Chapter 2
Generalized Black-Scholes Formulae for Martingales, in Terms of Last Passage Times

Abstract Let \((M_t, t \geq 0)\) be a positive, continuous local martingale such that
\[ M_t \xrightarrow{t \to \infty} M_\infty = 0 \text{ a.s.} \] In Section 2.1, we express the European put \(P(K, t) := \mathbb{E}[(K - M_t)^+]\) in terms of the last passage time \(G_K(M) := \sup\{t \geq 0; M_t = K\}\). In Section 2.2, under the extra assumption that \((M_t, t \geq 0)\) is a true martingale, we express the European call \(C(K, t) := \mathbb{E}[(M_t - K)^+]\) still in terms of the last passage time \(G_K(M)\). In Section 2.3, we shall give several examples of explicit computations of the law of \(G_K(M)\), and Section 2.4 will be devoted to the proof of a more general formula for this law. In Section 2.5, we recover, using the results of Section 2.1, Pitman-Yor’s formula for the law of \(G_K\) in the framework of transient diffusions.

The next sections shall extend these results in different ways:

- In Section 2.6, we present an example where \((M_t, t \geq 0)\) is no longer continuous, but only càdlàg without positive jumps,
- In Section 2.7, we remove the assumption \(M_\infty = 0\),
- Finally, in Section 2.8, we consider the framework of several orthogonal local martingales.

2.1 Expression of the European Put Price in Terms of Last Passage Times

2.1.1 Hypotheses and Notation

Let \((M_t, t \geq 0)\) be a local martingale defined on a filtered probability space \((\Omega, (\mathcal{F}_t, t \geq 0), \mathcal{F}_\infty, \mathbb{P})\). We assume that \(\mathcal{F}_t := \sigma(M_s; s \leq t), t \geq 0\) is the natural filtration of \((M_t, t \geq 0)\) and that \(\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t\). Let \(\mathcal{M}^{0,c}_+\) denote the set of local martingales such that:
for all $t \geq 0$, $M_t \geq 0$ a.s. \hfill (2.1)

• $(M_t, t \geq 0)$ is a.s. continuous \hfill (2.2)

• $M_0$ is a.s. constant and $\lim_{t \to \infty} M_t = 0$ a.s. \hfill (2.3)

Hence, a local martingale which belongs to $\mathcal{M}^{0,c}$ is a supermartingale.

For all $K, t \geq 0$, we define the put quantity $\Pi(K, t)$ associated to $M$:

$$\Pi(K, t) := \mathbb{E} \left[ (K - M_t)^+ \right].$$ \hfill (2.4)

We note that, since $x \mapsto (K - x)^+$ is a bounded convex function, $((K - M_t)^+, t \geq 0)$ is a submartingale, and therefore $\mathbb{E} \left[ (K - M_t)^+ \right]$ is an increasing function of $t$, converging to $K$ as $t \to \infty$. It is thus natural to try to express $\Pi(K, t)$ as $K$ times the distribution function of a positive random variable. This is the purpose of the next paragraph, which is a generalization of Point $(i)$ of Theorem 1.2.

### 2.1.2 Expression of $\Pi(K, t)$ in Terms of $\mathcal{G}_K^{(M)}$

Let $(M_t, t \geq 0) \in \mathcal{M}^{0,c}$ and define $\mathcal{G}_K^{(M)}$ by:

$$\mathcal{G}_K^{(M)} := \sup \{ t \geq 0; M_t = K \},$$ \hfill (2.5)

( = 0 if the set $\{ t \geq 0; M_t = K \}$ is empty.)

We shall often write $\mathcal{G}_K$ instead of $\mathcal{G}_K^{(M)}$ when there is no ambiguity.

**Theorem 2.1.** Let $K > 0$:

1) For any $\mathcal{T}_t$-stopping time $T$:

$$\left( 1 - \frac{M_T}{K} \right)^+ = \mathbb{P}(\mathcal{G}_K \leq T | \mathcal{F}_T).$$ \hfill (2.6)

2) Consequently:

$$\Pi(K, T) = \mathbb{E} \left[ (K - M_T)^+ \right] = K \mathbb{P}(\mathcal{G}_K \leq T).$$ \hfill (2.7)

### 2.1.3 Proof of Theorem 2.1

It hinges upon the following (classical) Lemma.
2.1 Expression of the European Put Price in Terms of Last Passage Times

Lemma 2.1 (Doob’s maximal identity).
If \((N_t, t \geq 0) \in \mathcal{M}^{0,c}_\infty\), then:
\[
\sup_{t \geq 0} N_t \xrightarrow{\text{law}} \frac{N_0}{U}
\]  
(2.8)

where \(U\) is a uniform r.v. on \([0, 1]\) independent of \(\mathcal{F}_0\).

Proof. Let \(a > N_0\) and \(T_a := \inf\{t \geq 0; N_t = a\} (= +\infty\) if this set is empty\). We use Doob’s optional stopping theorem to obtain:
\[
\mathbb{E}[N_{T_a} | \mathcal{F}_0] = a \mathbb{P}(T_a < \infty | \mathcal{F}_0) = N_0
\]
since \(N_{T_a} = 0\) if \(T_a = +\infty\) and \(N_{T_a} = a\) if \(T_a < +\infty\). Thus:
\[
\mathbb{P}\left(\sup_{t \geq 0} N_t > a \mid \mathcal{F}_0\right) = \frac{N_0}{a}
\]
since \(\{T_a < \infty\} = \left\{\sup_{t \geq 0} N_t > a\right\}\).

We now prove Theorem 2.1
We note that, since \(\lim_{t \to \infty} M_t = 0\) a.s.:
\[
\{\mathcal{G}_K < T\} = \left\{\sup_{s \geq T} M_s < K\right\}.
\]
We apply Lemma 2.1 to the local martingale \((M_{T+u}, u \geq 0)\), in the filtration \((\mathcal{F}_{T+u}, u \geq 0)\). Conditionally on \(\mathcal{F}_T:\)
\[
\sup_{s \geq T} M_s \xrightarrow{\text{law}} \frac{M_T}{U}
\]  
(2.9)

where \(U\) is uniform on \([0, 1]\), and independent of \(\mathcal{F}_T\). Consequently:
\[
\mathbb{P}\left(\sup_{s \geq T} M_s < K \mid \mathcal{F}_T\right) = \mathbb{P}\left(\frac{M_T}{U} < K \mid \mathcal{F}_T\right) = \left(1 - \frac{M_T}{K}\right) + .
\]  
(2.10)

Remark 2.1. Theorem 2.1 and Lemma 2.1 remain valid if we replace the hypothesis: \((M_t, t \geq 0)\) is a.s. continuous by the weaker one: \((M_t, t \geq 0)\) has càdlàg paths and no positive jumps. This relies on the fact that, in this new framework, we still have \(M_{T_a} = a\) on the event \(\{T_a < \infty\}\).

Of course, Theorem 2.1 has a practical interest only if we can explicitly compute the law of \(\mathcal{G}_K\). We shall tackle this computation in Section 2.3 below, but, before that, we study the way Theorem 2.1 is modified when we replace the Put price by a Call price.
Exercise 2.1 (Another proof of Doob’s maximal identity).
Let \((B_t, t \geq 0)\) denote a Brownian motion started at \(x\), and denote by 
\[ T_k := \inf\{t \geq 0; B_t = k\} \]
its first hitting time of level \(k\).

1) Let \(a < x < b\). Prove that:
\[
\mathbb{P}_x(T_a < T_b) = \frac{x-b}{a-b}.
\]

2) We assume that \(x = 1\). Deduce then that:
\[
\sup_{t \geq 0} B_{t \land T_0} = \frac{1}{U}
\]
where \(U\) is a uniform r.v. on \([0,1]\) independent from \((B_t, t \geq 0)\).

3) Let \((M_t, t \geq 0) \in \mathcal{M}^{0,c}_+\) such that \(M_0 = 1\) a.s. Apply question 2) and the Dambis, Dubins, Schwarz’s Theorem to recover Doob’s maximal identity. (Note that \(\langle M \rangle_\infty < \infty\) a.s. since \((M_t, t \geq 0)\) converges a.s.)

2.2 Expression of the European Call Price in Terms of Last Passage Times

2.2.1 Hypotheses
Let \((M_t, t \geq 0) \in \mathcal{M}^{0,c}_+\). For all \(K, t \geq 0\), we defined the call quantity \(C(K, t)\) associated to \(M\) by:
\[
C(K, t) := \mathbb{E}[(M_t - K)^+].
\]
In order to state the counterpart of Theorem 2.1 for the call price, we add the extra assumption:

\((T)\) \quad \((M_t, t \geq 0)\) is a (true) martingale such that \(M_0 = 1\).

In particular, \(\mathbb{E}[M_t] = 1\) for all \(t \geq 0\).

Let \(\mathbb{P}^{(M)}\) be the probability on \((\Omega, \mathcal{F}_\infty)\) such that, for all \(t \geq 0\):
\[
\mathbb{P}^{(M)} = M_t \cdot \mathbb{P}|_{\mathcal{F}_t}.
\]
In the framework of financial mathematics, this probability is called a change of numéraire probability.

We note that:
\- \(\mathbb{P}^{(M)}(T_0 < +\infty) = 0\), where \(T_0 := \inf\{t \geq 0; M_t = 0\}\). Indeed,
\[
\mathbb{P}^{(M)}(T_0 \leq t) = \mathbb{E}[M_t 1\{T_0 \leq t\}] = \mathbb{E}[M_{T_0} 1\{T_0 \leq t\}] = 0.
\]
Hence, \(T_0 = +\infty\) \(\mathbb{P}^{(M)}\)-a.s.
• It is easily seen that, under $\mathbb{P}(M_t, t \geq 0)$, $\frac{1}{M_t}$ is a positive local martingale. It is thus a supermartingale, which converges a.s. In fact:

$$\frac{1}{M_t} \xrightarrow{t \to \infty} 0 \quad \mathbb{P}(M_t)\text{-a.s.} \quad (2.14)$$

Indeed, for every $\varepsilon > 0$:

$$\mathbb{P}(M_t) \left( \frac{1}{M_t} > \varepsilon \right) = \mathbb{P}(M_t) \left( M_t < \frac{1}{\varepsilon} \right) = \mathbb{E} \left[ M_t 1_{ \{ M_t < 1/\varepsilon \} } \right] \xrightarrow{t \to \infty} 0$$

from the dominated convergence theorem. Thus (2.14) holds.

### 2.2.2 Price of a European Call in Terms of Last Passage Times

We state the counterpart of Theorem 2.1 for the call price.

**Theorem 2.2.** Let $(M_t, t \geq 0)$ be a positive, continuous martingale which converges to 0 a.s and such that $M_0 = 1$ a.s. Then:

i) For every bounded, $\mathcal{F}_t$-measurable r.v. $F_t$, and all $K \geq 0$:

$$\mathbb{E} \left[ F_t (M_t - K)^+ \right] = \mathbb{E}(M) \left[ F_t 1_{ \{ \mathcal{G}_K^{(M)} \leq t \} } \right]. \quad (2.15)$$

In particular:

$$\mathbb{E} \left[ (M_t - K)^+ \right] = \mathbb{P}(M) \left( \mathcal{G}_K^{(M)} \leq t \right). \quad (2.16)$$

ii) For every bounded, $\mathcal{F}_t$-measurable r.v. $F_t$:

$$\mathbb{E} \left[ F_t |M_t - K| \right] = K \mathbb{E} \left[ F_t 1_{ \{ \mathcal{G}_K^{(M)} \leq t \} } \right] + \mathbb{E}(M) \left[ F_t 1_{ \{ \mathcal{G}_K^{(M)} \leq t \} } \right] \quad (2.17)$$

iii) For every $K, t \geq 0$:

$$\mathbb{P}(M) \left( \mathcal{G}_K^{(M)} \leq t \right) = 1 - K + K \mathbb{P} \left( \mathcal{G}_K^{(M)} \leq t \right) \quad (2.18)$$

$$= 1 - K \mathbb{P}(\mathcal{G}_K^{(M)} > t)$$

and

$$\mathbb{P}(M) \left( \mathcal{G}_K^{(M)} > t \right) = K \mathbb{P}(\mathcal{G}_K^{(M)} > t).$$

Therefore, if $K \geq 1$:

$$\mathbb{P}(M) \left( \mathcal{G}_K^{(M)} \geq t | \mathcal{G}_K^{(M)} > 0 \right) = \mathbb{P}(\mathcal{G}_K^{(M)} \geq t),$$
2.2.3 Proof of Theorem 2.2

We first prove Point (i)

We have:
\[
\mathbb{E} \left[ F_t (M_t - K)^+ \right] = \mathbb{E} \left[ F_t (M_t - K)^+ 1_{\{M_t > 0\}} \right] \quad \text{(since } (M_t - K)^+ = 0 \text{ on } \{M_t = 0\})
\]
\[
= \mathbb{E} \left[ F_t M_t \left( 1 - \frac{K}{M_t} \right)^+ 1_{\{M_t > 0\}} \right]
\]
\[
= \mathbb{E}^{(M)} \left[ F_t \left( 1 - \frac{K}{M_t} \right)^+ 1_{\{M_t > 0\}} \right] \quad \text{(from the definition of } \mathbb{P}^{(M)})
\]
\[
= \mathbb{E}^{(M)} \left[ F_t 1_{\{\mathcal{G}^{(1/M)}_{1/K} \leq t\}} \right]
\]
by applying Theorem 2.1 with $1/M$ and $1/K$ instead of $M$ and $K$, and since $\mathbb{P}^{(M)}(M_t > 0) = 1$ from (2.13). But, by its very definition, $\mathcal{G}^{(1/M)}_{1/K} = \mathcal{G}^{(M)}_K$, and therefore:
\[
\mathbb{E} \left[ F_t (M_t - K)^+ \right] = \mathbb{E}^{(M)} \left[ F_t 1_{\{\mathcal{G}^{(M)}_K \leq t\}} \right].
\]
This is Point (i).

We now prove Point (iii)

We have,
\[
\mathbb{E} \left[ (M_t - K)^+ \right] - \mathbb{E} \left[ (K - M_t)^+ \right] = \mathbb{E}[M_t - K] = 1 - K.
\]
Hence, from Point (ii) of Theorem 2.1 and Point (i) of Theorem 2.2:
\[
\mathbb{P}^{(M)} \left( \mathcal{G}^{(M)}_K \leq t \right) - K \mathbb{P} \left( \mathcal{G}^{(M)}_K \leq t \right) = 1 - K.
\]
This is Point (iii).

Finally, Point (ii) is an easy consequence of the identity:
\[
|M_t - K| = (M_t - K)^+ + (K - M_t)^+,
\]
applying once again Point (ii) of Theorem 2.1 and Point (i) of Theorem 2.2. □
2.3 Some Examples of Computations of the Law of $\mathcal{G}^{(M)}_K$

In fact, formula (2.16) can be improved in the following way: let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a Borel and locally integrable function, and $\Phi(x) := \int_0^x \phi(y)dy$. Then, for any $t \geq 0$ and $F_t \in \mathcal{F}_t$, one has:

$$
\mathbb{E}[F_t \Phi(M_t)] = \mathbb{E}^{(M)}[F_t \phi \left( \inf_{s \geq t} M_s \right)].
$$

(Of course, (2.16) is also a particular case of (2.20) with $\Phi(x) = (x - K)^+$ and $\phi(x) = 1_{\{x > K\}}$.)

We prove (2.20)

We have:

$$
\mathbb{E}[F_t \Phi(M_t)] = \mathbb{E} \left[ F_t \int_0^{M_t} \phi(y)dy \right]
= \mathbb{E} [F_t M_t \phi(U M_t)]
= \mathbb{E}^{(M)} [F_t \phi(U M_t)]
= \mathbb{E}^{(M)} \left[ F_t \phi \left( \frac{1}{UM_t} \right) \right]
= \mathbb{E}^{(M)} \left[ F_t \phi \left( \frac{1}{\sup_{s \geq t} M_s} \right) \right].
$$

(applying Lemma 2.1 to the $\mathbb{P}^{(M)}$-local martingale $(1/M_t, t \geq 0)$)

$$
= \mathbb{E}^{(M)} \left[ F_t \phi \left( \inf_{s \geq t} M_s \right) \right].
$$

2.3 Some Examples of Computations of the Law of $\mathcal{G}^{(M)}_K$

Example 2.3.a. We get back to the classical Black-Scholes formula, with $\mathcal{G}_t := \exp (B_t - \frac{t}{2})$ where $B$ is a Brownian motion started from 0. From (2.7), the identity:

$$
\mathbb{E} \left[ \left( 1 - \frac{\mathcal{G}_t}{K} \right)^+ \right] = \mathbb{P}(\mathcal{G}^{(\mathcal{G})}_K \leq t)
$$

holds. Taking $K = 1$, it suffices to obtain the identity:

$$
\mathcal{G}_1^{(law)} = 4B_1^2
$$

(2.21)
to recover formula (1.19). In fact, this identity (2.21) may be obtained simply by
time inversion, since:

$$G_1 := \sup\{t \geq 0; \varepsilon_t = 1\} = \sup\{t \geq 0; B_t - \frac{t}{2} = 0\},$$

hence, with the notation of Subsection 1.1.3:

$$G_1 = G^{-1/2}(0) = \frac{1}{T^{(0)}_{-1/2}} \quad \text{(from (1.9))}$$

$$= \frac{4}{T^{(0)}_1} \quad \text{(by scaling)}$$

$$= 4B^2_1.$$

**Example 2.3.b.** Here \((M_t, t \geq 0)\) is the martingale defined by: \((M_t = B_t \wedge T_0, t \geq 0)\) where \((B_t, t \geq 0)\) is a Brownian motion started from 1, and \(T_0 := \inf\{t \geq 0; B_t = 0\}\). Then, for \(K \leq 1\):

$$G^{(M)}_{K} \quad \text{(law)} = \frac{(U_K)^2}{N^2} \quad \text{(law)} = T_{UK},$$

where \((T_x, x \geq 0)\) is the first hitting time process of a Brownian motion \((\beta_t, t \geq 0)\) starting from 0, \(N\) is a standard Gaussian r.v. and \(U_K\) is uniform on \([1 - K, 1 + K]\), independent from \(T\) and \(N\).

**Proof of (2.22)**

Applying Williams’ time reversal Theorem (see [91]), we have:

$$\left(B_{T_0 - u}, u \leq T_0\right) \quad \text{(law)} = \left(R_u, u \leq G_1(R)\right) \quad \text{(2.23)}$$

where \((R_u, u \geq 0)\) is a Bessel process of dimension 3 starting from 0 and \(G_1(R) := \sup\{u \geq 0; R_u = 1\}\). Hence:

$$T_0 \quad \text{(law)} = T_K(R) + G^{(M)}_K,$$

where on the RHS, \(T_K(R)\) and \(G^{(M)}_K\) are independent and

$$T_K(R) := \inf\{u \geq 0; R_u = K\}.$$

Taking the Laplace transform in \(\frac{\lambda^2}{2}\) on both sides of (2.24) gives:

$$e^{-\lambda} = \frac{\lambda K}{\sinh(\lambda K)} \mathbb{E}\left[e^{-\frac{\lambda^2}{2}G^{(M)}_K}\right],$$

since \(\mathbb{E}\left[e^{-\frac{\lambda^2}{2}T_0}\right] = e^{-\lambda}\) and \(\mathbb{E}\left[e^{-\frac{\lambda^2}{2}T_K(R)}\right] = \frac{\lambda K}{\sinh(\lambda K)}\).
Then, formula (2.25) becomes:

\[ \mathbb{E} \left[ e^{-\frac{\lambda^2}{2} \varphi_K^M} \right] = \frac{e^{-\lambda(1-K)} - e^{-\lambda(1+K)}}{2\lambda K} = \frac{1}{2K} \int_{1-K}^{1+K} e^{-\lambda x} dx \]

Thus:

\[ \mathbb{E} \left[ e^{-\frac{\lambda^2}{2} \varphi_K^M} \right] = \frac{1}{2K} \int_{1-K}^{1+K} \mathbb{E} \left[ e^{-\frac{\lambda^2}{2} x^2} \right] dx = \mathbb{E} \left[ e^{-\frac{\lambda^2}{2} \left( \frac{U_K}{N} \right)^2} \right] \]

since \( T_x \overset{\text{law}}{=} \frac{x^2}{N^2} \). Hence,

\[ \varphi_K^M \overset{\text{(law)}}{=} \frac{(U_K)^2}{N^2} \overset{\text{(law)}}{=} T_{\tilde{U}K}. \]

**Example 2.3.c.** \((M_t, t \geq 0)\) is the (strict) local martingale defined by:

\[
\left( M_t := \frac{1}{R_t}, t \geq 0 \right) \text{ where } (R_t, t \geq 0) \text{ is a 3-dimensional Bessel process starting from 1. Then, for every } K < 1: \\
\varphi_K^M \overset{\text{(law)}}{=} \frac{(\tilde{U}_K)^2}{N^2} \overset{\text{(law)}}{=} T_{\tilde{U}K} \quad (2.26)
\]

where \((T_x, x \geq 0)\) is the first hitting time process of a Brownian motion \((\beta_t, t \geq 0)\) starting from 0, \(N\) is a standard Gaussian r.v. and \(\tilde{U}_K\) is a uniform r.v. on \([-1, 1]\), independent from the process \(T\) and \(N\).

**Proof of (2.26)**

We observe that \( \varphi_K^M := \sup \left\{ u \geq 0; \frac{1}{R_u} = K \right\} = \sup \left\{ u \geq 0; R_u = \frac{1}{K} \right\} \). We consider the process \(R\) as obtained by time reversal from the Brownian motion \((\beta_t, t \geq 0)\) starting from 1/K and killed when it first hits 0. Hence, with the same notation as in Example 2.3.b, we have:

\[ T_0 \overset{\text{law}}{=} \varphi_K^M + T_1(R), \quad (2.27) \]

where on the RHS, \( \varphi_K^M \) and \( T_1(R) \) are assumed to be independent. Taking once again the Laplace transform in \(\frac{\lambda^2}{2}\) of both sides, one obtains:

\[ e^{-\lambda/K} = \frac{\lambda}{\sinh(\lambda)} \mathbb{E} \left[ e^{-\frac{\lambda^2}{2} \varphi_K^M} \right], \quad (2.28) \]
which can be rewritten:

\[
\mathbb{E} \left[ e^{-\frac{1}{2} \mathcal{G}^{(M)}_1} \right] = \frac{e^{-\lambda \left( \frac{1}{\lambda} - 1 \right)} - e^{-\lambda \left( \frac{1}{\lambda} + 1 \right)}}{2\lambda} = \frac{1}{2} \int_{\frac{1}{\lambda} - 1}^{\frac{1}{\lambda} + 1} e^{-\lambda x} dx
\]

\[
= \frac{1}{2} \int_{\frac{1}{\lambda} - 1}^{\frac{1}{\lambda} + 1} \mathbb{E} \left[ e^{-\frac{1}{2} x} \right] dx
\]

\[
= \frac{1}{2} \int_{\frac{1}{\lambda} - 1}^{\frac{1}{\lambda} + 1} \mathbb{E} \left[ e^{-\frac{1}{2} x^2} \right] dx = \mathbb{E} \left[ e^{-\frac{1}{2} \left( \frac{1}{\lambda} \right)^2} \right].
\]

\[\square\]

**Example 2.3.d.** \((M_t, t \geq 0)\) is the martingale defined by:

\[
\left( M_t = |B_t| h(L_t) + \int_{L_t}^{\infty} h(x) dx, t \geq 0 \right)
\]

where \((B_t, t \geq 0)\) is a Brownian motion starting from 0, \((L_t, t \geq 0)\) its local time at level 0 and \(h : \mathbb{R}^+ \to \mathbb{R}^+\) a strictly positive Borel function such that \(\int_0^\infty h(x) dx = 1\). \((M_t, t \geq 0)\) is the Azéma-Yor martingale associated with \(h\) (see [4]). We then have:

\[
\mathcal{G}^{(M)}_1 \overset{(l.w.)}{=} \left( H^{-1}(U) + \frac{U}{h \circ H^{-1}(U)} \right)^2 \cdot T_1
\]

(2.29)

where, on the RHS, \(T_1\) and \(U\) are assumed to be independent, \(U\) is uniform on \([0, 1]\), \(T_1\) is the first hitting time of 1 by a Brownian motion starting from 0, and \(H(u) := \int_0^u h(y) dy\). In particular, if \(h(l) = \frac{\lambda}{2} e^{-\lambda l^2}\), we have:

\[
\mathcal{G}^{(M)}_1 \overset{(l.w.)}{=} T_1 \left( \log \left( \frac{1}{\lambda} \right) + \frac{1}{\lambda} - 1 \right)
\]

(2.30)

where \(T_x\) is the first hitting time of \(x\) by a Brownian motion starting from 0, and \(U\) is uniform on \([0, 1]\), independent from \((T_x, x \geq 0)\).

**Proof of (2.29)**

We use the fact that \(\mathcal{G}^{(M)}_1\) has the same distribution under \(\mathbb{P}\) and \(\mathbb{P}^{(M)}\) (see (2.19)). The law of the canonical process \((X_t, t \geq 0)\) under \(\mathbb{P}^{(M)}\) is fully described in [72]. In particular, under \(\mathbb{P}^{(M)}\):

- \(L_\infty < \infty\) \(\mathbb{P}^{(M)}\)-a.s., and admits \(h\) as density function,
- Conditionally on \(\{L_\infty = l\}\), \((X_t, t \leq \tau_l)\) is a Brownian motion stopped at \(\tau_l\) (with \(\tau_l := \inf \{t \geq 0; L_t > l\}\)), independent from the process \((X_{\tau_l + t}, t \geq 0)\), and \((|X_{\tau_l + t}|, t \geq 0)\) is a 3-dimensional Bessel process started at 0.

Then, under \(\mathbb{P}^{(M)}\), conditionally on \(\{L_\infty = l\}\):

\[
\mathcal{G}^{(M)}_1 = \tau_l + \sup \left\{ t \geq 0; \int_t^\infty h(x) dx + h(l)|X_t| = 1 \right\}
\]

\[
= \tau_l + \sup \left\{ t \geq 0; |X_t| = \frac{\int_t^1 h(x) dx}{h(l)} \right\}.
\]
2.3 Some Examples of Computations of the Law of $\mathcal{G}^{(M)}_k$

Denoting $k(l) = \frac{1}{h(l)}H(l)$ we have, by time reversal (see Example 2.3.b and 2.3.c above):

$$\mathcal{G}^{(M)}_1 \overset{\text{law}}{=} \tau + T_{k(l)} \quad (2.31)$$

where $T_x$ is the first hitting time at level $x$ of a Brownian motion started from 0, and where, on the RHS, $\tau$ and $T_{k(l)}$ are assumed to be independent. But, $(\tau_x, x \geq 0)$ and $(T_x, x \geq 0)$ being independent and having the same law, we have:

$$\mathcal{G}^{(M)}_1 \overset{\text{law}}{=} T_{k(l)} \quad (2.32)$$

Then, since $L_\infty$ admits $h$ as density, $H(L_\infty)$ is uniformly distributed on $[0, 1]$. Hence:

$$\mathcal{G}^{(M)}_1 \overset{\text{law}}{=} T_{H^{-1}(U) + \frac{U}{h \circ H^{-1}(U)}} \quad \text{(by scaling).}$$

\[ \Box \]

Example 2.3.e. We end this series of examples by examining a situation which is no longer in the scope of Theorem 2.1 or Theorem 2.2: we shall compute the price of a call where $(M_t, t \geq 0)$ is only a local martingale, and not a (true) martingale. (In other words, we remove assumption (T)). More precisely, let $(X_t, t \geq 0, \mathbb{P}^{(3)}, a > 0)$ the canonical Bessel process of dimension 3, defined on the space $\Omega := C(\mathbb{R}^+, \mathbb{R}^+)$. Let $(M_t, t \geq 0)$ be the local martingale $(M_t = \frac{1}{X_t}, t \geq 0)$. Then:

$$\mathbb{E}^{(3)} \left[ F_t \left( \frac{1}{X_t} - 1 \right)^+ \right] = W_1 \left[ F_t 1_{\{\gamma \leq t \leq T_0\}} \right] \quad (2.33)$$

where $F_t$ is a generic bounded $\mathcal{F}_t$-measurable r.v., $W_1$ is the Wiener measure (with $W_1(X_0 = 1) = 1$), $T_0 := \inf\{t \geq 0; X_t = 0\}$ and $\gamma := \sup\{t < T_0; X_t = 1\}$. In particular:

$$\mathbb{E}^{(3)} \left[ \left( \frac{1}{X_t} - 1 \right)^+ \right] = W_1(\gamma \leq t \leq T_0) \quad (2.34)$$

Proof of (2.33)

From the well-known Doob’s $h$-transform relationship:

$$\mathbb{P}^{(3)}_{a} = \frac{X_{T_0}}{a} \cdot W_{a} \quad (2.35)$$

we have:
by applying Theorem 2.1, relation (2.6), with $K = 1$ and $M_t = X_t \wedge T_0$.

We observe that the LHS of (2.34) is not an increasing function of $t$. Indeed, the RHS converges to 0 as $t \to \infty$ as a consequence of Lebesgue’s dominated convergence Theorem. In fact, we can compute explicitly this RHS, which equals:

$$r(t) = W_1(T_0 \geq t) - W_1(\gamma \geq t) \quad \text{(since, by definition, } T_0 \geq \gamma).$$

Recall that, under $W_1$: $T_0 \overset{\text{(law)}}{=} \frac{1}{B_1^2}$ and, from Example 2.3.b with $K = 1$, $\gamma \overset{\text{(law)}}{=} \frac{(U_1)^2}{B_1^2}$ where $U_1$ is uniform on $[0, 2]$ and independent of $B_1^2$. Therefore:

$$r(t) = \mathbb{P}\left(|B_1| \leq \frac{1}{\sqrt{t}}\right) - \mathbb{P}\left(|B_1| \leq \frac{U_1}{\sqrt{t}}\right),$$

that is:

$$r(t) = \sqrt{\frac{2}{\pi}} \int_0^{1/\sqrt{t}} e^{-x^2/2} dx - \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2/2} \left(1 - \frac{x\sqrt{t}}{2}\right)^+ dx.$$  

(2.37)

In particular, $r$ starts to increase, reaches its overall maximum, and then decreases. Moreover, it easily follows from (2.37) that:

$$r(t) \sim \frac{1}{6} \sqrt{\frac{2}{\pi t}}.$$  

(2.38)

It is also easily proven that:

$$r(t) \sim \sqrt{\frac{t}{2\pi}}.$$  

(2.39)

This example will be taken up and developed in Section A.1 of the Complements.

2.4 A More General Formula for the Computation of the Law of $G_K^{(M)}$

2.4.1 Hypotheses

Let $(M_t, t \geq 0) \in \mathcal{M}_{+0}^c$. (See Section 2.1). Our purpose in this Section is to give a general formula for the law of $G_K^{(M)}$. To proceed, we add extra hypotheses on $M$:
2.4 A More General Formula for the Computation of the Law of $\mathcal{G}_K^{(M)}$

i) For every $t > 0$, the law of the r.v. $M_t$ admits a density $(m_t(x), x \geq 0)$, and $(t,x) \mapsto m_t(x)$ may be chosen continuous on $(]0, +\infty[)^2$.

ii) Let us denote by $(\langle M \rangle_t, t \geq 0)$ the increasing process of $(M_t, t \geq 0)$. We suppose that $d \langle M \rangle_t = \sigma_t^2 dt$, and that there exists a jointly continuous function:

$$(t,x) \mapsto \theta_t(x) := \mathbb{E} [\sigma_t^2 | M_t = x] \quad \text{on } (]0, +\infty[)^2.$$  \hspace{1cm} (2.40)

2.4.2 Description of the Law of $\mathcal{G}_K^{(M)}$

Theorem 2.3. Under the preceding hypotheses, the law of $\mathcal{G}_K^{(M)}$ is given by:

$$\mathbb{P} \left( \mathcal{G}_K^{(M)} \in dt \right) = \left( 1 - \frac{a}{K} \right)^+ \delta_0(dt) + \frac{1}{2K} \theta_t(K) m_t(K) 1_{\{t > 0\}} dt$$ \hspace{1cm} (2.41)

where, in (2.41), $a = M_0$ and $\delta_0$ denotes the Dirac measure at 0.

Proof. Using Tanaka’s formula, one obtains:

$$\mathbb{E} \left[ \langle (K-M_t)^+ \rangle \right] = (K-a)^+ + \frac{1}{2} \mathbb{E} [L^K_t(M)]$$ \hspace{1cm} (2.42)

where $(L^K_t(M), t \geq 0, K \geq 0)$ denotes the bicontinuous family of local times of the martingale $(M_t, t \geq 0)$. Thus, from Theorem 2.1, there is the relationship:

$$\mathbb{P} \left( \mathcal{G}_K^{(M)} \in dt \right) = \left( 1 - \frac{a}{K} \right)^+ \delta_0(dt) + \frac{1}{2K} d_t \mathbb{E} [L^K_t(M)]$$ \hspace{1cm} (2.43)

and formula (2.41) is now equivalent to the following expression for $d_t \mathbb{E} [L^K_t(M)]$:

$$d_t \mathbb{E} [L^K_t(M)] = \theta_t(K) m_t(K) dt \quad (t > 0).$$ \hspace{1cm} (2.44)

We now prove (2.44).

For every $f : \mathbb{R}^+ \to \mathbb{R}^+$ Borel, the density of occupation formula

$$\int_0^t f(M_s) d \langle M \rangle_s = \int_0^\infty f(K) L^K_t dK$$ \hspace{1cm} (2.45)

becomes, under hypothesis (2.40),

$$\int_0^t f(M_s) \sigma_s^2 ds = \int_0^\infty f(K) L^K_t dK.$$ \hspace{1cm} (2.46)

Thus, taking expectation on both sides of (2.46), we obtain:

$$\mathbb{E} \left[ \int_0^t f(M_s) \sigma_s^2 ds \right] = \int_0^\infty f(K) \mathbb{E} [L^K_t] dK.$$ \hspace{1cm} (2.47)
and, the LHS of (2.47), thanks to (2.40), equals:

\[
\int_0^t \mathbb{E} \left[ f(M_s) \sigma_s^2 \right] ds = \int_0^t \mathbb{E} \left[ f(M_s) \mathbb{E} \left[ \sigma_s^2 M_s \right] \right] ds
\]

\[
= \int_0^\infty f(K) dK \int_0^t \theta_s(K) m_s(K) ds.
\]

(2.48)

Comparing (2.47) and (2.48), we obtain:

\[
\mathbb{E} \left[ L^K_t (M) \right] = \int_0^t \theta_s(K) m_s(K) ds,
\]

and Theorem 2.3 is proven.

\[\Box\]

2.4.3 Some Examples of Applications of Theorem 2.3

Example 2.4.3.a. Here, \( (M_t := \mathcal{E}_t = \exp \left( B_t - \frac{t^2}{2} \right), t \geq 0) \) where \( (B_t, t \geq 0) \) is a Brownian motion started at 0. From Itô’s formula, \( \mathcal{E}_t = 1 + \int_0^t \mathcal{E}_s dB_s \), thus \( d \langle \mathcal{E} \rangle_t = \mathcal{E}_t^2 dt \) and we may apply Theorem 2.3 with \( \theta_t(x) = x^2 \) and \( m_t(x) = \frac{1}{x \sqrt{2\pi t}} \exp \left( -\frac{1}{2t} \left( \log(x) + \frac{t}{2} \right)^2 \right) \). We obtain:

\[
\mathbb{P} \left( \mathcal{G}_K^x \in dt \right) = \left( 1 - \frac{1}{K} \right)^+ \delta_0(dt) + \frac{1_{\{t > 0\}}}{2\sqrt{2\pi t}} \exp \left( -\frac{1}{2t} \left( \log(K) + \frac{t}{2} \right)^2 \right) dt
\]

(2.49)

This formula (2.49) agrees with formulae (1.10) and (1.14) since \( \mathcal{G}_K^x \) (law) = \( G_{\log(K)}^{-1/2} \).

Example 2.4.3.b. Let \( (M_t, 0 \leq t < 1) \) be the martingale defined by:

\[
\left( M_t = \frac{1}{\sqrt{1-t}} \exp \left( -\frac{B_t^2}{2(1-t)} \right) , t < 1 \right).
\]

This martingale is the Girsanov density of the law of the Brownian bridge \( (b_u, 0 \leq u < 1) \) with respect to the Wiener measure on the \( \sigma \)-field \( (\mathcal{F}_t) \) (see Exercise 2.2).

We have here:

\[
m_t(x) = \frac{1}{\sqrt{2\pi t}} \frac{2(1-t)}{x} \frac{1}{\sqrt{\Delta(x)}} \exp \left( -\frac{\Delta(x)}{x} \right) \mathbb{1}_{\{x < \frac{1}{\sqrt{1-t}}\}}
\]

(2.50)

with \( \Delta(x) := -2(1-t) \log \left( x \sqrt{1-t} \right) \) and:
\[ \theta_t(x) = \frac{x^2}{(1-t)^2} \Delta(x). \] (2.51)

Hence, from Theorem 2.3:

\[
P \left( G_k^{(M)} \in dt \right) = \left( 1 - \frac{1}{K} \right)^+ \delta_0(dt) + \frac{\sqrt{\Delta(K)}}{(1-t)\sqrt{2\pi t}} e^{-\frac{\Delta(K)}{2}} 1_{\left\{ K < \frac{1}{\sqrt{1-t}} \right\}} 1_{\{ t < 1 \}} dt
\] (2.52)

**Example 2.4.3.c.** Here, \( (M_t, t \geq 0) \) is the martingale defined by:

\[ M_t = \cosh(B_t)e^{-t/2}, t \geq 0 \]. We have from Itô’s formula: \( \sigma_t = \sinh(B_t)e^{-t/2} \), hence \( \sigma_t^2 = M_t^2 - e^{-t} \) and \( \theta_t(x) = x^2 - e^{-t} \). On the other hand:

\[
m_t(x) = \sqrt{\frac{2}{\pi t}} \exp \left( -\frac{1}{2t} \left( \text{Argcosh}(xe^{t/2}) \right)^2 \right) \frac{1}{\sqrt{x^2 - e^{-t}}} 1_{\{ x > e^{-t/2} \}}.
\] (2.53)

Hence, from Theorem 2.3:

\[
P \left( G_k^{(M)} \in dt \right) = \left( 1 - \frac{1}{K} \right)^+ \delta_0(dt) + \frac{1}{2K}(K^2 - e^{-t})m_t(K) 1_{\{ t > 0 \}} dt,
\]

where \( m_t(K) \) is given by (2.53).

**Example 2.4.3.d.** We consider Feller’s martingale, i.e. the solution of the stochastic equation:

\[ M_t = l + 2 \int_0^t \sqrt{M_s} dB_s. \]

\( (M_t, t \geq 0) \) is a square Bessel process of dimension 0 started from \( l \). From Itô’s formula, \( \theta_t(x) = 4x \), and it is known that the law of the r.v. \( M_t \) is given by:

\[ q_t^0(l, dK) = \exp(-l/2t) \delta_0(dK) + \frac{1}{2t} \sqrt{\frac{t}{K}} \exp \left( -\frac{l+K}{2t} \right) I_1 \left( \sqrt{\frac{l}{t}} \right) dK,
\]

where \( I_1 \) is the modified Bessel function of index 1 (see Appendix B.1). Hence, from Theorem 2.3:

\[
P \left( G_k^{(M)} \in dt \right) = \left( 1 - \frac{l}{K} \right)^+ \delta_0(dt) + \frac{1}{t} \sqrt{\frac{t}{K}} \exp \left( -\frac{l+K}{2t} \right) I_1 \left( \sqrt{\frac{l}{t}} \right) 1_{\{ t > 0 \}} dt.
\] (2.54)
Exercise 2.2 (Girsanov’s density of the Brownian bridge with respect to the Brownian motion).

Let \((B_t, 0 \leq t \leq 1)\) denote the canonical Brownian motion started at 0, \(\mathbb{P}\) the Wiener measure, and \((b_t, 0 \leq t \leq 1)\) the standard Brownian bridge (with \(b_0 = b_1 = 0\)). \((b_t, 0 \leq t < 1)\) is the strong solution of the SDE:

\[
\begin{cases}
    db_t = -\frac{b_t}{1-t} dt + dB_t \\
    b_0 = 0
\end{cases}
\quad (0 \leq t < 1).
\]

Let \(\Pi\) be the law on \(\mathcal{C}([0,1], \mathbb{R})\) of \((b_t, 0 \leq t \leq 1)\).

1) Prove, applying Girsanov’s Theorem, that:

\[
\Pi |_{\mathcal{F}_t} = M_t \cdot \mathbb{P} |_{\mathcal{F}_t} \quad (0 \leq t < 1)
\]

with

\[
M_t := \exp \left( - \int_0^t B_s dB_s - \frac{1}{2} \int_0^t \frac{B_s^2 ds}{(1-s)^2} \right), \quad 0 \leq t < 1.
\]

2) Use Itô’s formula to show that:

\[
M_t = \frac{1}{\sqrt{1-t}} \exp \left( - \frac{B_t^2}{2(1-t)} \right), \quad 0 \leq t < 1.
\]

Show that \(M_1 = 0\) a.s.

3) In the case of the \(n\)-dimensional Brownian motion in \(\mathbb{R}^n\), prove that the analogous martingale writes:

\[
M_t^{(n)} = \frac{1}{(1-t)^{n/2}} \exp \left( - \frac{\|B_t\|^2}{2(1-t)} \right), \quad 0 \leq t < 1
\]

where \(\|B_t\|^2 = (B_1^t)^2 + \ldots + (B_n^t)^2\).

4) More generally, let \((X_t, t \geq 0; \mathcal{F}_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R})\) be the canonical realization of a regular diffusion on \(\mathbb{R}\). We denote by \((P_t, t \geq 0)\) and \((p_t(x,y); t > 0, x, y \in \mathbb{R})\) the associated semi-group and its density kernel (with respect to the Lebesgue measure), which we assumed to be regular.

Let \(l > 0\). We denote by \(\Pi_{x \to y}^{(l)}\) the law, on \(\mathcal{C}([0,l], \mathbb{R})\) of the bridge of length \(l\) \((x_u, 0 \leq u \leq l)\) such that \(x_0 = x, x_l = y\). Let \(0 \leq t < l\). For every \(F : \mathcal{C}([0,l], \mathbb{R}) \to \mathbb{R}\) bounded and measurable and every \(f : \mathbb{R} \to \mathbb{R}\) Borel and bounded, we have:

\[
\mathbb{E}_x[F(X_u, u \leq t) f(X_t)] = \int_{\mathbb{R}} \mathbb{E}_x[F(X_u, u \leq t) | X_l = y] f(y) p_t(x,y) dy.
\]

i) Prove that:

\[
\mathbb{E}_x[F(X_u, u \leq t) f(X_t)] = \mathbb{E}_x[F(X_u, u \leq t) P_{l-t} f(X_t)].
\]
2.5 Computation of the Law of $G_K$ in the Framework of Transient Diffusions

ii) Deduce that:
$$\mathbb{E}_x [F(X_{u}, u \leq t) | X_t = y] = \mathbb{E}_x \left[ F(X_{u}, u \leq t) \frac{p_{l-t}(X_t, y)}{p_l(x, y)} \right]$$

and
$$\Pi_{x \to y}^{(l)} = M_t \cdot \mathbb{P}_x | \mathcal{F}_t \quad \text{with} \quad M_t := \frac{p_{l-t}(X_t, y)}{p_l(x, y)}. \quad (2)$$

Prove that (1) is a particular case of (2).

Comment: Relation (2) makes it possible to derive the expression of the bridge of a diffusion as the solution of a SDE, thanks to Girsanov’s Theorem.

2.5 Computation of the Law of $G_K$ in the Framework of Transient Diffusions

2.5.1 General Framework

Theorem 2.1 and Theorem 2.2 cast some light on our ability to compute explicitly the law of $G_K(M)$ when $M$ is a positive (local) martingale. We temporarily leave this framework and give (following Pitman-Yor, [65]) a general formula for the law of $G_K(X)$ when $(X_t, t \geq 0)$ is a transient diffusion taking values in $\mathbb{R}^+$. We consider the canonical realization of a transient diffusion $(X_t, t \geq 0; \mathbb{P}_x, x > 0)$ on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ (See [12]). For simplicity, we assume that:

i) $\mathbb{P}_x (T_0 < \infty) = 0$ for every $x > 0$, with $T_0 := \inf\{t \geq 0; X_t = 0\}$.

ii) $\mathbb{P}_x \left( \lim_{t \to \infty} X_t = +\infty \right) = 1, x > 0$.

As a consequence of (i) and (ii), there exists a scale function $s$ for this diffusion which satisfies:

$$s(0^+) = -\infty, \quad s(+\infty) = 0 \quad (s \text{ is increasing}). \quad (2.55)$$

Let $\Gamma$ be the infinitesimal generator of $(X_t, t \geq 0)$, and take the speed measure $m$ to be such that:

$$\Gamma = \frac{\partial}{\partial m} \frac{\partial}{\partial s}. \quad (2.56)$$

Let, for $K > 0$:

$$\mathcal{G}_K^{(X)} := \sup\{t \geq 0; X_t = K\}. \quad (2.57)$$

Let us also denote by $q(t, x, y) \ (= q(t, y, x))$ the density of the r.v. $X_t$ under $\mathbb{P}_x$, with respect to $m$; thus

$$\mathbb{P}_x (X_t \in A) = \int_A q(t, x, y)m(dy), \quad (2.58)$$

for every Borel set $A$. 

2.5.2 A General Formula for the Law of $G^X_K$

**Theorem 2.4 (Pitman-Yor, [65]).** For all $x, K > 0$:

$$\mathbb{P}_x \left( G^X_K \in dt \right) = \left( 1 - \frac{s(x)}{s(K)} \right)^+ \delta_0(dt) - \frac{1}{s(K)} q(t, x, K) dt \quad (2.59)$$

where $\delta_0(dt)$ denotes the Dirac measure at 0.

In particular, if $K \geq x$, since $s$ is increasing and negative, formula (2.59) reduces to:

$$\mathbb{P}_x \left( G^X_K \in dt \right) = -\frac{1}{s(K)} q(t, x, K) dt.$$  

**Proof.** We apply Theorem 2.1 to the local martingale $(M_t = -s(X_t), t \geq 0)$. This leads to:

$$\mathbb{P}_x \left( G^X_K \leq t \right) = \mathbb{P}_x \left( G^{-s(X)}_K \leq t \right) = \mathbb{E} \left[ \left( 1 - \frac{s(X_t)}{s(K)} \right)^+ \right]. \quad (2.60)$$

As we apply Tanaka’s formula to the submartingale $\left( 1 - \frac{s(X_t)}{s(K)} \right)^+$, we obtain:

$$\mathbb{P}_x \left( G^X_K \leq t \right) = \left( 1 - \frac{s(x)}{s(K)} \right)^+ - \frac{1}{2s(K)} \mathbb{E}_x \left[ L^{-s(K)}_t(M) \right] \quad (2.61)$$

where $(L^a_t(M), t \geq 0)$ denotes the local time of the local martingale $(M_t, t \geq 0)$ at level $a$.

We now prove:

$$\frac{\partial}{\partial t} \mathbb{E}_x \left[ L^{-s(K)}_t(M) \right] = 2q(t, x, K) \quad (2.62)$$

which obviously, together with (2.61), implies Theorem 2.4. In fact, (2.62) follows from the density of occupation formula for our diffusion $(X_t, t \geq 0)$:

for any $f : \mathbb{R}^+ \to \mathbb{R}^+$, Borel,

$$\int_0^t f(X_s) ds = \int_0^\infty f(K) t^K m(dK) \quad (2.63)$$

where $(l^K_t, t \geq 0, a > 0)$ is the family of diffusion local times. (See [12]). On the LHS of (2.63), taking the expectation, we have:

$$\mathbb{E}_x \left[ \int_0^t f(X_s) ds \right] = \int_0^\infty f(K) \left( \int_0^t q(s, x, K) ds \right) m(dK). \quad (2.64)$$

Thus, (2.63) and (2.64) imply:
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\[ \mathbb{E}_x \left[ t^K_i \right] = \int_0^t q(s, x, K) ds. \]  

(2.65)

On the other hand, there is the following relationship between the diffusion and martingale local times:

\[ 2l^K_t = L^{-s(K)}(M). \]  

(2.66)

Hence, from (2.61), (2.65) and (2.66):

\[
\frac{\partial}{\partial t} \mathbb{P}_x \left( G_K^{(X)} \leq t \right) = \left( 1 - \frac{s(x)}{s(K)} \right)^+ \delta_0 - \frac{1}{2s(K)} \frac{\partial}{\partial t} \left( \mathbb{E}_x \left[ L^{-s(K)}(M) \right] \right) \\
= \left( 1 - \frac{s(x)}{s(K)} \right)^+ \delta_0 - \frac{1}{s(K)} \frac{\partial}{\partial t} \left( \mathbb{E}_x \left[ l^K_t \right] \right) \\
= \left( 1 - \frac{s(x)}{s(K)} \right)^+ \delta_0 - \frac{1}{s(K)} q(t, x, K),
\]

where $\delta_0$ is the derivative of the Heaviside step function, i.e. the Dirac measure at 0.

Remark 2.2. Formula (2.59) still holds in the more general framework of a transient diffusion $(X_t, t \geq 0)$ taking values in $\mathbb{R}$, such that for example $X_t \xrightarrow{t \to \infty} +\infty$ a.s. Indeed, introducing $(\overline{X}_t := \exp(X_t), t \geq 0)$, it is easily seen that $\overline{X}$ satisfies the hypotheses of Subsection 2.5.1, and therefore Theorem 2.4 applies (with obvious notation):

\[ \mathbb{P}_x \left( G_K^{(\overline{X})} \in dt \right) = \left( 1 - \frac{s(\overline{X})}{s(K)} \right)^+ \delta_0(dt) - \frac{1}{s(K)} \overline{q}(t, \overline{x}, K) dt. \]

Then, using the identities: $G_K^{(\overline{X})} = G_K^{(X)}$, $\overline{s}(\overline{x}) = s(\log(\overline{x})) = s(x)$ and $\overline{q}(t, \overline{x}, \overline{y}) = q(t, \log(x), \log(y)) = q(t, x, y)$, we obtain:

\[ \mathbb{P}_x \left( G_K^{(X)} \in dt \right) = \left( 1 - \frac{s(x)}{s(\log(K))} \right)^+ \delta_0(dt) - \frac{1}{s(\log(K))} q(t, x, \log(K)) dt \]

which is (2.59).

2.5.3 Case Where the Infinitesimal Generator is Given by its Diffusion Coefficient and its Drift

In practice, it may be useful to write formula (2.59) in terms of the density $p(t, x, y)$ of the r.v. $X_t$ with respect to the Lebesgue measure $dy$ (and not $m(dy)$). We shall give
this new expression in the following. Let us assume that the infinitesimal generator \( \Gamma \) is of the form:
\[
\Gamma = \frac{1}{2} a(x) \frac{\partial^2}{\partial x^2} + b(x) \frac{\partial}{\partial x}.
\]
(2.67)
Consequently:
\[
\frac{dm(y)}{dy} = \frac{2}{s'(y)a(y)}
\]
(2.68)
and
\[
q(t,x,y) = \frac{1}{2} p(t,x,y)s'(y)a(y)
\]
(2.69)
so that formula (2.59) becomes:
\[
P_x \left( G^x_K \in dt \right) = \left( 1 - \frac{s(x)}{s(K)} \right)^+ \delta_0(dt) - \frac{s'(K)a(K)}{2s(K)} p(t,x,K) dt.
\]
(2.70)
We now give several examples of application of Theorem 2.4, and of relation (2.70).

- Let us go back to Example 2.4.3.d where we computed \( \mathbb{P} \left( G^M_K \in dt \right) \) for \((M_t, t \geq 0)\) a square Bessel process of dimension 0 started from \(l\). We then define the diffusion \((X_t, t \leq G^x_K)\) to be the time reversed process of the martingale \((M_t, t \leq T_0)\). \((X_t, t \geq 0)\) is a square Bessel process of dimension 4 started from 0 and therefore satisfies the hypotheses of Theorem 2.4. In this set-up, we have:
\[
s(x) = -\frac{1}{x}, \quad a(x) = 4x,
\]
and
\[
p(t,x,y) = \frac{1}{2t} \sqrt{\frac{y}{x}} \exp \left( -\frac{x + y}{2t} \right) I_1 \left( \frac{\sqrt{xy}}{t} \right).
\]
Now, applying the Markov property, we see that the law of \( G^M_K \) under \( \mathbb{P} \) is the same as the law of \( G^x_K \) under \( \mathbb{Q}^4_K \), where \( \mathbb{Q}^4_K \) denotes the law of a square Bessel process of dimension 4 started at \(K\). Then, relation (2.70) yields:
\[
\mathbb{P} \left( G^M_K \in dt \right) = \mathbb{Q}^4_K \left( G^x_K \in dt \right)
= \left( 1 - \frac{s(K)}{s(l)} \right)^+ \delta_0(dt) - \frac{s'(l)a(l)}{2s(l)} p(t,K,l) dt
= \left( 1 - \frac{l}{K} \right)^+ \delta_0(dt) + \frac{1}{t} \sqrt{\frac{K}{l}} \exp \left( -\frac{l + K}{2t} \right) I_1 \left( \frac{\sqrt{KL}}{t} \right) dt,
\]
which is (2.54).
2.6 Computation of the Put Associated to a Càdlàg Martingale Without Positive Jumps

- If \( X_t = B_t + \nu t \) (i.e. \( a = 1 \) and \( b = \nu \)), we have: \( s'(x) = e^{-2\nu x} \),

\[
p(t,0,K) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2t} (K - \nu t)^2 \right) \quad \text{and, from (2.70), for } \nu \text{ and } K > 0:
\]

\[
P_0 \left( G_{K}^{(X)} \in dt \right) = \frac{\nu}{\sqrt{2\pi t}} \exp \left( -\frac{1}{2t} (K - \nu t)^2 \right) dt,
\]

and this formula agrees with (1.12).

- If \((X_t, t \geq 0)\) is a transient Bessel process, i.e. if \( a = 1 \) and \( b(x) = \frac{2\nu + 1}{2x} \), with index \( \nu > 0 \) (i.e. with dimension \( d = 2\nu + 2 > 2 \)), we have: \( s(x) = -x^{-2\nu} \) and

\[
p(t,0,K) = \frac{2^{-\nu}}{\Gamma(\nu+1)} t^{-(\nu+1)} K^{2\nu+1} \exp \left( -\frac{K^2}{2t} \right) .
\]

Hence, from (2.70):

\[
P^{(\nu)}_0 \left( G_{K}^{(X)} \in dt \right) = \frac{\nu^{2-\nu}}{\Gamma(\nu+1)} \frac{1}{K^{\nu+1}} \frac{K^{2\nu+1}}{t^{\nu+1}} \exp \left( -\frac{K^2}{2t} \right) dt
\]

\[
= \frac{2^{-\nu}}{\Gamma(\nu)} \frac{K^{2\nu}}{t^{\nu+1}} \exp \left( -\frac{K^2}{2t} \right) dt. \quad (2.71)
\]

Note that, by time reversal, we recover Getoor’s result (see [25]):

\[
P^{(\nu)}_0 \left( G_{K}^{(X)} \in dt \right) = P^{(-\nu)}_K \left( T_0 \in dt \right) = \mathbb{P} \left( K^{2} \gamma_{\nu} \in dt \right) \quad (2.72)
\]

where \( P^{(-\nu)}_K \) denotes the law of a Bessel process of index \(-\nu\) for \( 0 < \nu < 1 \) (i.e. of dimension \( \delta = 4 - d \) for \( 2 < d < 4 \)) started at \( K \) and where \( \gamma_{\nu} \) is a gamma r.v. with parameter \( \nu \). See also Problem 4.1 in the present monograph, and e.g., [98], Paper # 1, for some closely related computations and references.

2.6 Computation of the Put Associated to a Càdlàg Martingale Without Positive Jumps

2.6.1 Notation

Let \((B_t, t \geq 0)\) be a Brownian motion started from 0, and \( \nu > 0 \). As in Chapter 1, we denote by \((B_t^{(\nu)} := B_t + \nu t, t \geq 0)\) the Brownian motion with drift \( \nu \) and \( T^{(\nu)}_a := \inf \{ t \geq 0; B_t + \nu t = a \} \). We have:

\[
\mathbb{E} \left[ e^{-\frac{t^2}{2} T^{(\nu)}_a} \right] = \exp \left( -a \left( \sqrt{\nu^2 + \lambda^2} - \nu \right) \right) \quad (2.73)
\]
and in particular, for $\lambda^2 = 1 + 2\nu$:

$$\mathbb{E} \left[ e^{-\left( \frac{1}{2} + \nu \right) T_a^{(\nu)}} \right] = \exp(-a) \quad (a \geq 0). \quad (2.74)$$

It follows from the fact that $(B_t^{(\nu)}, t \geq 0)$ is a Lévy process (which implies $T_a^{(\nu)} + T_b^{(\nu)}$ with $T_a^{(\nu)}$ and $T_b^{(\nu)}$ independent), together with (2.74) that:

$$\left( M_a^{(\nu)} := \exp \left\{ a - \left( \frac{1}{2} + \nu \right) T_a^{(\nu)} \right\}, \; a \geq 0 \right)$$

is a martingale. \quad (2.75)

In fact, this is a positive martingale, without positive jumps, and such that

$$\lim_{a \to \infty} M_a^{(\nu)} = 0 \; \text{a.s.} \quad (2.76)$$

Indeed, from the law of large numbers:

$$\frac{B_t + \nu t}{\left( \frac{1}{2} + \nu \right) t} \; \xrightarrow{t \to \infty} \; \frac{\nu}{\frac{1}{2} + \nu} \; \text{a.s.}$$

Hence, since $T_a^{(\nu)} \to \infty$ a.s.:

$$\frac{B_{T_a^{(\nu)}} + \nu T_a^{(\nu)}}{\left( \frac{1}{2} + \nu \right) T_a^{(\nu)}} = \frac{a}{\left( \frac{1}{2} + \nu \right) T_a^{(\nu)}} \; \xrightarrow{a \to \infty} \; \frac{\nu}{\frac{1}{2} + \nu} < 1 \; \text{a.s.}$$

From (2.75), this implies that $M_a^{(\nu)} \to 0$ a.s.

### 2.6.2 Computation of the Put Associated to the Martingale

$\left( M_a^{(\nu)}, a \geq 0 \right)$

**Proposition 2.1.** Let $K > 0$.

i) If $K < e^a$:

$$\mathbb{E} \left[ (K - M_a^{(\nu)})^+ \right] = K \mathbb{E} \left[ (\xi_t - e^{2a\nu})^+ \right] - \mathbb{E} \left[ (\xi_t - e^{2a(\nu+1)})^+ \right], \quad (2.77)$$

with $t = t(a, \nu, K) = \frac{2a^2(2\nu + 1)}{a - \log(K)}$. \quad (2.78)

ii) If $K \geq e^a$:

$$\mathbb{E} \left[ (K - M_a^{(\nu)})^+ \right] = K - 1. \quad (2.79)$$
Proof. (2.79) is obvious. We prove (2.77):

\[ E \left[ (K - M_\nu^{(\nu)})^+ \right] = \int_{a - \log(K)}^{\infty} \left( K - e^{a - (\frac{1}{2} + \nu)u} \right) \frac{a}{\sqrt{2\pi u^3}} \exp \left( - \frac{(a - \nu u)^2}{2u} \right) du \]

(from the explicit formula for the density of $T_a^{(\nu)}$ given by (1.11))

\[ = e^{\nu} \int_0^{\frac{a^2(2\nu + 1)}{a - \log(K)}} \left( K - e^{a - (\frac{1}{2} + \nu)\frac{a^2}{2}} \right) \frac{e^{\nu}}{\sqrt{2\pi u^3}} \exp \left( - \frac{s - a^2\nu^2}{2s} \right) ds \]

(after the change of variable $\frac{a^2}{u} = s$)

\[ = Ke^{\nu} \mathbb{E} \left[ \left\{ B_1^{(\nu+1)} \leq \frac{a^2(2\nu + 1)}{2(a - \log(K))} \right\} \frac{a^3\nu}{2B_1^{(\nu+1)}} \right] - e^{\nu(\nu+1)} \mathbb{E} \left[ \left\{ B_1^{(\nu+1)} \leq \frac{a^2(2\nu + 1)}{2(a - \log(K))} \right\} \frac{-a^2(\nu+1)^2}{2B_1^{(\nu+1)}} \right], \quad (2.80) \]

i.e., with $t = \frac{2a^2(2\nu + 1)}{a - \log(K)}$, $A = e^{2a\nu}$ and $B = e^{2a(\nu+1)}$:

\[ \mathbb{E} \left[ (K - M_\nu^{(\nu)})^+ \right] = K \sqrt{A} \mathbb{E} \left[ \left\{ 4B_1^{(\nu+1)} \leq t \right\} e^{-\frac{\log^2(A)}{8B_1^{(\nu+1)}}} \right] - \sqrt{B} \mathbb{E} \left[ \left\{ 4B_1^{(\nu+1)} \leq t \right\} e^{-\frac{\log^2(B)}{8B_1^{(\nu+1)}}} \right]. \quad (2.81) \]

We now apply formula (1.58), Theorem 1.4:

\[ \mathbb{E}[(\xi_t - K)\pm] = (1 - K)\pm + \sqrt{K} \mathbb{E}\left[ \left\{ 4B_1^{(\nu+1)} \leq t \right\} e^{-\frac{\log^2(K)}{8B_1^{(\nu+1)}}} \right] \]

successively with $K = e^{2a\nu}$ and $K = e^{2a(\nu+1)}$. We obtain:

\[ \mathbb{E}\left[ (\xi_t - e^{2a\nu})^+ \right] = e^{a\nu} \mathbb{E}\left[ \left\{ 4B_1^{(\nu+1)} \leq t \right\} e^{-\frac{a^2\nu^2}{2B_1^{(\nu+1)}}} \right], \quad (2.82) \]

\[ \mathbb{E}\left[ (\xi_t - e^{2a(\nu+1)})^+ \right] = e^{a(\nu+1)} \mathbb{E}\left[ \left\{ 4B_1^{(\nu+1)} \leq t \right\} e^{-\frac{a^2(\nu+1)^2}{2B_1^{(\nu+1)}}} \right]. \quad (2.83) \]

Gathering (2.80), (2.82) and (2.83) ends the proof of (2.77) and of Proposition 2.1. \qed
2.6.3 Computation of the Law of $G^{(M^{(\nu)})}_K$

We shall apply Proposition 2.1 to get the law of the r.v. $G^{(M^{(\nu)})}_K := \sup\{a \geq 0; M^{(\nu)}_a = K\}$.

**Proposition 2.2.** The r.v. $G^{(M^{(\nu)})}_K$ admits as probability density the function $f^{(M^{(\nu)})}_K$ given, for $K < e^a$, by:

$$f^{(M^{(\nu)})}_K(a) = \nu \left\{ \mathbb{P}\left(G^{(1/2)}_{2a\nu} \leq t\right) - \mathbb{P}\left(T^{(1/2)}_{2a\nu} < t\right) \right\}$$

$$+ \frac{\nu + 1}{K} \left\{ \mathbb{P}\left(T^{(1/2)}_{2a(\nu+1)} \leq t\right) - \mathbb{P}\left(G^{(1/2)}_{2a(\nu+1)} < t\right) \right\}$$

$$= 2^{\nu + 1} e^{2a(\nu+1)} \mathbb{P}\left(B_t^{(-1/2)} > 2a(\nu+1)\right) - 2\nu e^{2a\nu} \mathbb{P}\left(B_t^{(-1/2)} > 2a\nu\right)$$

(2.84)

with the notations of Chapter 1 and where $t = \frac{2a^2(2\nu + 1)}{a - \log(K)}$.

**Proof.** We first prove (2.84).

Since $\left(M^{(\nu)}_a, a \geq 0\right)$ has no positive jumps, we may apply Theorem 2.1 (see Remark 2.1) to obtain:

$$\mathbb{P}\left(G^{(M^{(\nu)})}_K \leq a\right) = \frac{1}{K} \mathbb{E}\left[\left(K - M^{(\nu)}_a\right)^+\right]$$

$$= \frac{1}{K} \left( \mathbb{E}\left[ K \left( \delta_t - e^{2a\nu}\right)^+ \right] - \mathbb{E}\left[ \left( \delta_t - e^{2a(\nu+1)}\right)^+ \right] \right)$$

(from Proposition 2.1, with $t = \frac{2a^2(2\nu + 1)}{a - \log(K)}$)

$$= \mathbb{P}\left(G^{(1/2)}_{2a\nu} \leq t\right) - \frac{1}{K} \mathbb{P}\left(G^{(1/2)}_{2a(\nu+1)} \leq t\right)$$

(from Theorem 1.2)

$$= \int_0^{\frac{2a^2(2\nu+1)}{a - \log(K)}} \frac{1}{2\sqrt{2\pi u}} \exp\left(-\frac{1}{2}\left(2a\nu - u\right)^2\right) du$$

(2.87)

$$- \frac{1}{K} \int_0^{\frac{2a^2(2\nu+1)}{a - \log(K)}} \frac{1}{2\sqrt{2\pi u}} \exp\left(-\frac{1}{2}\left(2a(\nu+1) - u\right)^2\right) du.$$
\[
\frac{\partial}{\partial a} P\left( g_K^{(M^{(\nu)})} \leq a \right)
= \int_0^t \frac{1}{2\sqrt{2\pi u}} \left( \nu - \frac{4a\nu^2}{u} \right) \exp\left( -\frac{1}{2u} \left( 2a\nu - \frac{u}{2} \right)^2 \right) \, du
\]
\[
- \frac{1}{K} \int_0^t \frac{1}{2\sqrt{2\pi u}} \left( \nu + 1 - \frac{4a(\nu+1)^2}{u} \right) \exp\left( -\frac{1}{2u} \left( 2a(\nu+1) - \frac{u}{2} \right)^2 \right) \, du
\]
\[
= \nu \left\{ P\left( G_{2a\nu}^{(1/2)} \leq t \right) - P\left( T_{2a\nu}^{(1/2)} \leq t \right) \right\}
- \frac{\nu + 1}{K} \left\{ P\left( G_{2a(\nu+1)}^{(1/2)} \leq t \right) - P\left( T_{2a(\nu+1)}^{(1/2)} \leq t \right) \right\}.
\]
This is relation (2.84).
We now prove (2.85)
From Theorem 1.3, identities (1.54) and (1.55) for \( K > 1 \) and \( \nu > 0 \), we deduce:
\[
P\left( T_{\log(K)}^{(\nu)} \leq t \right) = E\left[ \delta_t^{(2\nu)} 1_{\left\{ \delta_t^{(2\nu)} > K^{2\nu} \right\}} \right] + K^{2\nu} P\left( \delta_t^{(2\nu)} > K^{2\nu} \right)
\]
\[
P\left( G_{\log(K)}^{(\nu)} \leq t \right) = E\left[ \delta_t^{(2\nu)} 1_{\left\{ \delta_t^{(2\nu)} > K^{2\nu} \right\}} \right] - K^{2\nu} P\left( \delta_t^{(2\nu)} > K^{2\nu} \right).
\]
By subtracting:
\[
P\left( T_{\log(K)}^{(\nu)} \leq t \right) - P\left( G_{\log(K)}^{(\nu)} \leq t \right) = 2K^{2\nu} P\left( \delta_t^{(2\nu)} > K^{2\nu} \right)
= 2K^{2\nu} P\left( B_t^{(\nu)} > \log(K) \right). \quad (2.88)
\]
Finally, we obtain (2.85) from (2.84) by applying (2.88) first with \( \nu = 1/2 \) and \( K = e^{2a\nu} \), and then with \( K = e^{2a(\nu+1)} \) noting that this is allowed since \( e^{2a\nu} \) and \( e^{2a(\nu+1)} \) are larger than 1.

\[\boxdot\]

**2.6.4 A More Probabilistic Approach of Proposition 2.2**

We shall now prove again Proposition 2.2 via a more probabilistic method, when \( \nu = 0 \). More precisely, we will show:

**Proposition 2.3 (\( \nu = 0 \)).**

i) \( P\left( \mathcal{G}_K^{(M^{(0)})} \leq a \right) = P\left( G_{\log(K)}^{(-1/2)} \leq 2(a - \log(K)) \right) \)
\[ - P\left( G_{2a - \log(K)}^{(1/2)} \leq 2(a - \log(K)) \right). \quad (2.89)\]
ii) \( f_{\mathcal{G}^{(0)}}(a) = \mathbb{P}\left(T_{2a-\log(K)}^{(1/2)} \leq 2(a - \log(K))\right) - \mathbb{P}\left(G^{(1/2)}_{2a-\log(K)} \leq 2(a - \log(K))\right) \)

\[
= 2e^{2a} \frac{K}{K} \mathbb{P}\left(B_{2(a-\log(K))} > 3a - 2\log(K)\right). 
\]

(2.90)

In the following, we shall first prove Proposition 2.3, and then, we will check that relations (2.89) and (2.90) coincide with those of Proposition 2.2.

**Proof.** We first give a geometric representation of the two events \( \{G^{(0)} \leq a\} \) and \( \{G^{(-1/2)}_{\log(K)} \leq 2(a - \log(K))\} \). We have,

\[
\mathcal{G}^{(0)}(M^{(0)}) := \sup \left\{ a \geq 0; e^{a - \frac{T_{a}}{2}} \geq K \right\} = \sup \left\{ a \geq 0; a \geq \frac{1}{2} T_{a} + \log(K) \right\},
\]

hence the event \( \{\mathcal{G}^{(0)} \leq a\} \) is composed of the Brownian paths which do not leave the hatched area of Fig. 1:

![Fig. 1 \( \{\mathcal{G}^{(0)} \leq a\} \)](image)

Similarly:

\[
\mathcal{G}^{(-1/2)}_{\log(K)} := \sup \left\{ s \geq 0; B_{s}^{(-1/2)} = \log(K) \right\} = \sup \left\{ s \geq 0; B_{s} = \frac{1}{2} s + \log(K) \right\}
\]

and the event \( \{G^{(-1/2)}_{\log(K)} \leq 2(a - \log(K))\} \) is composed of the Brownian paths which do not leave the hatched area of Fig. 2:
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![Diagram](https://via.placeholder.com/150)

**Fig. 2** \( \{ G_{\log(K)}^{(-1/2)} \leq 2(a - \log(K)) \} \)

Consequently:

\[
\mathbb{P} \left( \mathcal{G}_K(M^{(0)}) \leq a \right) = \mathbb{P} \left( \left\{ G_{\log(K)}^{(-1/2)} \leq 2(a - \log(K)) \right\} \cap \left\{ \sup_{s \leq 2(a - \log(K))} B_s \leq a \right\} \right).
\]

We put \( T = 2(a - \log(K)) \), and this identity becomes:

\[
\mathbb{P} \left( \mathcal{G}_K(M^{(0)}) \leq a \right) = \mathbb{P} \left( G_{\log(K)}^{(-1/2)} \leq T \right) - \mathbb{P} \left( \left\{ G_{\log(K)}^{(-1/2)} \leq T \right\} \cap \left\{ \sup_{s \leq T} B_s > a \right\} \right).
\]

(2.91)

Let us now study the event:

\[
\left\{ G_{\log(K)}^{(-1/2)} \leq T \right\} \cap \left\{ \sup_{s \leq T} B_s > a \right\} = \left\{ \inf_{s \geq T} \left( \frac{s}{2} + \log(K) - B_s \right) > 0 \right\} \cap \left\{ \sup_{s \leq T} B_s > a \right\}.
\]

On this event, the Brownian paths hit a.s. level \( a \) before time \( T \), hence, applying Désiré André’s reflection principle, we can replace the piece of path after \( T \) by its symmetric with respect to the horizontal line of \( y \)-coordinate \( a \). We then obtain:

\[
\mathbb{P} \left( \left\{ \inf_{s \geq T} \left( \frac{s}{2} + \log(K) - B_s \right) > 0 \right\} \cap \left\{ \sup_{s \leq T} B_s > a \right\} \right)
\]

\[
= \mathbb{P} \left( \left\{ \inf_{s \geq T} B_s - \left( 2a - \log(K) - \frac{s}{2} \right) > 0 \right\} \cap \left\{ \sup_{s \leq T} B_s > a \right\} \right)
\]

\[
= \mathbb{P} \left( \inf_{s \geq T} B_s - \left( 2a - \log(K) - \frac{s}{2} \right) > 0 \right) = \mathbb{P} \left( G_{2a - \log(K)}^{(1/2)} \leq T \right).
\]
Plugging back this expression in (2.91), we finally get:

\[ \mathbb{P} \left( G_{K}^{(M^{(0)})} \leq a \right) = \mathbb{P} \left( G_{\log(K)}^{(-1/2)} \leq 2(a - \log(K)) \right) - \mathbb{P} \left( G_{2a-\log(K)}^{(1/2)} \leq 2(a - \log(K)) \right), \]

which is Point (i) of Proposition 2.3.

Let us now compute the density of \( G_{K}^{(M^{(0)})} \). We have:

\[ \mathbb{P} \left( G_{K}^{(M^{(0)})} \leq a \right) = \int_{0}^{2(a-\log(K))} \frac{1}{2\sqrt{2\pi u}} \left( e^{-\frac{1}{2u} (\log(K) + \frac{u}{2})^2} - e^{-\frac{1}{2u} (2a-\log(K) - \frac{u}{2})^2} \right) du. \]

We differentiate this identity with respect to \( a \). Once again, the terms coming from the differentiation of the upper bound in both integrals on the RHS cancel, and it finally remains:

\[
\begin{align*}
\frac{d}{da} \mathbb{P} \left( G_{K}^{(M^{(0)})} \leq a \right) &= \int_{0}^{2(a-\log(K))} \frac{1}{2\sqrt{2\pi u}} \left( e^{-\frac{1}{2u} (\log(K) + \frac{u}{2})^2} - e^{-\frac{1}{2u} (2a-\log(K) - \frac{u}{2})^2} \right) du \\
&= 2 \frac{e^{2a}}{K} \mathbb{P} \left( B_{2(a-\log(K))} > 3a - 2\log(K) \right),
\end{align*}
\]

the last equality coming from (2.88) with \( \nu = 1/2 \), and where we have replaced \( K \) by \( \frac{e^{2a}}{K} \) (\( \frac{e^{2a}}{K} > 1 \) since \( K < e^{a} \)).

\[
\square
\]

We now show that the two expressions (2.84) and (2.90) of the density of \( G_{K}^{(M^{(0)})} \) coincide. With \( \nu = 0 \), (2.84) becomes:

\[
\begin{align*}
\frac{d}{da} \mathbb{P} \left( G_{K}^{(M^{(0)})} \leq a \right) &= \frac{1}{K} \left\{ \mathbb{P} \left( T_{2a}^{1/2} \leq \frac{2a^2}{a-\log(K)} \right) - \mathbb{P} \left( G_{2a}^{1/2} \leq \frac{2a^2}{a-\log(K)} \right) \right\} \\
&= \frac{1}{K} \int_{0}^{\frac{2a^2}{a-\log(K)}} \left( \frac{2a}{\sqrt{2\pi u}} - \frac{1}{2\sqrt{2\pi u}} \right) e^{-\frac{1}{2u} (2a - \frac{u}{2})^2} du \\
&= \frac{1}{K} \int_{\frac{2a}{a-\log(K)}}^{\infty} \left( \frac{1}{\sqrt{2\pi s}} - \frac{a}{2\sqrt{2\pi s^3}} \right) e^{-\frac{1}{2s} (a - s)^2} ds
\end{align*}
\]

(after the change of variable \( s = \frac{4a^2}{u} \))

\[
\begin{align*}
&= \frac{1}{K} \left\{ \mathbb{P} \left( G_{a}^{(1)} \geq 2(a - \log(K)) \right) - \mathbb{P} \left( T_{a}^{(1)} \geq 2(a - \log(K)) \right) \right\} \\
&= \frac{1}{K} \left\{ \mathbb{P} \left( T_{a}^{(1)} \leq 2(a - \log(K)) \right) - \mathbb{P} \left( G_{a}^{(1)} \leq 2(a - \log(K)) \right) \right\}.
\end{align*}
\]

(2.92)
Then, from (2.88), one obtains:
\[
\begin{align*}
\mathbb{P}\left(T^{(1)}_a \leq 2(a - \log(K))\right) - \mathbb{P}\left(G^{(1)}_a \leq 2(a - \log(K))\right) &= 2(e^a)^2 \mathbb{P}\left(B_2(a - \log(K)) > a\right) \\
&= 2e^{2a} \mathbb{P}\left(B_2(a - \log(K)) \geq 2(a - \log(K)) > a\right) \\
&= 2e^{2a} \mathbb{P}\left(B_2(a - \log(K)) > 3a - 2\log(K)\right). 
\end{align*}
\]
(2.93)

Hence, plugging (2.93) in (2.92):
\[
\begin{align*}
\mathbb{P}
\end{align*}
\]
which is (2.84).

2.6.5 An application of Proposition 2.1 to the Local Times of the Martingale \((\mathbb{E}_t, t \geq 0)\)

We end this Section 2.6 by an application of Proposition 2.1 to the local times \((L^K_t, t \geq 0, K \geq 0)\) of the martingale \((\mathbb{E}_t, t \geq 0)\).

Proposition 2.4. Let \((L^K_t, t \geq 0, K \geq 0)\) denote the family of local times of the martingale \((\mathbb{E}_t, t \geq 0)\). Then, for all \(0 < K < e^a\) and \(\nu > 0\):
\[
(K - 1)^+ + \frac{1}{2} \mathbb{E}\left[L^K_{T_\nu^{(\nu)}}\right] = \frac{K}{2} \mathbb{E}\left[L^{2\nu}_{T}\right] - \frac{1}{2} \mathbb{E}\left[L^{2\nu(\nu+1)}_{T}\right].
\]
(2.94)

where \(t = \frac{2a^2(2\nu + 1)}{a - \log(K)}\).

Proof. We start by applying Tanaka’s formula to the martingale \((\mathbb{E}_t, t \geq 0)\) stopped at the time \(s \land T_\nu^{(\nu)}\). Using the identity \(B_t - \frac{t}{2} = B_t^{(\nu)} - \left(\frac{1}{2} + \nu\right)t\):
\[
\mathbb{E}\left[K - \mathbb{E}_s\mathbb{E}_T\right] = \mathbb{E}\left[K - \exp\left\{B_{s \land T_\nu^{(\nu)}}^{(\nu)} - \left(\frac{1}{2} + \nu\right)(s \land T_\nu^{(\nu)})\right\}\right] \\
= (K - 1)^+ + \frac{1}{2} \mathbb{E}\left[L^K_{s \land T_\nu^{(\nu)}}\right].
\]

We then let \(s\) tend to \(+\infty\), using the dominated convergence theorem for the term \(\mathbb{E}\left[K - \mathbb{E}_s\mathbb{E}_T\right]\), and the monotone convergence theorem for \(\mathbb{E}\left[L^K_{s \land T_\nu^{(\nu)}}\right]\). This leads to:
\[(K - 1)^+ + \frac{1}{2} \mathbb{E} \left[ L_{T_a^{(\nu)}}^K \right] = \mathbb{E} \left[ \left( K - \exp \left\{ a - \left( \frac{1}{2} + \nu \right) T_a^{(\nu)} \right\} \right)^+ \right]
\]
\[= K \mathbb{E} \left[ \left( \mathcal{E}_t - e^{2a\nu} \right)^+ \right] - \mathbb{E} \left[ \left( \mathcal{E}_t - e^{2a(\nu+1)} \right)^+ \right], \]
where \( t = \frac{2a^2(2\nu + 1)}{a - \log(K)} \) from Proposition 2.1. It only remains to apply once again Tanaka’s formula to obtain the result (2.94), since \( e^{2a\nu} \) and \( e^{2a(\nu+1)} \) are larger than 1 (= \( \mathcal{E}_0 \)).

\[\square\]

**Remark 2.3.** Since the application \( \phi: x \mapsto e^x \) is a \( C^1 \)-diffeomorphism from \( \mathbb{R} \) onto \( [0, +\infty] \), we have, with obvious notation:

\[L_t^{\phi(x)}(\phi(B^{(-1/2)})) = \phi'(x)L_t^x(B^{(-1/2)}).\]

Thus, denoting by \( \left( L_t^x(B^{(-1/2)}), t \geq 0, x \in \mathbb{R} \right) \) the family of local times of the semi-martingale \( \left( B_t^{(-1/2)}, t \geq 0 \right) \), Proposition 2.4 can be rewritten:

\[(K - 1)^+ + \frac{K}{2} \mathbb{E} \left[ L_{T_a^{(\nu)}}(B^{(-1/2)}) \right]
\]
\[= \frac{K}{2} e^{2a\nu} \mathbb{E} \left[ L_t^{2a\nu}(B^{(-1/2)}) \right] - \frac{e^{2a(\nu+1)}}{2} \mathbb{E} \left[ L_t^{2a(\nu+1)}(B^{(-1/2)}) \right]. \quad (2.95)\]

In the same spirit as that of Proposition 2.3, we could give, when \( \nu = 0 \), a probabilistic proof of identity (2.95). We leave this proof to the interested reader. Formula (2.95) writes then:

\[\mathbb{E} \left[ L_{T_a}^{\log(K)}(B^{(-1/2)}) \right] = \mathbb{E} \left[ L_{T_a}^{\log(K)}(B^{(-1/2)}) \right] - \mathbb{E} \left[ L_{T_a}^{2a-\log(K)}(B^{(1/2)}) \right]. \quad (2.96)\]

## 2.7 The case \( M_\infty \neq 0 \)

### 2.7.1 Hypotheses

Our aim in this Section is to give a generalization of Theorem 2.1 when we remove the assumption \( M_\infty = 0 \) a.s. More precisely, we still consider a positive and continuous local martingale \( (M_t, t \geq 0) \) defined on a filtered probability space \( (\Omega, (\mathcal{F}_t, t \geq 0), \mathcal{F}_\infty, \mathbb{P}) \). We assume that \( (\mathcal{F}_t := \sigma(M_s, s \leq t), t \geq 0) \) is the natural filtration of \( (M_t, t \geq 0) \) and that \( \mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \). As a positive local martingale,
(\(M_t, t \geq 0\)) is a positive supermartingale, which therefore converges a.s. towards a r.v \(M_\infty\) as \(t \to \infty\). But, as opposed to the previous sections, we no longer assume that \(M_\infty = 0\) a.s. Then, in this new framework, Theorem 2.1 extends in the following way:

### 2.7.2 A Generalization of Theorem 2.1

**Theorem 2.5.** Let \((M_t, t \geq 0)\) a positive continuous local martingale. For every \(K \geq 0\):

\[
\mathbb{E}\left[1_{\{\Phi^M_K \leq t\}} (K - M_\infty)^+ \mid \mathcal{F}_t\right] = (K - M_t)^+.
\] (2.97)

Of course, if \(M_\infty = 0\) a.s., we recover Theorem 2.1.

### 2.7.3 First Proof of Theorem 2.5

It hinges on the balayage formula which we first recall (see [70], Chapter VI, p.260):

**Balayage formula:**

Let \((Y_t, t \geq 0)\) be a continuous semi-martingale and \(\mathcal{G}_Y(t) = \sup\{s \leq t; Y_s = 0\}\). Then, for any bounded predictable process \((\Phi_s, s \geq 0)\), we have:

\[
\Phi_{\mathcal{G}_Y(t)} Y_t = \Phi_0 Y_0 + \int_0^t \Phi_{\mathcal{G}_Y(s)} dY_s.
\] (2.98)

We note \(\mathcal{G}_K(s) := \sup\{u \leq s; M_u = K\}\). The balayage formula, applied to \((Y_t = (K - M_t)^+, t \geq 0)\), becomes, for every bounded and predictable process \((\Phi_s, s \geq 0)\) and \(t \geq 0\):

\[
\Phi_{\mathcal{G}_K(t)} (K - M_t)^+ = \Phi_0 (K - M_0)^+ - \int_0^t \Phi_{\mathcal{G}_K(s)} 1_{\{M_s < K\}} dM_s + \frac{1}{2} \int_0^t \Phi_{\mathcal{G}_K(s)} dL^K_s
\]

(where \((L^K_s, s \geq 0)\) denotes the local time at level \(K\) of the local martingale \((M_s, s \geq 0)\)),

\[
= \Phi_0 (K - M_0)^+ - \int_0^t \Phi_{\mathcal{G}_K(s)} 1_{\{M_s < K\}} dM_s + \frac{1}{2} \int_0^t \Phi_s dL^K_s
\] (2.99)

since \(dL^K_s\) charges only the set of times for which \(M_s = K\) i.e. for which \(\mathcal{G}_K(s) = s\).

We now apply (2.99) between \(t\) and \(+\infty\) to obtain:

\[
\mathbb{E}\left[\Phi^M_{\mathcal{G}_K(t)} (K - M_\infty)^+ \mid \mathcal{F}_t\right] = \Phi_{\mathcal{G}_K(t)} (K - M_t)^+ + \frac{1}{2} \mathbb{E}\left[\int_t^\infty \Phi_s dL^K_s \mid \mathcal{F}_t\right]
\] (2.100)
since $\mathcal{G}_K(\infty) = \mathcal{G}_K(M)$. Taking in (2.100) $\Phi_s = \varphi(s)1_{\{s \leq t\}}$ for a bounded Borel function $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}$ and observing that $\mathcal{G}_K(t) \leq t$ and $\int_t^\infty \varphi(s)1_{\{s \leq t\}}dL^K_t = 0$, we obtain:

$$
\mathbb{E} \left[ \varphi \left( \mathcal{G}_K(M) \right) 1_{\{\mathcal{G}_K(M) \leq t\}} \right] = \varphi \left( \mathcal{G}_K(t) \right) \left( M_t - K \right)^+.
$$

$$
\square
$$

### 2.7.4 A Second Proof of Theorem 2.5

Let $T$ be a $\mathcal{F}_t$-stopping time. It is clear that:

$$
\mathbb{E} \left[ 1_{\{\mathcal{G}_K(M) \leq T\}} \left( K - M_\infty \right)^+ \right] = \mathbb{E} \left[ 1_{\{d^K_T = \infty\}} \left( K - M_\infty \right)^+ \right] \tag{2.101}
$$

with $d^K_T := \inf\{t > T; M_t = K\}$, since $\{\mathcal{G}_K(M) \leq T\} = \{d^K_T = \infty\}$. Then:

$$
\mathbb{E} \left[ 1_{\{d^K_T = \infty\}} \left( K - M_\infty \right)^+ \right] = \mathbb{E} \left[ 1_{\{d^K_T = \infty\}} \left( K - M_{d^K_T} \right)^+ \right] = \mathbb{E} \left[ \left( K - M_{d^K_T} \right)^+ \right] \tag{2.102}
$$

since, on the set $(d^K_T < \infty)$, $\left( K - M_{d^K_T} \right)^+ = 0$. We now note that, for $t$ between $T$ and $d^K_T$, $L^K_t$ is constant since $M_t \neq K$. Hence, from Tanaka’s formula, the RHS of (2.102) equals:

$$
\mathbb{E} \left[ \left( K - M_{d^K_T} \right)^+ \right] = \mathbb{E} \left[ (K - M_T)^+ \right]. \tag{2.103}
$$

Gathering (2.101), (2.102) and (2.103), we obtain:

$$
\mathbb{E} \left[ 1_{\{\mathcal{G}_K(M) \leq T\}} \left( K - M_\infty \right)^+ \right] = \mathbb{E} \left[ (K - M_T)^+ \right]. \tag{2.104}
$$

This identity may be reinforced as:

$$
\mathbb{E} \left[ 1_{\{\mathcal{G}_K(M) \leq T\}} \left( K - M_\infty \right)^+ \right] = (K - M_T)^+ \tag{2.104}
$$

by replacing in (2.104) $T$ by $T_\Lambda := \begin{cases} T \text{ on } \Lambda, \\ +\infty \text{ on } \Lambda^c \end{cases}$ for any generic set $\Lambda \in \mathcal{F}_T$. 

$$
\square
$$
2.7 The case $M_\infty \neq 0$

2.7.5 On the Law of $S_\infty := \sup_{t \geq 0} M_t$

We have seen in Section 2.1 that, when $M_t \xrightarrow{t \to \infty} 0$ a.s., then:

$$S_\infty \overset{(law)}{=} \frac{M_0}{U}$$ (2.105)

where $U$ is uniform on $[0, 1]$ and independent from $M_0$. What can be said about the law of $S_\infty$ if we remove the assumption $M_\infty = 0$? Going back to the proof of Lemma 2.1 and applying Doob’s optional stopping Theorem, we obtain, for $b > a = M_0$ and $T_b := \inf\{t \geq 0; M_t = b\}$:

$$\mathbb{E}[M_{T_b}] = a$$ (2.106)

i.e.

$$b \mathbb{P}(S_\infty \geq b) + \mathbb{E}[M_\infty 1_{\{S_\infty < b\}}] = a.$$ (2.107)

Let us remark that, applying the monotone convergence theorem:

$$\lim_{b \to \infty} b \mathbb{P}(S_\infty \geq b) = a - \mathbb{E}[M_\infty].$$ (2.108)

Furthermore, relation (2.107) leads us to introduce the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by:

$$\Phi(S_\infty) = \mathbb{E}[M_\infty | S_\infty].$$ (2.109)

It is clear that $\Phi(x) \leq x$ and (2.107) becomes:

$$b \mathbb{P}(S_\infty \geq b) + \mathbb{E}[\Phi(S_\infty) 1_{\{S_\infty < b\}}] = a.$$ (2.110)

Assuming $\Phi$ is given, we may consider (2.110) as an equation for the distribution of $S_\infty$, and we obtain:

**Proposition 2.5.** For simplicity, we assume that for every $x > 0$, $\Phi(x) < x$. Then, the law of $S_\infty$ is given by:

$$\mathbb{P}(S_\infty \geq b) = \exp\left(-\int_{a}^{b} \frac{dx}{x - \Phi(x)}\right).$$ (2.111)

Observe that, since $S_\infty < \infty$ a.s., it follows from (2.111) that:

$$\int_{a}^{\infty} \frac{dx}{x - \Phi(x)} = +\infty.$$ (2.112)

**Proof.** From formula (2.110), denoting $\overline{\mu}(b) := \mathbb{P}(S_\infty \geq b)$, we obtain:

$$b \overline{\mu}(b) - \int_{a}^{b} \Phi(x) d\overline{\mu}(x) = a.$$ (2.113)
Consequently, by differentiation:

\[ b d \overline{\mu}(b) + \overline{\mu}(b) db - \Phi(b) d\overline{\mu}(b) = 0 \]

i.e.

\[ (b - \Phi(b)) d\overline{\mu}(b) = -\overline{\mu}(b) db \]  \hspace{1cm} (2.114)

The above equation yields:

\[ \overline{\mu}(b) = C \exp \left( - \int_a^b \frac{dx}{x - \Phi(x)} \right) \]  \hspace{1cm} (2.115)

which implies \( C = 1 \) by taking \( b = a \), since \( \overline{\mu}(a) = 1 \).

Example 2.7.5. Let \( (B_t, t \geq 0) \) be a Brownian motion started at \( a > 0 \), and \( S^B_t := \sup_{s \leq t} B_s \). For \( 0 < \alpha < 1 \), we define the stopping time:

\[ T^{(\alpha)}_a := \inf \{ t \geq 0; B_t = \alpha S^B_t \} \]  \hspace{1cm} (2.116)

to which we associate the martingale \( (M_t = B_{t \wedge T^{(\alpha)}_a}, t \geq 0) \). Then, \( \Phi(x) = \alpha x \), and consequently, we have:

\[ \int_a^b \frac{dx}{x - \Phi(x)} = \frac{1}{\alpha} \log \left( \frac{b}{a} \right) \quad (b \geq a) \]  \hspace{1cm} (2.117)

and

\[ \mu(b) = \exp \left( - \frac{1}{\alpha} \log \left( \frac{b}{a} \right) \right) = \left( \frac{a}{b} \right)^{\frac{1}{\alpha}} \quad (b \geq a). \] \hspace{1cm} (2.118)

(Observe that \( \mu \) is the tail of a Pareto distribution.)

Remark 2.4.

a) Under which condition(s) is \( (M_t, t \geq 0) \) uniformly integrable? This question had of course a negative answer when \( M_\infty = 0 \) a.s. But now? Uniform integrability is equivalent to \( \mathbb{E}[M_\infty] = \mathbb{E}[M_0] = a \), which is satisfied if and only if (see (2.109)):

\[ \mathbb{E}[\Phi(S_\infty)] = a. \] \hspace{1cm} (2.119)

From (2.108), uniform integrability of \( (M_t, t \geq 0) \) is also equivalent to:

\[ \lim_{b \to \infty} b \Pr(S_\infty \geq b) = 0. \] \hspace{1cm} (2.120)

From (2.119) and (2.111), uniform integrability of \( (M_t, t \geq 0) \) is also equivalent to:

\[ \int_a^\infty \frac{\Phi(x)}{x - \Phi(x)} \left( \exp \left( - \int_a^x \frac{dy}{y - \Phi(y)} \right) \right) dx = a. \] \hspace{1cm} (2.121)
Let us check that (2.120) and (2.121) coincide. Indeed, we have, from (2.121):

\[-a = \int_a^\infty \left( x - \Phi(x) - \frac{x}{x - \Phi(x)} \right) \left( \exp \left( - \int_a^x \frac{dy}{y - \Phi(y)} \right) \right) dx\]

\[= \int_a^\infty \bar{\mu}(x) dx - \int_a^\infty \frac{x}{x - \Phi(x)} \exp \left( - \int_a^x \frac{dy}{y - \Phi(y)} \right) dx\]

\[= \int_a^\infty \bar{\mu}(x) dx + [x\bar{\mu}(x)]_a^\infty - \int_a^\infty \bar{\mu}(x) dx\]

(after an integration by parts)

\[= \lim_{b \to \infty} b \mathbb{P}(S_\infty \geq b) - a.\]

Going back to Example 2.7.5, we have: \(\Phi(x) = \alpha x\) and \(\bar{\mu}(b) = \mathbb{P}(S_\infty \geq b) = \left( \frac{a}{b} \right)^{\frac{1}{1-a}}\). Hence: \(b \mathbb{P}(S_\infty \geq b) = a \frac{1}{1-a} b^{-\frac{a}{1-a}} \to 0\) as \(b \to \infty\). Then, Example 2.7.5 is a case of uniform integrability.

**b)** Can we describe all the laws of \((M_t, t \geq 0)\) which satisfy (2.109) for a given \(\Phi\)? See Rogers [71] where the law of \((S_\infty, M_\infty)\) is described in all generality. See also Vallois [87]. However, these authors assume a priori that \((M_t, t \geq 0)\) is uniformly integrable. We shall study later (see Chapter 3) the law of \(\mathcal{G}^{(M)}_K\) when \(M_\infty \neq 0\).

### 2.8 Extension of Theorem 2.1 to the Case of Orthogonal Local Martingales

#### 2.8.1 Statement of the Main Result

In this section, we shall extend Theorem 2.1 to the case of orthogonal local martingales. Let \((M_t^{(i)}, t \geq 0; i = 1, \cdots, n)\) be a set of \(n\) positive, continuous local martingales such that, for all \(i = 1, \cdots, n:\)

\[\lim_{t \to \infty} M_t^{(i)} = 0 \quad \text{a.s.} \tag{2.122}\]

We assume moreover that these martingales are orthogonal, i.e., for all \(1 \leq i < j \leq n:\)

\[\left\langle M^{(i)}, M^{(j)} \right\rangle_t = 0 \quad (0 \leq t < \infty) \tag{2.123}\]

where \(\left\langle M^{(i)}, M^{(j)} \right\rangle_t, t \geq 0\) denotes the bracket of the martingales \(M^{(i)}\) and \(M^{(j)}\). The counterpart of Theorem 2.1 writes then:
Theorem 2.6. Under the preceding hypotheses:

i) For every \( K_i \geq 0 \) and every bounded \( \mathcal{F}_t \)-measurable r.v. \( F_t \):

\[
\mathbb{E} \left[ F_t \prod_{i=1}^{n} (K_i - M_t^{(i)})^+ \right] = \mathbb{E} \left[ F_t \left( \prod_{i=1}^{n} K_i \right) 1_{\{ \bigwedge_{i=1}^{n} \mathcal{G}_t^{(i)} \leq t \}} \right]
\]

(2.124)

where \( \bigwedge_{i=1}^{n} \mathcal{G}_t^{(i)} := \sup_{i=1}^{n} \mathcal{G}_t^{(M_t^{(i)})} \).

(2.125)

ii) In other words, the submartingale \( \mathbb{P} \left( \bigwedge_{i=1}^{n} \mathcal{G}_t^{(i)} \leq t \mid \mathcal{F}_t \right) \) equals:

\[
\mathbb{P} \left( \bigwedge_{i=1}^{n} \mathcal{G}_t^{(i)} \leq t \mid \mathcal{F}_t \right) = \prod_{i=1}^{n} \left( 1 - \frac{M_t^{(i)}}{K_i} \right)^+
\]

(2.126)

\[
= \prod_{i=1}^{n} \mathbb{P} \left( \mathcal{G}_t^{(i)} \leq t \mid \mathcal{F}_t \right).
\]

(2.127)

We shall discuss the law of \( \bigwedge_{i=1}^{n} \mathcal{G}_t^{(i)} \) later (see Theorem 2.7). For the moment, we give two proofs of Theorem 2.6. For clarity’s sake, we shall assume that \( n = 2 \), i.e. we are considering two orthogonal local martingales \( (M_t, t \geq 0) \) and \( (M_t', t \geq 0) \). Clearly, our arguments extend easily to the general case.

2.8.2 First Proof of Theorem 2.6, via Enlargement Theory

(2.124) writes:

\[
\mathbb{E} \left[ F_t (K - M_t)^+ (K' - M_t')^+ \right] = \mathbb{E} \left[ F_t KK' 1_{\{ \mathcal{G}_t \wedge \mathcal{G}_t' \leq t \}} \right].
\]

(2.128)

To prove (2.128), we first use Point (i) of Theorem 2.1 which allows to write the LHS of (2.128) as:

\[
\mathbb{E} \left[ F_t (K - M_t)^+ (K' - M_t')^+ \right] = \mathbb{E} \left[ F_t (K' - M_t')^+ K 1_{\{ \mathcal{G}_t \leq t \}} \right].
\]

(2.129)

Let us define \( (\mathcal{F}_t^K, t \geq 0) \) the smallest filtration containing \( (\mathcal{F}_t, t \geq 0) \) and making \( \mathcal{G}_K \) a stopping time. From Lemma 2.2 below, \( (M_t', t \geq 0) \) remains a local martingale in this new filtration. Then, we may use again Theorem 2.1, this time with respect to \( (M_t', t \geq 0) \) which is a local martingale in \( (\mathcal{F}_t^K, t \geq 0) \) to obtain:
\[ \mathbb{E} \left[ F_t (K - M_t)^+ (K' - M'_t)^+ \right] = \mathbb{E} \left[ F_t K 1_{\{G_K \leq t\}} (K' - M'_t)^+ \right] \\
= \mathbb{E} \left[ F_t K 1_{\{G_K \leq t\}} K' 1_{\{G_{K'} \leq t\}} \right] \\
= \mathbb{E} \left[ F_t KK' 1_{\{G_K \vee G_{K'} \leq t\}} \right]. \]

This is Theorem 2.6.

**Lemma 2.2.** \((M'_t, t \geq 0)\) is a local martingale in the enlarged filtration \((\mathcal{F}^K_t, t \geq 0)\).

**Proof.** For the reader’s convenience, we recall some enlargement formulae (see Mansuy-Yor [51], and Subsection 3.2.1). Let \(Z_t^{(K)} := P(G_K > t | \mathcal{F}_t)\) the Azéma supermartingale associated with \(G_K\). Then, if \((M'_t, t \geq 0)\) is a continuous \((\mathcal{F}_t, t \geq 0)\)-local martingale, there exists \((\tilde{M}'_t, t \geq 0)\) a \((\mathcal{F}^K_t, t \geq 0)\) local martingale such that:

\[ M'_t = \tilde{M}'_t + \int_0^{1 \land G_K} \frac{d \langle M', Z^{(K)} \rangle_s}{Z^{(K)}_s} - \int_t^t \langle M', Z^{(K)} \rangle_s 1 - Z^{(K)}_s. \quad (2.130) \]

In our situation, we have, from Theorem 2.1:

\[ Z_t^{(K)} = \left( 1 - \frac{M_t}{K} \right)^+. \quad (2.131) \]

This implies that:

\[ d \langle M', Z^{(K)} \rangle_s = -1_{\{M_t < K\}} \frac{d \langle M', M \rangle_s}{K} = 0 \quad (2.132) \]

since \(M\) and \(M'\) are orthogonal. Hence, Lemma 2.2 and Theorem 2.6 are proven.

\(\square\)

### 2.8.3 Second Proof of Theorem 2.6, via Knight’s Representation of Orthogonal Continuous Martingales

It hinges upon the following:

**Lemma 2.3.** For every \(t \geq 0\), conditionally on \(\mathcal{F}_t\):

\[ \left( \sup_{u \geq t} M_u, \sup_{u \geq t} M'_u \right) \xrightarrow{(law)} \left( M_t, M'_t \right) \]

where, on the RHS of (2.133), \(U\) and \(U'\) are two independent r.v’s which are uniform on \([0, 1]\), and independent from \(\mathcal{F}_t\).

Let us admit for a moment Lemma 2.3, and let us prove Theorem 2.6. We have, since \(M_t \xrightarrow{t \to \infty} 0\) a.s. and \(M'_t \xrightarrow{t \to \infty} 0\) a.s.:
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\[ \mathbb{P}(\mathcal{G}_K \cup \mathcal{G}'_{K'} \leq t | \mathcal{F}_t) = \mathbb{P}\left( \left( \sup_{u \geq t} M_u < K \right) \cap \left( \sup_{u \geq t} M'_u < K' \right) | \mathcal{F}_t \right) \]

\[ = \mathbb{P}\left( \left( \frac{M_t}{U} < K \right) \cap \left( \frac{M'_t}{U'} < K' \right) | \mathcal{F}_t \right) \quad \text{(from Lemma 2.3)} \]

\[ = \mathbb{P}\left( \left( \frac{M_t}{K} < U \right) \cap \left( \frac{M'_t}{K'} < U' \right) | \mathcal{F}_t \right) \]

\[ = \left( 1 - \frac{M_t}{K} \right)^+ \left( 1 - \frac{M'_t}{K'} \right)^+ \]

which is Theorem 2.6.

We now give two proofs of Lemma 2.3.

**First proof of Lemma 2.3**

Replacing \((M_s, s \geq 0)\) and \((M'_s, s \geq 0)\) by \((M_t, t \geq 0)\) and \((M'_t, t \geq 0)\), it is enough to prove Lemma 2.3 for \(t = 0\). Let \(b \geq a, b' \geq a'\), \(T_b := \inf\{t \geq 0; M_t = b\}\) and \(T'_b := \inf\{t \geq 0; M'_t = b'\}\). Since \(M\) and \(M'\) are orthogonal, \((M_t \wedge T_b, M'_t \wedge T'_b, t \geq 0)\) is a bounded martingale. By Doob’s optional stopping Theorem, we have:

\[ a a' = \mathbb{E}\left[ M_{T_b} M'_{T'_b} \right] = b b' \mathbb{P}(T_b < \infty, T'_b < \infty) \]

since \(M_{T_b} = 0\) (resp. \(M'_{T'_b} = 0\)) on the set \(\{T_b = +\infty\}\) (resp. \(\{T'_b = +\infty\}\)). Thus:

\[ \mathbb{P}\left( \left\{ \sup_{s \geq 0} M_s > b \right\} \cap \left\{ \sup_{s \geq 0} M'_s > b' \right\} \right) = \mathbb{P}(T_b < \infty, T'_b < \infty) = \frac{a a'}{b b'} \]

which is Lemma 2.3.

**Second proof of Lemma 2.3**

From the Dambis, Dubins, Schwarz’s Theorem, there exist two Brownian motions \((\beta_u, u \leq T_0(\beta))\) and \((\beta'_u, u \leq T_0(\beta'))\) with \(T_0(\beta) := \inf\{t \geq 0; \beta_t = 0\}\) (resp. \(T_0(\beta') := \inf\{t \geq 0; \beta'_t = 0\}\)), started at \(a\) and \(a'\) such that:

\[ M_u = \beta_{(M)u}, \quad M'_u = \beta'_{(M')u}, \quad (2.134) \]

Moreover:

\[ \langle M \rangle_\infty = T_0(\beta), \quad \langle M' \rangle_\infty = T_0(\beta'). \quad (2.135) \]

Let us admit for a moment that the orthogonality of \(M\) and \(M'\) implies:

\( (\beta_u, u \leq T_0(\beta)) \) and \( (\beta'_u, u \leq T_0(\beta')) \) are independent, \( (2.136) \)

and let us show that (2.136) implies Lemma 2.3. As we have already mentioned it, it is sufficient to prove Lemma 2.3 for \(t = 0\). But, since from Lemma 2.1:
\[
\sup_{u \geq 0} M_u = \sup_{u \leq T_0(\beta)} \beta_u \overset{(law)}{=} \frac{a}{U},
\]

(2.136) implies then:

\[
\left( \sup_{u \geq 0} M_u, \sup_{u \geq 0} M'_u \right) = \left( \sup_{u \leq T_0(\beta)} \beta_u, \sup_{u \leq T_0(\beta')} \beta'_u \right) \overset{(law)}{=} \left( \frac{a}{U}, \frac{a'}{U'} \right)
\]

where, on the RHS of (2.137), \(U\) and \(U'\) are two independent r.v.’s uniform on \([0, 1]\).

It remains to prove (2.136)

Of course, when \(\langle M \rangle_\infty = \langle M' \rangle_\infty = +\infty\), relation (2.136) is the celebrated Knight’s Theorem on the representation of orthogonal martingales. The proof we shall present below is a variant of P.A. Meyer’s proof [56] of Knight’s Theorem. We consider two square integrable variables \(H\) and \(H'\) which are measurable with respect to \(\sigma(\beta_u, u \leq T_0(\beta))\) and \(\sigma(\beta'_u, u \leq T_0(\beta'))\). From Itô’s representation Theorem, they may be written as:

\[
H = \mathbb{E}[H] + \int_0^{T_0(\beta)} h_u d\beta_u, \quad H' = \mathbb{E}[H'] + \int_0^{T_0(\beta')} h'_u d\beta'_u
\]

(2.138)

where \((h_u, u \geq 0)\) and \((h'_u, u \geq 0)\) are two predictable processes with respect to the natural filtration of \((\beta_u \wedge T_0(\beta), u \geq 0)\) and \((\beta'_u \wedge T_0(\beta'), u \geq 0)\) such that:

\[
\mathbb{E} \left[ \int_0^{T_0(\beta)} (h_u)^2 du + \int_0^{T_0(\beta')} (h'_u)^2 du \right] < \infty.
\]

Now, after the change of variable \(u = \langle M \rangle_t\) (resp. \(u = \langle M' \rangle_t\)) in (2.138), we obtain:

\[
H = \mathbb{E}[H] + \int_0^\infty h_{\langle M \rangle_t} dM_t, \quad H' = \mathbb{E}[H'] + \int_0^\infty h'_{\langle M' \rangle_t} dM'_t,
\]

(2.139)

and, because of the orthogonality of \(M\) and \(M'\), and Itô’s formula:

\[
\mathbb{E}[HH'] = \mathbb{E}[H]\mathbb{E}[H'],
\]

which is Lemma 2.3.

\(\square\)

2.8.4 On the Law of \(\bigvee_{i=1, \ldots, n} \mathcal{G}_{K_i}^{(i)}\)

We now give the counterpart of Theorem 2.3 in the situation of Section 2.8, where we deal with several orthogonal local martingales. As previously, to simplify, we assume that \(n = 2\) and we make the following hypotheses:
i) The r.v. $M_t$ (resp. $M'_t$) admits a density $x \mapsto m_t(x)$ (resp. $x \mapsto m'_t(x)$) which is jointly continuous in $t$ and $x$.

ii) There exist two previsible processes $(\sigma_t, t \geq 0)$ and $(\sigma'_t, t \geq 0)$ such that:

$$d\langle M \rangle_t = \sigma_t^2 dt, \quad d\langle M' \rangle_t = (\sigma'_t)^2 dt.$$ 

Then, there is the following.

**Theorem 2.7.**

$$\mathbb{P}\left((G_K \lor G'_{K'}) \in dt\right) = \left(1 - \frac{M_0}{K}\right)^{+} \left(1 - \frac{M'_0}{K'}\right)^{+} \delta_0(dt) + \gamma_{K,K'}(t)dt \quad (t \geq 0)$$  

(2.140)

with

$$\gamma_{K,K'}(t) = \frac{1}{2K} \mathbb{E}\left[\sigma_t^2 \left(1 - \frac{M'_t}{K'}\right)^{+} | M_t = K\right] m_t(K)$$

$$\quad + \frac{1}{2K'} \mathbb{E}\left[\left(\sigma'_t\right)^2 \left(1 - \frac{M'_t}{K'}\right)^{+} | M'_t = K'\right] m'_t(K')$$  

(2.141)

where all the functions of several variables appearing in (2.141) admit jointly continuous versions.

The proof of (2.140) is essentially the same as that of Theorem 2.3. We start by writing, applying Theorem 2.6:

$$\mathbb{P}\left((G_K \lor G'_{K'}) \leq t\right) = \mathbb{E}\left[\left(1 - \frac{M_t}{K}\right)^{+} \left(1 - \frac{M'_t}{K'}\right)^{+}\right]$$  

(2.142)

and we develop the RHS of (2.142) thanks to Tanaka’s formula to obtain:

$$\mathbb{P}\left((G_K \lor G'_{K'}) \leq t\right) = \left(1 - \frac{M_0}{K}\right)^{+} \left(1 - \frac{M'_0}{K'}\right)^{+}$$

$$\quad + \frac{1}{2} \left(\mathbb{E}\left[\int_{0}^{t} \left(1 - \frac{M'_s}{K'}\right)^{+} \frac{dL^K_s}{K}\right] + \mathbb{E}\left[\int_{0}^{t} \left(1 - \frac{M'_s}{K'}\right)^{+} \frac{dL^{K'}_{s}}{K'}\right]\right)$$

(since $M$ and $M'$ are orthogonal) where $(L^K_s, s \geq 0)$ (resp. $(L^{K'}_{s}, s \geq 0)$) denotes the local time of $M$ at level $K$ (resp. of $M'$ at level $K'$). And, as for Theorem 2.3, the proof can now be ended by applying the occupation density formula.

**Problem 2.1 (On time inversion of a Lévy process).**

1) Let $X_1, X_2, \ldots, X_n, \ldots$ denote a sequence of i.i.d. integrable r.v’s. Define:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \quad (n \geq 1).$$
i) Prove that, for every \( 1 \leq i \leq n \), \( \mathbb{E}[X_i | S_n] = \frac{S_n}{n} \).

ii) Let \( F_{n+1}^+ := \sigma(S_{n+1}, S_{n+2}, \ldots) \). Prove that, for every \( n \geq 1 \):

\[
\mathbb{E} \left[ \frac{S_n}{n} | F_{n+1}^+ \right] = \frac{S_{n+1}}{n+1}.
\]

iii) Define:\( \nu \)

iv) Prove that the process \( \Lambda_t, t \geq 0 \) is an integrable Lévy process, i.e., for any \( t \geq 0 \), \( \mathbb{E}[|\Lambda_t|] < \infty \).

i) Prove that \( \mathbb{E}[\Lambda_t] = t \mathbb{E}[\Lambda_1] \).

ii) Let \( F_t^+ := \sigma(\Lambda_s; s \geq t) \). Prove that \( \left( \frac{1}{t} \Lambda_t, t \geq 0 \right) \) is a \( F_t^+ \)-inverse martingale, i.e., for every \( s < t \):

\[
\mathbb{E} \left[ \frac{\Lambda_t}{s} | F_t^+ \right] = \frac{\Lambda_t}{t}.
\]

Hint: Use discretization and 1) (see [36]).

iii) Define:

\[
M_t = \begin{cases} 
  t \Lambda_t & \text{if } t > 0, \\
  \mathbb{E}[\Lambda_1] & \text{if } t = 0.
\end{cases}
\]  

(1)

Prove that \( (M_t, t \geq 0) \) is a martingale with respect to the filtration \( \mathcal{G} := \{ \sigma(\Lambda_u, u \geq \frac{1}{t}), t > 0 \} \), which is the natural filtration of \( (M_t, t \geq 0) \). (Note that \( \Lambda \) being a càdlàg process, the martingale \( M \) as defined by (1) is left-continuous (and right-limited). To get a right-continuous process, one should take \( M_t := t \Lambda_t(\xi_t^+) \).

iv) Prove that the process \( (M_t, t \geq 0) \) is an (inhomogeneous) Markov process.

v) Identify \( (M_t, t \geq 0) \) when \( \Lambda_t \) is a Brownian motion with drift.

3) We now assume that \( (\Lambda_t, t \geq 0) \) is an integrable subordinator, with Lévy measure \( \nu \), and without drift term. Thus, there is the Lévy-Khintchine formula:

\[
\mathbb{E}[e^{-\lambda \Lambda_t}] = \exp \left( -t \int_0^\infty (1 - e^{-x}) \nu(dx) \right) \quad \text{with} \quad \int_0^\infty (x \wedge 1) \nu(dx) < \infty. \quad (2)
\]

i) Prove that the integrability of \( \Lambda_t \) is equivalent to \( \int_0^\infty x \nu(dx) < \infty \).

ii) Observe that the martingale \( (M_t, t \geq 0) \) has no positive jumps.

iii) Prove that \( M_t \to 0 \) a.s.; to start with, one may show that \( M_t \to 0 \) in law.

iv) Prove that, for every \( K \leq \mathbb{E}[\Lambda_1] \) and every \( t \geq 0 \):

\[
\frac{1}{K} \mathbb{E} \left[ (K - M_t)^+ \right] = \mathbb{P} \left( \mathcal{G}_K^{(M)} \leq t \right)
\]

with \( \mathcal{G}_K^{(M)} := \sup \{ t \geq 0; M_t = K \} \). (Theorem 2.1 may be useful).

v) Let, for every \( K < \mathbb{E}[\Lambda_1] \), \( T_K^{(\Lambda)} := \inf \{ u \geq 0; \Lambda_u \geq Ku \} \). Prove that \( T_K^{(\Lambda)} \) is finite a.s. and that \( \frac{1}{T_K^{(\Lambda)}} \) a.s. Deduce that, for every \( t \geq 0 \):
\[ \mathbb{P} \left( \tilde{T}^{(A)}_K \geq t \right) = \frac{1}{K} \mathbb{E} \left[ (K - M_t)^+ \right]. \]

vi) Compute explicitly the law of \( \tilde{T}^{(A)}_K \) when \((\Lambda_t, t \geq 0)\) is the gamma subordinator. Recall that the density \( f_{\Lambda_t} \) of \( \Lambda_t \) is then equal to:

\[ f_{\Lambda_t}(x) = \frac{1}{\Gamma(t)} e^{-x^t} 1_{\{x > 0\}}. \tag{4} \]

Solve the same question when \((\Lambda_t, t \geq 0)\) is the Poisson process with parameter \( \lambda \).

4) We now complete the results of question 3) when \((\Lambda_t, t \geq 0)\) is the gamma subordinator. We recall that the density of \( \Lambda_t \) is given by (4), and that its Lévy measure \( \nu \) is given by:

\[ \nu(dx) = e^{-x} dx. \tag{5} \]

i) Exploit the fact that, conditionally upon \( \gamma_1 \mid s = y \), for \( s < t \), is given by:

\[ \gamma_1 \mid s = \gamma \beta \left( \frac{1}{t}, \frac{1}{s} - \frac{1}{t} \right) \tag{6} \]

where \( \beta \left( \frac{1}{t}, \frac{1}{s} - \frac{1}{t} \right) \) is a beta variable with parameters \( \frac{1}{t} \) and \( \frac{1}{s} - \frac{1}{t} \), in order to show that, for every \( \alpha > 0 \):

\[ \left( M^{(\alpha)}_t = \frac{\Gamma \left( \frac{1}{t} \right) \left( \gamma_1 \right)^{\alpha}}{\Gamma \left( \alpha + \frac{1}{t} \right)}, t \geq 0 \right) \] is a \((\mathcal{G}_t, t \geq 0)\) martingale. \( \tag{7} \)

ii) Using (7) with \( \alpha = 2 \), deduce that the bracket of \((M_t, t \geq 0)\) is given by:

\[ \langle M \rangle_t = \int_0^t \frac{M^2_s}{1 + s} ds. \]

iii) Prove more generally that:

\[ d\langle M^{(\alpha)}_t, M^{(\beta)}_t \rangle = -M^{(\alpha)}_t M^{(\beta)}_t \left. \frac{\theta_{\alpha, \beta}(t)}{\theta_{\alpha, \beta}(t)} \right| d\]

where \( \theta_{\alpha, \beta}(t) = \frac{\Gamma(\alpha + \frac{1}{t}) \Gamma(\beta + \frac{1}{t})}{\Gamma(\frac{1}{t}) \Gamma(\alpha + \beta + \frac{1}{t})} \).

iv) Prove that the infinitesimal generator \( \tilde{L} \) of the Markov process \((M_t, t \geq 0)\) satisfies:

\[ \tilde{L}f(s,x) = \frac{xf''(x)}{s} + \frac{1}{s^2} \int_0^1 (f(xz) - f(x)) \frac{z^{\frac{1}{2} - 1}}{1 - z} dz \tag{8} \]

for every \( C^1 \) function \( f \). Check that, for \( f_1(x) := x, \tilde{L}f_1 = 0 \).
v) Prove that, for every $C^2$ function $f$, $\lim_{s \to 0} \tilde{L}f(s,x) = \frac{x^2 f''(x)}{2}$.

vi) Use (8) to give another proof of (7) when $\alpha \in \mathbb{N}^*$.

vii) Use relation (7) to obtain, for any integer $n$, the law of $T_K^{(n)}$ where:

$$T_K^{(n)} := \inf \left\{ u \geq 0; \Lambda_u \geq [Ku(1 + u)(1 + 2u) \ldots (1 + (n - 1)u)]^{\frac{1}{2}} \right\}$$

(and $K < \mathbb{E}[\Lambda_1] = 1$).

5) We now improve upon the results of question 3) when $(\Lambda_t, t \geq 0)$ is the Poisson process of parameter 1 (and Lévy measure $\nu(dx) = \delta_1(dx)$).

i) Prove that, for $s < t$, conditionally upon $\Lambda_1 = y$, $\Lambda_t$ follows a binomial distribution $B(y, s/t)$ with parameters $y$ and $(s/t)$.

ii) Let $\tilde{L}$ denote the infinitesimal generator of the Markov process $(M_t, t \geq 0)$. Prove that:

$$\tilde{L}f(s,x) = \frac{xf'(x)}{s} + (f(x) - f(x+s)) \frac{x}{s^2}$$

for $f$ in $C^1$. Check that, for $f_1(x) = x$, $\tilde{L}f_1 = 0$.

iii) Prove that, for every $C^2$ function $f$, $\lim_{s \to 0} \tilde{L}f(s,x) = \frac{1}{2} xf''(x)$.

iv) Compute $\tilde{L}f_2(s,x)$, with $f_2(x) = x^2$. Deduce that $(M^2_t - tM_t, t \geq 0)$ is a $(\mathcal{G}_t, t \geq 0)$ martingale and that the bracket of $(M_t, t \geq 0)$ is given by:

$$\langle M \rangle_t = \int_0^t M_s ds.$$

v) Compute $\tilde{L}f_n(s,x)$, with $f_n(x) = x^n$ $(n \in \mathbb{N})$.

(Answer: $\tilde{L}f(s,x) = \sum_{j=1}^{n-1} \binom{n}{j-1} x^j (-s)^{n-1-j}$.)

### 2.9 Notes and Comments

The results of Sections 2.1, 2.2, 2.3, 2.4 and 2.7 are taken from the preprints of A. Bentata and M. Yor (see [5], [6] and [5, F], [6, F], [7, F]). The description of the European put and call in terms of last passage times also appears in D. Madan, B. Roynette and M. Yor (see [47] and [48]). The results of Section 2.5 are taken from J. Pitman and M. Yor ([65]). The study of the local martingale of Example 2.3.e, $(\frac{1}{X_t}, t \geq 0)$, where $(X_t, t \geq 0)$ is a Bessel process of dimension 3 is borrowed from Ju-Yi Yen and M. Yor ([93]) while the study of the càdlàg martingale in Section 2.6 is due to C. Profeta ([67]). The generalization of these results to the case of several orthogonal continuous local martingales is taken from B. Roynette, Y. Yano and M. Yor (see [74]).
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