Preface

In 1993, from 27 June to 1 July, I gave ten lectures for a CBMS–NSF conference, organized by Maria Schonbek at UCSC,\textsuperscript{1} Santa Cruz, CA. As I was asked to write lecture notes, I wrote the parts concerning homogenization and compensated compactness in the following years, but I barely started writing the part concerning $H$-measures.

In the fall of 1997, facing an increase in aggressiveness against me, I decided to put that project on hold, and I devised a new strategy to write lecture notes for the graduate courses that I was going to teach at CMU (\textit{Carnegie Mellon University})\textsuperscript{2,3} Pittsburgh, PA. After doing so for the courses that I taught in the spring of 1999 and in the spring of 2000, I made the texts available on the web page of CNA (Center for Nonlinear Analysis at CMU).

For the graduate course that I taught in the fall of 2001, I still needed to write the last four lectures, but I also prepared the last version of my CBMS–NSF course, from the summer of 1996, to make it also available on the web page of CNA, so that those who received a copy of various chapters would not be the only ones to know the content of those chapters that I wrote.

This led to a sharp increase of aggressiveness against me, so after putting my project on hold, I learned to live again in a hostile environment.

\textsuperscript{1} Maria Elena Schonbek, Argentinean-born mathematician. She worked at Northwestern University, Evanston, IL, at VPISU (Virginia Polytechnic Institute and State University), Blacksburg, VA, at University of Rhode Island, Kingston, RI, at Duke University, Durham, NC, and she now works at UCSC (University of California at Santa Cruz), Santa Cruz, CA.

\textsuperscript{2} Andrew Carnegie, Scottish-born businessman and philanthropist, 1835–1919. Besides endowing a technical school in Pittsburgh, PA, which became Carnegie Tech (Carnegie Institute of Technology) and then CMU (\textit{Carnegie Mellon University}) after it merged in 1967 with the Mellon Institute of Industrial Research, he funded about three thousand public libraries, and those in United States are named \textit{Carnegie libraries}.

\textsuperscript{3} Andrew William Mellon, American financier and philanthropist, 1855–1937. He founded the Mellon Institute of Industrial Research in Pittsburgh, PA, which merged in 1967 with Carnegie Tech (Carnegie Institute of Technology) to form CMU (\textit{Carnegie Mellon University}).
In the summer of 2002, I started revising my first two lecture notes by adding information about the persons whom I mention in the text, and for doing this I used footnotes, despite a warning by KNUTH [45] that footnotes tend to be distracting, but as he added “Yet Gibbon’s Decline and Fall would not have been the same without footnotes,” I decided not to restrain myself. I cannot say if my excessive use of footnotes resembles that of GIBBON, as I have not yet read The History of the Decline and Fall of the Roman Empire [34], but I wonder if the recent organized attacks on the western academic systems are following some of the reasons that GIBBON proposed for explaining the decline and the collapse of the mighty Roman empire.

Where should I publish my lecture notes once written? I found the answer in October 2002 at a conference at Accademia dei Lincei in Roma (Rome), Italy, when my good friends Carlo SBORDONE and Franco BREZZI mentioned their plan6,7 to have a series of lecture notes at UMI (Unione Matematica Italiana), published by Springer.8

I submitted my first lecture notes for publication in the summer of 2004, but I took a long time before making the requested corrections, and they appeared only in August 2006 as volume 1 of the UMI Lecture Notes series [116], An Introduction to Navier–Stokes Equation and Oceanography.9,10

I submitted my second lecture notes for publication in August 2006, and they appeared in June 2007 as volume 3 of the UMI Lecture Notes series [117], An Introduction to Sobolev Spaces and Interpolation Spaces.11

I submitted my third lecture notes for publication in January 2007 and they appeared in March 2008 as volume 6 of the UMI Lecture Notes series [119], From Hyperbolic Systems to Kinetic Theory, A Personalized Quest.

4 Donald Ervin KNUTH, American mathematician, born in 1938. He worked at Caltech (California Institute of Technology), Pasadena, CA, and at Stanford University, Stanford, CA.
5 Edward GIBBON, English historian, 1817–1877.
6 Carlo SBORDONE, Italian mathematician, born in 1948. He works at Università degli Studi di Napoli Federico II, Napoli (Naples), Italy. He was president of UMI (Unione Matematica Italiana) from 2000 to 2006.
7 Franco BREZZI, Italian mathematician, born in 1945. He works at Università degli Studi di Pavia, Pavia, Italy. He became president of UMI (Unione Matematica Italiana) in 2006.
8 Julius SPRINGER, German publisher, 1817–1877.
11 Sergei L’vovich SOBOLEV, Russian mathematician, 1908–1989. He worked in Leningrad, in Moscow, and in Novosibirsk, Russia. There is now a Sobolev Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk, Russia. I first met Sergei SOBOLEV when I was a student, in Paris in 1969, and conversed with him in French, which he spoke perfectly (all educated Europeans at the beginning of the twentieth century learned French).
In the summer of 2007, it was time for me to think again about my CBMS–NSF course. Because I already wrote lecture notes on how homogenization appears in optimal shape design [111] for lectures given during a CIME–CIM summer school, organized by Arrigo CELLINA and António ORNELAS,\textsuperscript{12,13} in Tróia, Portugal, in June 1998, I wrote this book in a different way, describing how my ideas in homogenization were introduced during my quest for understanding more about continuum mechanics and physics, so that chapters follow a loose chronological order.

As in my preceding lecture notes, I use footnotes for giving some biographical information about people related to what I mention, and in the text I use the first name of those whom I met. In my third lecture notes, I started putting at the end of each chapter the additional footnotes that are not directly related to the text but expand on some information found in previous footnotes; in this book, instead of presenting them in the order where the names appeared, I organized the additional footnotes in alphabetical order.

When one misses the footnote containing the information about someone, a chapter of biographical information at the end of the book permits one to find where the desired footnote is.

I may be wrong about some information that I give in footnotes, and I hope to be told about my mistakes, and that is true about everything that I wrote in the book, of course!

I want to thank my good friends Carlo SBORDONE and Franco BREZZI for their support, in general, and for the particular question of the publication of my lecture notes in a series of Unione Matematica Italiana.

I want to thank Carnegie Mellon University for according me a sabbatical period in the fall of 2007, and Politecnico di Milano for its hospitality during that time, at it was of great help for concentrating on my writing programme.

I want to thank Université Pierre et Marie Curie for a 1 month invitation at Laboratoire Jacques-Louis Lions,\textsuperscript{14,15} in May/June 2008, as it was during

\begin{footnotes}
\footnote{Arrigo CELLINA, Italian mathematician, born in 1941. He works at Università di Milano Bicocca, Milano (Milan), Italy.}
\footnote{António COSTA DE ORNELAS GONÇALVES, Portuguese mathematician, born in 1951. He works in Évora, Portugal.}
\footnote{Pierre CURIE, French physicist, 1859–1906, and his wife Marie SKŁODOWSKA-CURIE, Polish-born physicist, 1867–1934, received the Nobel Prize in Physics in 1903 in recognition of the extraordinary services they have rendered by their joint research on the radiation phenomena discovered by Professor Henri BECQUEREL, jointly with Henri BECQUEREL. Marie SKŁODOWSKA-CURIE also received the Nobel Prize in Chemistry in 1911 in recognition of her services to the advancement of chemistry by the discovery of the elements radium and polonium, by the isolation of radium, and the study of the nature and compounds of this remarkable element. They worked in Paris, France. Université Paris VI, Paris, is named after them, UPMC (Université Pierre et Marie Curie).}
\footnote{Jacques-Louis LIONS, French mathematician, 1928–2001. He received the Japan Prize in 1991. He worked in Nancy and in Paris, France; he held a chair (analyse}
this period that I wrote the last chapters of the book. I want to thank François Murat\textsuperscript{16} for his hospitality during my visits to Paris for almost 20 years and for his unfailing friendship for almost 40 years.

I could not publish my first three lecture notes and start the preparation of this fourth book without the support of Lucia Ostoni. I want to thank her for much more than providing the warmest possible atmosphere during my stays in Milano, because she gave me the stability that I lacked so much during a large portion of the last 30 years, so that I now feel safer for resuming my research, whose main goal is to give a sounder mathematical foundation to twentieth century continuum mechanics and physics.

Milano, June 2008

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PS: (Pittsburgh, August 2008) Although I finished writing the book at the end of June, while I was in Milano attending the last meeting of Instituto Lombardo before the summer, I still had to check the chapter on notation and to create an index, and while doing that, I realized that I should explain my choices in a better way, in particular the subject of Chap. 1.

My general goal is to understand in a better way the continuum mechanics and the physics of the twentieth century, that is, the questions where small scales appear, plasticity and turbulence on the one hand, atomic physics and phase transitions on the other, and I think that the General Theory of Homogenization (GTH) as I developed it is crucial for starting in the right direction, but as there are a few dogmas to change, if not to discard completely, in continuum mechanics and in physics, I need to explain why the difficulties are similar to those that appeared in religions, where the deadlocks still remain.

\textsuperscript{16}François Murat, French mathematician, born in 1947. He works at CNRS (Centre National de la Recherche Scientifique) and UPMC (Université Pierre et Marie Curie), in LJLL (Laboratoire Jacques-Louis Lions), Paris, France.
Describing my family background and my studies is a way to answer the question that should be asked in the future: among those who realized at the end of the twentieth century that some of the dogmas in continuum mechanics and physics had to be discarded as wrong and counter-productive, what explains how they could start thinking differently? Should I say that I do not know who else but myself fits in this category? I expect that by telling this story, more will be able to follow a path similar to mine in the future, that is, there will be more mathematicians interested in the other sciences than mathematics!

Because I use the words parables and gospels in the first sentence of Chap. 1, some may stop reading the book, but in the second sentence I explain why parables are like general theorems, and by the end of the second footnote at the bottom of the first page, one will already learn that I am no longer a Christian, so that any misunderstanding about my intentions should result from the prejudices of the reader against religions, which is not a scientific attitude, and at the end of the book it should be clear that many “scientists” behaved in the recent past like religious fundamentalists.

What I advocate is for all to use their brain in a critical way!

Additional footnotes: BECQUEREL,17 DUKE,18 Federico II,19 LUCAS H.,20 NOBEL,21 STANFORD,22

Detailed Description of Contents
a.b: refers to Corollary, Definition, Lemma, or Theorem # b in Chap. # a, while (a.b) refers to Eq. # b in Chap. # a.

Chapter 1: Why Do I Write?
   About my sense of duty.

Chapter 2: A Personalized Overview of Homogenization I
   About my understanding of homogenization in the 1970s.

17 Antoine Henri BECQUEREL, French physicist, 1852–1908. He received the Nobel Prize in Physics in 1903, in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity, jointly with Pierre CURIE and Marie SKlodowska-CURIE. He worked in Paris, France.
19 Friedrich VON HOHENSTAUFEN, German king, 1194–1250. Holy Roman Emperor, as Friedrich II, 1220–1250. He founded the first European state university in 1224, in Napoli (Naples), Italy, where he is known as Federico secondo, and Università degli Studi di Napoli is named after him.
21 Alfred Bernhard NOBEL, Swedish industrialist and philanthropist, 1833–1896. He created a fund to be used as awards for people whose work most benefited humanity.
22 Leland STANFORD, American businessman, 1824–1893. Stanford University is named after him (as is the city of Stanford, CA, where it is located).
**Chapter 3:** A Personalized Overview of Homogenization II

About my understanding of homogenization after 1980.

**Chapter 4:** An Academic Question of Jacques-Louis Lions

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My early difficulties about reading and writing, splitting some chapters into two parts, remarks on homogenization in optimal design, adapted microstructures for heat conduction and elasticity, remarks about three-point correlations, the difficulty of discovering useful generalizations, why periodicity assumptions are not so useful, when does the frequency of light play a role, the geometrical theory of diffraction (GTD) of Keller, about Bloch waves and the Bragg law for X-ray diffraction, about concentration effects, beyond partial differential equations and GTH.

35: Biographical Information

Basic biographical information for people whose name is associated with something mentioned in the book.

36: Abbreviations and Mathematical Notation

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