François Murat and myself were led to Theorem 6.5 and to the div–curl lemma 7.2 for purely mathematical reasons, but we learned from the work of Évariste Sanchez-Palencia that our work was related to an interesting question of continuum mechanics or physics, that of describing the relations between a “microscopic” level and a macroscopic level.

Some people thought that my use of microscopic was misleading, because it suggested the level of atoms, and that I should use the term mesoscopic instead (used to mean any intermediate scale between microscopic and macroscopic), but one should understand the difference between the point of view of a mathematician and that of a specialist of continuum mechanics, or of physics. If a mathematician studies oscillations and concentration effects compatible with a partial differential equation, it may happen that this equation was used for modeling something in continuum mechanics or physics; in that case, there is some physical intuition behind the equation, and this intuition may lead to conjectures about the mathematical properties of the solutions of the equation. The mathematician knows that some conjectures may be false, either because the equation is not a good model for the piece of reality that one thought, or because those who made the conjectures were misled for various reasons: the only way for a mathematician to be sure if the intuitions/conjectures are right is to prove theorems about the equation, and he/she should not be intimidated into confusing conjectures for theorems.¹

Actually, there were enough wrong guesses made on the side of specialists of continuum mechanics or physics for questions which belong to the mathematical theory of homogenization now that it exists (thanks to the pioneering work of Sergio Spagnolo, François Murat, and myself), that one should be careful not to accept conjectures without explaining if they are compatible with the actual state of the mathematical knowledge on the question.

Theorems in homogenization were showing me that a few things were wrongly guessed in continuum mechanics or physics. Using my approach to

¹ This mistake is common among mathematicians who try to become “applied” by speaking the language of engineers: they often show clearly to anyone who was well trained that they talk about questions that they do not understand.

different scales based on weak convergence for some quantities and other topologies like G/H-convergence for others, I understood what internal energy is, and what the div–curl lemma says for a model of electrostatics, and then about equipartition of hidden energy (which is not about counting degrees of freedom). I understood why the first principle of thermodynamics is obvious, and why thermodynamics is not about dynamics and is nonsense for describing what happens during evolution, because those who developed the theory did not understand what internal energy is, and how it could move around, and having called it heat and invoked probabilistic games was like having sacrificed to idols. In the late 1970s, looking at the homogenization problem for a first-order scalar hyperbolic equation, which shows nonlocal effects in the effective equation, I understood why quantum mechanics started wrongly, and what turbulence is about (which is not about playing probabilistic games).

In 1983, thanks to Robert Dautray who offered me a position at CEA,\textsuperscript{2} I read about the Dirac equation, and I understood a few things: how quantum mechanics went astray talking about nonexistent “particles” and letting the speed of light $c$ tend to $\infty$, what mass probably is, and what is wrong about the Boltzmann equation, because of completely unadapted ideas concerning “collisions.” I actually told Pierre-Louis Lions that the Boltzmann equation is not a good physical model,\textsuperscript{3} and I am surprised that he never mentioned the known defects of the equation.\textsuperscript{4} After that, I started thinking about a general programme for giving better mathematical foundations to twentieth century mechanics, plasticity and turbulence, and twentieth century physics, atomic physics and phase transitions; later, I called my programme beyond partial differential equations, after having already developed $H$-measures and variants, a hint of which I had in 1984, before an illuminating idea of Bostick about “electrons” [9].\textsuperscript{5}

\textsuperscript{2}I met with disaster in Orsay, France, opposing alone an infamous method of falsifying voting results, and unlike others who took the side of my political opponents against me (for reasons which they rarely explained), Robert Dautray gave me the possibility to escape this hell by giving me a job at CEA (Commissariat à l’Énergie Atomique), and he helped me a lot more by telling me what to read for my almost impossible task of understanding physics (in opposition to what physicists say, which is rarely the same thing!).


\textsuperscript{4}In 1983, I told Pierre-Louis Lions in too cryptic a way that the Boltzmann equation is not a good physical model, saying that only two mathematical questions remained, to let the “mean free path” go to 0, and to avoid the angular cut-off introduced by Harold Grad, and I told him in 1990 an idea for that, based on using restriction theorems on spheres.

\textsuperscript{5}I liked what Bostick wrote about “electrons” in his article of January 1985 [9], and it confirmed my idea about mass, and it gave me a more precise idea about other “particles,” but I could not guess what his idea about “photons” means.
For the basic example of a scalar second-order elliptic equation used for Theorem 6.5, I had already used a notation of *electrostatics*, despite the fact that real physical situations lead to symmetric cases. François MURAT and myself had no practical motivation for considering nonsymmetric cases, and we were just following a classical path for mathematicians, to understand more about our subject by finding the limitations of our framework, but in the mid 1980s, Graeme MILTON found a practical situation where nonsymmetric matrices arise, for \( N = 2 \), in connection with the (classical) Hall effect.\(^6\)

For *electrostatics*, a simplification of the Maxwell–Heaviside equation valid for stationary solutions with no magnetic field and no current, important quantities are the electrostatic potential \( U \), defined up to addition of a constant, the electric field \( E = -\nabla U \), the polarization field \( D \), the density of electric charge \( \rho \), and the density of electrostatic energy \( e = \frac{1}{2}(E,D) \). Moreover, the balance equation \( \text{div}(D) = \rho \), and the constitutive relation \( D = A E \) hold,\(^7\) where the *dielectric permittivity* \( A \) is a symmetric positive definite tensor.

The quantities indexed by \( n \), \( U_n \), \( E_n \), \( D_n \), \( \rho_n \), and \( e_n \), correspond to physical quantities at a small scale, which it is useful to call mesoscopic for recalling that a different set of equations is valid at the level of “atoms,”\(^8\) and at this mesoscopic level specialists of materials talk about grains with a given crystallographic orientations, arranged along grain boundaries to form a polycrystal, which may share interfaces with poly-crystals of different materials. Looking at this assembly from a macroscopic level, one would like to define macroscopic quantities, and understand what effective equations they satisfy. In my framework, macroscopic quantities are defined as weak limits (or strong limits) in natural spaces, here the weak convergence in \( H^1_{\text{loc}}(\Omega) \) for \( U \), the weak convergence in \( L^2(\Omega;\mathbb{R}^N) \) for \( E \) or \( D \), the weak convergence in \( M_b(\Omega) \) for \( e \), but for \( \rho \) the strong convergence in \( H^{-1}_{\text{loc}}(\Omega) \) appears as a technical constraint.

Actually, using weak convergence for relating different levels was not a new idea, and it was used implicitly each time a discrete distribution of masses or a discrete distribution of charges was replaced by a density of mass or a

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\(^6\) Edwin Herbert HALL, American physicist, 1855–1938. He worked at Harvard University, Cambridge, MA.

\(^7\) This is for a linear material, and nonlinear questions will be discussed later.

\(^8\) I conjecture that a good model is to consider the full Maxwell–Heaviside equations (in a vacuum, so that \( D = \varepsilon_0 E, B = \mu_0 H \) and \( \varepsilon_0 \mu_0 c^2 = 1 \)), coupled with the Dirac equation (without mass term). This is a semi-linear hyperbolic system, too difficult to study at the moment, and I conjecture that a homogenization process should give something like the models which are used, except that in the relation \( D = A E \), \( A \) is supposed to depend upon frequency, i.e., the operator is not local. After all, it is better to start at a mesoscopic level with the equations at hand and, after mentioning that they may not be such good physical models, to state clearly which mathematical questions are of interest!
density of charge, but in my framework, based on my joint work with François Murat on homogenization and compensated compactness, there is something new, related to handling some nonlinear effects.

As Sergio Spagnolo found before François Murat and myself, for \( N \geq 2 \), the effective dielectric permittivity \( A^{\text{eff}} \) is not obtained by computing weak limits of functions of \( A^n \), and one needs a new topology of weak type, G-convergence (or H-convergence for the nonsymmetric case); an intuitive reason for this difference is that one identifies \( A \) from measurements of \( E \) and \( D \) (by linear methods).\(^{9}\) The div–curl lemma provides another observation of a nonlinear character, that one has \( e^\infty = \frac{1}{2} (E^\infty, D^\infty) \), so that the same formula holds at mesoscopic level and at macroscopic level; in other words, in a context of electrostatics, it is not necessary to introduce an internal energy for keeping track of a part of the energy hidden at an intermediate level.

For fluids showing oscillations in their velocity field \( u^n \), converging weakly to \( u^\infty \), the limit of the density of kinetic energy \( \frac{1}{2} |u^n|^2 \) is then \( \frac{1}{2} |u^\infty|^2 + g\epsilon \), and the internal energy per unit of mass \( \epsilon \) is \( \geq 0 \), and may be \( > 0 \) at some places if the convergence is not strong.

When one understands that effective properties cannot always be computed from a few macroscopic averages/limits, one deduces that apart from the first principle of thermodynamics which is but the conservation of energy (when one does not forget about the different ways in which energy may be stored at an intermediate level), the rules of thermodynamics are rather misleading, because energy can be stored at various intermediate levels, in different modes, each one following its own rule for moving around, so that expecting a rule for the evolution of the sum is naive, if not rather silly!

Of course, specialists of continuum mechanics or physics dealt with effective quantities for a long time before precise mathematical definitions were proposed, and this is a perfectly normal behavior for engineers and physicists, because not much would be done in continuum mechanics, in physics, and in technology, if one waited for mathematicians to understand what equations to use for the phenomena which engineers and physicists were interested in. As one should expect for any game with unwritten rules, not all players understood the same thing, and a price paid is that a few guesses published in the engineering or physics literature were found to be incorrect when a

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\(^9\) For \( N = 1 \), \( \frac{1}{|A^n|} \to \frac{1}{|A^\infty|} \) in \( L^\infty(\Omega) \) weak *.

\(^{10}\) One measures values of \( U_n \) at a few points (actually averages on small sets), and by interpolation one obtains an approximation \( U_{\text{app}} \), which is near \( U^\infty \), from which one deduces an approximation \( E^{\text{app}} = -\text{grad}(U_{\text{app}}) \), which is near \( E^\infty \) in a weak topology. For \( D^n \), one measures fluxes through a few surfaces, and by interpolation one deduces an approximation \( D^{\text{app}} \), which is near \( D^\infty \) in a weak topology.
It also happens that wrong results are published in mathematical journals, since the refereeing process is not perfect, and mathematicians have not found a way to correct this type of mistake.

After François Murat and myself found that one cannot deduce the effective properties of a mixture from the proportions of materials used (which Sergio Spagnolo knew before us), apart from one-dimensional situations like the laminated cases that I shall discuss again in Chap. 12, I was quite puzzled to discover in the spring of 1974 that a book by Landau and Lifshitz contained a section giving a formula for the conductivity of a mixture. It was easy to guess why they implicitly considered that their mixtures would be isotropic, but they did not seem to know that when mixing two isotropic conductors (with conductivities $\alpha \neq \beta$) using given proportions, one may obtain various isotropic effective conductivities for the mixtures obtained, because they did not mention that their formula could only be an approximation! They should notice an obvious discrepancy, that their formula is not symmetric in $\alpha$ and $\beta$ if one uses a proportion of 50% for each material!

A year later, in May 1975, at a meeting at UMD, College Park, MD, I heard Ivo Babuška report on his checking the accuracy of the formulas published in the literature for a particular periodic design (where there is an effective value), and the range of answers was quite large. I shall describe in Chap. 21 the question of bounds for the effective conductivity of a mixture in terms of the proportions of its various components (valid for all possible arrangements at a small scale), and quite surprisingly, the formula derived by Landau and Lifshitz is a good approximation for the effective conductivity of an effectively isotropic mixture in the case of small amplitude oscillations for the conductivities of its isotropic components, a quite puzzling fact considering the complete absence of logical inference in their “derivation,” from an explicit computation of a sphere of one isotropic material embedded into an infinite medium made of the other isotropic material. This curious efficiency was my motivation for developing the theory of $H$-measures (Chap. 28), for proving formulas of small amplitude homogenization (Chap. 29).

Not much is known mathematically about what concerns realistic mixing processes, and grinding is understood by engineers to mean different things, depending upon grinding cereals or minerals with various degrees of hardness,

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11 In such cases, one should be careful to check if the mathematical framework proposed describes in a correct way all the cases which were considered.

12 It is more important in mathematics to avoid publishing wrong results, but I saw an example of a questionable behavior of rejecting a good paper on homogenization because an incompetent referee (who confused homogenization and $\Gamma$-convergence) thought that it contradicted a published paper on $\Gamma$-convergence, obviously published without a correct refereeing process.

13 Strangely enough, one person in the audience disagreed that there should be only one answer for this problem.
but no mathematician knows what it could mean.\textsuperscript{14} Having ground materials in fine powders, and poured them in given proportions into a container, other difficulties arise for defining terms like shaking and compressing the mixture, so that it becomes isotropic and traps no air.

The advantage of my framework, using various types of weak convergences, is that, unlike for the “ensemble averages” or the “thermodynamic limits” of the probabilistic methods, the notions that I use are adapted to partial differential equations, and to continuum mechanics or physics, and they help understand an adapted notion of distance between mixtures.

Electricity is another simplification of the Maxwell–Heaviside equation also valid for stationary solutions with no magnetic field, but now with a density of current satisfying the Ohm law \( j = \sigma E \) for a conductivity \( \sigma \), where \( \sigma \) is a symmetric positive definite tensor (whose inverse \( \sigma^{-1} \) is the resistivity tensor). Besides \( E = -\text{grad}(U) \), the equation of conservation of charge becomes \( \text{div}(j) = 0 \), and the div–curl lemma tells that one can pass to the limit in \( (j,E) \), which begs for a different physical interpretation.

The Ohm law comes out of the Lorentz force, that a “particle” or an “ion” of charge \( q_k \) and velocity \( v \) feels a force \( f = q_k (E + v \times B) \), which is \( q_k E \) because one assumes \( B = 0 \), but if one admits that this type of particle/ion is slowed down by a drag force \( -K_k v \), where \( K_k \) is a symmetric positive definite tensor, this type of particle/ion is accelerated to attain a limiting velocity \( v_k \),\textsuperscript{15} satisfying \( q_k E = K_k v_k \), i.e., \( v_k = q_k K_k^{-1} E \),\textsuperscript{16} and the density of current \( j \) is the average of \( q_k v_k \), which is then \( \sigma E \), with \( \sigma \) being the average of \( q_k^2 K_k^{-1} \).

Independent of this computation giving an intuition about the Ohm law, \( (j,E) \) is the average of \( (q_k v_k, E) \), which is the power of the Lorentz force (per unit volume).

Another application of the div–curl lemma concerns equipartition of hidden energy for a scalar wave equation in an open \( \Omega \subset \mathbb{R}^N \)

\[ \varrho \frac{\partial^2 u}{\partial t^2} - \text{div}_x (A \text{grad}_x(u)) = 0 \text{ in } \Omega \times (0,T), \quad (8.1) \]

where \( \varrho \) and \( A \) are independent of \( t \), and \( A \) is a symmetric tensor, so that any smooth solution of (8.1) satisfies the conservation law

\textsuperscript{14} Many mathematicians are already quite deluded about what concerns elastic behavior, so that going beyond an elastic range for considering plasticity and cracks is much beyond their grasp. Of course, using probabilistic ideas is just a way to sweep the dirt under the rug, and it cannot result in the cleaning process that mathematicians are responsible for, among models used by engineers or physicists.

\textsuperscript{15} As one mentions a limiting velocity, one must exclude high-frequency excitations and only consider either stationary situations or slow variations in time.

\textsuperscript{16} That force is related to acceleration is a crucial observation of Newton, and it might be because people observed limiting velocities, due to friction effects, that one thought earlier that force was proportional to velocity.
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \left| \frac{\partial u}{\partial t} \right|^2 + \frac{1}{2} (A \text{grad}_x(u), \text{grad}_x(u)) \right) - \text{div} \left( A \text{grad}_x(u) \frac{\partial u}{\partial t} \right) = 0, \quad (8.2) \]

which expresses the conservation of energy in the case where \( \rho \) is positive and \( A \) is positive definite, so that (8.1) is a wave equation. The density of energy is the sum of the density of kinetic energy \( \frac{1}{2} \rho \left| \frac{\partial u}{\partial t} \right|^2 \) and of the density of potential energy \( \frac{1}{2} (A \text{grad}_x(u), \text{grad}_x(u)) \). If a sequence of solutions \( u_n \) of (8.1) converges to 0 in \( H^1(\Omega \times (0,T)) \) weak, then one has

\[ \frac{1}{2} \rho \left| \frac{\partial u_n}{\partial t} \right|^2 - \frac{1}{2} (A \text{grad}_x(u_n), \text{grad}_x(u_n)) \rightharpoonup 0 \text{ in } L^1(\Omega \times (0,T)) \text{ weak *}, \quad (8.3) \]

i.e., in \( L^1(\Omega \times (0,T)) \) with the weak * topology of \( M_b(\Omega \times (0,T)) \), dual of \( C_0(\Omega \times (0,T)) \). One can construct such sequences of solutions of (8.1) by imposing Dirichlet conditions or Neumann conditions on \( \partial \Omega \) for instance, and initial data

\[ u_n(\cdot, 0) = v_n \rightharpoonup 0 \text{ in } H^1(\Omega) \text{ weak}, \quad \frac{\partial u_n}{\partial t}(\cdot, 0) = w_n \rightharpoonup 0 \text{ in } L^2(\Omega) \text{ weak}, \quad (8.4) \]

and (8.3) says that there is a macroscopic equipartition of hidden energy, i.e., the weak limits of the density of kinetic energy and of the density of potential energy are the same. Computing this common limit can be done with H-measures, which require more information on \( u_n \) and \( v_n \) than their weak limits.

In order to obtain (8.3) by applying the div–curl lemma, one replaces \( t \) by \( x_0 \), one defines \( E^n = \text{grad}_x u_n \) and one defines \( D^n \) by \( D^n_0 = \rho \frac{\partial u_n}{\partial t} \) and \( D^n_i = -(A \text{grad}_x(u^n)) \) for \( i = 1, \ldots, N \).

Of course, the form of equipartition of hidden energy that I obtained does not resemble that which I was taught by my physics teachers, where degrees of freedom are counted, but I finally understood what it really meant. Discussing the same question of equipartition of hidden energy for the Maxwell–Heaviside equation is quite instructive too, but it requires more than the div–curl lemma, and I shall describe it later, after the more general form of compensated compactness that I also developed with François Murat, and it will also use the framework of differential forms that I shall discuss now in Chap. 9.

Additional footnotes: Harold Grad.\(^{18}\)

\(^{17}\) One interpretation in the case \( N = 2 \) is to consider \( u \) as the vertical displacement of a membrane, \( \rho \) being its density and \( A \) being an elasticity tensor, so that in this model the density of potential energy has an elastic origin.

\(^{18}\) Harold Grad, American mathematician, 1923–1987. He worked at NYU (New York University), New York, NY.
The General Theory of Homogenization
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Tartar, L.
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