.20), the formula*

$$b_{\mu-2} = e^*_{-2}e_0 b_\mu + b_\mu e_0^* e_{-2} + e_0^* a_{\mu-2} e_0 - \frac{i}{2}\Big(e_0^*\{a_\mu, e_0\} + \{e_0^*, a_\mu e_0\}\Big), \tag{9.32}$$

where

$$e_{-2} = \frac{i}{4}\{e_0, e_0^*\}e_0 + \alpha_{-2}e_0,$$

with $\alpha^*_{-2} = -\alpha_{-2}$ determined by equation (9.24) through $\beta_{-2} = e_0^*\alpha_{-2}e_0$.

In the case $N=2$, supposing that $a_\mu = a_\mu^* > 0$ and that there exist positive smooth functions $\lambda_1, \lambda_2 \in S(m^\mu, g)$ (where we now write λ_j, $j=1,2$, for the eigenvalues of a_μ) such that

$$|\lambda_1(X) - \lambda_2(X)| \gtrsim m(X)^\mu, \ \forall X \in \mathbb{R}^{2n}, \tag{9.33}$$

whence the existence of a smooth unitary matrix e_0 such that

$$e_0(X)^* a_\mu(X) e_0(X) = \begin{bmatrix} \lambda_1(X) & 0 \\ 0 & \lambda_2(X) \end{bmatrix}, \forall X \in \mathbb{R}^{2n}, \tag{9.34}$$

we have the following corollary.

Corollary 9.2.6. *Suppose* $a_\mu = a_\mu^* > 0$ *possesses smooth eigenvalues* λ_1, λ_2 *satisfying (9.33). Let* $\{w_1, w_2\}$ *be the* **canonical** *basis of* $\mathbb{C}^2$, $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, *so that* $b_\mu(X) w_j = \lambda_j(X) w_j$, $j = 1,2$, *for all* $X \in \mathbb{R}^{2n}$. *Then the symbol of the* h^∞*-diagonalization* b *is a* **diagonal** 2×2 *matrix. Moreover, for the subprincipal symbol* $b_{\mu-2} = \begin{bmatrix} b_{\mu-2}^{(11)} & 0 \\ 0 & b_{\mu-2}^{(22)} \end{bmatrix}$ *we have*

$$\begin{aligned} b_{\mu-2}^{(jj)} &= \langle b_{\mu-2} w_j, w_j \rangle \\ &= \langle e_0^* a_{\mu-2} e_0 w_j, w_j \rangle + \frac{1}{2}\mathrm{Im}\Big(\langle \{e_0^*, \lambda_j\} e_0 w_j, w_j \rangle \Big) \\ &\quad + \frac{1}{2}\mathrm{Im}\Big(\langle e_0^* \{a_\mu, e_0\} w_j, w_j \rangle \Big), \quad j = 1,2. \end{aligned} \tag{9.35}$$

Proof. From the proof of Theorem 9.2.1 we have that the matrix $\beta = \beta_{-2}$ is skew-adjoint, whence

$$\langle \beta_{-2}^* b_\mu w_j, w_j \rangle + \langle b_\mu \beta_{-2} w_j, w_j \rangle = -\lambda_j \langle \beta_{-2} w_j, w_j \rangle + \lambda_j \langle \beta_{-2} w_j, w_j \rangle = 0. \tag{9.36}$$

Then, from (9.32), recalling that

$$e_0^* e_{-2} = \frac{i}{4} e_0^* \{e_0, e_0^*\} e_0 + \beta_{-2},$$

using $a_\mu e_0 = e_0 b_\mu$ and (9.36), and since $(\partial b_\mu) w_j = (\partial \lambda_j) w_j$, we obtain

$$\begin{aligned} \langle b_{\mu-2} w_j, w_j \rangle &= \mathrm{Re}\, \langle b_{\mu-2} w_j, w_j \rangle \\ &= \langle e_0^* a_{\mu-2} e_0 w_j, w_j \rangle + 2\,\mathrm{Re}\Big(\frac{i}{4} \langle e_0^* \{e_0, e_0^*\} e_0 b_\mu w_j, w_j \rangle \Big) \\ &\quad - \mathrm{Re}\Big(\frac{i}{2} \langle e_0^* \{a_\mu, e_0\} w_j, w_j \rangle \Big) - \mathrm{Re}\Big(\frac{i}{2} \langle \{e_0^*, e_0 b_\mu\} w_j, w_j \rangle \Big) \\ &= \langle e_0^* a_{\mu-2} e_0 w_j, w_j \rangle \\ &\quad - \frac{1}{2}\mathrm{Im}\Big(\langle e_0^* \{e_0, e_0^*\} e_0 b_\mu w_j, w_j \rangle \Big) + \frac{1}{2}\mathrm{Im}\Big(\langle e_0^* \{a_\mu, e_0\} w_j, w_j \rangle \Big) \\ &\quad + \frac{1}{2}\mathrm{Im}\Big(\langle \{e_0^*, e_0\} b_\mu w_j, w_j \rangle \Big) \\ &\quad + \frac{1}{2} \sum_{\ell=1}^{n} \mathrm{Im}\Big(\langle (\frac{\partial e_0^*}{\partial \xi_\ell} e_0 \frac{\partial \lambda_j}{\partial x_\ell} - \frac{\partial e_0^*}{\partial x_\ell} e_0 \frac{\partial \lambda_j}{\partial \xi_\ell}) w_j, w_j \rangle \Big). \end{aligned}$$

Now, since $e_0 e_0^* = I = e_0^* e_0$, we have that

$$(\partial e_0) e_0^* + e_0 (\partial e_0^*) = 0 \tag{9.37}$$

whence

$$\{e_0, e_0^* e_0\} = 0 = \{e_0, e_0^*\}e_0 + \sum_{\ell=1}^{n} \Big(\frac{\partial e_0}{\partial \xi_\ell} e_0^* \frac{\partial e_0}{\partial x_\ell} - \frac{\partial e_0}{\partial x_\ell} e_0^* \frac{\partial e_0}{\partial \xi_\ell} \Big)$$

(using (9.37))

$$= \{e_0, e_0^*\}e_0 - e_0\{e_0^*, e_0\},$$

that is

$$e_0^*\{e_0, e_0^*\}e_0 = \{e_0^*, e_0\}. \tag{9.38}$$

Using (9.38) in the above expression for $\langle b_{\mu-2}w_j, w_j\rangle$ and noting that

$$\frac{\partial e_0^*}{\partial \xi_\ell} e_0 \frac{\partial \lambda_j}{\partial x_\ell} - \frac{\partial e_0^*}{\partial x_\ell} e_0 \frac{\partial \lambda_j}{\partial \xi_\ell} = \frac{\partial e_0^*}{\partial \xi_\ell} \frac{\partial \lambda_j}{\partial x_\ell} e_0 - \frac{\partial e_0^*}{\partial x_\ell} \frac{\partial \lambda_j}{\partial \xi_\ell} e_0 = \{e_0^*, \lambda_j\}e_0$$

($\partial \lambda_j$ being a scalar), proves (9.35). □

We must now study the "transformation properties" (we are interested just in the 2×2 case) of the subprincipal terms depending on the choice of e_0. More precisely, we have the following proposition.

Proposition 9.2.7. *Let* $0 < a_\mu = a_\mu^* \in S(m^\mu, g; \mathsf{M}_2)$ *satisfy (9.33). Let* e_0 *and* $\tilde{e}_0$ *be smooth, unitary* 2×2 *matrices in* $S(1, g; \mathsf{M}_2)$ *such that*

$$e_0^* a_\mu e_0 = \tilde{e}_0^* a_\mu \tilde{e}_0 = b_\mu = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Denote by $b_{\mu-2}$ *and* $\tilde{b}_{\mu-2}$ *the subprincipal terms given in Corollary 9.2.6, associated respectively with* e_0 *and* $\tilde{e}_0$*. Let hence* $f \in S(1, g; \mathsf{M}_2)$ *be the unitary matrix*

$$f = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, \ \textit{such that}\ \ f^* f = f f^* = I \ \textit{and}\ e_0 = \tilde{e}_0 f,$$

so that the $f_j \in C^\infty(\mathbb{R}^{2n}; \mathbb{C})$ *belong to* $S(1, g)$ *and* $|f_j(X)| = 1$*, for all* $X \in \mathbb{R}^{2n}$*,* $j = 1, 2$*. Then, with* $\{w_1, w_2\}$ *the canonical basis of* $\mathbb{C}^2$ *as before,*

$$b_{\mu-2}^{(jj)} = \langle b_{\mu-2} w_j, w_j\rangle = \langle \tilde{b}_{\mu-2} w_j, w_j\rangle + \mathrm{Im}\big(f_j\{\bar{f}_j, \lambda_j\}\big), \ \ j = 1, 2. \tag{9.39}$$

Proof. In the first place we have, since $f w_j = f_j w_j$ and $|f_j| = 1$, $j = 1, 2$,

$$\begin{aligned}\langle e_0^* a_{\mu-2} e_0 w_j, w_j\rangle &= \langle f^* \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 f w_j, w_j\rangle \\ &= \langle f_j a_{\mu-2} \tilde{e}_0 w_j, f_j \tilde{e}_0 w_j\rangle = \langle a_{\mu-2} \tilde{e}_0 w_j, \tilde{e}_0 w_j\rangle. \end{aligned} \tag{9.40}$$

Next, using $(\partial f)w_j = (\partial f_j)w_j$, we compute

$$\begin{aligned}\langle \{e_0^*, \lambda_j\} e_0 w_j, w_j\rangle &= \langle \{f^* \tilde{e}_0^*, \lambda_j\} \tilde{e}_0 f_j w_j, w_j\rangle \\ &= \langle f_j \{\tilde{e}_0^*, \lambda_j\} \tilde{e}_0 w_j, f_j w_j\rangle \\ &\quad + \langle f_j \sum_{\ell=1}^{n} \Big(\frac{\partial f^*}{\partial \xi_\ell} \tilde{e}_0^* \frac{\partial \lambda_j}{\partial x_\ell} - \frac{\partial f^*}{\partial x_\ell} \tilde{e}_0^* \frac{\partial \lambda_j}{\partial \xi_\ell}\Big) \tilde{e}_0 w_j, w_j\rangle \\ &= \langle \{\tilde{e}_0^*, \lambda_j\} \tilde{e}_0 w_j, w_j\rangle \\ &\quad + \sum_{\ell=1}^{n} \Big(\langle f_j \tilde{e}_0^* \frac{\partial \lambda_j}{\partial x_\ell} \tilde{e}_0 w_j, \frac{\partial f_j}{\partial \xi_\ell} w_j\rangle - \langle f_j \tilde{e}_0^* \frac{\partial \lambda_j}{\partial \xi_\ell} \tilde{e}_0 w_j, \frac{\partial f_j}{\partial x_\ell} w_j\rangle\Big) \\ &= \langle \{\tilde{e}_0^*, \lambda_j\} \tilde{e}_0 w_j, w_j\rangle + \langle f_j \{\bar{f}_j, \lambda_j\} w_j, w_j\rangle \\ &= \langle \{\tilde{e}_0^*, \lambda_j\} \tilde{e}_0 w_j, w_j\rangle + f_j \{\bar{f}_j, \lambda_j\}, \quad j = 1, 2,\end{aligned}$$

that is

$$\langle \{e_0^*, \lambda_j\} e_0 w_j, w_j\rangle = \langle \{\tilde{e}_0^*, \lambda_j\} \tilde{e}_0 w_j, w_j\rangle + f_j \{\bar{f}_j, \lambda_j\},\ j = 1, 2.$$

Hence, for $j = 1, 2$,

$$\frac{1}{2}\mathrm{Im}\Big(\langle \{e_0^*, \lambda_j\} e_0 w_j, w_j\rangle\Big) = \frac{1}{2}\mathrm{Im}\Big(\langle \{\tilde{e}_0^*, \lambda_j\} \tilde{e}_0 w_j, w_j\rangle\Big) + \frac{1}{2}\mathrm{Im}\big(f_j \{\bar{f}_j, \lambda_j\}\big). \quad (9.41)$$

We now consider

$$\begin{aligned}\langle e_0^* \{a_\mu, e_0\} w_j, w_j\rangle &= \bar{f}_j \langle \{a_\mu, \tilde{e}_0 f\} w_j, \tilde{e}_0 w_j\rangle \\ &= \langle \{a_\mu, \tilde{e}_0\} w_j, \tilde{e}_0 w_j\rangle + \bar{f}_j \langle \{a_\mu, f_j\} \tilde{e}_0 w_j, \tilde{e}_0 w_j\rangle \\ &= \langle \tilde{e}_0^* \{a_\mu, \tilde{e}_0\} w_j, w_j\rangle + \bar{f}_j \langle \{a_\mu, f_j\} \tilde{e}_0 w_j, \tilde{e}_0 w_j\rangle.\end{aligned}$$

Since

$$(\partial a_\mu)\tilde{e}_0 + a_\mu(\partial \tilde{e}_0) = \tilde{e}_0(\partial b_\mu) + (\partial \tilde{e}_0) b_\mu,$$

we get

$$\partial(a_\mu - \lambda_j)\tilde{e}_0 w_j = -(a_\mu - \lambda_j)(\partial \tilde{e}_0) w_j,\ j = 1, 2. \quad (9.42)$$

It hence follows, with ∂ and ∂' generic first-order derivatives,

$$(\partial a_\mu)(\partial' f_j)\tilde{e}_0 w_j = (\partial' f_j)(\partial a_\mu)\tilde{e}_0 w_j = (\partial' f_j)\Big((\partial \lambda_j)\tilde{e}_0 w_j - (a_\mu - \lambda_j)(\partial \tilde{e}_0) w_j\Big),$$

whence

$$\{a_\mu, f_j\}\tilde{e}_0 w_j = \{\lambda_j, f_j\}\tilde{e}_0 w_j - (a_\mu - \lambda_j)\{\tilde{e}_0, f_j\} w_j,\ j = 1, 2,$$

so that, by using

$$\langle (a_\mu - \lambda_j)\{\tilde{e}_0, f_j\}w_j, \tilde{e}_0 w_j\rangle = \langle \{\tilde{e}_0, f_j\}w_j, (a_\mu - \lambda_j)\tilde{e}_0 w_j\rangle = 0,$$

we obtain

$$\bar{f}_j\langle \{a_\mu, f_j\}\tilde{e}_0 w_j, \tilde{e}_0 w_j\rangle = \bar{f}_j\langle \{\lambda_j, f_j\}\tilde{e}_0 w_j, \tilde{e}_0 w_j\rangle = \bar{f}_j\{\lambda_j, f_j\},\ j = 1,2.$$

Hence

$$\langle \tilde{e}_0^*\{a_\mu, \tilde{e}_0\}w_j, w_j\rangle + \bar{f}_j\langle \{a_\mu, f_j\}\tilde{e}_0 w_j, \tilde{e}_0 w_j\rangle = \langle \tilde{e}_0^*\{a_\mu, \tilde{e}_0\}w_j, w_j\rangle + \bar{f}_j\{\lambda_j, f_j\},$$

for $j = 1,2$, and thus

$$\frac{1}{2}\mathrm{Im}\Big(\langle e_0^*\{a_\mu, e_0\}w_j, w_j\rangle\Big) = \frac{1}{2}\mathrm{Im}\Big(\langle \tilde{e}_0^*\{a_\mu, \tilde{e}_0\}w_j, w_j\rangle\Big) + \frac{1}{2}\mathrm{Im}\big(\bar{f}_j\{\lambda_j, f_j\}\big), \tag{9.43}$$

for $j = 1,2$. We now observe that

$$0 = \{1, \lambda_j\} = \{|f_j|^2, \lambda_j\} = f_j\{\bar{f}_j, \lambda_j\} + \bar{f}_j\{f_j, \lambda_j\},$$

so that

$$\bar{f}_j\{\lambda_j, f_j\} = f_j\{\bar{f}_j, \lambda_j\},\ \ j = 1,2. \tag{9.44}$$

Plugging (9.40), (9.41), (9.43) and (9.44) in (9.35) gives

$$\begin{aligned}\langle b_{\mu-2}w_j, w_j\rangle &= \langle \tilde{b}_{\mu-2}w_j, w_j\rangle + \frac{1}{2}\mathrm{Im}\big(f_j\{\bar{f}_j, \lambda_j\}\big) + \frac{1}{2}\mathrm{Im}\big(\bar{f}_j\{\lambda_j, f_j\}\big)\\ &= \langle \tilde{b}_{\mu-2}w_j, w_j\rangle + \mathrm{Im}\big(f_j\{\bar{f}_j, \lambda_j\}\big),\ j = 1,2,\end{aligned}$$

which proves the proposition. □

Remark 9.2.8. It is useful to remark that Proposition 9.2.5, Corollary 9.2.6 and Proposition 9.2.7 all hold true in the case of **classical** symbols (with the usual Weyl-quantization). △

Remark 9.2.9. Notice that when $a = a^*$ is an $N \times N$ **globally positive elliptic differential system** of order μ (hence μ is even), by virtue of the homogeneity

$$\lambda_{j,\mu}(X) = |X|^\mu \lambda_{j,\mu}(\frac{X}{|X|}),\ \ \forall X \in \mathbb{R}^{2n} \setminus \{0\},\ \ j = 1, \ldots, N,$$

we have that condition (9.33) on the eigenvalues of a_μ becomes

$$|\lambda_{j,\mu}(\omega) - \lambda_{j',\mu}(\omega)| \gtrsim 1,\ \ \forall \omega \in \mathbb{S}^{2n-1},\ j \neq j'. \tag{9.45}$$

△

9.3 Some Estimates for Semiclassical Operators

Recall from Definition 3.2.25 and Proposition 3.2.26 that for $s \in \mathbb{Z}_+$,

$$B^s(\mathbb{R}^n) = \{u \in L^2;\, x^\alpha \partial_x^\beta u \in L^2,\ |\alpha| + |\beta| \le s\},$$

and $B^{-s} = (B^s)^*$. It is also useful to recall that

$$B^2(\mathbb{R}^n) = D(p_0^{\mathrm{w}}(x,D)),$$

where p_0^{w} is the usual harmonic oscillator $(|x|^2 + |D|^2)/2 = (|x|^2 - \Delta)/2$ in $\mathbb{R}^n$. Recall also that

$$B^s(\mathbb{R}^n;\mathbb{C}^N) = B^s(\mathbb{R}^n) \otimes \mathbb{C}^N,$$

and that, from (5.7), on B^{2s} we have the equivalent norms

$$\|u\|^2_{B^{2s},1} := \sum_{|\alpha|+|\beta|\le 2s} \|x^\alpha \partial_x^\beta u\|_0^2, \text{ and } \|u\|^2_{B^{2s}} := \|u\|_0^2 + \|p_0^{\mathrm{w}}(x,D)^s u\|_0^2.$$

We next wish to introduce the semiclassical parameter $h \in (0,1]$ into the game.

Consider the L^2-isometry, also automorphism of $\mathscr{S}'$ and $\mathscr{S}$,

$$U_h \colon u \longmapsto (U_h u)(x) = h^{-n/4} u(x/\sqrt{h}).$$

From (9.6) we have

$$U_h^{-1} p_0^{\mathrm{w}}(x,hD) U_h = p_0^{\mathrm{w}}(\sqrt{h}x, \sqrt{h}D) = h\, p_0^{\mathrm{w}}(x,D). \tag{9.46}$$

Since

$$\partial_x^\beta \Big((U_h^{-1} u)(x) \Big) = h^{n/4} h^{|\beta|/2} (\partial_x^\beta u)(\sqrt{h}x) = h^{|\beta|/2} (U_h^{-1}(\partial_x^\beta u))(x),$$

we get

$$\|x^\alpha \partial_x^\beta (U_h^{-1} u)\|_0^2 = h^{|\beta|-|\alpha|} \|U_h^{-1}(x^\alpha \partial_x^\beta u)\|_0^2 = h^{|\beta|-|\alpha|} \|x^\alpha \partial_x^\beta u\|_0^2, \tag{9.47}$$

since U_h^{-1} is an L^2-isometry as well.

Consider now the h-dependent norm

$$\|u\|^2_{B^{2s},h} := \sum_{|\alpha|+|\beta|\le 2s} h^{2|\beta|} \|x^\alpha \partial_x^\beta u\|_0^2,$$

which is an equivalent norm of B^{2s}, for one readily has

$$h^{4s} \|u\|^2_{B^{2s},1} \le \|u\|^2_{B^{2s},h} \le \|u\|^2_{B^{2s},1},\ \ \forall u \in B^{2s}.$$

Using (9.47) we get

$$\|u\|_{B^{2s},h}^2 = \sum_{|\alpha|+|\beta|\le 2s} h^{2|\beta|}\|U_h^{-1}(x^\alpha\partial_x^\beta u)\|_0^2 = \sum_{|\alpha|+|\beta|\le 2s} h^{|\alpha|+|\beta|}\|x^\alpha\partial_x^\beta(U_h^{-1}u)\|_0^2,$$

from which it follows, with constants independent of h, that

$$h^{2s}\Big(\|U_h^{-1}u\|_0^2+\|p_0^{\mathrm{w}}(x,D)^sU_h^{-1}u\|_0^2\Big)\lesssim\|u\|_{B^{2s},h}^2\lesssim\|U_h^{-1}u\|_0^2+\|p_0^{\mathrm{w}}(x,D)^sU_h^{-1}u\|_0^2,$$

and since

$$\|U_h^{-1}u\|_0^2+\|p_0^{\mathrm{w}}(x,D)^sU_h^{-1}u\|_0^2=\|U_h^{-1}u\|_0^2+\|U_hp_0^{\mathrm{w}}(x,D)^sU_h^{-1}u\|_0^2=$$

(by (9.46))

$$=\|u\|_0^2+h^{-2s}\|p_0^{\mathrm{w}}(x,hD)^su\|_0^2=h^{-2s}\Big(h^{2s}\|u\|_0^2+\|p_0^{\mathrm{w}}(x,hD)^su\|_0^2\Big),$$

we get, with constants independent of h,

$$\begin{cases} h^{2s}\Big(\|u\|_0^2+\|p_0^{\mathrm{w}}(x,hD)^su\|_0^2\Big)\lesssim\|u\|_{B^{2s},h}^2\lesssim h^{-2s}\Big(\|u\|_0^2+\|p_0^{\mathrm{w}}(x,hD)^su\|_0^2\Big),\\[2ex] h^{4s}\|u\|_{B^{2s},1}^2\le\|u\|_{B^{2s},h}^2\le\|u\|_{B^{2s},1}^2, \end{cases} \tag{9.48}$$

for all $u\in B^{2s}$. We may hence prove the following useful fact.

Proposition 9.3.1. *Let $r\in S_0^0(m^{-2N_0};\mathsf{M}_N)$, for some $N_0\in\mathbb{N}$. Then, given any integer k_0 with $0\le k_0\le N_0$,*

$$r^{\mathrm{w}}(x,hD)\colon L^2(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N)$$

is continuous, satisfying the following estimates: there exists $C>0$ (dependent on k_0 but **independent of** *h) such that*

$$\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_0},1}\le Ch^{-3k_0}\|u\|_0,\ \ \forall u\in L^2(\mathbb{R}^n;\mathbb{C}^N),$$

and

$$\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_0},h}\le Ch^{-k_0}\|u\|_0,\ \ \forall u\in L^2(\mathbb{R}^n;\mathbb{C}^N),$$

for all $h\in(0,1]$. In particular, when $0<k_0\le N_0$,

$$r^{\mathrm{w}}(x,hD)\colon L^2(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N)\hookrightarrow\hookrightarrow L^2(\mathbb{R}^n;\mathbb{C}^N)$$

is compact, for all $h\in(0,1]$.

Proof. We want to estimate $\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_0},h}$ (of course, we may suppose $k_0>0$, otherwise there is nothing to prove). Now, $r^{\mathrm{w}}(x,hD)$ is also a *bounded* operator $L^2 \longrightarrow L^2$ with norm bounded independently of $h\in(0,1]$. Take a sequence $\{u_j\}_{j\in\mathbb{Z}_+}\subset\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$ such that $u_j\to u$ in L^2 as $j\to+\infty$. Since $(\underbrace{p_0\sharp_h\dots\sharp_h p_0}_{k_0})\sharp_h r\in S^0_0(1;\mathsf{M}_N)$, we have, by the L^2-boundedness, with constants independent of h,

$$\|p_0^{\mathrm{w}}(x,hD)^{k_0}r^{\mathrm{w}}(x,hD)(u_j-u_{j'})\|_0\lesssim\|u_j-u_{j'}\|_0\longrightarrow 0,\text{ as } j,j'\to+\infty.$$

Hence, by (9.48), with constants independent of h,

$$\begin{aligned}h^{4k_0}\|r^{\mathrm{w}}(x,hD)(u_j-u_{j'})\|^2_{B^{2k_0},1}&\le\|r^{\mathrm{w}}(x,hD)(u_j-u_{j'})\|^2_{B^{2k_0},h}\\&\lesssim h^{-2k_0}\big(\|r^{\mathrm{w}}(x,hD)(u_j-u_{j'})\|_0^2+\|p_0^{\mathrm{w}}(x,hD)^{k_0}r^{\mathrm{w}}(x,hD)(u_j-u_{j'})\|_0^2\big)\\&\lesssim h^{-2k_0}\|u_j-u_{j'}\|_0^2\longrightarrow 0,\text{ as } j,j'\to+\infty.\end{aligned}$$

Hence $r^{\mathrm{w}}(x,hD)u_j\overset{B^{2k_0}}{\longrightarrow}v\in B^{2k_0}$ as $j\to+\infty$. But on the other hand we also have $r^{\mathrm{w}}(x,hD)u_j\overset{\mathscr{S}'}{\longrightarrow}r^{\mathrm{w}}(x,hD)u$ as $j\to+\infty$. It therefore follows that

$$r^{\mathrm{w}}(x,hD)u\in B^{2k_0},\ \ \forall u\in L^2,$$

and

$$\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_0},1}\le Ch^{-3k_0}\|u\|_0,\ \ \forall u\in L^2,$$

$$\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_0},h}\le Ch^{-k_0}\|u\|_0,\ \ \forall u\in L^2,$$

which concludes the proof. □

Corollary 9.3.2. *Let $r\in S^0_0(m^{-2N_0};\mathsf{M}_N)$, for some $N_0\in\mathbb{N}$. Then, given any integers k_0,k_1 with $0\le k_0\le k_1\le N_0$,*

$$r^{\mathrm{w}}(x,hD)\colon B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow B^{2k_1}(\mathbb{R}^n;\mathbb{C}^N)$$

is continuous, satisfying the following estimates: there exists $C>0$ (independent of h) such that

$$\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_1},1}\le Ch^{-3k_1}\|u\|_{B^{2k_0},1},\ \ \forall u\in B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N),$$

and

$$\|r^{\mathrm{w}}(x,hD)u\|_{B^{2k_1},h}\le Ch^{-k_1}\|u\|_{B^{2k_0},h},\ \ \forall u\in B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N),$$

for all $h\in(0,1]$. In particular, when $0\le k_0<k_1\le N_0$,

$$r^{\mathrm{w}}(x,hD)\colon B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow B^{2k_1}(\mathbb{R}^n;\mathbb{C}^N)\hookrightarrow\hookrightarrow B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N)$$

is compact, for all $h\in(0,1]$.

Proof. Given any $u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$, we have that $r^{\mathrm{w}}(x,hD)u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$. Hence it suffices to prove the inequalities for Schwartz functions u, with constants independent of u and h. Since $B^{2k_0}(\mathbb{R}^n;\mathbb{C}^N) \subset L^2(\mathbb{R}^n;\mathbb{C}^N)$ with

$$\|u\|_0^2 \le \|u\|^2_{B^{2k_0},h}, \quad \text{and} \quad \|u\|_0^2 \le \|u\|^2_{B^{2k_0},1},$$

from Proposition 9.3.1 we have

$$\|r^{\mathrm{w}}(x,hD)u\|^2_{B^{2k_1},h} \le Ch^{-2k_1}\|u\|_0^2 \le Ch^{-2k_1}\|u\|^2_{B^{2k_0},h},$$

and

$$\|r^{\mathrm{w}}(x,hD)u\|^2_{B^{2k_1},1} \le Ch^{-6k_1}\|u\|_0^2 \le h^{-6k_1}\|u\|^2_{B^{2k_0},1},$$

for a constant $C > 0$ independent of u and of h, which concludes the proof. □

9.4 Some Spectral Properties of Semiclassical GPDOs

We establish in this section a few useful results about spectral properties of h-Weyl quantizations of $N \times N$ *semiclassical* GPD systems which are positive elliptic. We start by giving the definition of semiclassical global polynomial differential system.

Definition 9.4.1 (Semiclassical GPD). We shall say that a classical semiclassical symbol $a \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ is a **semiclassical GPD system of order** μ if $\mu \in \mathbb{N}$ and

$$a = \sum_{j=0}^{[\mu/2]} h^j a_{\mu-2j}, \;\; a_{\mu-2j} \in S(m^{\mu-2j}, g; \mathsf{M}_N)$$

(with $[\mu/2]$ denoting, as usual, the integer part of $\mu/2$), where the entries of the $a_{\mu-2j}$ are **homogeneous polynomials** in $X \in \mathbb{R}^{2n}$ of degree $\mu - 2j$.

We say that a semiclassical GPD system $a \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ of order μ is **elliptic** (resp. **positive elliptic**, when $a = a^*$) if the principal part a_μ is a homogeneous globally elliptic (resp. globally positive elliptic) symbol.

The h-Weyl quantization of a semiclassical GPD will be called an h-GPDO.

Suppose

$$a = \sum_{j=0}^{[\mu/2]} h^j a_{\mu-2j}$$

is an $N \times N$ semiclassical GPD symbol of order $\mu \in \mathbb{N}$ in $\mathbb{R}^n$. Since (9.5) holds also on $\mathscr{S}'(\mathbb{R}^n;\mathbb{C}^N)$ and $\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$, we have the following lemma.

Lemma 9.4.2. *Let $E > 0$ and let $a \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ be an $N \times N$* **semiclassical** GPD *system of order $\mu \in \mathbb{N}$. Let U_E be the isometry introduced in Remark 9.1.2. Then*

$$U_E^{-1} a^{\mathrm{w}}(x, hD) U_E = E^{\mu/2} a^{\mathrm{w}}(x, \tilde{h}D), \tag{9.49}$$

where $\tilde{h} = h/E$ and

$$a^{\mathrm{w}}(x, \tilde{h}D) = \sum_{j=0}^{[\mu/2]} \tilde{h}^j a^{\mathrm{w}}_{\mu-2j}(x, \tilde{h}D).$$

In particular, when $E = h$, we have

$$U_h^{-1} a^{\mathrm{w}}(x, hD) U_h = h^{\mu/2} a^{\mathrm{w}}(x, D), \tag{9.50}$$

where

$$a^{\mathrm{w}}(x, D) = \sum_{j=0}^{[\mu/2]} a^{\mathrm{w}}_{\mu-2j}(x, D).$$

Proof. The proof follows immediately from Remark 9.1.2. In fact, it suffices to observe that for $\tilde{h} = h/E$ we have

$$\begin{aligned} a^{\mathrm{w}}(\sqrt{E}x, \sqrt{E}\tilde{h}D) &= U_E^{-1} a^{\mathrm{w}}(x, hD) U_E \\ &= \sum_{j=0}^{[\mu/2]} h^j U_E^{-1} a^{\mathrm{w}}_{\mu-2j}(x, hD) U_E = \sum_{j=0}^{[\mu/2]} \tilde{h}^j E^j a^{\mathrm{w}}_{\mu-2j}(\sqrt{E}x, \sqrt{E}\tilde{h}D) \\ &= \sum_{j=0}^{[\mu/2]} \tilde{h}^j E^j E^{\mu/2-j} a^{\mathrm{w}}_{\mu-2j}(x, \tilde{h}D) = E^{\mu/2} a^{\mathrm{w}}(x, \tilde{h}D), \end{aligned} \tag{9.51}$$

in view of the fact that $h = E\tilde{h}$ and that the $a_{\mu-2j}$ have entries which are all homogeneous of degree $\mu - 2j$. □

We therefore get the following scaling properties of eigenvalues.

Lemma 9.4.3. *Let $E > 0$ and let $a \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ be an $N \times N$* **semiclassical** GPD *system of order $\mu \in \mathbb{N}$. Let U_E be the isometry introduced in Remark 9.1.2. Then*

$$a^{\mathrm{w}}(x, hD) u(h) = \lambda(h) u(h), \quad u(h) \in L^2(\mathbb{R}^n; \mathbb{C}^N),\ u(h) \neq 0,$$

that is $u(h)$ is an eigenfunction of $a^{\mathrm{w}}(x, hD)$ belonging to the eigenvalue $\lambda(h)$, iff, with $\tilde{u}(\tilde{h}) := U_E^{-1} u(h) \in L^2(\mathbb{R}^n; \mathbb{C}^N)$,

$$a^{\mathrm{w}}(x, \tilde{h}D) \tilde{u}(\tilde{h}) = \frac{\lambda(E\tilde{h})}{E^{\mu/2}} \tilde{u}(\tilde{h}), \ \text{where} \ \tilde{h} = \frac{h}{E},$$

that is, $\tilde{u}(\tilde{h})$ is an eigenfunction of $a^{\mathrm{w}}(x,\tilde{h}D)$ belonging to the eigenvalue $\dfrac{\lambda(E\tilde{h})}{E^{\mu/2}}$. Hence,

$$\lambda(h) \in \operatorname{Spec}(a^{\mathrm{w}}(x,hD)) \Longleftrightarrow \frac{\lambda(E\tilde{h})}{E^{\mu/2}} \in \operatorname{Spec}(a^{\mathrm{w}}(x,\tilde{h}D)),\ \tilde{h} = \frac{h}{E}.$$

In particular, when $E = h$, one obtains that

$$\lambda(h) \in \operatorname{Spec}(a^{\mathrm{w}}(x,hD)) \Longleftrightarrow \frac{\lambda(h)}{h^{\mu/2}} \in \operatorname{Spec}(a^{\mathrm{w}}(x,D)).$$

Proof. The proof follows immediately from Lemma 9.4.2. □

As a (by now elementary) consequence of the results of Section 3.2, namely Remark 9.1.2, Lemma 9.4.2 and Theorem 3.3.13 we have the following fact concerning the spectrum of semiclassical GPD systems.

Proposition 9.4.4. *Let $A = A^* \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ be an $N \times N$ positive elliptic semiclassical* GPD *system of order $\mu \in 2\mathbb{N}$. Consider the unbounded operator $A(h)$ defined by*

$$A(h)\colon B^\mu(\mathbb{R}^n;\mathbb{C}^N) \subset L^2(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow L^2(\mathbb{R}^n;\mathbb{C}^N),$$
$$A(h)u = A^{\mathrm{w}}(x,hD)u,\ \forall u \in B^\mu(\mathbb{R}^n;\mathbb{C}^N).$$

Then $A(h)$ is semi-bounded from below for all $h \in (0,1]$. Hence, $\operatorname{Spec}(A(h))$ *is made of a sequence of eigenvalues $\{\lambda_j(h)\}_{j\geq 1} \subset \mathbb{R}$ with finite multiplicities, such that*

$$-\infty < \lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h) \leq \ldots \longrightarrow +\infty,$$

with repetitions according to the multiplicity. As before, the eigenfunctions of $A(h)$ all belong to the Schwartz space and form, possibly after an orthonormalization procedure, a basis of $L^2(\mathbb{R}^n;\mathbb{C}^N)$.

Proof. In fact, by Lemma 9.4.2 we have

$$U_h^{-1} A(h) U_h = h^{\mu/2} A(1),$$

so that

$$\begin{aligned}(A(h)u,u) &= (U_h^{-1}A(h)U_hU_h^{-1}u, U_h^{-1}u) = h^{\mu/2}(A(1)U_h^{-1}u, U_h^{-1}u)\\ &\geq -h^{\mu/2}C\|U_h^{-1}u\|_0^2 = -h^{\mu/2}C\|u\|_0^2,\ \forall u \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N).\end{aligned}$$

This concludes the proof of the lemma. □

Hence, when A is an $N \times N$ *positive elliptic semiclassical* GPD system of order μ we immediately obtain from the Minimax Principle, Lemma 9.4.3 and Proposition 9.4.4 the following corollary.

Corollary 9.4.5. *Let $A = A^* \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ be an $N \times N$ positive elliptic semiclassical* GPD *system of order $\mu \in 2\mathbb{N}$. Let $\phi_j \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$, $j \in \mathbb{N}$, be an eigenfunction of $A^{\mathrm{w}}(x,D)$ (i.e. with $h = 1$) belonging to the eigenvalue λ_j. Then*

$$\varphi_j(h;x) := (U_h\phi_j)(x) = h^{-n/4}\phi_j(\frac{x}{\sqrt{h}}) \tag{9.52}$$

belongs to the eigenvalue $\lambda_j(h) := h^{\mu/2}\lambda_j$ of $A^{\mathrm{w}}(x,hD)$. In particular

$$\varphi_j(\frac{h}{E};x) = E^{n/4}\varphi_j(h;\sqrt{E}x), \;\; j \geq 1, \tag{9.53}$$

belongs to the eigenvalue $\lambda_j(\frac{h}{E}) = \left(\frac{h}{E}\right)^{\mu/2}\lambda_j$ of $A^{\mathrm{w}}(x,\frac{h}{E}D)$. Hence, using the Minimax, for every $j \in \mathbb{N}$

$$\lambda_j \in \mathrm{Spec}(A^{\mathrm{w}}(x,D)) \Longleftrightarrow h^{\mu/2}\lambda_j = \lambda_j(h) \in \mathrm{Spec}(A^{\mathrm{w}}(x,hD)).$$

We finally consider the following situation, that will be very useful later on. Let $A = A^* \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ be an $N \times N$ semiclassical positive elliptic GPD system. Let $R = R^* \in S^0_0(1, g; \mathsf{M}_N)$ with $\mathrm{supp}(R) \subset K$, where K is a compact set *independent of* h. Then

$$R^{\mathrm{w}}(x,hD)\colon \mathscr{S}'(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$$

is **continuous** and, as an operator in L^2, $\|R^{\mathrm{w}}(x,hD)\|_{L^2\to L^2} = O(1)$ for all $h \in (0,1]$ (see Dimassi-Sjöstrand [7]). We have the following important result.

Proposition 9.4.6. *Let $A = A^* \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ be an $N \times N$ semiclassical positive elliptic* GPD *system, and let $R = R^* \in S^0_0(1, g; \mathsf{M}_N)$ with* $\mathrm{supp}(R) \subset K$*, where K is a compact set* **independent of** *h. Let us consider $A^{\mathrm{w}}_0(x,hD) = A^{\mathrm{w}}(x,hD) + R^{\mathrm{w}}(x,hD)$. Then for all $h \in (0,1]$ the unbounded operator $A_0(h)$ defined by*

$$A_0(h)\colon B^\mu(\mathbb{R}^n;\mathbb{C}^N) \subset L^2(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow L^2(\mathbb{R}^n;\mathbb{C}^N),$$

$$A_0(h)u = A^{\mathrm{w}}_0(x,hD)u, \; \forall u \in B^\mu(\mathbb{R}^n;\mathbb{C}^N),$$

is self-adjoint with a discrete spectrum bounded from below, made of a sequence of eigenvalues $\{\lambda_j(h)\}_{j\geq 1} \subset \mathbb{R}$ with finite multiplicities, such that

$$-\infty < \lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h) \leq \ldots \longrightarrow +\infty,$$

with repetitions according to the multiplicity.

Proof. That $A_0(h) = A_0(h)^*$ with the same domain B^μ of $A(h)$ is trivial, for $R^{\mathrm{w}}(x,hD)$ is bounded in L^2 and symmetric. By Theorem 10.1.1 below the resolvent set of $A_0(h)$ is non-empty, and since B^μ is compactly embedded into L^2, $A_0(h)$

also has a discrete spectrum, that must be bounded from below, for the spectrum of $A(h)$ is bounded from below and $R^{\mathrm{w}}(x,hD)$ is bounded in L^2. □

Remark 9.4.7. One may prove the discreteness of $\mathrm{Spec}(A_0(h))$ also as follows. Since the Schwartz-kernel

$$\begin{aligned}
\mathsf{K}_R(x,y;h) &= (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}R(\frac{x+y}{2},h\xi)d\xi \\
&= (2\pi h)^{-n}\int e^{ih^{-1}\langle x-y,\xi\rangle}R(\frac{x+y}{2},\xi)d\xi \\
&= h^{-n}(\mathscr{F}_{\xi\to t}R)(\frac{x+y}{2},t)\big|_{t=h^{-1}(x-y)}
\end{aligned}$$

of $R^{\mathrm{w}}(x,hD)$ belongs to $\mathscr{S}(\mathbb{R}^n\times\mathbb{R}^n;\mathsf{M}_N)$, the operator $R^{\mathrm{w}}(x,hD)$ is actually Hilbert-Schmidt. Hence $A_0(h)$ and $A(h)$ have the same essential spectrum. Since the essential spectrum of $A(h)$ is empty, the same is true for that of $A_0(h)$. △

Remark 9.4.8. More generally, one may prove that if a classical semiclassical symbol $A=A^*\in S^0_{0,\mathrm{cl}}(m^\mu,g;\mathsf{M}_N)$, with $\mu\geq 1$, has **globally positive elliptic** principal symbol, then there exists $h_0\in(0,1]$ such that the unbounded operator $A(h)$ defined by

$$\begin{gathered}
A(h)\colon B^\mu(\mathbb{R}^n;\mathbb{C}^N)\subset L^2(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow L^2(\mathbb{R}^n;\mathbb{C}^N), \\
A(h)u=A^{\mathrm{w}}(x,hD)u,\ \forall u\in B^\mu(\mathbb{R}^n;\mathbb{C}^N),
\end{gathered}$$

is **self-adjoint**, **semi-bounded** from below and with a **discrete spectrum**, for all $h\in(0,h_0]$. In addition, from the elliptic regularity we have that its eigenfunctions are in $\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$, and, possibly after an orthonormalization procedure, they form a basis of $L^2(\mathbb{R}^n;\mathbb{C}^N)$.

In fact, one uses (9.5) of Remark 9.1.2 (with $E=h$), namely

$$U_h^{-1}A^{\mathrm{w}}(x,hD)U_h=A^{\mathrm{w}}(\sqrt{h}x,\sqrt{h}D)$$

to reduce matters to Theorem 3.3.13. △

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