Chapter 2
Polynomial Solutions of Eigenvalue Problems

2.1 Hahn's q-Operator

Let $\mathscr{P}$ denote the space of polynomials over $\mathbb{C}$.

In [261] W. Hahn introduced the linear operator $\mathscr{A}_{q,\omega}$ defined by

$$(\mathscr{A}_{q,\omega}p)(x) := \frac{p(qx+\omega)-p(x)}{qx+\omega-x}, \quad p \in \mathscr{P}, \quad x \in \mathbb{R} \setminus \left\{\frac{\omega}{1-q}\right\} \tag{2.1.1}$$

for all $q \in \mathbb{R} \setminus \{-1,0\}$[1], $\omega \in \mathbb{R}$ and $(q,\omega) \neq (1,0)$. This class of operators includes the q-derivative operator $\mathscr{D}_q$ ($\omega = 0$), the difference operator Δ ($q = 1$ and $\omega = 1$) and also the differentiation operator D as a limit case ($q \to 1$ and $\omega = 0$). In order to avoid the latter limiting process, we introduce the operator $\mathscr{A}_{q,\omega}$ in a second way. From (2.1.1) we obtain that $\mathscr{A}_{q,\omega}(1) = 0$, $\mathscr{A}_{q,\omega}(x) = 1$ and the product rule

$$(\mathscr{A}_{q,\omega}(p_1p_2))(x) = (\mathscr{A}_{q,\omega}p_1)(x)p_2(x) + p_1(qx+\omega)(\mathscr{A}_{q,\omega}p_2)(x) \tag{2.1.2}$$

for $p_1, p_2 \in \mathscr{P}$. Now we have

Theorem 2.1. *For all $q \in \mathbb{R} \setminus \{-1,0\}$ and $\omega \in \mathbb{R}$ there exists a unique linear operator $\mathscr{A}_{q,\omega}$ on $\mathscr{P}$ satisfying $\mathscr{A}_{q,\omega}(x) = 1$ and the product rule (2.1.2).*

Proof. The product rule (2.1.2) implies that $\mathscr{A}_{q,\omega}(1\cdot 1) = \mathscr{A}_{q,\omega}(1)\cdot 1 + 1\cdot \mathscr{A}_{q,\omega}(1)$. Hence we have $\mathscr{A}_{q,\omega}(1) = 0$. So $\mathscr{A}_{q,\omega}$ is uniquely defined on the basis $\{1, x, x^2, \ldots\}$ of the space $\mathscr{P}$ by

$$\mathscr{A}_{q,\omega}(x^{n+1}) = \mathscr{A}_{q,\omega}(x)x^n + (qx+\omega)\mathscr{A}_{q,\omega}(x^n), \quad n = 1,2,3,\ldots$$

and the initial values $\mathscr{A}_{q,\omega}(1) = 0$ and $\mathscr{A}_{q,\omega}(x) = 1$. □

[1] An essential property of the operator $\mathscr{A}_{q,\omega}$ is that its action on a polynomial of degree n leads to a polynomial of degree $n-1$ for all $n = 1,2,3,\ldots$. For $q = -1$ this property does not hold (for instance $\mathscr{A}_{-1,\omega}(x^2) = \omega$). That is why the case $q = -1$ is excluded.

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As a consequence of this theorem, (2.1.2) together with $\mathscr{A}_{q,\omega}(x) = 1$ can be seen as a definition of $\mathscr{A}_{q,\omega}$. This definition holds for all $q \in \mathbb{R} \setminus \{-1,0\}$ and $\omega \in \mathbb{R}$, and therefore includes the differentiation operator $D = \mathscr{A}_{1,0}$.

2.2 Eigenvalue Problems

We consider the eigenvalue problem[2]

$$\varphi(x)\left(\mathscr{A}_{q,\omega}^2 y_n\right)(x) + \psi(x)\left(\mathscr{A}_{q,\omega} y_n\right)(x) = \lambda_n y_n(qx+\omega) \tag{2.2.1}$$

for polynomials y_n of degree n, where $\mathscr{A}_{q,\omega}^2 y_n = \mathscr{A}_{q,\omega}\left(\mathscr{A}_{q,\omega} y_n\right)$ with $\lambda_n \in \mathbb{C}$ and $n \in \{0,1,2,\ldots\}$. In [106], [372] and [379], for instance, it is shown that polynomial solutions of any degree n, where $n \in \{0,1,2,\ldots\}$, can only exist if φ is a polynomial of degree at most 2 and ψ is a polynomial of degree 1, say

$$\varphi(x) = ex^2 + 2fx + g, \quad \psi(x) = 2\varepsilon x + \gamma, \quad e,f,g,\varepsilon,\gamma \in \mathbb{C}, \quad \varepsilon \neq 0. \tag{2.2.2}$$

In order to calculate the eigenvalues λ_n, we define (cf. (1.8.2))

$$[-1] = -\frac{1}{q}, \quad [0] = 0, \quad [n] = \sum_{k=0}^{n-1} q^k, \quad n = 1,2,3,\ldots.$$

Note that this definition holds for all $q \in \mathbb{R} \setminus \{-1,0\}$. Now we obtain

$$[n] - q^{n-k}[k] = [n-k], \quad n \geq k, \quad n,k \in \{0,1,2,\ldots\} \tag{2.2.3}$$

and

$$[n][n-1] - q^{n-k}[k][k-1] = [n-k][n+k-1], \quad n \geq k, \quad n,k \in \{0,1,2,\ldots\}. \tag{2.2.4}$$

Further we have

$$\begin{aligned} \mathscr{A}_{q,\omega}(x^n) &= \frac{(qx+\omega)^n - x^n}{qx+\omega-x} = \sum_{k=0}^{n-1} (qx+\omega)^{n-1-k} x^k \\ &= x^{n-1} \sum_{k=0}^{n-1} q^k + r(x) = [n]x^{n-1} + r(x), \quad n = 2,3,4,\ldots, \end{aligned} \tag{2.2.5}$$

where r is a polynomial of degree at most $n-2$. The eigenvalues λ_n can now be obtained by comparing the coefficients of x^n in (2.2.1):

$$\lambda_n = \frac{[n]}{q^n}\left(e[n-1] + 2\varepsilon\right), \quad n = 0,1,2,\ldots. \tag{2.2.6}$$

[2] It will turn out to be convenient to take $y_n(qx+\omega)$ on the right-hand side.

So we have: if (2.2.1) has a polynomial solution of degree n, then λ_n given by (2.2.6) is the corresponding eigenvalue.

Note that this result is also valid for $q = 1$ and $\omega = 0$.

So the eigenvalue problem (2.2.1) can be written in the form

$$\begin{aligned}(ex^2+2fx+g)\left(\mathscr{A}_{q,\omega}^2 y_n\right)(x) + (2\varepsilon x+\gamma)\left(\mathscr{A}_{q,\omega} y_n\right)(x)\\ = \frac{[n]}{q^n}(e[n-1]+2\varepsilon)y_n(qx+\omega), \quad n = 0,1,2,\ldots.\end{aligned} \tag{2.2.7}$$

In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = 1$) this reads

$$(ex^2+2fx+g)\left(\Delta^2 y_n\right)(x) + (2\varepsilon x+\gamma)\left(\Delta y_n\right)(x) = n(e(n-1)+2\varepsilon)y_n(x+1) \tag{2.2.8}$$

for $n = 0,1,2,\ldots$ and in the case of the differentiation operator D (i.e. $q = 1$ and $\omega = 0$)

$$(ex^2+2fx+g)y_n''(x) + (2\varepsilon x+\gamma)y_n'(x) = n(e(n-1)+2\varepsilon)y_n(x) \tag{2.2.9}$$

for $n = 0,1,2,\ldots$. We remark that the operator $\mathscr{A}_{q,\omega}$ is not invariant under translations for $q \neq 1$, unlike the operators Δ and D. This can be seen as follows. If we apply the operator $\mathscr{A}_{q,\omega}$ to the polynomial $p(\cdot + c)$, where $c \in \mathbb{R}$ is a constant, we obtain

$$\left(\mathscr{A}_{q,\omega} p(\cdot + c)\right)(x) = \frac{p(qx+\omega+c) - p(x+c)}{qx+\omega-x} \tag{2.2.10}$$

and, if we apply the operator $\mathscr{A}_{q,\omega}$ to the polynomial p first and then replace the argument x by $x + c$, we have

$$\begin{aligned}\left(\mathscr{A}_{q,\omega} p(\cdot)\right)(x+c) &= \frac{p(q(x+c)+\omega) - p(x+c)}{q(x+c)+\omega-(x+c)}\\ &= \frac{p(qx+\overline{\omega}+c) - p(x+c)}{qx+\overline{\omega}-x},\end{aligned} \tag{2.2.11}$$

where $\overline{\omega} = \omega + c(q-1)$. Both results coincide only if $q = 1$.

Now we will give another version of the operator equation (2.2.7) in the case that $(q,\omega) \neq (1,0)$. In order to do this, we write

$$\left(\mathscr{A}_{q,\omega} y_n\right)(x) = \frac{y_n(qx+\omega) - y_n(x)}{qx+\omega-x} =: p_n(x)$$

and

$$\begin{aligned}\left(\mathscr{A}_{q,\omega}^2 y_n\right)(x) = \left(\mathscr{A}_{q,\omega} p_n\right)(x) &= \frac{p_n(qx+\omega) - p_n(x)}{qx+\omega-x}\\ &= \frac{y_n(q^2x+[2]\omega) - (1+q)y_n(qx+\omega) + qy_n(x)}{q(qx+\omega-x)^2}.\end{aligned}$$

Then the operator equation (2.2.7) can be written in the form

$$\begin{aligned} C(qx+\omega)y_n(q^2x+[2]\omega) - \{C(qx+\omega)+D(qx+\omega)\}\, y_n(qx+\omega) \\ + D(qx+\omega)y_n(x) = \lambda_n y_n(qx+\omega), \end{aligned}$$

where

$$C(qx+\omega) = \frac{ex^2+2fx+g}{q(qx+\omega-x)^2} \quad \text{and} \quad D(qx+\omega) = qC(qx+\omega) - \frac{2\varepsilon x+\gamma}{qx+\omega-x}.$$

If we now replace x by $(x-\omega)/q$, we obtain the so-called *symmetric* form

$$C(x)y_n(qx+\omega) - \{C(x)+D(x)\}\, y_n(x) + D(x)y_n\left((x-\omega)/q\right) = \lambda_n y_n(x) \quad (2.2.12)$$

for $n = 0,1,2,\ldots$ with

$$\begin{aligned} C(x) = \frac{e(x-\omega)^2+2fq(x-\omega)+gq^2}{q(qx+\omega-x)^2} \\ \text{and} \quad D(x) = qC(x) - \frac{2\varepsilon(x-\omega)+\gamma q}{qx+\omega-x}. \end{aligned} \quad (2.2.13)$$

Finally, we will derive a third version of the operator equation (2.2.7), which involves the operators $\mathscr{A}_{q,\omega}$ and $\mathscr{A}_{1/q,-\omega/q}$. First of all we have

$$\begin{aligned} \left(\mathscr{A}_{q,\omega}y_n\right)\left((x-\omega)/q\right) &= \frac{y_n\left(q((x-\omega)/q)+\omega\right) - y_n((x-\omega)/q)}{q((x-\omega)/q)+\omega-(x-\omega)/q} \\ &= \frac{y_n((x-\omega)/q) - y_n(x)}{(x-\omega)/q - x} = \left(\mathscr{A}_{1/q,-\omega/q}y_n\right)(x). \end{aligned}$$

In the case of the q-derivative operator $\mathscr{D}_q$ (i.e. $\omega = 0$), this reads

$$\left(\mathscr{D}_q y_n\right)(x/q) = \left(\mathscr{A}_{q,0}y_n\right)(x/q) = \left(\mathscr{A}_{1/q,0}y_n\right)(x) = \mathscr{D}_{1/q}y_n(x).$$

In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = 1$), this reads

$$\left(\Delta y_n\right)(x-1) = \left(\mathscr{A}_{1,1}y_n\right)(x-1) = \left(\mathscr{A}_{1,-1}y_n\right)(x) =: \nabla y_n(x).$$

Now we have

$$\begin{aligned} \left(\mathscr{A}_{q,\omega}\left(\mathscr{A}_{1/q,-\omega/q}y_n\right)\right)(x) &= \frac{\left(\mathscr{A}_{1/q,-\omega/q}y_n\right)(qx+\omega) - \left(\mathscr{A}_{1/q,-\omega/q}y_n\right)(x)}{qx+\omega-x} \\ &= \frac{y_n(qx+\omega) - (1+q)y_n(x) + qy_n((x-\omega)/q)}{(qx+\omega-x)^2} \\ &= q^{-1}\left(\mathscr{A}^2_{q,\omega}y_n\right)\left((x-\omega)/q\right). \end{aligned}$$

Hence we obtain for $n = 0,1,2,\ldots$

$$q\varphi((x-\omega)/q)\left(\mathscr{A}_{q,\omega}\left(\mathscr{A}_{1/q,-\omega/q}y_n\right)\right)(x) \\ +\psi((x-\omega)/q)\left(\mathscr{A}_{1/q,-\omega/q}y_n\right)(x)=\lambda_n y_n(x). \tag{2.2.14}$$

In the case of the q-derivative operator $\mathscr{D}_q$ (i.e. $\omega = 0$), this reads

$$q\varphi(x/q)\left(\mathscr{D}_q\left(\mathscr{D}_{1/q}y_n\right)\right)(x)+\psi(x/q)\left(\mathscr{D}_{1/q}y_n\right)(x)=\lambda_n y_n(x) \tag{2.2.15}$$

for $n = 0,1,2,\ldots$. In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = 1$), this reads

$$\varphi(x-1)\left(\Delta\left(\nabla y_n\right)\right)(x)+\psi(x-1)\left(\nabla y_n\right)(x)=\lambda_n y_n(x), \quad n=0,1,2,\ldots. \tag{2.2.16}$$

2.3 The Regularity Condition

In this section we will point out in which cases the eigenvalue problem (2.2.1) has essentially unique polynomial solutions $y_n(x)$ of degrees $n = 0,1,2,\ldots,N$ for some positive integer N with possibly $N \to \infty$. Solutions are called essentially unique if they are determined up to a factor independent of x. We have

Theorem 2.2. *Let N denote a positive integer (possibly $N \to \infty$). Then the following statements are equivalent:*

1. *For each $n = 0,1,2,\ldots,N$ there exists a solution y_n of the eigenvalue problem (2.2.1) and all eigenspaces are one dimensional.*
2. *For $m,n \in \{0,1,2,\ldots,N\}$ with $m \neq n$ we have $\lambda_m \neq \lambda_n$.*

Proof. Assume that $\lambda_m = \lambda_n$ for $m \neq n$. Then there is either no polynomial solution for one of the degrees m and n or the solutions y_m and y_n belong to the same eigenspace. This shows that the first statement implies the second.

Now we use induction to show that the second statement implies the first. For $n = 0$ we have $\lambda_0 = 0$ and the one-dimensional eigenspace generated by $y_0(x) = 1$. Now we assume that $n \in \{1,2,3,\ldots\}$. Suppose that the polynomials $y_\nu(x)$ are solutions of degree ν for $\nu = 0,1,2,\ldots,n-1$. Then the (monic) polynomial $y_n(x)$ of degree n given by

$$y_n(x)=x^n+\sum_{\nu=0}^{n-1}\alpha_\nu y_\nu(x) \quad \text{with} \quad \alpha_\nu \in \mathbb{C} \tag{2.3.1}$$

is a solution of (2.2.1) if

$$\begin{aligned}
&\varphi(x)\mathscr{A}_{q,\omega}^2(x^n)+\psi(x)\mathscr{A}_{q,\omega}(x^n)\\
&\qquad+\varphi(x)\left(\mathscr{A}_{q,\omega}^2\sum_{\nu=0}^{n-1}\alpha_\nu y_\nu\right)(x)+\psi(x)\left(\mathscr{A}_{q,\omega}\sum_{\nu=0}^{n-1}\alpha_\nu y_\nu\right)(x)\\
&=\lambda_n\left((qx+\omega)^n+\sum_{\nu=0}^{n-1}\alpha_\nu y_\nu(qx+\omega)\right)
\end{aligned}$$

holds. The polynomial $\varphi(x)\mathscr{A}_{q,\omega}^2(x^n)+\psi(x)\mathscr{A}_{q,\omega}(x^n)$ has degree at most n. Hence we may write

$$\varphi(x)\mathscr{A}_{q,\omega}^2(x^n)+\psi(x)\mathscr{A}_{q,\omega}(x^n)=\beta_n(qx+\omega)^n+\sum_{\nu=0}^{n-1}\beta_\nu y_\nu(qx+\omega)$$

with $\beta_n,\beta_\nu\in\mathbb{C}$. Combining the last two equations, we get

$$\beta_n(qx+\omega)^n+\sum_{\nu=0}^{n-1}(\beta_\nu+\lambda_\nu\alpha_\nu)y_\nu(qx+\omega)=\lambda_n\left((qx+\omega)^n+\sum_{\nu=0}^{n-1}\alpha_\nu y_\nu(qx+\omega)\right)$$

and therefore

$$(\beta_n-\lambda_n)(qx+\omega)^n+\sum_{\nu=0}^{n-1}(\alpha_\nu(\lambda_\nu-\lambda_n)+\beta_\nu)y_\nu(qx+\omega)=0.$$

Since $\lambda_\nu\neq\lambda_n$, this implies that the numbers α_ν are uniquely determined by this equation. So in fact this means that the (monic) polynomial solution y_n given by (2.3.1) is uniquely determined, which implies that the corresponding eigenspace is one dimensional. □

Now we use (2.2.3) and (2.2.4) to find from (2.2.6):

$$\begin{aligned}
q^n(\lambda_n-\lambda_m)&=e([n][n-1]-q^{n-m}[m][m-1])+2\varepsilon([n]-q^{n-m}[m])\\
&=e[n-m][n+m-1]+2\varepsilon[n-m]\\
&=[n-m](e[n+m-1]+2\varepsilon),\quad n\geq m,\quad m,n\in\{0,1,2,\ldots\}.
\end{aligned}$$

Hence we have

$$\lambda_n-\lambda_m=\frac{[n-m]}{q^n}(e[n+m-1]+2\varepsilon),\quad n\geq m,\quad m,n\in\{0,1,2,\ldots\}. \tag{2.3.2}$$

Since $q\neq-1$, it follows that $[n-m]\neq0$ for $m\neq n$, so $\lambda_m\neq\lambda_n$ is equivalent to $e[n+m-1]+2\varepsilon\neq0$. Therefore, theorem 2.2 leads to:

Corollary 2.3. *Let N denote a positive integer (possibly $N\to\infty$). Then the eigenvalue problem (2.2.1) has polynomial solutions y_n of degree n for all $n=0,1,2,\ldots,N$ with one-dimensional eigenspaces if and only if the regularity condition*

$$e[n] + 2\varepsilon \neq 0, \quad n = 0, 1, 2, \ldots, 2N-2 \tag{2.3.3}$$

holds.

2.4 Determination of the Polynomial Solutions

We want to obtain a two-term recurrence relation for the coefficients of the polynomial solutions y_n of (2.2.1). In order to achieve this, we introduce so-called generalized binomial coefficients $\begin{bmatrix} x\,;\,c \\ n \end{bmatrix}$ with $x \in \mathbb{R}$, $c \in \mathbb{C}$ and $n \in \{0, 1, 2, \ldots\}$, such that

$$\left(\mathscr{A}_{q,\omega} \begin{bmatrix} \cdot\,;\,c \\ 0 \end{bmatrix}\right)(x) = 0 \quad \text{and} \quad \left(\mathscr{A}_{q,\omega} \begin{bmatrix} \cdot\,;\,c \\ n \end{bmatrix}\right)(x) = \begin{bmatrix} x\,;\,c \\ n-1 \end{bmatrix} \tag{2.4.1}$$

for $n = 1, 2, 3, \ldots$. This can be done by

$$\begin{bmatrix} x\,;\,c \\ 0 \end{bmatrix} := 1 \quad \text{and} \quad \begin{bmatrix} x\,;\,c \\ n \end{bmatrix} := \prod_{i=1}^{n} \frac{x + cq^{i-1} - [i-1]\omega}{[i]}, \quad n = 1, 2, 3, \ldots. \tag{2.4.2}$$

Note that these generalized binomial coefficients depend on q and ω. However, for ease of expression we omit these in the notation of the symbol $\begin{bmatrix} \; \end{bmatrix}$. In the case of the difference operator (i.e. $q = 1$ and $\omega = 1$) these generalized binomial coefficients reduce to the ordinary binomial coefficients and

$$\Delta \binom{x+c}{0} = 0 \quad \text{and} \quad \Delta \binom{x+c}{n} = \binom{x+c}{n-1}, \quad n = 1, 2, 3, \ldots.$$

In the case of the differentiation operator D (i.e. $q = 1$ and $\omega = 0$), we have

$$D(1) = 0 \quad \text{and} \quad D\frac{(x+c)^n}{n!} = \frac{(x+c)^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots.$$

2.4.1 First Approach

To simplify the expressions, we write $v = \omega + c(1-q)$. Then we have

$$\begin{bmatrix} x\,;\,c \\ n \end{bmatrix} = \prod_{i=1}^{n} \frac{x + c - [i-1]v}{[i]}, \quad n = 1, 2, 3, \ldots.$$

Now we write

$$y_n(x) = \sum_{k=0}^{n} a_{n,k} \begin{bmatrix} x\,;\,c \\ k \end{bmatrix}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \ldots. \tag{2.4.3}$$

For a given $a_{n,n} \neq 0$ we want to determine the other coefficients $a_{n,k}$ in such a way that $y_n(x)$ given by (2.4.3) satisfies (2.2.7). By using (2.2.3), (2.2.4), (2.4.1),

$$x \begin{bmatrix} x\,;c \\ k-1 \end{bmatrix} = [k] \begin{bmatrix} x\,;c \\ k \end{bmatrix} + ([k-1]v-c) \begin{bmatrix} x\,;c \\ k-1 \end{bmatrix}, \quad k=1,2,3,\ldots,$$

$$x^2 \begin{bmatrix} x\,;c \\ k-2 \end{bmatrix} = [k][k-1] \begin{bmatrix} x\,;c \\ k \end{bmatrix} + [k-1]\{([k-1]+[k-2])\,v-2c\} \begin{bmatrix} x\,;c \\ k-1 \end{bmatrix} + ([k-2]v-c)^2 \begin{bmatrix} x\,;c \\ k-2 \end{bmatrix}, \quad k=2,3,4,\ldots,$$

$$\begin{bmatrix} qx+\omega\,;c \\ k \end{bmatrix} = q^k \begin{bmatrix} x\,;c \\ k \end{bmatrix} + vq^{k-1} \begin{bmatrix} x\,;c \\ k-1 \end{bmatrix}, \quad k=1,2,3,\ldots,$$

by substituting (2.4.3) into (2.2.7) and by comparing the coefficients, we find

$$\begin{aligned}&[n-k]\,(e[n+k-1]+2\varepsilon)\,a_{n,k} + \Big\{e[n][n-1]v+2\varepsilon[n-k]v \\ &\quad - e[k]\,(([k]+[k-1])v-2c)\,q^{n-k} + (2\varepsilon c-2f[k]-\gamma)q^{n-k}\Big\}a_{n,k+1} \\ &\quad - \big(e([k]v-c)^2+2f([k]v-c)+g\big)\,q^{n-k}a_{n,k+2} = 0 \end{aligned} \tag{2.4.4}$$

for $k = n-1, n-2, n-3, \ldots, 0$ with the convention that $a_{n,n+1} := 0$. Hence, if the regularity condition (2.3.3) holds, then (2.4.4) gives us the coefficients $a_{n,k}$ for $k = n-1, n-2, n-3, \ldots, 0$ in terms of $a_{n,n} \neq 0$.

The three-term recurrence relation (2.4.4) can be rewritten in a such a way that the possibility of a two-term recurrence relation becomes apparent:

$$\begin{aligned}&[n-k]\,(e[n+k-1]+2\varepsilon)\,a_{n,k} - (e[k-1]v-2ec+2f)\,[k-1]q^{n-k+1}a_{n,k+1} \\ &\quad + [n-k-1]\,(e[n+k]+2\varepsilon)\,va_{n,k+1} - (e[k]v-2ec+2f)\,[k]vq^{n-k}a_{n,k+2} \\ &\quad + ((e+2\varepsilon)v+\{2c(e+\varepsilon)-\gamma-2f\}\,q)\,q^{n-k-1}a_{n,k+1} \\ &\quad - (ec^2-2fc+g)q^{n-k}a_{n,k+2} = 0.\end{aligned}$$

This implies that if

$$(e+2\varepsilon)v^2 + \{2c(e+\varepsilon)-\gamma-2f\}\,vq = -(ec^2-2fc+g)q^2 \tag{2.4.5}$$

holds, then the recurrence relation (2.4.4) can be written in the form

$$s(k)a_{n,k} + t(k)a_{n,k+1} + v\,\big(s(k+1)a_{n,k+1} + t(k+1)a_{n,k+2}\big) = 0$$

with

$$s(k) := [n-k]\,(e[n+k-1]+2\varepsilon)$$

and

$$t(k) := \left((e+2\varepsilon)v + \left\{2\varepsilon c - \gamma + 2(ec-f)[k] - e[k-1]^2 vq\right\} q\right) q^{n-k-1}.$$

Now we consider the two-term recurrence relation

$$s(k)a_{n,k} + t(k)a_{n,k+1} = 0, \quad k = n-1, n-2, n-3, \ldots, 0. \tag{2.4.6}$$

If there exists a number c satisfying (2.4.5) and if the coefficients $a_{n,k}$ satisfy (2.4.6), then they also satisfy (2.4.4). Therefore, the coefficients $a_{n,k}$ can be determined by (2.4.6) in terms of $a_{n,n} \neq 0$ provided that there exists a number c such that (2.4.5) holds.

In the case that $v \neq 0$, and c satisfies (2.4.5), we can write the two-term recurrence relation for the coefficients $a_{n,k}$ in a different form. If we multiply (2.4.6) by v and use (2.4.5), we obtain

$$\begin{aligned}&[n-k]\,(e[n+k-1]+2\varepsilon)\,v a_{n,k}\\ &\quad - \left\{e([k-1]v-c)^2 + 2f([k-1]v-c) + g\right\} q^{n-k+1} a_{n,k+1} = 0\end{aligned} \tag{2.4.7}$$

for $k = n-1, n-2, n-3, \ldots, 0$.

With this first approach the case of the differentiation operator D (i.e. $q = 1$ and $\omega = v = 0$) can be treated completely. If there exists a c satisfying (2.4.5), id est $ec^2 - 2fc + g = 0$, then the coefficients of the polynomial solutions

$$y_n(x) = \sum_{k=0}^{n} a_{n,k} \frac{(x+c)^k}{k!}, \quad a_{n,n} \neq 0, \quad n = 0, 1, 2, \ldots \tag{2.4.8}$$

of the second-order differential equation (2.2.9) satisfy the two-term recurrence relation (2.4.6), id est

$$(n-k)\,(e(n+k-1)+2\varepsilon)\,a_{n,k} + (2(ec-f)k + 2\varepsilon c - \gamma)\,a_{n,k+1} = 0 \tag{2.4.9}$$

for $k = n-1, n-2, n-3, \ldots, 0$. If (2.4.5) has no solution for c, id est $e = f = 0$ and $g \neq 0$, then we find from (2.4.4) with $c = \gamma/2\varepsilon$ the two-term recurrence relation

$$2\varepsilon(n-k)a_{n,k} - g a_{n,k+2} = 0, \quad a_{n,n+1} = 0, \quad k = n-1, n-2, n-3, \ldots, 0, \tag{2.4.10}$$

which leads to the (symmetric) Hermite polynomials.

2.4.2 Second Approach

In order to deal with the cases where c cannot be determined by (2.4.5), we use a second approach. For this purpose we will modify (2.4.3) in such a way that in the numerator of the generalized binomial coefficient $x + c$ occurs as the penultimate factor. By using $v = \omega + c(1-q)$ and the definition (2.4.2), we obtain

$$\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix} = \prod_{i=1}^{k} \frac{x+cq^{i-1}+[k-2]vq^{i+1-k}-[i-1]\omega}{[i]}$$
$$= \prod_{i=1}^{k} \frac{x+c+[k-i-1]vq^{i+1-k}}{[i]} \qquad (2.4.11)$$

for $k = 1,2,3,\ldots$. Note that

$$\left(\mathscr{A}_{q,\omega}\begin{bmatrix} \cdot\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix}\right)(x) = \begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-1 \end{bmatrix}, \quad k = 1,2,3,\ldots.$$

These alternative generalized binomial coefficients can be used to build a second form for the solutions:

$$y_n(x) = \sum_{k=0}^{n} b_{n,k} \begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix}, \quad b_{n,n} \neq 0, \quad n = 0,1,2,\ldots. \qquad (2.4.12)$$

For a given $b_{n,n} \neq 0$ we want to determine the other coefficients $b_{n,k}$ in such a way that $y_n(x)$ given by (2.4.12) satisfies (2.2.7). By using

$$x\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-1 \end{bmatrix} = [k]\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix} + (v-c)\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-1 \end{bmatrix}$$

for $k = 1,2,3,\ldots$,

$$x^2\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-2 \end{bmatrix} = [k][k-1]\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix}$$
$$+ [k-1](v-2c)\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-1 \end{bmatrix} + c^2\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-2 \end{bmatrix}$$

for $k = 2,3,4,\ldots$,

$$x\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-2 \end{bmatrix} = [k-1]\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-1 \end{bmatrix} - c\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-2 \end{bmatrix}$$

for $k = 2,3,4,\ldots$ and

$$\begin{bmatrix} qx+\omega\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix} = q^k\begin{bmatrix} x\,;\,c+[k-1]vq^{1-k} \\ k \end{bmatrix}$$
$$= q^k\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k \end{bmatrix} + vq\begin{bmatrix} x\,;\,c+[k-2]vq^{2-k} \\ k-1 \end{bmatrix}$$

for $k = 1,2,3,\ldots$, we find by substituting (2.4.12) into (2.2.7)

$$\sum_{k=0}^{n}[n-k]\,(e[n+k-1]+2\varepsilon)\,q^k\begin{bmatrix}x\,;\,c+[k-1]vq^{1-k}\\ k\end{bmatrix}b_{n,k}$$
$$+\sum_{k=1}^{n}\left\{(e[k-1]+2\varepsilon)\left([k]q^{1-k}-1\right)vq^n+(2[k-1](ec-f)+2\varepsilon c-\gamma)\,q^n\right\}$$
$$\times\begin{bmatrix}x\,;\,c+[k-2]vq^{2-k}\\ k-1\end{bmatrix}b_{n,k}$$
$$-\sum_{k=2}^{n}(ec^2-2fc+g)q^n\begin{bmatrix}x\,;\,c+[k-2]vq^{2-k}\\ k-2\end{bmatrix}b_{n,k}=0.$$

Hence, if $ec^2-2fc+g=0$ can be solved for c, only two sums remain. These lead to a two-term recurrence relation for the coefficients $b_{n,k}$:

$$[n-k]\,(e[n+k-1]+2\varepsilon)\,b_{n,k}+\Big\{(e[k]+2\varepsilon)([k+1]q^{-k}-1)v$$
$$+\,(2[k](ec-f)+2\varepsilon c-\gamma)\Big\}\,q^{n-k}b_{n,k+1}=0 \tag{2.4.13}$$

for $k=n-1,n-2,n-3,\ldots,0$.

The case of the difference operator Δ (i.e. $q=\omega=v=1$) can be treated by using both approaches. The first approach leads to solutions of the form

$$y_n(x)=\sum_{k=0}^{n}a_{n,k}\binom{x+c}{k},\quad a_{n,n}\neq 0,\quad n=0,1,2,\ldots, \tag{2.4.14}$$

where c is a solution of (2.4.5), id est

$$e(c+1)^2-2f(c+1)+g=-2\varepsilon(c+1)+\gamma$$

and the coefficients satisfy (2.4.6), which reads

$$(n-k)\,(e(n+k-1)+2\varepsilon)\,a_{n,k}$$
$$-\left(e(k-1-c)^2+2f(k-1-c)+g\right)a_{n,k+1}=0 \tag{2.4.15}$$

for $k=n-1,n-2,n-3,\ldots,0$. The second approach leads to solutions of the form

$$y_n(x)=\sum_{k=0}^{n}b_{n,k}\binom{x+c+k-2}{k},\quad b_{n,n}\neq 0,\quad n=0,1,2,\ldots. \tag{2.4.16}$$

If c satisfies $ec^2-2fc+g=0$, then the coefficients satisfy the two-term recurrence relation (2.4.13), which reads

$$(n-k)\,(e(n+k-1)+2\varepsilon)\,b_{n,k}$$
$$+\left(ek^2+2(ec-f+\varepsilon)k+2\varepsilon c-\gamma\right)b_{n,k+1}=0 \tag{2.4.17}$$

for $k=n-1,n-2,n-3,\ldots,0$.

2.5 Existence of a Three-Term Recurrence Relation

In this section we will show that the monic polynomial solutions y_n of the operator equation (2.2.1) satisfy a three-term recurrence relation of the form

$$y_{n+1}(x) = (x - c_n)y_n(x) - d_n y_{n-1}(x), \quad c_n, d_n \in \mathbb{C}, \quad n = 1, 2, 3, \ldots. \tag{2.5.1}$$

For simplification we introduce the operator $\mathscr{S}_{q,\omega}$ on the space $\mathscr{P}$ of polynomials defined by

$$\left(\mathscr{S}_{q,\omega} p\right)(x) := p(qx + \omega), \quad p \in \mathscr{P}, \quad x \in \mathbb{R}. \tag{2.5.2}$$

Then the following commutation relations hold:

$$\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega} = q\mathscr{S}_{q,\omega}\mathscr{A}_{q,\omega} \quad \text{and} \quad \mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega} = q\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}^{-1}. \tag{2.5.3}$$

Further we will use the notation

$$\widehat{p}(x) := \left(\mathscr{S}_{q,\omega}^{-1} p\right)(x) = p((x - \omega)/q), \quad p \in \mathscr{P}, \quad x \in \mathbb{R} \tag{2.5.4}$$

for convenience.

Theorem 2.4. *Let Λ be a linear functional on $\mathscr{P}$ defined by*

$$\Lambda[1] = 1 \quad \textit{and} \quad \Lambda[y_n] = 0, \quad n = 1, 2, 3, \ldots, \tag{2.5.5}$$

where y_n denotes a polynomial solution of the operator equation (2.2.1). Then the following distributional equation holds for every polynomial $p \in \mathscr{P}$:

$$\Lambda[q\widehat{\varphi}(\mathscr{A}_{q,\omega} p) + \widehat{\psi} p] = 0. \tag{2.5.6}$$

Furthermore, for all $m, n \in \{0, 1, 2, \ldots\}$ we have

$$\Lambda\left[q\widehat{\varphi}\left(\mathscr{A}_{q,\omega} y_m\right)\left(\mathscr{A}_{q,\omega} y_n\right)\right] = -\lambda_n \Lambda[y_m y_n]. \tag{2.5.7}$$

Proof. By using (2.5.2), we obtain from (2.2.1) for every polynomial

$$p^*(x) = \sum_{k=0}^{n} \alpha_k y_k(x), \quad \alpha_k \in \mathbb{C}$$

that

$$\begin{aligned}
&\varphi(x)\left(\mathscr{A}_{q,\omega}^2 p^*\right)(x) + \psi(x)\left(\mathscr{A}_{q,\omega} p^*\right)(x) \\
&= \sum_{k=0}^{n} \alpha_k \left\{\varphi(x)\left(\mathscr{A}_{q,\omega}^2 y_k\right)(x) + \psi(x)\left(\mathscr{A}_{q,\omega} y_k\right)(x)\right\} \\
&= \sum_{k=0}^{n} \alpha_k \lambda_k \left(\mathscr{S}_{q,\omega} y_k\right)(x).
\end{aligned}$$

Applying $\mathscr{S}_{q,\omega}^{-1}$ to both sides and using the commutation relation (2.5.3), we obtain by using the notation (2.5.4)

$$q\widehat{\varphi}(x)\left(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}p^*\right)(x)+\widehat{\psi}(x)\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}p^*\right)(x)=\sum_{k=0}^{n}\alpha_k\lambda_k y_k(x).$$

From (2.2.5) it can be deduced that for each polynomial $p\in\mathscr{P}$ there exists a polynomial p^* such that

$$p(x)=\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}p^*\right)(x).$$

Now we use the fact that $\lambda_0=0$ and $\Lambda[y_k]=0$ for $k=1,2,3,\ldots$ to conclude that

$$\Lambda\left[q\widehat{\varphi}\left(\mathscr{A}_{q,\omega}p\right)+\widehat{\psi}p\right]=\sum_{k=0}^{n}\alpha_k\lambda_k\Lambda[y_k]=0,$$

which proves (2.5.6).

To prove (2.5.7), we apply $\mathscr{S}_{q,\omega}^{-1}$ to the operator equation (2.2.1), use the commutation relation (2.5.3) and multiply the result by y_m:

$$q\widehat{\varphi}(x)\left(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)(x)y_m(x)+\widehat{\psi}(x)\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)(x)y_m(x)=\lambda_n y_m(x)y_n(x).$$

If we apply the product rule (2.1.2) to $p_1(x)=\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)(x)$ and $p_2(x)=y_m(x)$, then we find

$$\begin{aligned}&\left(\mathscr{A}_{q,\omega}\left(\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)y_m\right)\right)(x)\\&=\left(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)(x)y_m(x)+\left(\mathscr{A}_{q,\omega}y_n\right)(x)\left(\mathscr{A}_{q,\omega}y_m\right)(x).\end{aligned}$$

Combining the last two results, we find that

$$\begin{aligned}q\widehat{\varphi}(x)&\left\{\left(\mathscr{A}_{q,\omega}\left(\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)y_m\right)\right)(x)-\left(\mathscr{A}_{q,\omega}y_n\right)(x)\left(\mathscr{A}_{q,\omega}y_m\right)(x)\right\}\\&+\widehat{\psi}(x)\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)(x)y_m(x)=\lambda_n y_m(x)y_n(x).\end{aligned}$$

Finally, we apply Λ on both sides of this equation to obtain

$$\begin{aligned}&\Lambda\left[q\widehat{\varphi}\left(\mathscr{A}_{q,\omega}\left(\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)y_m\right)\right)+\widehat{\psi}\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)y_m\right]\\&\quad-\Lambda\left[q\widehat{\varphi}\left(\mathscr{A}_{q,\omega}y_n\right)\left(\mathscr{A}_{q,\omega}y_m\right)\right]=\lambda_n\Lambda[y_m y_n].\end{aligned}$$

The first term equals zero in view of (2.5.6) with $p(x)=\left(\mathscr{S}_{q,\omega}^{-1}\mathscr{A}_{q,\omega}y_n\right)(x)y_m(x)$. This completes the proof of (2.5.7). □

The following theorem states the ”weak” orthogonality of the polynomial solutions of the eigenvalue problem (2.2.1).

Theorem 2.5. *Let the regularity condition (2.3.3) hold for the operator equation (2.2.1) with polynomial solutions y_n with $n=0,1,2,\ldots$. Then the linear functional Λ given by (2.5.5) satisfies*

$$\Lambda[y_m y_n] = 0 \quad \textit{for} \quad m \neq n, \quad m,n \in \{0,1,2,\ldots\}. \tag{2.5.8}$$

Proof. From (2.5.7) we get for $m,n \in \{0,1,2,\ldots\}$

$$\begin{cases} \Lambda\left[q\widehat{\varphi}\left(\mathscr{A}_{q,\omega}y_n\right)\left(\mathscr{A}_{q,\omega}y_m\right)\right] = -\lambda_n \Lambda[y_m y_n] \\ \\ \Lambda\left[q\widehat{\varphi}\left(\mathscr{A}_{q,\omega}y_m\right)\left(\mathscr{A}_{q,\omega}y_n\right)\right] = -\lambda_m \Lambda[y_m y_n]. \end{cases}$$

Subtracting these two equations, we find

$$(\lambda_n - \lambda_m)\Lambda[y_m y_n] = 0, \quad m,n \in \{0,1,2,\ldots\}.$$

The regularity condition (2.3.3) implies that $\lambda_m \neq \lambda_n$ for $m \neq n$, which implies (2.5.8). □

If the linear functional Λ in the preceding theorem is quasi-definite, it is well known that a recurrence relation of the form (2.5.1) exists for all $n = 0,1,2,\ldots$. See for instance [146]. Now we will prove:

Theorem 2.6. *With the assumptions of the preceding theorem, assume that there exists a number $N \in \{1,2,3\ldots\}$ such that*

$$\Lambda[y_n^2] \neq 0, \quad \textit{for} \quad n = 0,1,2,\ldots,N-1 \quad \textit{and} \quad \Lambda[y_N^2] = 0. \tag{2.5.9}$$

Then there exists a three-term recurrence relation of the form

$$y_{n+1}(x) = (x - c_n)y_n(x) - d_n y_{n-1}(x), \quad n = 1,2,3,\ldots,N+1 \tag{2.5.10}$$

with $d_N = 0$.

Proof. The monic polynomial y_{n+1} can be written as

$$y_{n+1}(x) = x y_n(x) + \sum_{k=0}^{n} \alpha_k^{(n)} y_k(x), \quad \alpha_k^{(n)} \in \mathbb{C}, \quad n = 0,1,2,\ldots. \tag{2.5.11}$$

Now we want to show that $\alpha_k^{(n)} = 0$ for $k = 0,1,2,\ldots,n-2$. To do this we multiply this equation by y_ν for $\nu \in \{0,1,2,\ldots,n-2\}$ and apply the linear functional Λ to both sides, which leads to

$$\alpha_k^{(n)} \Lambda[y_k^2] = 0, \quad k = 0,1,2,\ldots,n-2.$$

Now we use (2.5.9) to conclude that $\alpha_k^{(n)} = 0$ for $k = 0,1,2,\ldots,n-2$ for all $n = 2,3,4,\ldots,N+1$, which proves (2.5.10) with $c_n = -\alpha_n^{(n)}$ and $d_n = -\alpha_{n-1}^{(n)}$.

In order to show that $d_N = 0$, we start with (2.5.10) for $n = N$, multiply by y_{N-1} and apply the linear functional Λ to find

$$\Lambda[y_{N+1}y_{N-1}] = \Lambda[x y_N y_{N-1}] - c_N \Lambda[y_N y_{N-1}] - d_N \Lambda[y_{N-1}^2].$$

Since $\Lambda[xy_N y_{N-1}] = \Lambda[y_N^2] = 0$ and $\Lambda[y_{N-1}^2] \neq 0$, we get

$$d_N = \frac{\Lambda[y_N^2]}{\Lambda[y_{N-1}^2]} = 0.$$

This completes the proof. □

Now we will show that a finite system $\{y_n\}_{n=0}^N$ of polynomial solutions of the eigenvalue problem (2.2.1) which satisfies a three-term recurrence relation of the form (2.5.10) can be extended to an infinite system $\{y_n\}_{n=0}^\infty$ of polynomials which satisfies a three-term recurrence relation of the form (2.5.1). First we will prove:

Theorem 2.7. *Let the monic solutions $\{y_n\}_{n=0}^N$ of the eigenvalue problem (2.2.1) satisfy the three-term recurrence relation (2.5.10) and let the regularity condition (2.3.3) with $N \to \infty$ hold. Then we may write*

$$y_{N+k} = \widetilde{y}_k y_N, \quad k = 0, 1, 2, \ldots \tag{2.5.12}$$

with monic polynomials $\widetilde{y}_k$ of degree k which are solutions of the eigenvalue problem

$$\widetilde{\varphi}\left(\mathscr{A}_{q,\omega}^2 \widetilde{y}_k\right) + \widetilde{\psi}\left(\mathscr{A}_{q,\omega}\widetilde{y}_k\right) = \widetilde{\lambda}_k\left(\mathscr{S}_{q,\omega}\widetilde{y}_k\right), \quad k = 0, 1, 2, \ldots, \tag{2.5.13}$$

where $\widetilde{\lambda}_k = \lambda_{N+k} - \lambda_N$, $\widetilde{\varphi}$ is a polynomial of degree at most 2 with $\widetilde{\varphi} \neq 0$ and $\widetilde{\psi}$ is a polynomial of degree 1 exactly.

Proof. Since $d_N = 0$, we have from (2.5.10)

$$y_{N+1}(x) = (x - c_N)y_N(x) = \widetilde{y}_1(x)y_N(x) \quad \text{with} \quad \widetilde{y}_1(x) = x - c_N$$

and

$$\begin{aligned} y_{N+2}(x) &= (x - c_{N+1})y_{N+1}(x) - d_{N+1}y_N(x) \\ &= \left((x - c_{N+1})(x - c_N) - d_{N+1}\right)y_N(x) = \widetilde{y}_2(x)y_N(x) \end{aligned}$$

with $\widetilde{y}_2(x) = (x - c_{N+1})(x - c_N) - d_{N+1}$. Together with $\widetilde{y}_0(x) = 1$ this proves (2.5.12) for $k = 0, 1, 2$.

Substitution of $y_{N+k} = \widetilde{y}_k y_N$ in the eigenvalue problem (2.2.1) gives

$$\varphi\left(\mathscr{A}_{q,\omega}^2\left(\widetilde{y}_k y_N\right)\right) + \psi\left(\mathscr{A}_{q,\omega}\left(\widetilde{y}_k y_N\right)\right) = \lambda_{N+k}\left(\mathscr{S}_{q,\omega}\left(\widetilde{y}_k y_N\right)\right).$$

By using the product rule (2.1.2), we find

$$\mathscr{A}_{q,\omega}\left(\widetilde{y}_k y_N\right) = \left(\mathscr{A}_{q,\omega}y_N\right)\left(\mathscr{S}_{q,\omega}\widetilde{y}_k\right) + y_N\left(\mathscr{A}_{q,\omega}\widetilde{y}_k\right),$$

$$\mathscr{A}_{q,\omega}\left(\widetilde{y}_k y_N\right) = \left(\mathscr{A}_{q,\omega}y_N\right)\widetilde{y}_k + \left(\mathscr{S}_{q,\omega}y_N\right)\left(\mathscr{A}_{q,\omega}\widetilde{y}_k\right)$$

and

$$\mathscr{A}_{q,\omega}^2(\widetilde{y}_k y_N) = \left(\mathscr{A}_{q,\omega}^2 y_N\right)(\mathscr{S}_{q,\omega}\widetilde{y}_k) + (\mathscr{A}_{q,\omega}y_N)(\mathscr{A}_{q,\omega}\widetilde{y}_k) \\ + \left(\mathscr{S}_{q,\omega}^2 y_N\right)\left(\mathscr{A}_{q,\omega}^2\widetilde{y}_k\right) + (\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}y_N)(\mathscr{A}_{q,\omega}\widetilde{y}_k).$$

Hence

$$\varphi\left(\mathscr{S}_{q,\omega}^2 y_N\right)\left(\mathscr{A}_{q,\omega}^2\widetilde{y}_k\right) + (\varphi(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}y_N) + \varphi(\mathscr{A}_{q,\omega}y_N) + \psi y_N)(\mathscr{A}_{q,\omega}\widetilde{y}_k) \\ + \left(\varphi\left(\mathscr{A}_{q,\omega}^2 y_N\right) + \psi(\mathscr{A}_{q,\omega}y_N)\right)(\mathscr{S}_{q,\omega}\widetilde{y}_k) = \lambda_{N+k}(\mathscr{S}_{q,\omega}y_N)(\mathscr{S}_{q,\omega}\widetilde{y}_k).$$

Together with

$$\varphi\left(\mathscr{A}_{q,\omega}^2 y_N\right) + \psi(\mathscr{A}_{q,\omega}y_N) = \lambda_N(\mathscr{S}_{q,\omega}y_N),$$

we obtain

$$\varphi\left(\mathscr{S}_{q,\omega}^2 y_N\right)\left(\mathscr{A}_{q,\omega}^2\widetilde{y}_k\right) + (\varphi(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}y_N) + \varphi(\mathscr{A}_{q,\omega}y_N) + \psi y_N)(\mathscr{A}_{q,\omega}\widetilde{y}_k) \\ = (\lambda_{N+k} - \lambda_N)(\mathscr{S}_{q,\omega}y_N)(\mathscr{S}_{q,\omega}\widetilde{y}_k).$$

This equation holds for $k = 0,1,2$. For $k = 1$ this reads

$$(\varphi(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}y_N) + \varphi(\mathscr{A}_{q,\omega}y_N) + \psi y_N)C_1 = (\lambda_{N+1} - \lambda_N)(\mathscr{S}_{q,\omega}y_N)(\mathscr{S}_{q,\omega}\widetilde{y}_1)$$

with $C_1 = \mathscr{A}_{q,\omega}\widetilde{y}_1 (\neq 0)$. Since $\lambda_{N+1} - \lambda_N \neq 0$, the polynomial on the left-hand side of this equation contains $\mathscr{S}_{q,\omega}y_N$ as a factor. Hence the polynomial $\widetilde{\psi}$ can be defined by

$$\widetilde{\psi} := \frac{\varphi(\mathscr{A}_{q,\omega}\mathscr{S}_{q,\omega}y_N) + \varphi(\mathscr{A}_{q,\omega}y_N) + \psi y_N}{\mathscr{S}_{q,\omega}y_N}.$$

For $k = 2$ we find

$$\varphi C_2\left(\mathscr{S}_{q,\omega}^2 y_N\right) + (\mathscr{A}_{q,\omega}\widetilde{y}_2)\,\widetilde{\psi}(\mathscr{S}_{q,\omega}y_N) = (\lambda_{N+2} - \lambda_N)(\mathscr{S}_{q,\omega}y_N)(\mathscr{S}_{q,\omega}\widetilde{y}_2)$$

with $C_2 = \mathscr{A}_{q,\omega}^2\widetilde{y}_2 (\neq 0)$. Since $\lambda_{N+2} - \lambda_N \neq 0$, this implies that $\mathscr{S}_{q,\omega}y_N$ divides $\varphi\left(\mathscr{S}_{q,\omega}^2 y_N\right)$. Hence the polynomial $\widetilde{\varphi}$ can be defined by

$$\widetilde{\varphi} := \frac{\varphi\left(\mathscr{S}_{q,\omega}^2 y_N\right)}{\mathscr{S}_{q,\omega}y_N}.$$

So we have found that

$$\widetilde{\varphi}\left(\mathscr{A}_{q,\omega}^2\widetilde{y}_k\right) + \widetilde{\psi}(\mathscr{A}_{q,\omega}\widetilde{y}_k) = \widetilde{\lambda}_k(\mathscr{S}_{q,\omega}\widetilde{y}_k), \quad \widetilde{\lambda}_k = \lambda_{N+k} - \lambda_N.$$

If (2.3.2) is used, the regularity condition (2.3.3) implies that

$$\widetilde{\lambda}_k = \lambda_{N+k} - \lambda_N = \frac{[k]}{q^{N+k}}(e[2N+k-1] + 2\varepsilon)(\neq 0), \quad k = 1,2,3,\ldots.$$

This proves that for (2.5.13) polynomial solutions of all degrees exist. □

The preceding theorem allows us to prove the existence of a recurrence relation of the form (2.5.1). Let $d_n \neq 0$ for $n = 1,2,3,\ldots,N-1$ and $d_N = 0$. Then we have the recurrence relation (2.5.10). The preceding theorem shows that this recurrence relation can be continued as long as it is possible to continue the recurrence relation

$$\widetilde{y}_{k+1}(x) = (x-\widetilde{c}_k)\widetilde{y}_k(x) - \widetilde{d}_k\widetilde{y}_{k-1}(x)$$

for $\widetilde{y}_k$. This recurrence relation either holds for all $k = 0,1,2,\ldots$ or there is a number $K \in \{1,2,3,\ldots\}$ such that $\widetilde{d}_k \neq 0$ for $k = 0,1,2,\ldots,K-1$ and $\widetilde{d}_K = 0$. Then the preceding theorem can be applied again. This process can be continued until we arrive at:

Theorem 2.8. *If the regularity condition (2.3.3) with $N \to \infty$ holds, then there exist numbers $c_n, d_n \in \mathbb{C}$ such that the polynomial solutions $\{y_n\}_{n=0}^{\infty}$ of the eigenvalue problem (2.2.1) satisfy a three-term recurrence relation of the form (2.5.1).*

2.6 Explicit Form of the Three-Term Recurrence Relation

To determine the polynomial solutions in section 2.4, we needed a two-term recurrence relation for the coefficients. In this section we will not need such a two-term recurrence relation. Therefore we only need the representation (2.4.3) and the three-term recurrence relation (2.4.4). We will not need (2.4.5) here.

In the case of the differentiation operator D (i.e. $q = 1$ and $\omega = \nu = 0$), we simply use the form

$$y_n(x) = \sum_{k=0}^{n} a_{n,k}\frac{x^k}{k!}, \quad a_{n,n} \neq 0, \quad n = 0,1,2,\ldots.$$

Since we do not need a two-term recurrence relation for the coefficients, it suffices to take $c = 0$. In that case (2.4.4) reduces to the three-term recurrence relation

$$(n-k)\left(e(n+k-1)+2\varepsilon\right)a_{n,k} - (2fk+\gamma)a_{n,k+1} - g a_{n,k+2} = 0 \qquad (2.6.1)$$

for $k = n-1, n-2, n-3, \ldots, 0$ with $a_{n,n+1} := 0$.

In the case of the q-derivative operator $\mathscr{D}_q$ (i.e. $q \neq 1$ and $\omega = 0$), we need an extra observation in order to deal with the lack of translation invariance. Recall that (2.2.10) and (2.2.11) imply that

$$\left(\mathscr{A}_{q,\omega}p(\cdot)\right)(x+c) = \left(\mathscr{A}_{q,\overline{\omega}}p(\cdot+c)\right)(x)$$

with $\overline{\omega} = \omega + c(q-1)$ and $c \in \mathbb{R}$. For $q \neq 1$ we can put $c = \omega/(1-q)$, which yields $\overline{\omega} = 0$. Hence, since $\mathscr{D}_q := \mathscr{A}_{q,0}$, the operator equation (2.2.1) can be written in terms of the q-derivative operator as

$$\varphi(x+c)\left(\mathscr{D}_q^2 y_n(\cdot+c)\right)(x) + \psi(x+c)\left(\mathscr{D}_q y_n(\cdot+c)\right)(x) = \lambda_n y_n(qx+c), \quad n = 0,1,2,\ldots \qquad (2.6.2)$$

with $c = \omega/(1-q)$. This translation does not affect the possible orthogonality. In fact the regularity condition (2.3.3) is preserved since the leading coefficients of $\varphi(x)$ and $\psi(x)$ are equal to the leading coefficients of $\varphi(x+c)$ and $\psi(x+c)$, respectively. Similarly, the recurrence relation (2.5.1) for the polynomials y_n is transformed into the recurrence relation

$$y_{n+1}(x+c) = (x-c_n+c)y_n(x+c) - d_n y_{n-1}(x+c), \quad n = 0,1,2,\ldots \tag{2.6.3}$$

for the polynomials $y_n(x+c)$. While d_n remains unchanged, c_n is replaced by $c_n - c$. However, c_n is real if and only if $c_n - c$ is real. Together with the theorem by Favard (see the next chapter) this implies that the polynomials y_n are orthogonal in the positive-definite or quasi-definite sense if and only if this is the case for the polynomials $y_n(x+c)$. In the case of the q-derivative operator $\mathscr{D}_q$, we have $\omega = 0$ and therefore $c = \omega/(1-q) = 0$ and $v = \omega + c(1-q) = 0$. If we set $\omega = c = v = 0$ into (2.4.4), we obtain the three-term recurrence relation

$$[n-k]\,(e[n+k-1]+2\varepsilon)\,a_{n,k} - (2f[k]+\gamma)\,q^{n-k}a_{n,k+1} - gq^{n-k}a_{n,k+2} = 0 \tag{2.6.4}$$

for $k = n-1, n-2, n-3, \ldots, 0$ with $a_{n,n+1} := 0$. Note that (2.6.4) for $q = 1$ equals (2.6.1). This implies that the case of the differentiation operator D needs no special treatment.

As a generalization of the difference operator for $q = 1$ and $\omega \neq 0$, we use another observation. From the definition (2.1.1) it follows that

$$\begin{aligned}(\mathscr{A}_{q,\omega}p(\,\cdot\,))\,(\rho x) &= \frac{p(q\rho x+\omega) - p(\rho x)}{q\rho x + \omega - \rho x} \\ &= \frac{p(\rho(qx+\omega/\rho)) - p(\rho x)}{\rho(qx+\omega/\rho - x)} = \frac{1}{\rho}\left(\mathscr{A}_{q,\omega/\rho}p(\rho\,\cdot)\right)(x)\end{aligned}$$

for $\rho \neq 0$ and $p \in \mathscr{P}$. For $q = 1$ and $\rho = \omega(\neq 0)$ this yields $\mathscr{A}_{q,\omega/\rho} = \mathscr{A}_{1,1} = \Delta$ and the operator equation (2.2.1) reads

$$\frac{1}{\omega^2}\varphi(\omega x)\left(\Delta^2 y_n(\omega\,\cdot)\right)(x) + \frac{1}{\omega}\psi(\omega x)\left(\Delta y_n(\omega\,\cdot)\right)(x) = \lambda_n y_n(\omega x + \omega). \tag{2.6.5}$$

Similar to the case $q \neq 1$ and $\overline{\omega} = 0$ above, it can easily be seen by means of the three-term recurrence relation that the possible orthogonality is not affected by the dilatation $x \mapsto \omega x$.

In the case of the difference operator Δ (i.e. $q = 1$ and $\omega = v = 1$), we may use the form

$$y_n(x) = \sum_{k=0}^{n} a_{n,k}\binom{x}{k}, \quad a_{n,n} \neq 0, \quad n = 0,1,2,\ldots. \tag{2.6.6}$$

Since we do not need a two-term recurrence relation for the coefficients, we may simply take $c = 0$. In that case (2.4.4) reads the three-term recurrence relation

$$
\begin{aligned}
&(n-k)\,(e(n+k-1)+2\varepsilon)\,a_{n,k}\\
&\quad+\{e\,(n(n-1)-k(2k-1))+2\varepsilon(n-k)-2fk-\gamma\}\,a_{n,k+1}\\
&\quad-(ek^2+2fk+g)a_{n,k+2}=0
\end{aligned}
\tag{2.6.7}
$$

for $k=n-1,n-2,n-3,\ldots,0$ with $a_{n,n+1}:=0$.

In order to obtain the explicit form of the three-term recurrence relation (2.5.1), we substitute the monic polynomials

$$
y_n(x)=\sum_{k=0}^{n}\alpha_k^{(n)}x^k,\quad \alpha_n^{(n)}=1,\quad n=0,1,2,\ldots \tag{2.6.8}
$$

in the recurrence relation (2.5.1) and $y_1(x)=x-c_0$. Comparison of the coefficients of x^n and x^{n-1} yields

$$
c_0=-\alpha_0^{(1)}\quad\text{and}\quad c_n=\alpha_{n-1}^{(n)}-\alpha_n^{(n+1)},\quad n=1,2,3,\ldots \tag{2.6.9}
$$

and

$$
d_1=-\alpha_0^{(2)}-c_1\alpha_0^{(1)}\quad\text{and}\quad d_n=\alpha_{n-2}^{(n)}-\alpha_{n-1}^{(n+1)}-c_n\alpha_{n-1}^{(n)} \tag{2.6.10}
$$

for $n=2,3,4,\ldots$. Hence it suffices to compute $\alpha_{n-1}^{(n)}$ for $n=1,2,3,\ldots$ and $\alpha_{n-2}^{(n)}$ for $n=2,3,4,\ldots$.

In the case of the q-derivative operator $\mathscr{D}_q$, we find from (2.6.4) for $k=n-1$

$$
(e[2n-2]+2\varepsilon)\,a_{n,n-1}=q\,(2f[n-1]+\gamma)\,a_{n,n},\quad n=1,2,3,\ldots
$$

and for $k=n-2$ and $n=2,3,4,\ldots$

$$
[2]\,(e[2n-3]+2\varepsilon)\,a_{n,n-2}-q^2\,(2f[n-2]+\gamma)\,a_{n,n-1}-q^2ga_{n,n}=0.
$$

Hence, if the regularity condition (2.3.3) holds for $n=1,2,3,\ldots$, we find

$$
a_{n,n-1}=\frac{q\,(2f[n-1]+\gamma)}{e[2n-2]+2\varepsilon}a_{n,n},\quad n=1,2,3,\ldots
$$

and

$$
a_{n,n-2}=\frac{q^3\,(2f[n-1]+\gamma)\,(2f[n-2]+\gamma)+q^2g\,(e[2n-2]+2\varepsilon)}{[2]\,(e[2n-3]+2\varepsilon)\,(e[2n-2]+2\varepsilon)}a_{n,n}
$$

for $n=2,3,4,\ldots$. Comparing (2.4.3) and (2.6.8), we find by using (2.4.2) that

$$
a_{n,n}=[n]!,\quad \alpha_{n-1}^{(n)}=\frac{a_{n,n-1}}{[n-1]!}=[n]\frac{a_{n,n-1}}{a_{n,n}},\quad n=1,2,3\ldots
$$

with

$$
[0]!:=1,\quad [n]!:=\prod_{i=1}^{n}[i],\quad n=1,2,3,\ldots
$$

and

$$\alpha_{n-2}^{(n)} = \frac{a_{n,n-2}}{[n-2]!} = [n][n-1]\frac{a_{n,n-2}}{a_{n,n}}, \quad n = 2,3,4,\ldots.$$

Hence we have

$$\alpha_{n-1}^{(n)} = \frac{q[n]\,(2f[n-1]+\gamma)}{e[2n-2]+2\varepsilon}, \quad n = 1,2,3,\ldots$$

and

$$\alpha_{n-2}^{(n)} = \frac{q^2[n][n-1]\,\{q\,(2f[n-1]+\gamma)\,(2f[n-2]+\gamma)+g\,(e[2n-2]+2\varepsilon)\}}{[2]\,(e[2n-3]+2\varepsilon)\,(e[2n-2]+2\varepsilon)}$$

for $n = 2,3,4,\ldots$. By using (2.6.9), we conclude that $c_0 = -\gamma q/2\varepsilon$ and

$$\begin{aligned} c_n &= \frac{q[n]\,(2f[n-1]+\gamma)}{e[2n-2]+2\varepsilon} - \frac{q[n+1]\,(2f[n]+\gamma)}{e[2n]+2\varepsilon} \\ &= \frac{q\,\{[n]\,(2f[n-1]+\gamma)\,(e[2n]+2\varepsilon) - [n+1]\,(2f[n]+\gamma)\,(e[2n-2]+2\varepsilon)\}}{(e[2n-2]+2\varepsilon)\,(e[2n]+2\varepsilon)} \\ &= -\frac{q^n\left\{(e[n-1]+2\varepsilon)\,(2f[n](1+q)+\gamma q) - e\gamma[n+1]q^{n-1}\right\}}{(e[2n-2]+2\varepsilon)\,(e[2n]+2\varepsilon)} \end{aligned} \tag{2.6.11}$$

for $n = 1,2,3,\ldots$. By using (2.6.10), we obtain

$$\begin{aligned} d_1 &= -\frac{q^2[2]\,\{\gamma q(2f+\gamma)+g\,(e[2]+2\varepsilon)\}}{[2]\,(e+2\varepsilon)\,(e[2]+2\varepsilon)} + \frac{q\,\{2\varepsilon\,(2f(1+q)+\gamma q) - e\gamma[2]\}}{2\varepsilon\,(e[2]+2\varepsilon)} \cdot \frac{\gamma q}{2\varepsilon} \\ &= \frac{q^2\,(e[2]+2\varepsilon)\left(4\varepsilon(f\gamma - g\varepsilon) - e\gamma^2\right)}{(2\varepsilon)^2\,(e+2\varepsilon)\,(e[2]+2\varepsilon)} = \frac{q^2\left(4\varepsilon(f\gamma - g\varepsilon) - e\gamma^2\right)}{4\varepsilon^2\,(e+2\varepsilon)} \end{aligned}$$

and for $n = 2,3,4,\ldots$

$$\begin{aligned} d_n &= \frac{q^2[n][n-1]\,\{q\,(2f[n-1]+\gamma)\,(2f[n-2]+\gamma)+g\,(e[2n-2]+2\varepsilon)\}}{[2]\,(e[2n-3]+2\varepsilon)\,(e[2n-2]+2\varepsilon)} \\ &\qquad - \frac{q^2[n][n+1]\,\{q\,(2f[n]+\gamma)\,(2f[n-1]+\gamma)+g\,(e[2n]+2\varepsilon)\}}{[2]\,(e[2n-1]+2\varepsilon)\,(e[2n]+2\varepsilon)} \\ &\qquad + \frac{q[n]\,(2f[n-1]+\gamma)}{e[2n-2]+2\varepsilon} \\ &\qquad\qquad \times \frac{q^n\left\{(e[n-1]+2\varepsilon)\,(2f[n](1+q)+\gamma q) - e\gamma[n+1]q^{n-1}\right\}}{(e[2n-2]+2\varepsilon)\,(e[2n]+2\varepsilon)} \\ &= \frac{q^{n+1}[n]\,(e[n-2]+2\varepsilon)}{(e[2n-3]+2\varepsilon)\,(e[2n-2]+2\varepsilon)^2\,(e[2n-1]+2\varepsilon)} \\ &\qquad \times \Big\{q^{n-1}\,(2f[n-1]+\gamma)\left(2f\,\{e[n-1]+2\varepsilon\} - q^{n-1}e\gamma\right) \\ &\qquad\qquad - g\,(e[2n-2]+2\varepsilon)^2\Big\}. \end{aligned} \tag{2.6.12}$$

Note that the latter formula also holds for $n = 1$.

In the case of the differentiation operator D, we have $q = 1$. In that case we get

$$c_n = -\frac{2fn\,(e(n-1)+2\varepsilon) - \gamma(e-\varepsilon)}{2\,(e(n-1)+\varepsilon)\,(en+\varepsilon)}, \quad n = 0,1,2,\ldots \tag{2.6.13}$$

and

$$\begin{aligned} d_n = {} & \frac{n\,(e(n-2)+2\varepsilon)}{4\,(e(2n-3)+2\varepsilon)\,(e(n-1)+\varepsilon)^2\,(e(2n-1)+2\varepsilon)} \\ & \times \Big\{ \{2f(n-1)+\gamma\}\,\{2f\,(e(n-1)+2\varepsilon) - e\gamma\} \\ & - 4g\,(e(n-1)+\varepsilon)^2 \Big\}, \quad n = 1,2,3,\ldots. \end{aligned} \tag{2.6.14}$$

In the case of the difference operator Δ, we find from (2.6.7) with $k = n-1$

$$(e(2n-2)+2\varepsilon)\,a_{n,n-1} = \big(e(n^2-4n+3) - 2\varepsilon + 2f(n-1) + \gamma\big)\,a_{n,n}$$

for $n = 1,2,3,\ldots$, and with $k = n-2$

$$\begin{aligned} & 2\,(e(2n-3)+2\varepsilon)\,a_{n,n-2} - \big(e(n^2-8n+10) - 4\varepsilon + 2f(n-2) + \gamma\big)\,a_{n,n-1} \\ & \quad - \big(e(n-2)^2 + 2f(n-2) + g\big)\,a_{n,n} = 0 \end{aligned}$$

for $n = 2,3,4,\ldots$. Hence, if the regularity condition (2.3.3) holds for $n = 1,2,3,\ldots$, we find

$$a_{n,n-1} = \frac{e(n-1)(n-3) - 2\varepsilon + 2f(n-1) + \gamma}{2\,(e(n-1)+\varepsilon)}\,a_{n,n}, \quad n = 1,2,3,\ldots$$

and for $n = 2,3,4,\ldots$

$$\begin{aligned} a_{n,n-2} = \Big\{ & \frac{e(n-2)^2 + 2f(n-2) + g}{2\,(e(2n-3)+2\varepsilon)} \\ & + \big(e(n^2-8n+10) - 4\varepsilon + 2f(n-2) + \gamma\big) \\ & \quad \times \frac{(e(n-1)(n-3) - 2\varepsilon + 2f(n-1) + \gamma)}{4\,(e(n-1)+\varepsilon)\,(e(2n-3)+2\varepsilon)} \Big\}\,a_{n,n}. \end{aligned}$$

Comparing (2.6.8) and (2.6.6), we find that

$$a_{n,n} = n!, \quad \alpha_{n-1}^{(n)} = n\left\{\frac{a_{n,n-1}}{a_{n,n}} - \frac{n-1}{2}\right\} = \frac{n\,(2(n-1)(f-e) - (n+1)\varepsilon + \gamma)}{2\,(e(n-1)+\varepsilon)}$$

for $n = 1,2,3,\ldots$ and

$$
\begin{aligned}
\alpha_{n-2}^{(n)} &= \binom{n}{3}\frac{3n-1}{4} - \frac{n-2}{2}\frac{a_{n,n-1}}{(n-2)!} + \frac{a_{n,n-2}}{(n-2)!} \\
&= n(n-1)\left\{\frac{(n-2)(3n-1)}{24} - \frac{n-2}{2}\frac{a_{n,n-1}}{a_{n,n}} + \frac{a_{n,n-2}}{a_{n,n}}\right\} \\
&= \frac{n(n-1)}{4\,(e(n-1)+\varepsilon)\,(e(2n-3)+2\varepsilon)} \\
&\qquad \times \left\{\frac{1}{6}(n+1)(3n+2)\,(e(n-1)+\varepsilon)\,(e(2n-3)+2\varepsilon)\right. \\
&\qquad\quad - en(n-1)^2\,(e(2n-3)+2\varepsilon) + e^2(n-1)^2(n-2)^2 \\
&\qquad\quad - 2\,(2f(n-1)+\gamma)\,(e(2n-3)+n\varepsilon) - \gamma(e(n-1)+2\varepsilon) \\
&\qquad\quad \left. + (2f(n-1)+\gamma)\,(2f(n-2)+\gamma) + 2g\,(e(n-1)+\varepsilon)\right\}
\end{aligned}
$$

for $n = 2, 3, 4, \ldots$. By using (2.6.9), we conclude that $c_0 = 1 - \gamma/2\varepsilon$ and

$$
\begin{aligned}
c_n &= \frac{n\,(2(n-1)(f-e) - (n+1)\varepsilon + \gamma)}{2\,(e(n-1)+\varepsilon)} \\
&\qquad - \frac{(n+1)\,(2n(f-e) - (n+2)\varepsilon + \gamma)}{2(en+\varepsilon)} \\
&= \frac{n\,(e(n-1)+2\varepsilon)\,(2(e-f)+\varepsilon) + (e-\varepsilon)(\gamma-2\varepsilon)}{2\,(e(n-1)+\varepsilon)\,(en+\varepsilon)}
\end{aligned} \tag{2.6.15}
$$

for $n = 1, 2, 3, \ldots$. By using (2.6.10), we find

$$
\begin{aligned}
d_1 &= -\frac{1}{2(e+\varepsilon)(e+2\varepsilon)}\Big\{4(e+\varepsilon)(e+2\varepsilon) - 2e(e+2\varepsilon) \\
&\qquad - 2(2f+\gamma)(e+2\varepsilon) - \gamma(e+2\varepsilon) + (2f+\gamma)\gamma + 2g(e+\varepsilon)\Big\} \\
&\qquad + \frac{2\varepsilon\,(2(e-f)+\varepsilon) + (e-\varepsilon)(\gamma-2\varepsilon)}{2\varepsilon(e+\varepsilon)} \cdot \frac{2\varepsilon-\gamma}{2\varepsilon} \\
&= \frac{4\varepsilon(f\gamma - g\varepsilon) - e\gamma^2}{4\varepsilon^2(e+2\varepsilon)}.
\end{aligned}
$$

and for $n = 2, 3, 4, \ldots$

$$
\begin{aligned}
d_n = {} & \frac{n(n-1)}{4\,(e(n-1)+\varepsilon)\,(e(2n-3)+2\varepsilon)} \\
& \quad \times \Bigg\{ \frac{1}{6}(n+1)(3n+2)\,(e(n-1)+\varepsilon)\,(e(2n-3)+2\varepsilon) \\
& \qquad - en(n-1)^2\,(e(2n-3)+2\varepsilon) + e^2(n-1)^2(n-2)^2 \\
& \qquad - 2\,(2f(n-1)+\gamma)\,(e(2n-3)+n\varepsilon) - \gamma\,(e(n-1)+2\varepsilon) \\
& \qquad + (2f(n-1)+\gamma)\,(2f(n-2)+\gamma) + 2g\,(e(n-1)+\varepsilon) \Bigg\} \\
& - \frac{n(n+1)}{4(en+\varepsilon)\,(e(2n-1)+2\varepsilon)} \\
& \quad \times \Bigg\{ \frac{1}{6}(n+2)(3n+5)(en+\varepsilon)\,(e(2n-1)+2\varepsilon) \\
& \qquad - en^2(n+1)\,(e(2n-1)+2\varepsilon) + e^2n^2(n-1)^2 \\
& \qquad - 2(2fn+\gamma)\,(e(2n-1)+(n+1)\varepsilon) - \gamma(en+2\varepsilon) \\
& \qquad + (2fn+\gamma)\,(2f(n-1)+\gamma) + 2g(en+\varepsilon) \Bigg\} \\
& - \frac{n\,(2(e-f)+\varepsilon)\,(e(n-1)+2\varepsilon)+(e-\varepsilon)(\gamma-2\varepsilon)}{2\,(e(n-1)+\varepsilon)\,(en+\varepsilon)} \\
& \quad \times \frac{n\,(2(n-1)(f-e)-(n+1)\varepsilon+\gamma)}{2\,(e(n-1)+\varepsilon)} \\
= {} & -\frac{n\,(e(n-2)+2\varepsilon)}{4\,(e(2n-3)+2\varepsilon)\,(e(n-1)+\varepsilon)^2\,(e(2n-1)+2\varepsilon)} \\
& \quad \times \Big\{ e(n-1)^2\,(e(n-1)+2\varepsilon)^2 \\
& \qquad + 2(n-1)\,(e(n-1)+2\varepsilon)\,(2eg+2f(\varepsilon-f)-e\gamma) \\
& \qquad + 4\varepsilon(g\varepsilon-f\gamma)+e\gamma^2 \Big\}.
\end{aligned}
\tag{2.6.16}
$$

Note that the latter formula also holds for $n=1$.

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