Chapter 2
1D Models of Ekman Layers

Abstract This chapter introduces the reader to one-dimensional water-column models using fixed vertical levels. Such a model is applied to study the dynamics of surface and bottom Ekman layers in the ocean.

2.1 Useful Background Knowledge

2.1.1 Inertial Oscillations

Before exploring the Ekman-layer dynamics, it is useful to revisit features inherent with inertial oscillations. Flows under the sole influence of the Coriolis force are described by the momentum equations:

\[
\frac{\partial u}{\partial t} + f v = 0 \quad (2.1)
\]
\[
\frac{\partial v}{\partial t} - fu = 0 \quad (2.2)
\]

where \( f \) is the Coriolis parameter, given by \( f = 2\Omega \sin(\varphi) \), where \( \Omega = 7.27 \times 10^{-5} \text{s}^{-1} \) is the rotation frequency of Earth, and \( \varphi \) is geographical latitude in radians. For an initial flow in the \( x \)-direction of speed \( u_o \), the solution of the latter equations is given by:

\[
\begin{align*}
    u(t) &= +u_o \cos (ft) \\
    v(t) &= -u_o \sin (ft)
\end{align*}
\]

The resultant flow trajectories are circles of a radius of \( u_o / |f| \), called inertial radius. The period of one complete cycle is \( 2\pi / f \), called inertial period. The inertial period is 12 hrs at the poles and infinite directly at the equator where the Coriolis force vanishes. Figure 2.1 shows flow paths associated with inertial oscillations for \( u_o = 0.1 \text{ m/s} \) and \( f = 1 \times 10^{-4} \text{s}^{-1} \) with and without ambient uniform flow.
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2.1.2 Semi-implicit Treatment of the Coriolis Force

Adequate formulation of the Coriolis force in a finite-difference model can be achieved by means of a semi-implicit approach. For the momentum equations governing inertial oscillations (Eqs. 2.1 and 2.2), this approach gives:

\[
\begin{align*}
    u^{n+1} &= u^n + 0.5 \alpha (v^n + v^{n+1}) \\
    v^{n+1} &= v^n - 0.5 \alpha (u^n + u^{n+1})
\end{align*}
\]

where \( n \) is the current time level, \( n + 1 \) refers to the future value (one time step \( \Delta t \) ahead), and \( \alpha = \Delta t f \). Cross-combination of the latter equations yields the final form:

\[
\begin{align*}
    u^{n+1} &= \left[(1 - \beta)u^n + \alpha v^n\right]/(1 + \beta) \\
    v^{n+1} &= \left[(1 - \beta)v^n - \alpha u^n\right]/(1 + \beta)
\end{align*}
\]

where \( \beta = 0.25 \alpha^2 \). This scheme requires numerical time steps small compared with the rotation period; that is, \( |\alpha| \ll 1 \), otherwise the period of the parcel’s circular motion will differ from the true value. This semi-implicit scheme is widely used by modellers.
2.2 The Surface Ekman Layer

2.2.1 Boundary-Layer Equations

This section explores the dynamics of frictional boundary layers in the ocean, called *Ekman layers* (Ekman, 1905). For simplicity, we assume horizontal homogeneity of all variables and an ocean of uniform density, so that the Navier-Stokes equations take the reduced form:

\[
\frac{\partial u}{\partial t} - f v = \frac{\partial}{\partial z} \left( A_z \frac{\partial u}{\partial z} \right) \tag{2.3}
\]

\[
\frac{\partial v}{\partial t} + f u = \frac{\partial}{\partial z} \left( A_z \frac{\partial v}{\partial z} \right) \tag{2.4}
\]

where \( f \) is the Coriolis parameter, and the terms on the right-hand side of these *boundary-layer equations* represent vertical turbulent diffusion of momentum with \( A_z \) being vertical eddy viscosity.

Wind stress operates as a tangential frictional force at the sea surface, and the associated boundary conditions read:

\[
\left( A_z \frac{\partial u}{\partial z} \right)_{z=0} = \frac{\tau_{\text{wind}}^x}{\rho_o} \quad \text{and} \quad \left( A_z \frac{\partial v}{\partial z} \right)_{z=0} = \frac{\tau_{\text{wind}}^y}{\rho_o} \tag{2.5}
\]

where \( \rho_o \) is surface density. The components of the wind-stress vector are given by:

\[
\tau_{\text{wind}}^x = \rho_{\text{air}} C_d U \sqrt{U^2 + V^2} \quad \text{and} \quad \tau_{\text{wind}}^y = \rho_{\text{air}} C_d V \sqrt{U^2 + V^2} \tag{2.6}
\]

where \( \rho_{\text{air}} \) is air density, \( C_d \) is the nondimensional *wind-drag coefficient* with values in a range of \( 1.1 - 1.5 \times 10^{-3} \), and \( U \) and \( V \) are horizontal components of the wind vector measured at a height of 10 m above sea level. The wind stress vector field has the same direction as the wind, but its magnitude is proportional to the square of the wind speed. Hence, the stronger the wind the greater are its impacts on surface flows.

The wind stress creates tangential friction along the sea surface and, thus, transfers momentum into the ocean. On time scales of days, the resultant oceanic motion becomes influenced by the Coriolis force. In the absence of other influences, final steady state consists of a dynamical balance between the Coriolis force and the friction force. What is the structure of the resultant steady-state flow pattern?

2.2.2 Scaling: The Temporal Rossby Number

Consider an oscillatory flow of a maximum speed of \( U_o \) on a period of \( T \). On the basis of this, the Coriolis force attains a maximum value of \( f U_o \). On the other hand,
the order of magnitude of the temporal derivative in Eq.(2.3) can be estimated at:

\[
\frac{\partial u}{\partial t} \approx \frac{U_o}{T}
\]

The ratio of this estimate with that of the Coriolis force in the boundary-layer equations is given by:

\[
Ro_t = \frac{1}{f T}
\]  

(2.7)

and is called the \textit{temporal Rossby number}. This comparative ratio implies that the Coriolis force can no longer be neglected in the momentum equations if \( Ro_t \approx 1 \), or, in other words, if the time scale of a process (establishment of a frictional boundary layer here) is of the order of the inertial period, given by \( 2\pi/f \). Hence, except for the equatorial regions, where the inertial period becomes long, the Coriolis force becomes important if a flow lasts longer than a few days. Considerations based on typical \textit{scales of motion} and comparative ratios of terms in the momentum equations are called \textit{scaling} considerations.

### 2.2.3 Scaling: The Ekman Number

For a steady state, the Coriolis force is balanced by the frictional force associated with vertical diffusion of momentum. The form of the boundary equations (Eqs. 2.3 and 2.4) implies that lateral velocity varies exponentially with depth. With such a velocity profile; that is,

\[
u(z) = U_o \exp(z/D)
\]

where \( U_o \) is the surface value and \( D \) is a depth scale, the magnitude of the frictional force is \( A_z U_o / D^2 \), assuming vertical eddy viscosity to be uniform. The ratio between this magnitude with that of the Coriolis force is called the \textit{Ekman number} and is given by:

\[
Ek = \frac{A_z}{D^2 f}
\]  

(2.8)

Accordingly, a steady state of the boundary-layer equations (Eqs. 2.3 and 2.4) implies that \( Ek \approx 1 \), which corresponds to a depth scale of:

\[
D = \sqrt{\frac{A_z}{f}}
\]  

(2.9)
Consequently, surface Ekman layers in the ocean are limited in their vertical extent. Typical thicknesses are 50–150 m with increasing values toward the equator.

### 2.2.4 Solutions of the Boundary-Layer Equations

The analytical solutions of the boundary-layer equations are derived and discussed in many textbooks (e.g. Cushman-Roisin, 1994). These solutions are a first benchmark for verification of a one-dimensional finite-difference water-column model being constructed in the next section. This approach also gives us the opportunity to explore situations of nonuniform values of vertical eddy viscosity for which analytical solutions are more difficult to derive.

### 2.2.5 Finite-Difference Equations

The water column is represented by grid cells stacked on top of each other with a vertical grid spacing of $\Delta z$ and a grid index $i$ with $i = 1$ pointing to the surface cell and $i = nz$ to the bottom cell (Fig. 2.2). In this one-dimensional application, values of velocity components and eddy viscosity are calculated at the same grid point. The uppermost grid point is located at a distance of $0.5\Delta z$ below the sea surface.

![Fig. 2.2 Grid index, grid points and grid spacing for Ekman-layer modelling](image)

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2.2 The Surface Ekman Layer

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Use of the semi-implicit approach for the Coriolis force leads to the finite-difference equations:

\[ u_{n+1}^i = u_n^i + 0.5 \alpha (v_n^i + v_{n+1}^i) + \text{Diff}_u \]  
\[ v_{n+1}^i = v_n^i - 0.5 \alpha (u_n^i + u_{n+1}^i) + \text{Diff}_v \]  

where \( n \) is the current time level, \( n + 1 \) refers to the future value (one time step \( \Delta t \) ahead), \( \alpha = \Delta t f \), and \( \text{Diff}_u \) and \( \text{Diff}_v \) are the diffusion terms. Cross-combination of the latter equations gives:

\[ u_{n+1}^i = \frac{[(1 - \beta)u_n^i + \alpha v_n^i + 0.5\alpha \text{Diff}_v + \text{Diff}_u]/(1 + \beta)}{\Delta z} \]  
\[ v_{n+1}^i = \frac{[(1 - \beta)v_n^i - \alpha u_n^i - 0.5\alpha \text{Diff}_u + \text{Diff}_v]/(1 + \beta)}{\Delta z} \]  

with \( \beta = 0.25 \alpha^2 \). Accurate representation of the Coriolis force requires \( \Delta t \ll 1/|f| \).

### 2.2.6 Formulation of Diffusion Terms

In finite-difference form, the diffusion terms can be written as:

\[ \text{Diff}_u = \Delta t \frac{A_z^+(u_{i-1}^n - u_i^n)}{\Delta z} - A_z^-(u_i^n - u_{i+1}^n) / \Delta z \]  
\[ \text{Diff}_v = \Delta t \frac{A_z^+(v_{i-1}^n - v_i^n)}{\Delta z} - A_z^-(v_i^n - v_{i+1}^n) / \Delta z \]  

where \( A_z \) is vertical eddy viscosity with \( A_z^+ = 0.5(A_{z,i-1} + A_{z,i}) \) and \( A_z^- = 0.5(A_{z,i} + A_{z,i+1}) \).

### 2.2.7 Stability Criterion for Diffusion Terms

The above finite-difference form of the diffusion terms is associated with the stability criterion:

\[ \Delta t \leq \frac{(\Delta z)^2}{A_{z,\text{max}}} \]  

where \( A_{z,\text{max}} \) is the maximum value that vertical eddy viscosity attains during a simulation. The time step chosen has to satisfy Eq. 2.16, otherwise the prediction becomes numerically unstable causing the computer code to crash.
2.3 Exercise 1: The Surface Ekman Layer

2.3.1 Task Description

We consider a water column, 500 m in depth, represented by an equidistant vertical grid spacing of 1 m. The Coriolis parameter is chosen as $f = 1 \times 10^{-4} \text{s}^{-1}$ corresponding to mid-latitudes in the Northern Hemisphere. The water column is initially at rest. The model is forced via prescription of a southerly wind of a wind stress of $\tau_y = 0.5 \text{ Pa}$ in magnitude.

The total simulation time is 5 days. To avoid the appearance of strong inertial oscillations, the wind stress is linearly adjusted from zero to its final value over the first 2 days of simulation. The time step is set to $\Delta t = 5 \text{ s}$.

The surface wind stress enters the finite-difference equations implicitly via the boundary values $u_0^n$ and $v_0^n$. Using Eq. (2.5), these values are calculated from:

$$u_0^n = u_1^n + \frac{\tau_{\text{wind}}^x}{\rho_o A_z^+} \Delta z$$  \hspace{1cm} (2.17)

$$v_0^n = v_1^n + \frac{\tau_{\text{wind}}^y}{\rho_o A_z^+} \Delta z$$  \hspace{1cm} (2.18)

where $A_z^+ = 0.5(A_{z,0} + A_{z,1})$ with $A_{z,0}$ representing vertical eddy viscosity near the sea surface. The following three different eddy-viscosity scenarios are considered in this exercise:

1. Eddy viscosity is uniform with a constant value of $A_z = 5 \times 10^{-2} \text{ m}^2 \text{s}^{-1}$;
2. Same as before, but with a local minimum of $A_z = 4 \times 10^{-3} \text{ m}^2 \text{s}^{-1}$ around a depth of 20 m mimicking a reduction of turbulence levels by an assumed strong local density stratification;
3. Eddy viscosity is calculated from Prandtl’s mixing-length approach (Prandtl, 1925) according to:

$$A_z = L^2 \sqrt{(\partial u/\partial z)^2 + (\partial v/\partial z)^2}$$

where, for simplicity, the mixing length is set to a constant value of $L = 2 \text{ m}$.

Only results of the first scenario are presented here. The other scenarios are included as options in the FORTRAN 95 code and remain for the reader to be tested. The resultant steady-state flow pattern is visualised via displacements of neutrally buoyant floats. To this end, a prediction scheme for neutrally buoyant floats is added to the code. Initially, floats form a vertical line and lateral displacements are predicted with:

$$X^{n+1} = X^n + \Delta t \ u$$
$$Y^{n+1} = Y^n + \Delta t \ v$$
where \((X, Y)\) is the location of a float, and \((u, v)\) is horizontal velocity at the depth horizon of this float.

### 2.3.2 Results

The wind-stress forcing imposed creates a lateral current in the ocean being strongest at the surface and decreasing rapidly with depth (Fig. 2.3). The surface current is directed 45° to the right with respect to the wind direction in the Northern Hemisphere. The flow direction turns clockwise with increasing distance from the sea surface. The final shape is called the *Ekman spiral*. The *Ekman-layer depth* can be defined as the depth at which the speed of the Ekman flow has decreased to \(\exp(-\pi) \approx 0.04\) (4%) of its surface value. According to theory, this takes place at a depth of:

\[
\delta_E = \pi \sqrt{\frac{2A_c}{|f|}}
\]

yielding 100 m for the setting of this exercises. This is in excellent agreement with the simulation result. Note that some textbooks define the Ekman-layer depth without the \(\pi\) multiplier.

![Fig. 2.3 Exercise 1. Structure of the surface Ekman layer (Northern Hemisphere). Small arrows indicate lateral float displacements shown from the surface to a depth of 100 m at steps of 10 m. The thick arrow indicates the wind direction](image-url)
2.3.3 Explanation of the Ekman-Layer Structure

Imagine that the ocean consists of multiple thin layers piled up on top of each other (Fig. 2.4a). Each layer is forced by the overlying layer by a tangential stress and, itself, is subject to friction with the layer underneath. Friction operates opposite to the drift direction of a layer. The Coriolis force acts perpendicular to this direction, to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. A balance of forces (Fig. 2.4b) implies now that the direction of movement of a layer is turned by a certain fraction clockwise (Northern Hemisphere) with respect to the overlying layer.

The friction force at the bottom of a layer acts as a surface stress (in the opposite direction) for the next deeper layer. Since this surface stress forms the longest side (hypotenuse) of the right-angled force triangle, this tangential stress has to decrease from one layer to the next deeper one such that the drift speed decreases with depth. Consequently, the resultant layer motions make up an Ekman spiral.

2.3.4 Additional Exercises for the Reader

Repeat the simulation for the other eddy-viscosity scenarios outlined above. Calculate depth-averaged values of the components of horizontal velocity and produce data outputs on hourly intervals. Theory suggest that the depth-averaged flow in the Ekman layer is at right angle with respect to the wind direction, to the right in the Northern Hemisphere and to the left in the Southern Hemisphere (e.g. Pond and Pickard, 1983). Does the model yield the same result? If not, explore the time-average values of the results and consider what you have learned about inertial oscillations.
2.4 The Bottom Ekman Layer

2.4.1 Boundary-Layer Equations

A bottom Ekman layer develops owing to frictional effects experienced by a flow in contact with the sea floor. For simplicity, this ambient (geostrophic) flow of vector components $u_{geo}$ and $v_{geo}$ is assumed to be horizontally uniform and the sea floor is assumed to be plane. With horizontal homogeneity of all variables, the boundary-layer equations can then be formulated as:

\[
\frac{\partial \hat{u}}{\partial t} - f \hat{v} = \frac{\partial}{\partial z} \left( A_z \frac{\partial \hat{u}}{\partial z} \right) \quad (2.20)
\]

\[
\frac{\partial \hat{v}}{\partial t} + f \hat{u} = \frac{\partial}{\partial z} \left( A_z \frac{\partial \hat{v}}{\partial z} \right) \quad (2.21)
\]

where $\hat{u} = u - u_{geo}$ and $\hat{v} = v - v_{geo}$ are flow variations with respect to the ambient flow. Forcing is indirectly provided by the condition of vanishing flow at the sea floor; that is, $\hat{u} = -u_{geo}$ and $\hat{v} = -v_{geo}$. Mathematically, this problem is equivalent to the situation of a fluid at rest but a sea floor moving horizontally at a velocity of $(-u_{geo}, -v_{geo})$.

2.5 Exercise 2: The Bottom Ekman Layer

2.5.1 Task Description

Take the same grid configuration and initial conditions as in Exercise 1. Forcing is provided by prescription of an ambient uniform geostrophic flow of a speed of $v_{geo} = 0.1 \text{ m/s}$ directed to the north. For simplicity, uniform vertical eddy viscosity is used with a value of $A_z = 5 \times 10^{-3} \text{ m}^2 \text{s}^{-1}$. The time step is set to $\Delta t = 5 \text{ s}$. The total simulation time is 5 days with data output of the flow field at the end of the simulation. Note that implementation of a bottom-friction law is not necessary here, since bottom friction arises implicitly via prescription of a near-bottom value of eddy viscosity in conjunction with the assumption of vanishing flow at the sea floor.

2.5.2 Results

Like in the surface Ekman layer, the Coriolis force produces an Ekman spiral near the sea floor (Fig. 2.5). In the bottom layer, however, the flow turns toward the left with respect to the direction of the ambient flow as we move closer to the sea floor. Here, it is the sea floor that imposes a frictional stress on the fluid. In this sense, the dynamics that make up the bottom Ekman layer is very similar to features of the
surface Ekman layer when turning the water column upside down and when imagining a sea floor moving at a speed of 0.1 m/s along the “surface” of a fluid being initially at rest. As for the surface Ekman layer, (2.19) determines the thickness of the Ekman layer. Bottom Ekman layers in the ocean attain thicknesses of 5–50 m. In this exercise, we yield a thickness of around 20 m.

### 2.5.3 Additional Exercises for the Reader

Consider a situation of a southerly wind of a wind-stress magnitude of $\tau_y = 0.5 \text{ Pa}$ in magnitude in conjunction with a northward ambient geostrophic flow of $v_{geo} = 0.1 \text{ m/s}$ in speed. Conduct a sequence of experiments with total water depth $h$ varying between 20 and 200 m. For which value of $h$ do the surface and bottom Ekman layers appear as separate features without overlapping? The reader is also encouraged to simulate Ekman-layer dynamics for the Southern-Hemisphere situation.
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