Preface

In 1929 S. Bochner (see [106]) found all families of polynomials satisfying a second-order differential equation with polynomial coefficients. This led to the continuous classical orthogonal polynomials named after Jacobi, Laguerre and Hermite. The Bessel polynomials also appear in this study by S. Bochner. However, these polynomials can only be seen as continuous classical orthogonal polynomials in the case of a *finite* system (in case of positive-definite orthogonality). The continuous classical orthogonal polynomials are treated in chapter 4.

In 1949 W. Hahn (see [261]) found orthogonal polynomial solutions of second-order q-difference equations. This class of families of orthogonal polynomials is known as the Hahn class of orthogonal polynomials.

Many other families of orthogonal polynomials such as the discrete classical orthogonal polynomials have been very well known for a long time, but a classification of all of these families did not exist. A first attempt to combine both the continuous and discrete classical orthogonal polynomials was made in 1985 by R. Askey and J.A. Wilson (see [72]) by introducing the so-called Askey scheme of hypergeometric orthogonal polynomials. In [72] the continuous classical orthogonal polynomials were introduced as limiting cases of the Wilson polynomials and the discrete classical orthogonal polynomials as limiting cases of the Racah polynomials. These polynomials are treated in chapter 7.

In [72] R. Askey and J.A. Wilson also introduced q-analogues of the Wilson polynomials, which are known as the Askey-Wilson polynomials.

We also mention the books [417] by A.F. Nikiforov and V.B. Uvarov (1988) and [416] by A.F. Nikiforov, S.K. Suslov and V.B. Uvarov (1991) in this perspective.

Furthermore we refer to [256], [275] and [507] for characterizations of the Askey-Wilson polynomials.

In 1994 the first and third author (see [318]) published a preliminary version of their report on *the Askey scheme of hypergeometric orthogonal polynomials and its q-analogue*. All known q-analogues of the families of orthogonal polynomials belonging to the Askey scheme were arranged into a q-analogue of this Askey scheme. In 1998 (see [319]) a completely revised and updated version of this report appeared, containing more formulas for these families of orthogonal polynomials. However, a classification of the orthogonal polynomials was still missing.

In the meanwhile, the second author studied the classification of several kinds of both continuous and discrete orthogonal polynomials. This led to several publications on orthogonal polynomial solutions of several kinds of eigenvalue problems. In [385] four types of q-orthogonal polynomials in x and three types of q-orthogonal polynomials in q^{-x} were introduced. These seven types form a comprehensive basis for the classical q-orthogonal polynomials in x and in q^{-x}. These polynomials are treated in chapter 10 and chapter 11.

The intention of this book is to give a classification of all families of classical orthogonal polynomials and their q-analogues, the classical q-orthogonal polynomials. In order to do this we make the following observations:

- We only consider *positive-definite orthogonality* in terms of an inner product.
- We consider both *infinite* and *finite* systems of orthogonal polynomials.

The characterization of classical orthogonal polynomials can be written in terms of *eigenvalue problems* as follows: the polynomial solutions of all degrees $n = 0,1,2,\ldots$ satisfy a three-term recurrence relation (with respect to the degree n) from which we can deduce necessary and sufficient conditions by use of the theorem of Favard. These conditions lead to all possible weight functions for orthogonality. It is also possible to have finite systems of orthogonal polynomials.

We consider the following cases:

1. *Second-order differential equations* (chapter 4).

The polynomial solutions lead to the continuous classical orthogonal polynomials; three infinite systems of Hermite, Laguerre and Jacobi polynomials and three finite systems of Jacobi, Bessel and pseudo Jacobi polynomials.

2. *Second-order difference equations with real coefficients* (chapter 5).

The polynomial solutions lead to the first part of the discrete classical orthogonal polynomials; two infinite systems of Charlier and Meixner polynomials and two finite systems of Krawtchouk and Hahn polynomials.

3. *Second-order difference equations with complex coefficients* (chapter 6).

The polynomial solutions lead to the second part of the discrete classical orthogonal polynomials; two infinite and finite systems of Meixner-Pollaczek and continuous Hahn polynomials.

Second-order difference equations can also be seen as three-term recurrence relations in terms of the argument x. The connection between the three-term recurrence relation with respect to the degree n and the one with respect to the argument x leads to the concept of *duality*.

4. The concept of *duality* leads to polynomial solutions in $x(x+u)$, with $x \in \mathbb{R}$ and $u \in \mathbb{R}$ a constant, of *second-order difference equations with real coefficients* (chapter 7).

The polynomial solutions lead to the third part of the discrete classical orthogonal polynomials; two infinite and finite systems of dual Hahn and Racah polynomials.

5. The concept of *duality* also leads to polynomial solutions in $z(z+u)$, with $z \in \mathbb{C}$ and $u \in \mathbb{R}$ a constant, of *second-order difference equations with complex coefficients* (chapter 8).

The polynomial solutions lead to the fourth part of the discrete classical orthogonal polynomials; two infinite and finite systems of continuous dual Hahn and Wilson polynomials.

In chapter 9 we list all families of hypergeometric orthogonal polynomials belonging to the Askey scheme. In each case we use the most common notation and we list the most important properties of the polynomials such as a representation as hypergeometric function, orthogonality relation(s), the three-term recurrence relation, the second-order differential or difference equation, the forward shift (or degree lowering) and backward shift (or degree raising) operator, a Rodrigues-type formula and some generating functions. Moreover, in each case we mention the connection between various families by given the appropriate limit relations.

6. Hahn's q-operator leads to eigenvalue problems in terms of *second-order q-difference equations* (chapter 10).

The polynomial solutions lead to the first part of the classical q-orthogonal polynomials, containing the Stieltjes-Wigert, the q-Laguerre, the little q-Jacobi, the little q-Laguerre, the q-Bessel, the big q-Jacobi, the big q-Laguerre, the Al-Salam-Carlitz and the discrete q-Hermite polynomials.

7. By changing x into q^{-x}, we obtain even more *second-order q-difference equations* (chapter 11).

The polynomial solutions lead to the second part of the classical q-orthogonal polynomials, containing the q-Meixner, the q-Krawtchouk, the quantum q-Krawtchouk, the affine q-Krawtchouk, the q-Hahn and the q-Charlier polynomials.

8. The concept of *duality* can also be applied to the case of q-orthogonal polynomials, which leads to polynomial solutions in $q^{-x}+uq^{x}$, with $x \in \mathbb{R}$ and $u \in \mathbb{R}$ a constant, of *second-order q-difference equations with real coefficients* (chapter 12).

The polynomial solutions lead to the third part of the classical q-orthogonal polynomials, containing the q-Racah, the dual q-Racah, the dual q-Hahn, the dual q-Krawtchouk and the dual q-Charlier polynomials.

9. By changing q^x into $\frac{z}{a}$, we also obtain polynomial solutions in $\frac{a}{z}+\frac{uz}{a}$, with $z \in \mathbb{C}$ and $u \in \mathbb{R}$ and $a \in \mathbb{C}$ constants, of *second-order q-difference equations with complex coefficients* (chapter 13).

Real polynomial solutions can only exist for $|z| = 1$. These lead to the fourth part of the classical q-orthogonal polynomials, containing the Askey-Wilson, the continuous q-Hahn, the continuous q-Jacobi, the continuous dual q-Hahn, the q-Meixner-Pollaczek, the Al-Salam-Chihara, the continuous q-Laguerre and the continuous (big) q-Hermite polynomials.

In chapter 14 we list all families of basic hypergeometric orthogonal polynomials belonging to the q-analogue of the Askey scheme. Again, in each case we use the most common notation and we list the most important properties of the polynomials such as a representation as basic hypergeometric function, orthogonality relation(s), the three-term recurrence relation, the second-order q-difference equation, the forward shift (or degree lowering) and backward shift (or degree raising) operator, a Rodrigues-type formula and some generating functions. Moreover, in each case we also indicate the limit relations between various families of q-orthogonal polynomials and the limit relations ($q \to 1$) to the classical hypergeometric orthogonal polynomials belonging to the Askey scheme.

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