Chapter 2
Some Notions of Electromagnetism

A lot of this book is about electromagnetism, carrying out calculations with rather standard bits of Maxwell’s theory. This is why we begin with an overview of the main results used later. The aim is not to produce a textbook account, with complete proofs and explanations, only to exhibit the sequence of results a reader would need to know in order to follow the rest and fill in some of the more apposite details. To help things along, and make it easier for someone to pick up what is required, the overview given in this chapter is largely based on the account in *The Feynman Lectures on Physics*, by Feynman, Leighton, and Sands [1, 2]. The discussion in these books is remarkable anyway, and strongly recommended to anyone who does not know it.

2.1 Maxwell’s Equations and Their Solution

We shall write Maxwell’s equations in the form [2, Chap. 21]

\[
\begin{align*}
\nabla \cdot E &= \frac{\rho}{\varepsilon_0}, \\
\nabla \cdot B &= 0, \\
\n\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\n\nabla \times B &= \frac{\mathbf{j}}{\varepsilon_0} + \frac{\partial E}{\partial t}, \\
\end{align*}
\]

(2.1)

where \( E \) is the electric field, \( B \) is the magnetic field, \( \mathbf{j} \) is the current density, \( \rho \) is the charge density, \( c \) is the speed of light in vacuum, and \( \varepsilon_0 \) is a constant.

Because \( \nabla \cdot B = 0 \) and \( \nabla \times E = -\partial B/\partial t \), there is a 3-vector field \( \mathbf{A} \) and a scalar field \( \phi \) such that

\[
\begin{align*}
\mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \\
\mathbf{B} &= \nabla \times \mathbf{A}.
\end{align*}
\]

(2.2)
\( \phi \) and \( A \) are called the scalar and vector potential, respectively. Together they form the 4-potential \( A^\mu = (\phi / c, A) \), a 4-vector under Lorentz transformations (see below) [2, Chap. 25].

When \( E \) and \( B \) have the form given in (2.2), the two Maxwell equations that do not mention the sources \( \rho \) and \( j \) are automatically satisfied. Then it turns out that, with one proviso, the other two Maxwell equations take the form

\[
\Box^2 \phi = \frac{\rho}{\varepsilon_0}, \quad \Box^2 A = \frac{j}{\varepsilon_0 c^2},
\]

(2.3)

where \( \Box^2 \) is the differential operator

\[
\Box^2 := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2.
\]

(2.4)

The proviso here is that one must put a constraint on \( \phi \) and \( A \), viz.,

\[
\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}.
\]

(2.5)

This is possible because several choices of \( \phi \) and \( A \) actually lead to (2.2), something known as gauge freedom. Then (2.5) is called a gauge condition, in fact the Lorenz gauge condition. It constrains the potentials somewhat, but not totally. There remains some gauge freedom, i.e., one can simultaneously change \( A \) to \( A' := A + \nabla \psi \) and \( \phi \) to \( \phi' := \phi - \partial \psi / \partial t \) and obtain the same \( E \) and \( B \) from (2.2), and even maintain the condition (2.5) applied now to \( A' \) and \( \phi' \), provided only that \( \psi \) satisfies \( \Box \psi = 0 \).

The great thing about the step from \( E \) and \( B \) to \( \phi \) and \( A \) is that (2.3) and (2.5) can be solved by

\[
\phi(\mathbf{r}_0, t_0) = \int \frac{\rho(\mathbf{r}_1, t_0 - r_{01}/c)}{4\pi \varepsilon_0 r_{01}} \, dV_1, \quad A(\mathbf{r}_0, t_0) = \int \frac{j(\mathbf{r}_1, t_0 - r_{01}/c)}{4\pi \varepsilon_0 c^2 r_{01}} \, dV_1,
\]

(2.6)

where \( (\mathbf{r}_0, t_0) \) is the field point, i.e., the event of spacetime at which we are evaluating the fields, while the dummy variable \( \mathbf{r}_1 \) in the integral ranges over all values in \( \mathbb{R}^3 \), and \( \mathbf{r}_{01} := \mathbf{r}_0 - \mathbf{r}_1 \) is the vector from the volume element \( dV_1 \) to the field point, with length \( r_{01} \).

The time \( t_{01} := t_0 - r_{01}/c \) is called the retarded time, somewhat vaguely. It is the time when there would have had to have been some charge at the integration point \( \mathbf{r}_1 \) in order for something at the field point to be affected by a field due to that charge at that integration point. The little diagram in Fig. 2.1 is designed to illustrate this trivial point, but anyone with any doubts about this is advised to go through the account in [2] in detail, or consult some other elementary textbook.

One also speaks of the retarded point \( \mathbf{r}_{01} \) for a given field point and a given charge element. In terms appropriate to the special theory of relativity (SR), the retarded point is the intersection of the past light cone of the field point with the worldline of the charge element. It is the spatial position of the charge element when it produced
2.2 Relativistic Notation

Fig. 2.1 A picture to show why the retarded time \( t_+ := t_0 - r_{01}/c \) turns up in the solution to Maxwell’s equations. The curve denotes the spatial trajectory of a charge element \( C \). The latter would have had to have been at the retarded point \( r_+ = r_1 \) to affect the field point \( (r_0, t_0) \), because the distance \( r_{01} := |r_0 - r_1| \) is precisely the distance that light would cover in the time \( t_0 - t_+ \).

the fields that affect the field point. For given field point and given charge element, the retarded point is unique.

Note in passing that charge is conserved by any system satisfying Maxwell’s equations, because they imply

\[
\nabla \cdot j = -\frac{\partial \rho}{\partial t},
\]

(2.7)

In integral form, this states that the flux of current out through any closed surface is equal to minus the rate of change of the amount of charge within the surface. Together the charge and current densities form the 4-current density \( j^\mu = (c\rho, j) \), a 4-vector under Lorentz transformations (see below) [2, Chap. 18].

Note that (2.6) is a very general solution to Maxwell’s equations (2.3), for any charge distribution with any motion. There are other solutions involving advanced times, rather than retarded times, and combinations of both, but in the present discussion, we shall stick with the retarded solutions in (2.6) and the old-fashioned notion of causality. One then obtains the electric and magnetic fields from (2.2).

2.2 Relativistic Notation

All this can be phrased in terms of four-vectors and other tensors. If we consider homogeneous coordinates, that is, each with the same physical dimensions,

\[
x^\mu = (ct, x, y, z), \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial (ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),
\]

and the Minkowski metric in the form
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\[ \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}, \]

then \( A^\mu = (\phi/c, A) \) is a homogeneous 4-vector under Lorentz transformations [2, Chap. 25], since \( \phi/c \) and \( A \) have the same physical dimensions, and the Lorenz gauge constraint (2.5) takes the elegant form

\[ \partial_\mu A^\mu = 0. \quad (2.8) \]

The charge density \( \rho \) and current density \( j \) form another homogeneous four-vector

\[ j^\mu = (c\rho, j) \quad [2, \text{Chap. 18}], \]

and charge conservation (2.7) assumes the equally elegant form

\[ \partial_\mu j^\mu = 0. \quad (2.9) \]

The differential operator \( \Box^2 \) in (2.4) has the form \( \Box^2 = \partial_\mu \partial^\mu \), and Maxwell’s equations (2.3) for the potential take the form

\[ \Box^2 A^\mu = \frac{j^\mu}{\varepsilon_0 c^2}. \quad (2.10) \]

Furthermore, if we define

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.11) \]

where \( A_\mu := \eta_{\mu\nu} A^\nu \), then (2.2) implies that

\[ F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \quad (2.12) \]

This is also physically homogeneous in the sense that each component of the tensor has the same physical dimensions. The contravariant version is

\[ F^{\mu\nu} := \eta^{\mu\sigma} \eta^{\nu\tau} F_{\sigma\tau} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \quad (2.13) \]

The two Maxwell equations among (2.1) that do not mention the sources \( \rho \) and \( j \) take the form

\[ F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0. \quad (2.14) \]

This can also be written
where the square brackets on indices indicate antisymmetrisation, whence the latter says that

\[ F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} - F_{\nu\mu,\sigma} - F_{\mu\sigma,\nu} - F_{\sigma\nu,\mu} = 0 , \]

and, since \( F_{\mu\nu} \) is antisymmetric, this is just the same as (2.14). The two Maxwell equations mentioning \( \rho \) and \( j \) take the form

\[ F_{\mu\nu,\nu} = -\frac{j^\mu}{\varepsilon_0c^2} . \]

**Note.** The reader should be warned that this is only one set of conventions for writing everything in relativistic notation. The Minkowski metric is sometimes the negative of the form shown here, and it contains a factor of \( c^2 \) if non-homogeneous coordinates \((t,x,y,z)\) are chosen. Unfortunately, one encounters many variants, so it is better to just get used to that.

### 2.3 Lorentz Force Law

In a certain sense, the Lorentz force law tells us what the electric and magnetic fields actually do out in the real world when there is a charge there to probe them, because they give the force that those fields will exert on the charge, viz.,

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) , \]

where \( q \) is the value of the charge in coulombs and \( \mathbf{u} \) is its 3-velocity. This is a non-relativistic version. We would like to look here at the Lorentz force law in its special relativistic formulation. We need the 4-velocity

\[ u^\mu := \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau} (c, \mathbf{u}) , \]

where the coordinate 3-velocity \( \mathbf{u} \) in the given inertial frame is defined by

\[ \mathbf{u} := \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) , \]

and the proper time \( \tau \) by

\[ c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 , \]

whence
\[
\left( \frac{d\tau}{dt} \right)^2 = 1 - \frac{u^2}{c^2}, \quad \frac{dt}{d\tau} = \gamma.
\]

(2.20)

defining \( \gamma(u) \) in the usual way. Hence, for the 4-velocity,

\[
\nu^\mu = \gamma(c, u).
\]

(2.21)

In special relativity, the Lorentz force law takes the form

\[
m_0 \frac{d^2 x^\mu}{d\tau^2} = q F^\mu_\nu \nu^\nu
\]

(2.22)

where \( m_0 \) and \( q \) are the particle mass and charge, respectively.

Let us see what these four equations tell us in terms of \( E \) and \( B \). The left-hand side is

\[
m_0 \frac{d^2 x^\mu}{d\tau^2} = m_0 \frac{dt}{d\tau} \frac{d\nu^\mu}{dt} = m_0 \gamma \frac{d\nu^\mu}{dt}.
\]

(2.23)

On the right-hand side, we have

\[
F^\mu_\nu \nu^\nu = \gamma \left( \frac{E \cdot u}{c} \right).
\]

(2.24)

Now the Lorentz force equation consists of one vector equation (components \( i = 1, 2, 3 \)) and one scalar equation (the 0 component), but in fact the scalar equation follows from the vector equation as we shall see in a moment. The vector equation states that

\[
m_0 \frac{d}{dt} (\gamma u) = q (E + u \times B)
\]

(2.25)

This is the usual version of the relativistic Lorentz force law [2, Sect. 26.4]. It can be written more specifically in the form

\[
\frac{d}{dt} \left[ \frac{m_0}{(1 - u^2/c^2)^{1/2}} u \right] = q (E + u \times B).
\]

(2.26)

The left-hand side is the coordinate time rate of change of the relativistic 3-momentum (which has to multiplied by \( \gamma \) to give the spatial components of the 4-force).

Now in special relativity, the 4-force and 4-velocity are not independent. In fact, they are orthogonal in the Minkowski geometry. Let us see how this relationship comes out when the 4-force is equated with \( q F^\mu_\nu \nu^\nu \). The scalar equation requires

\[
m_0 c \frac{d\gamma}{dt} = q E \cdot u / c.
\]

(2.27)
2.4 Electromagnetic Energy–Momentum Tensor

But the vector equation already implies that

\[ qE \cdot u = m_0 u \frac{d}{dt} (\gamma u) \, . \quad (2.28) \]

This is because the magnetic field does no work, i.e., the magnetic force \( u \times B \) is orthogonal to the velocity \( u \) (in 3-space). This in turn means that

\[ qE \cdot u = m_0 u^2 \frac{d\gamma}{dt} + \gamma m_0 u \cdot \dot{u} \, . \quad (2.29) \]

The right-hand side can be simplified here using

\[ \frac{d\gamma}{dt} = \frac{u \cdot \dot{u}}{c^2} \gamma^3 \, , \quad (2.30) \]

whence the vector equation in the form (2.29) does indeed imply that

\[ qE \cdot u = m_0 c^2 \frac{d\gamma}{dt} \, . \]

In fact, we have the following identity:

\[ u^2 \frac{d\gamma}{dt} + \gamma u \cdot \dot{u} = c^2 \frac{d\gamma}{dt} \, , \quad (2.31) \]

which is precisely the relation which says that the 4-force and 4-velocity are orthogonal (a completely general result).

The point of the last short discussion is just to make it clear that the whole content of the relativistic relation (2.22) is expressed by the 3-vector relation (2.25).

2.4 Electromagnetic Energy–Momentum Tensor

The definition we shall use is

\[ T^{\mu\nu} = -\varepsilon_0 c^2 \left( F^{\mu\sigma} F^{\sigma\nu} + \frac{1}{4} F_{\sigma\tau} F^{\sigma\tau} \eta^{\mu\nu} \right) \, , \quad (2.32) \]

with the above definition for \( F_{\mu\nu} \). If we work out the components of \( T^{\mu\nu} \) in terms of the electric and magnetic fields, we obtain

\[ T^{\mu\nu} = \varepsilon_0 \begin{pmatrix} -\frac{1}{2} (E^2 + c^2 B^2) & -c E \times B \\ -c E \times B & EE + c^2 BB - \frac{1}{2} (E^2 + c^2 B^2) \end{pmatrix} \, . \quad (2.33) \]

The \( 3 \times 3 \) matrix in the bottom right is
\[ T := \varepsilon_0 \left[ EE + c^2 BB - \frac{1}{2} (E^2 + c^2 B^2) \right], \]  

where

\[ EE := \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} (E_x E_y E_z) = \begin{pmatrix} E_x^2 & E_x E_y & E_x E_z \\ E_y E_x & E_y^2 & E_y E_z \\ E_z E_x & E_z E_y & E_z^2 \end{pmatrix}, \]

and similarly for \( BB \).

We can now bring in the energy density \( u \) and its flow \( S \) with the same conventions as Feynman in [2, Sect. 27.5]. Hence,

\[ u = \frac{\varepsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{\varepsilon_0 c^2}{2} \mathbf{B} \cdot \mathbf{B} = \frac{\varepsilon_0}{2} (E^2 + c^2 B^2) \]  

and

\[ S := \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B}. \]

We now have

\[ T^{00} = -u, \quad T^{0k} = -\frac{1}{c} S^k. \]

The vector quantity \( S \) is the energy flux of the field, i.e., the flow of energy per unit time across a unit area perpendicular to the flow. It is known as the Poynting vector. One can obtain the field momentum density (momentum per unit volume) from it in the form [2, Sect. 27.6]

\[ g := \frac{1}{c^2} S = \varepsilon_0 \mathbf{E} \times \mathbf{B}, \]

a formula that will be put to use later.

The above quantities \( u \) and \( S \) are chosen because they satisfy the constraint of energy conservation, viz.,

\[ \frac{\partial u}{\partial t} = -\nabla \cdot S - \mathbf{E} \cdot \mathbf{j}, \]

which expresses the idea that the total field energy in a given volume decreases either because field energy flows out of the volume (the term \( \nabla \cdot S \)) or because it loses energy to matter by doing work on it (the term \( \mathbf{E} \cdot \mathbf{j} \)). In relativistic language, this becomes \( T^{0\nu} v = F^{0\nu} j_\nu \).

It is important to see that the field does work on each unit volume of matter at the rate \( \mathbf{E} \cdot \mathbf{j} \). The force on a particle is \( \mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \), and the rate of doing work is \( \mathbf{F} \cdot \mathbf{v} = q \mathbf{E} \cdot \mathbf{v} \). If there are \( N \) particles per unit volume, the rate of doing work per unit volume is \( N q \mathbf{E} \cdot \mathbf{v} \), and \( N q v \) is of course what we have called \( \mathbf{j} \). The quantity \( \mathbf{E} \cdot \mathbf{j} \) is also called the Lorentz force density.
With the definition (2.32) for $T^\mu{}^\nu$, Maxwell’s equations in the relativistic form (2.14) and (2.16) in fact imply

$$T^\mu{}^\nu,\nu = F^\mu{}^\nu j^\nu$$

(2.40)

It is instructive to see this derivation. We begin with Maxwell’s equations in the form

$$F^\mu{}^\nu,\nu = -\frac{j^\mu}{\varepsilon_0 c^2}, \quad F_{[\mu\nu,\sigma]} = 0.$$  

(2.41)

Now, from

$$T^\mu{}^\nu = -\varepsilon_0 c^2 \left( F^{\mu}{}_{\sigma} F^{\sigma\nu} + \frac{1}{4} F_{\sigma\tau} F^{\sigma\tau} \eta^{\mu\nu} \right),$$  

(2.42)

we have

$$T^{\mu}{}^{\nu},\nu = -\varepsilon_0 c^2 \left( F^{\mu}{}_{\sigma,\nu} F^{\sigma\nu} + F^{\mu}{}_{\sigma} F^{\sigma\nu},\nu + \frac{1}{4} F_{\sigma\tau,\nu} F^{\sigma\tau} \eta^{\mu\nu} + \frac{1}{4} F_{\sigma\tau} F^{\sigma\tau},\nu \eta^{\mu\nu} \right).$$  

(2.43)

Now note that, using the Maxwell equation in (2.41) that refers to the sources, the second term on the right-hand side is

$$\text{second term} = -\varepsilon_0 c^2 F^{\mu}{}_{\sigma} F^{\sigma\nu},\nu = F^{\mu}{}_{\sigma} j^\sigma,$$

(2.44)

and this is the 4-force density. One can then show that the other terms in (2.43) sum to zero, i.e.,

$$F^{\mu}{}_{\sigma,\nu} F^{\sigma\nu} + \frac{1}{4} F_{\sigma\tau,\nu} F^{\sigma\tau} \eta^{\mu\nu} + \frac{1}{4} F_{\sigma\tau} F^{\sigma\tau},\nu \eta^{\mu\nu} = 0,$$

using the antisymmetry of $F^{\sigma\tau}$ and the source-free Maxwell equations in (2.41). Our conclusion here is therefore

$$T^{\mu}{}^{\nu},\nu = F^{\mu}{}^{\nu} j^\nu,$$

(2.45)

as claimed. In words, Maxwell’s equations ensure that the divergence $T^{\mu}{}^{\nu},\nu$ of the electromagnetic energy–momentum tensor is equal to the electromagnetic force density on the charge distribution according to the Lorentz force law. We conclude that the electromagnetic energy–momentum tensor is conserved at a point if and only if the electromagnetic force density on the charges there is zero. One situation where this happens is if there are no charges!

Consider for a moment how this analysis would fit in with a charged dust model [8, pp. 104–118], where one would have a total energy–momentum tensor of the form
\[ T_{\text{charged dust}} = \frac{m}{q} \rho u \otimes u + T_{\text{em}}, \]

where \( u \) is the 4-velocity field of the dust, \( m \) and \( q \) are the mass and charge of each dust particle, respectively, and \( \rho \) is the proper charge density distribution. When the mass part of the tensor (proportional to \( u \otimes u \) here) is included, conservation of the total tensor says two separate things. The component of the resulting equation parallel to the 4-velocity field \( u \) gives conservation of mass, whilst the component orthogonal to the 4-velocity field says that charges follow non-geodesic curves as given by the Lorentz force law.

Of course, in most situations, other forces will be involved, i.e., the distribution could not be treated as a charged dust. However, considering this model for a moment, let us just ask what has changed by adding the mass term. In fact, when we obtained (2.45) above, we concluded that the electromagnetic energy–momentum tensor alone is conserved at a point if and only if the electromagnetic force density on the charges there is zero. Of course, this never happens unless there are no charges there! In that case, we can define a field energy density and momentum flow, and interpret the resulting equation. When there are charges, we have to equate the right-hand side of (2.45) with the missing bit of the global conservation equation, including the mass terms, and this delivers the Lorentz force law dictating how the charged dust must flow in the given fields.

As a final point regarding the quantities \( u \) and \( S \), it is important to note that there is some ambiguity here, because there are other definitions that lead to the right relations. However, they are more complex [2, Sect. 27.4]. There is also some strangeness in the way energy is conserved. A good discussion and examples can be found in [2, Chap. 27].

### 2.5 Solution for Point Charge with Arbitrary Motion

One case that interests us in this book is a point charge with arbitrary motion. The notion of point charge is clearly idealistic and one of the themes here is that there are no point particles. However, this does not mean that the point charge is not a useful mathematical approximation. It is in this sense that it is presented here.

#### 2.5.1 Fields Due to a Single Point Charge

**Four-Current Vector**

The first step is to obtain the 4-vector current describing such a source. As usual in the relativistic context, we begin by arbitrarily choosing some inertial frame to describe things in. The trajectory of the source is described in Minkowski spacetime by \( x(\tau) \), where \( \tau \) is the proper time. The trajectory can also be parametrised by
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\[ t = x^0(\tau)/c. \] Then the 4-vector current density is

\[ j^\mu(y, t) = q \frac{dx^\mu}{dt} \delta^3(y - x(\tau)) \bigg|_{ct=x^0(\tau)}, \quad (2.46) \]

where \( \delta^3 \) is the 3D Dirac delta distribution.

It is worth being quite clear about this formula. In a simplistic model of a charge distribution,

\[ j(y) = \rho_0(y)V(y), \quad (2.47) \]

where

\[ V(y) := \frac{dx}{d\tau} = \gamma(v)(e, v), \quad v := |v|, \quad \gamma(v) := (1 - v^2/c^2)^{-1/2}, \]

is the 4-velocity field of the charge, with \( \tau \) the proper time of the charge, and \( \rho_0(y) \) is the local charge density as measured at each point in an instantaneous rest frame of the charge. Now \( dt = \gamma(v) d\tau \) so in the simplistic model (2.47),

\[ j^\mu(y) = \rho_0 \gamma(v) \frac{dx^\mu}{dt}. \]

But \( \rho_0 \gamma(v) = \rho \), the local charge density as measured from our chosen inertial frame, so our formula (2.46) will be right if the term

\[ q \delta^3(y - x(\tau)) \bigg|_{ct=x^0(\tau)} \quad (2.48) \]

corresponds to the charge density distribution as measured in that inertial frame. Notice that the condition \( ct = x^0(\tau) \) says what \( \tau \) is supposed to be, given \( t \), and not the other way round. Then the delta function is a point distribution located at the space point \( x(\tau) \), the point of the trajectory corresponding to the given \( t \), so it is intuitively reasonable to suggest that (2.48) is a good model for the charge density distribution of our point charge with worldline \( x(\tau) \).

Note that the distribution (2.46) can be rewritten in covariant form:

\[ j^\mu(y, t) = q \int d\tau \frac{dx^\mu}{d\tau} \delta^4(y - x(\tau)), \quad (2.49) \]

as can be seen by changing the variable to \( t \) and doing the integration over this variable.

Since the charge density distribution in space is supposed to be

\[ \rho = \frac{j^0(y, t)}{e} = q \delta^3(y - x(\tau)), \]

at any instant of time \( t \), the total charge at that time is
\[
\int d^3y \, j^0(y, t) = q ,
\]
which is constant in time. We ought to check that \( \partial_\mu j^\mu = 0 \), as required by (2.9). We need to see why the flux of current out of any closed surface is equal to the rate of change of charge contained within it. The current density is
\[
q v(x(\tau)) \delta^3(y - x(\tau)) \bigg|_{ct = x^0(\tau)} = v \rho ,
\]
and this is intuitively sufficient.

**Potential Equation**

The equation for the 4-vector potential is (2.10), viz.,
\[
\Box A^\mu = \frac{j^\mu}{\varepsilon_0 c^2} ,
\]
and we will maintain the Lorenz gauge, so that
\[
A^\mu,\mu = 0 .
\]

We use the retarded Green function
\[
G_{ret}(x) = \frac{c}{2\pi} \theta(+x^0) \delta(x^2)
\]
to solve for \( A^\mu \), where \( \theta \) is the step function, equal to zero for negative values of the argument and +1 for positive values. This has the property that
\[
\Box^2 G_{ret} = \delta^4(x) .
\]
The solution for the 4-potential is thus
\[
A^\mu(y) = \frac{1}{2\pi \varepsilon_0 c} \int d^4z \, \theta(y^0 - z^0) \delta((y - z)^2) j^\mu(z) , \tag{2.50}
\]
by the standard property of Green functions. We should check that this satisfies the Lorenz gauge condition. Taking the relativistic divergence, the operator \( \partial_\mu \) can be transferred to the 4-current inside the integral, and this time charge conservation implies the Lorenz gauge condition.

Then, inserting our 4-current density (2.49) into (2.50) and carrying out the integral over \( z \) using the 4D delta function from (2.49), we obtain
\[
A^\mu(y) = \frac{q}{2\pi \varepsilon_0 c} \int_{-\infty}^{\infty} d\tau \theta(y^0 - x^0(\tau)) \delta\left(\left[y - x(\tau)\right]^2\right) \dot{x}^\mu(\tau) , \tag{2.51}
\]
where the dot on \( x^\mu \) indicates the proper time derivative. This formula clearly exhibits the fact that our \( A^\mu \) depends only on the worldline at earlier times.
Retarded Points

As mentioned earlier, for all \( y \), there exists a unique \( x_+ = x(\tau_+) \) on the trajectory of the charged particle, such that

\[
(y - x_+)^2 = 0, \quad x_+^0 < y^0.
\]

This point is referred to as the retarded point. In relativistic jargon, it is the intersection of the worldline with the past null cone through \( y \) (see Fig. 2.2).

Let us understand the significance of these points via the delta function approach. Consider first a standard piece of distribution theory. Suppose we have

\[
I = \int d\tau h(\tau) \delta(g(\tau)),
\]

where \( g(\tau) = (y - x(\tau))^2 \) only has the one zero for \( y^0 > x^0(\tau) \), namely at \( \tau = \tau_+ \). Expanding out

\[
g(\tau) = g(\tau_+ + \tau - \tau_+) \approx g(\tau_+) + (\tau - \tau_+)g'(\tau_+),
\]

the first term is zero and the second is supposed small (for \( \tau \) close to \( \tau_+ \)). Consequently,

\[
I = \int d\tau h(\tau) \delta((\tau - \tau_+)g'(\tau_+)),
\]

and we can now use

\[
g'(\tau) = 2[y - x(\tau)] \cdot \dot{x}(\tau)
\]

Fig. 2.2 A spacetime picture of the retarded point, to be contrasted with the space picture in Fig. 2.1
to deduce

\[ I = \frac{h(\tau_+)}{|g'(\tau_+)|} . \]

The modulus comes in from the variable change

\[ \tau' := g'(\tau_+)\tau . \]

However, it is unnecessary, because it can be shown that

\[ \dot{x}_+ \cdot (y - x_+) > 0 , \]

by observing that \( \dot{x}_+ \) is timelike, with \( \dot{x}_+^0 > 0 \), and \( y - x_+ \) is null, with \( y^0 - x_+^0 > 0 \).

### Lienard–Wiechert Retarded Potential

The above study of retarded points can be applied to the potential (2.51) we obtained earlier. The result is the Lienard–Wiechert retarded potential

\[ A_{\text{ret}}^\mu(y) = \frac{q}{4\pi \varepsilon_0 c} \frac{\dot{x}_+^\mu}{\dot{x}_+^0 (y - x_+)} . \] (2.52)

This is the relativistic generalisation of the Coulomb potential. The above derivation displays the power of the distribution approach using the step function and Dirac delta. A longer but much more physical derivation can be found in [2, Chap. 21], and is strongly recommended.

We would like to write this in more familiar terms. Define \( r_+ := y - x_+ \), the vector from the retarded point to the field point (recalling that the field point is associated with a unique retarded point). Define also

\[ \mathbf{v}_+ = \frac{\mathbf{d}x}{\mathbf{d}t} \bigg|_{\tau_+} , \]

the 3-velocity of the charge at the retarded point associated with the field point.

We observe that

\[ y - x_+ = (y^0 - x_+^0, \mathbf{r}_+) \]

is null, and since \( y^0 > x_+^0 \),

\[ y^0 - x_+^0 = |\mathbf{r}_+| = r_+ . \]

Furthermore,
\[ \dot{x}_+ = \left. \frac{dx}{d\tau} \right|_{\tau} = \left. \frac{dx}{dt} \right|_{\tau} \frac{dt}{d\tau}, \]

and the derivative \( dt/d\tau \) will cancel in the ratio of terms.

Finally, we obtain the equations

\[ A^0_{\text{ret}}(y) = \frac{q}{4\pi \varepsilon_0 c (r_+ - r_+ \cdot v_+/c)}, \quad A_{\text{ret}}(y) = \frac{q v_+}{4\pi \varepsilon_0 c^2 (r_+ - r_+ \cdot v_+/c)}. \] (2.53)

**Retrieving the Coulomb Potential**

Choose the frame in which

\[ x^\mu_+ = (1, 0, 0, 0), \]

namely, the instantaneous rest frame of the source charge when it was at the retarded point. Then \( v_+ = 0 \). In this frame,

\[ \phi_{\text{ret}} = c A^0_{\text{ret}}(y) = \frac{q}{4\pi \varepsilon_0 r_+}, \quad A_{\text{ret}}(y) = 0, \]

where \( r_+ \) is the distance between the test point \( y \) and the retarded point related to \( y \). Notice that the velocity of our frame depends on the field point, because it is equal to the velocity of the charge at the retarded point for that field point. We cannot say that all effects due to acceleration of the source charge disappear everywhere in space in this frame, not even instantaneously, but we can say that the scalar potential \( \phi \) goes as \( 1/r_+ \), and not as the reciprocal of the distance to the point where the charge is now. (The word ‘now’ refers to simultaneity in whatever inertial frame we have selected at the outset.)

**Electromagnetic Fields**

The electromagnetic fields due to the charge can now be calculated, using the relations (2.2) on p. 5, viz.,

\[ E = -c \nabla A^0 - \frac{\partial A}{\partial t}, \quad B = \nabla \times A. \]

This deduction will be sketched here because heavy use will be made of the resulting formulas for \( E \) and \( B \) later on, and because the kind of calculation arising here illustrates something about the physics.

First for the notation, we drop the subscript indicating that we are talking about the retarded solutions, and stick with the notation \( y^\mu = (y^0, y) \) for the (homogeneous) coordinates of the field point and \( x^\mu(t_+) \) for the event coinciding with the charge at the appropriate retarded time \( t_+ \) for the given field point. The retarded dis-
placement vector from the charge at the appropriate retarded time to the field point is then denoted by \( \mathbf{r}_+ := \mathbf{y} - \mathbf{x}(t_+) \), as in the discussion of the Lienard–Wiechert potential above.

This immediately reminds us that the key thing about \( t_+ \), or indeed \( \mathbf{x}_+ := \mathbf{x}(t_+) \), \( \mathbf{r}_+ \), and \( \mathbf{v}_+ \), is that they are functions of the field point. If we are to carry out the derivatives in

\[
\frac{4\pi\varepsilon_0}{q} \mathbf{E} = -\nabla \frac{1}{r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c} - \frac{\partial}{\partial y^0} \frac{\mathbf{v}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} ,
\]

then first of all, we need to find the partial derivatives of \( t_+ \) with respect to \( y^\mu \). This is the point about the physics just mentioned: everything here hinges on the retarded time.

So the key relation here is the one defining the retarded time for the given field point, viz.,

\[
r_+ = |\mathbf{y} - \mathbf{x}(t_+)| = y^0 - ct_+ ,
\]

which can also be written explicitly in the form

\[
\left[ y^1 - x^1(t_+) \right]^2 + \left[ y^2 - x^2(t_+) \right]^2 + \left[ y^3 - x^3(t_+) \right]^2 = (y^0 - ct_+)^2 .
\]

Taking partial derivatives of this relation with respect to \( y^\mu \), we soon arrive at

\[
\frac{\partial t_+}{\partial y^0} = \frac{r_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} , \quad \nabla t_+ = -\frac{\mathbf{r}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} .
\]

It is then a simple matter to obtain

\[
\frac{\partial \mathbf{r}_+}{\partial y^0} = -\frac{\mathbf{r}_+ \mathbf{v}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} , \quad \frac{\partial \mathbf{r}_+}{\partial y^0} = -\frac{\mathbf{r}_+ \cdot \mathbf{v}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} ,
\]

\[
\frac{\partial \mathbf{v}_+}{\partial y^0} = \frac{\mathbf{r}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} , \quad \frac{\partial \mathbf{v}_+}{\partial y^0} = -\frac{\mathbf{r}_+ \cdot \mathbf{v}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)} .
\]

Now with a few applications of the Leibniz and chain rules and a certain amount of book-keeping, (2.54) leads to the result
### 2.5 Solution for Point Charge with Arbitrary Motion

We said in Sect. 2.4 that the Poynting vector \( \varepsilon_0 c E \times B \) determines the flux of energy in the EM fields, i.e., the flow of energy per unit time across a unit area perpendicular to the flow [2, Chap. 27]. In the case of a point charge source, we can find the flux of energy across a sphere \( S \) centred on the retarded point, all points on such a sphere having the same retarded point, namely the centre of \( S \). Taking \( r \) as the vector from the common retarded point, and using \( dS = r r d\Omega \),

\[
\frac{d\Phi}{dt} = \varepsilon_0 c^2 \int_S dS \cdot E \times B = \varepsilon_0 c \int_S d\Omega (r \times E)^2 > 0.
\]

An approximation can be made in the case where the source charge has a small velocity. The result will be the well known Larmor formula for radiation by an accelerating charge.

Both \( E \) and \( B \) contain terms in \( 1/r^2 \) contributing only at short distances, and terms in \( 1/r \) which are called radiative terms. We shall keep only the latter here. Then, by (2.61) and (2.62), respectively, dropping terms in \( v/c \) and dropping the subscript + that indicates that all quantities are evaluated relative to the retarded time,

\[
E_{\text{rad}} = \frac{q}{4\pi \varepsilon_0 c^2 r^3} \left[ r \times \left( r \times \frac{dv}{dt} \right) \right],
\]

and

\[
B_{\text{rad}} = -\frac{q}{4\pi \varepsilon_0 c^3 r^2} r \times \frac{dv}{dt}.
\]

Making this small velocity approximation, the radiated power is
Choosing a polar axis instantaneously parallel to the 3-acceleration of the source charge, and denoting by $\theta$ the angle between this axis and $\mathbf{r}$,

$$\frac{d\mathbf{E}}{dt} = \frac{q^2}{8\pi\varepsilon_0 c^3} \left( \frac{d\mathbf{v}}{dt} \right)^2 \int d\Omega \sin^3 \theta = \frac{4}{3} \frac{q^2}{8\pi\varepsilon_0 c^3} \left( \frac{d\mathbf{v}}{dt} \right)^2 .$$

This is the Larmor formula for the power radiated by an accelerating source charge.

The formula can be made to look considerably neater by making the replacement $e^2 := q^2 / 4\pi\varepsilon_0$, whence

$$\frac{d\mathbf{E}}{dt} = \frac{2e^2a^2}{3c^3} , \quad (2.63)$$

where $a$ is the magnitude of the acceleration. If $q = q_e = 1.60206 \times 10^{-19}$ C is the electron charge in coulombs, and since $1/4\pi\varepsilon_0 = 8.98748 \times 10^9$ in the mks system of units, it turns out that $e$ is numerically equal to $1.5188 \times 10^{-14}$ [1, Chap. 32].

### 2.5.3 Alternative Formula for Fields Due to a Point Charge

This section is really a digression to advertise an elegant version of (2.61) which Feynman apparently devised himself as a way of explaining synchrotron radiation [2, Sect. 21.4]. Indeed, the formula (2.61) for the electric fields due to a point source charge can be rewritten in a revealing way:

$$\mathbf{E} = -\frac{q}{4\pi\varepsilon_0} \left[ \frac{\mathbf{e}_+}{r_+^2} + \frac{r_+}{c} \frac{d}{dt} \left( \frac{\mathbf{e}_+}{r_+^2} \right) + \frac{1}{c^2} \frac{d^2}{dt^2} \mathbf{e}_+ \right] , \quad (2.64)$$

reverting to the form $q$ for the charge and introducing the unit vector $\mathbf{e}_+$ from the field point to the corresponding retarded point, and the retarded distance $r_+$.

This can be interpreted as follows. Firstly, there is a term which looks just like the Coulomb field, but relating to the retarded point. Then there is a term in which nature appears to allow for the fact that the effect is retarded, by means of a correction equal to the rate of change of the main term multiplied by the retarded time $r_+/c$. In other words, we add something like the change which has taken place in the Coulomb term whilst the information is being transferred to the field point. Then there is yet another term, which turns out to be the one describing radiation, and which Feynman uses in [1] to derive all the physical effects related to EM radiation, viz., interference, diffraction, refraction, light scattering, polarisation, and so on.

The magnetic field is just

$$\mathbf{B} = -\frac{\mathbf{e}_+ \wedge \mathbf{E}}{c} ,$$
as we found in (2.62), noting that
\[ \mathbf{e}_+ := -\frac{\mathbf{r}_+}{r_+}. \] (2.65)

Although this section is frankly a digression, the above formulas for $\mathbf{E}$ and $\mathbf{B}$ illustrate something about the classical theory of electromagnetism which supports one of the main themes in this book. Hopefully, the reader will agree that there is something thoroughly remarkable about the EM fields produced by a point charge, which is fully expressed by (2.64), in particular the second term which adjusts for the retardation in the sense explained, and the third term which clearly brings out the role of the acceleration in producing radiation effects, among other things (see below for a further note on that).

It is one of the themes here that Maxwell’s theory, although pre-quantum, is still extremely rich. Just as it told us to move on to the special theory of relativity, which in a sense could be said to explain some of the mysteries of uniform velocity motion, maybe it can also tell us something about accelerating motions and inertia. This is not to say that one could then ignore what quantum theory has done to improve on things in QED. The idea put forward here is not to try to do away with that and promote a rebirth of classical theory, but to look back at the classical theory and ask whether it cannot still teach us something that could then be recognised also within a quantum theoretical version of that.

### Relating the Two Formulas

This is not done explicitly in [2], but it is not difficult. The best approach is to start with the new formula and convert it to the original one. This will involve converting the time derivatives of the direction vector $\mathbf{e}_+$ into the notation $\mathbf{r}_+ := y - x_+$ and $t = y^0 / c$ used in Sect. 2.5.1, although preferably with some simplifications such as dropping the + subscript and adopting the common trick of setting $c = 1$. Most of the exercise is straightforward book-keeping of terms generated by the derivatives and it would not be useful to display all that here. The following is therefore just a pointer.

From (2.65), we have
\[ \frac{d\mathbf{e}_+}{dr} = -\frac{1}{r_+} \frac{d\mathbf{r}_+}{dr} + \frac{\mathbf{r}_+}{r_+^2} \frac{dr_+}{dr}. \]

Of course, $d/dt$ corresponds to $c \partial / \partial y^0$ in the notation of Sect. 2.5.1, and we have the results (2.57), viz.,
\[ \frac{\partial \mathbf{r}_+}{\partial y^0} = -\frac{\mathbf{r}_+ \cdot \mathbf{v}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)}, \quad \frac{\partial \mathbf{r}_+}{\partial y^0} = -\frac{\mathbf{r}_+ \cdot \mathbf{v}_+}{c(r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+/c)}, \]

which imply easily that
\[
\frac{de_+}{dt} = \frac{v_+ - (r_+ \cdot v_+)r_+ / r_+^2}{r_+ - r_+ \cdot v_+ / c} = \frac{v_+ - (e_+ \cdot v_+)e_+}{r_+ - r_+ \cdot v_+ / c}.
\]

A lot of terms are generated in taking the next derivative to obtain \(d^2 e_+ / dt^2\), where one also requires the expression (2.60) for \(\partial v_+ / \partial y^0\), viz.,

\[
\frac{\partial v_+}{\partial y^0} = \frac{r_+ a_+}{c(r_+ - r_+ \cdot v_+ / c)}.
\]

The book-keeping is left to the reader. The point of doing this calculation is to illustrate once again the great care needed to manipulate time derivatives in this context.

**Using the New Formula**

As mentioned, the remarkable relation (2.64) can be used to derive all the basic results concerning EM radiation by accelerating charges [1, 2]. The idea is to select the piece of \(E\) which varies inversely as the distance, and neglect the terms varying inversely as the square of the distance. Indeed, this could be taken as a definition of radiative terms. The point is that the energy density of this part of the field, proportional to \(E^2\), will go as \(1/r^2\), whence its flux through a series of spheres centered on some retarded point will not diminish with distance from the retarded point (radius of the spheres).

It turns out after a little analysis that the radiative part of the electric field is

\[
E_{\text{rad}} = -\frac{q}{4\pi\varepsilon_0 c^2} \frac{d^2 e_+}{dt^2}.
\]  

(2.66)

The picture we thus get is as follows. We look at the charge, in its apparent position, and note the direction of the unit vector (projecting the direction vector onto the unit sphere centred on ourselves). As the charge moves around, the unit vector wiggles, and the acceleration of that unit vector is what gives the radiative field. Now this unit vector will have both a transverse and a radial component of acceleration. The latter is due to the fact that the end point must stay on the surface of a sphere. One can argue that this radial component of acceleration is inversely proportional to the square of the distance of the source charge, and hence does not contribute to the radiation. So finally, one only need consider the transverse component of the field in (2.66), because this is the only component of the fields that escapes to infinity in the usual sense that physicists understand it.
2.5.4 Point Charge with Constant Velocity

It is very instructive indeed to examine the potentials and fields due to a point charge with constant velocity, and they will be used in what follows, so here are the details [2, Sect. 21.6]. We choose the \( x \) axis along the trajectory of the charge, so that this trajectory is given by \( x = vt, y = 0, \) and \( z = 0 \) (see Fig. 2.3).

We choose a field point \( (t, x, y, z) \) at which to evaluate the Lienard–Wiechert potentials (2.53) on p. 19, viz.,

\[
A_\text{ret}^0(y) = \frac{q}{4\pi\varepsilon_0 c (r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+ / c)} , \quad A_\text{ret}^0(y) = \frac{q\mathbf{v}_+}{4\pi\varepsilon_0 c^2 (r_+ - \mathbf{r}_+ \cdot \mathbf{v}_+ / c)} .
\] (2.67)

In the present case, \( \mathbf{v}_+ = \mathbf{v} = (v, 0, 0) \). We need to find the position of the charge at the retarded time

\[
t_+ = t - \frac{r_+}{c} ,
\] (2.68)

where \( r_+ \) is the distance from the field point to the charge at the retarded time. Now because the charge motion is so simple, we know that it was at \( x_+ = vt_+ \) at the retarded time, so we know that

\[
r_+^2 = (x - vt_+)^2 + y^2 + z^2 ,
\]

which combines with (2.68) to give the usual condition

\[
c^2(t - t_+)^2 = (x - vt_+)^2 + y^2 + z^2 .
\]

Solving this quadratic equation in \( t_+ \), we find

\[
\gamma^{-2}t_+ = t - \frac{vx}{c^2} - \frac{1}{c} \left[ (x - vt)^2 + \gamma^{-2}(y^2 + z^2) \right]^{1/2} , \quad \gamma^{-2} := \left( 1 - \frac{v^2}{c^2} \right) .
\] (2.69)

Then \( r_+ \) is obtained from \( r_+ = c(t - t_+) \).

The scalar potential \( \phi := cA^0 \) is now found from

\[
\phi(t, x, y, z) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r_+ - \mathbf{r}_+ \cdot \mathbf{v} / c} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r_+ - (x - vt_+)v / c} .
\]

The denominator here is

\[
c(t - t_+) - \frac{v}{c}(x - vt_+) = c \left[ t - \frac{vx}{c^2} - \gamma^{-2}t_+ \right] ,
\]

and substituting in the formula (2.69) for \( \gamma^{-2}t_+ \), this gives

\[
c(t - t_+) - \frac{v}{c}(x - vt_+) = \left[ (x - vt)^2 + \gamma^{-2}(y^2 + z^2) \right]^{1/2} .
\]
Finally then, the scalar potential due to a charge moving with constant speed \( v \) along the \( x \) axis is

$$\phi = cA^0 = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{\left( (x-vt)^2 + \gamma^{-2}(y^2 + z^2) \right)^{1/2}} \right)$$  \hspace{1em} (2.70)$$

Another way to put this is

$$\phi = \frac{q}{4\pi\varepsilon_0} \left( \frac{\gamma}{\gamma^2(x-vt)^2 + y^2 + z^2} \right)^{1/2}$$  \hspace{1em} (2.71)$$

From (2.67), \( A = v\phi/c^2 \), whence

$$A = \frac{q}{4\pi\varepsilon_0 c^2} \left( \frac{\gamma v}{\gamma^2(x-\nu t)^2 + y^2 + z^2} \right)^{1/2}$$  \hspace{1em} (2.72)$$

Note that these can be obtained from the Coulomb potential, with \( A = 0 \), by a Lorentz transformation, because in the rest frame of the charge, that would be the potential. However, Lorentz actually obtained the form of the Lorentz transformation by looking at the way \((\phi/c, A)\) changes in going from one inertial frame to another [2, Sect. 21.6].

It is interesting to note that the potentials given here at \( x, y, z \) and at time \( t \), relative to some inertial frame, for a charge whose present position in this frame is \((\nu t, 0, 0)\), are neatly expressed in terms of the coordinates \((x - \nu t, y, z)\) of the field...
point as measured from the current position of the moving charge, despite the fact that it is the behaviour of the charge back at the appropriate retarded position that really counts (see Fig. 2.3). This has a consequence when we come to look at the electric and magnetic fields.

It also allows one to get back the general formulas (2.67) in a remarkable way. This account is adapted from [2, Sect. 26.1]. In Fig. 2.4, the charge is moving in an arbitrary way. We seek the potentials at \((x, y, z)\) at time \(t\). We first identify the retarded point \(P_+\) and retarded time \(t_+\), because we know that it is the doings of the charge at the retarded time that determine what happens now at our field point. If \(v_+\) is the velocity of the charge at the retarded time, then we construct what Feynman calls the projected position \(P_{\text{proj}}\), where the charge would be now (at time \(t\)) if it had continued on without acceleration since time \(t_+\). In fact, its real position now is \(P\) on the diagram. Then the potentials at \((x, y, z)\) at time \(t\) are just what (2.71) and (2.72) would give for the imaginary charge at \(P_{\text{proj}}\), because the potentials depend solely on what the charge was doing at the retarded time, so they will be the same at \((x, y, z)\) now whether the charge continued moving at constant velocity \(v_+\), or however else it may actually move thereafter.

The whole of electromagnetism as expressed by Maxwell’s equations thus follows from the three axioms:

- \(A^\mu\) is a four-vector.
- The potential for a stationary charge in an inertial frame is the Coulomb potential \(\phi = q/4\pi\varepsilon_0 r, A = 0\).

Fig. 2.4 Charge with arbitrary motion. The potentials at the chosen field point and time are determined by the position \(P_+\) and velocity \(v_+\) at the retarded time \(t_+ = t - r_+/c\), but they are neatly expressed in terms of the coordinates relative to what Feynman calls the projected position \(P_{\text{proj}}\). Note that the distance from \(P_+\) to \(P_{\text{proj}}\) is just the length of the vector \(v_+(t - t_+)\), which is just \(r_+v_+/c\).
• The potentials produced by a charge with arbitrary motion depend only on the velocity and position of the charge at the retarded time.

Knowing that $A^\mu$ is a four-vector, we transform the Coulomb potential to get the potentials (2.71) and (2.72) for a charge with constant velocity. We then use the rule that, even for an arbitrary motion of the charge, the potentials depend only on the position and velocity at the retarded time, together with the argument above, to find the potentials in that case.

As so clearly explained in [2, Sect. 26.1], this does not mean that the whole of electrodynamics can be deduced solely from the Lorentz transformation and Coulomb’s law. We do need to know that the scalar and vector potentials form a four-vector, and we do need to know that the potentials for the arbitrarily moving charge depend only on the position and velocity, and not for example the acceleration, at the retarded time. We see from (2.61) and (2.62) on p. 21 that the electric and magnetic fields do in fact depend on the acceleration of the charge at the retarded time, as well as the position and velocity then.

But let us return to the electric and magnetic fields for a charge moving with constant velocity. Naturally, we apply the relations (2.2) on p. 5, and this leads to

$$E = \frac{q\gamma}{4\pi\varepsilon_0 \left[ \gamma^2 (x - vt)^2 + y^2 + z^2 \right]^{3/2}} \begin{pmatrix} x - vt \\ y \\ z \end{pmatrix}$$  \hspace{1cm} (2.73)

and

$$B = \frac{\mathbf{v} \times \mathbf{E}}{c^2}.$$  \hspace{1cm} (2.74)

The first observation is that, although the influence of the charge at the given field point at the given time comes from the retarded position of the charge, the electric fields are actually radial from the present position of the charge. The word ‘present’ refers to simultaneity in the chosen inertial frame. It is a rather remarkable fact that this will be true whatever inertial frame we choose. The discussion in [2, Sect. 26.2] is highly recommended once again.

A second point is this. Equations (2.73) and (2.74) were the starting point for a remarkable paper by Bell entitled How to Teach Special Relativity [5]. For those who find Einstein’s approach to special relativity somewhat aphysical, and Minkowski’s rather too geometrical, Bell shows how the readjusting orbit of an electron in the EM fields of a moving nucleus will change the shape and period of a gently accelerating atom, in just the way decreed by relativistic contraction and time dilation. In short, it is a return to a physical way of understanding why, given that there are special (Minkowski) coordinate systems for describing spacetime, adapted to certain (inertially moving) observers, these coordinate systems should be related by Lorentz transformations. This is not just deduced in a pseudo-axiomatic way as is often the case in sophisticated textbooks, nor imputed without further ado to the Minkowskian geometry of spacetime.
This reading is once again highly recommended and more will be said about the subject later on. It is relevant here, where one of the themes is that our theories of the fundamental forces (mainly the theory of electromagnetism in this book) are telling us things that we may not have heard. It was Maxwell’s electromagnetism that told us about the special theory of relativity, although there is a clear tendency to turn things around and start with relativity in a dry and mathematical way. We should not forget where the theory of relativity came from. The view in this book is that Maxwell’s electromagnetism is telling us more, if only we would listen.

The reader should be warned regarding Bell’s paper that some scientists, and in particular philosophers of science it seems, are radically opposed to Bell’s approach in [5]. This may just be because their epistemological concerns are radically different, but the debate is interesting. See for example [9].
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