Chapter 2
Partitions With Distinct Evens

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Abstract Partitions with no repeated even parts (DE-partitions) are considered. A DE-rank for DE-partitions is defined to be the integer part of half the largest part minus the number of even parts. $\Delta(n)$ denotes the excess of the number of DE-partitions with even DE-rank over those with odd DE-rank. Surprisingly $\Delta(n)$ is (1) always non-negative, (2) almost always zero, and (3) assumes every positive integer value infinitely often. The main results follow from the work of Corson, Favero, Liesinger and Zubairy. Companion theorems for DE-partitions counted by exceptional parts conclude the paper.

2.1 Introduction


$$R(q) = \sum_{n=0}^{\infty} q^{(n+1)/2} \frac{(-q;q)_n}{(-q;q)_n} = \sum_{n=0}^{\infty} S(n)q^n$$

was examined. Here

$$(A;q)_n = (1-A)(1-Aq) \cdots (1-Aq^{n-1}).$$

It was shown [4] that $S(n)$ is almost always equal to zero and also assumes every integral value infinitely often. Combinatorially $S(n)$ is the excess of the number of partitions of $n$ into distinct parts with even rank over those with odd rank. The rank

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of a partition is the largest part minus the number of parts \[7\], \[5\]. A similar theorem was proven \[4\, \text{Sec. 5}\] for partitions into odd parts without gaps.

The results for \(S(n)\) rely crucially on the identity \[4\, \text{p. 392}\]

\[
R(q) = \sum_{n \geq 0} \left( -1 \right)^{n+j} q^{n(3n+1)/2-j^2} \left( 1 - q^{2n+1} \right).
\tag{2.3}
\]

It was noted at the end of \[4\] that there are numerous series similar in form to the right-hand expression in (2.3).

Indeed, results of this nature were given for Ramanujan’s fifth order mock theta functions \[2\] (c.f. \[17\]), and such identities formed the basis for pathbreaking work by Zwegers \[18\] and Bringmann, Ono and Rhoades \[6\].

The object of this paper is to reveal a similar phenomenon connected to DE-partitions, i.e. partitions with no repeated even parts. Now DE-partitions have been examined previously. R. Honsberger \[13\] proved the following Euler-type theorem.

**Theorem 2.1.** Let \(P_{DE}(n)\) denote the number of partitions of \(n\) with no repeated even parts. Let \(P_{<4}(n)\) denote the number of partitions of \(n\) in which no part appears more than thrice. Let \(P_{\not|4}(n)\) denote the number of partitions of \(n\) into parts not divisible by 4. Then

\[P_{DE}(n) = P_{<4}(n) = P_{\not|4}(n)\]

for each \(n \geq 0\).

Honsberger’s proof is immediate from the following identification of the related generating functions

\[
\sum_{n \geq 0} P_{DE}(n)q^n = \frac{(-q^2;q^2)_{\infty}}{(q;q^2)_{\infty}} = \frac{(q^4;q^4)_{\infty}}{(q^2;q^2)_{\infty}(q^2;q^2)_{\infty}} = \frac{(q^4;q^4)_{\infty}}{(q;q)_{\infty}} \sum_{n \geq 0} P_{\not|4}(n)q^n = \prod_{n=1}^{\infty} \left( 1 + q^n + q^{2n} + q^{3n} \right) = \sum_{n \geq 0} P_{<4}(n)q^n.
\]

The fact that \(P_{<4}(n) = P_{\not|4}(n)\) is due to J. W. L. Glaisher \[10\], and the asymptotics of these partition functions has been completely examined by P. Hagis \[11\].

From here on, our focus will be on the DE-rank of DE-partitions which is defined to be the integer part of half the largest part minus the number of even parts.

We let \(\delta(m,n)\) denote the number of DE-partitions of \(n\) with DE-rank \(m\).

**Theorem 2.2.**

\[
\sum_{m,n \geq 0} \delta(m,n)z^mq^n = 1 + \sum_{j \geq 0} \frac{(-z^{-1}q^2;q^2)_j z^j q^{2j+1}}{(q;q^2)_{j+1}} (1 + q)
\tag{2.4}
\]
Next we write
\[
\Delta(n) = \sum_{m \geq 0} (-1)^m \delta(m,n). \tag{2.5}
\]

**Theorem 2.3.**
\[
\sum_{n \geq 0} \Delta(n)(-q)^n = \sum_{n \geq 0} \frac{(-1)^n q^n(q^{n+1}/2)(q;q)_n}{(-q)_n}. \tag{2.6}
\]

Fortunately, the expression on the right-hand side of (2.6) is, in fact, \(W_1(-q)\), a function studied extensively by Corson et al. in [7]. In particular, their Theorem 2.3 combined with our Theorem 2.3 yields

**Theorem 2.4.**
\[
\sum_{n \geq 0} \Delta(n)q^n = \sum_{n=0}^{\infty} \left( q^{(2n+1)/2} + q^{(2n+2)/2} \right) \sum_{j=-n}^{n} q^{-j^2}. \tag{2.7}
\]

**Theorem 2.5.** \(\Delta(n)\) is the number of inequivalent elements of the ring of integers of \(Q(\sqrt{2})\) with norm \(8n + 1\).

It immediately follows that

**Corollary 2.1.** \(\Delta(n)\) is always non-negative.

Finally, Corson et al. [7] in the Remark just before their Corollary 5.3 make an assertion equivalent to

**Corollary 2.2.** \(\Delta(n)\) is almost always equal to zero.

The Corollary 5.3 of Corson et al. [7] is equivalent to

**Corollary 2.3.** \(\Delta(n)\) is equal to any given positive integer infinitely often.

The above results in some sense relate only half of the Corson et al. [7] paper to DE-partitions. In order to consider their companion function \(W_2(q)\), we need a new definition related to DE-partitions. We shall say that a part of a DE-partition is *exceptional* if it is either even or one of the smallest parts or both. For example, \(5 + 4 + 2 + 1 + 1\) is a DE-partition with four exceptional parts.

We let \(\varepsilon(m,n)\) denote the number of DE-partitions of \(n\) with \(m\) exceptional parts, and we write
\[
E(n) = \sum_{m \geq 0} (-1)^{m-1} \varepsilon(m,n). \tag{2.8}
\]

Our main result for \(E(n)\) requires the \(W_2(q)\) of Corson et al. [7]:
\[
W_2(q) = \sum_{n=1}^{\infty} \frac{(-1;q^2)_n(-q)_n}{(q;q^2)_n}. \tag{2.9}
\]
Theorem 2.6. 
\[ \sum_{n=1}^{\infty} E(n)q^n = W_2(-q) - \sum_{n=1}^{\infty} q^{\left(\frac{n+1}{2}\right)}. \tag{2.10} \]

This assertion allows us to utilize Theorem 3.3 of Corson et al. [7] to establish immediately that

Theorem 2.7. 
\[ \sum_{n=1}^{\infty} E(n)q^n = \sum_{n \geq 1} \left( q^{\left(\frac{2n}{2}\right)} + q^{\left(\frac{2n+1}{2}\right)} \right) \sum_{j=-n}^{n-1} q^{-j^2 - j}. \]

The three results following Theorem 2.4 now have perfect analogs as consequences of Theorem 2.7. These follow for Theorem 3.3 of [7], the Remark preceding Corollary 5.3 and the proof of Corollary 5.3.

**Theorem 2.8.** \( E(n) \) is the number of inequivalent elements of the ring of integers of \( \mathbb{Q}(\sqrt{2}) \) with norm \( 8n - 1 \) or one less if \( n \) is a triangular number.

**Corollary 2.4.** \( E(n) \) is always non-negative.

**Corollary 2.5.** \( E(n) \) is almost always equal to 0.

The analog of Corollary 2.3 is quite plausible but it does not follow directly because of the second term in (2.10).

The remainder of the paper will be devoted to proofs of Theorems 2.2, 2.3 and 2.6. All the other results are, as noted, direct consequences of these three results and results in Corson et al. [7].

I thank Dean Hickerson for an extensive set of comments on this paper. In particular he has noted that \( \Delta(n) \) is also the number of divisors of \( 8n + 1 \) which are congruent to \( \pm 1 \) modulo 8 minus the number which are congruent to 3 or 5 modulo 8. Consequently, \( \Delta(n) \) is the coefficient of \( 8n + 1 \) in

\[ \sum_{n=1}^{\infty} \frac{\left(\frac{2}{n}\right) q^n}{1 - q^n}, \]

where \( \left(\frac{2}{n}\right) \) is the Legendre symbol.

Finally I note that A. Patkowski [15] has recently found two related theorems for DE-partitions. His theorems provide other lacunary series arising from DE-partition statistics other than the rank.

### 2.2 Proof of Theorem 2.2

For those DE-partitions with largest part \( 2j + 1 \), the DE-rank generating function is
\[
\frac{(1 + z^{-1}q^2)(1 + z^{-1}q^4) \cdots (1 + z^{-1}q^{2j})z^j q^{j+1}}{(1 - q)(1 - q^3) \cdots (1 - q^{2j+1})}.
\]

For those DE-partitions with largest part $2j + 2$, the DE-rank generating function is
\[
\frac{(1 + z^{-1}q^2)(1 + z^{-1}q^4) \cdots (1 + z^{-1}q^{2j})z^j q^{j+1}}{(1 - q)(1 - q^3) \cdots (1 - q^{2j+1})}.
\]

We take the empty partition to have DE-rank 0, and so adding together the empty case, the odd case and the even case we find
\[
\sum_{m,n \geq 0} \delta(m,n) z^m q^n = 1 + \sum_{j=0}^{\infty} \frac{(-z^{-1}q^2; q^2)_j z^j ((q^2 j + 1 + q^{2j+2})}{(q; q^2)_j},
\]
which is equivalent to Theorem 2.2. \(\Box\)

### 2.3 Proof of Theorem 2.3

By Theorem 2.2 with $z$ replaced by $-1$ and $q$ replaced by $-q$, we see that
\[
\sum_{n \geq 0} \Delta(n)(-q)^n = \sum_{m,n \geq 0} \delta(m,n)(-1)^{m+n} q^n
\]
\[
= 1 + \sum_{j \geq 0} \frac{(q^2; q^2)_j (-1)^j q^{2j+1}(1 - q)}{(q^2; q^2)_j}
\]
\[
= 1 - \frac{q(1 - q)}{1 + q} \sum_{j \geq 0} \frac{(q^2; q^2)_j (q^2 q^2)_j (-q^2)^j}{(q^2; q^2)_j (-q^3; q^2)_j}
\]
\[
= 1 + \sum_{j=0}^{\infty} \frac{(q^2 q^2)_j (-q)^{j+1}}{(-q^2; q^2)_j}
\]
\[
= \sum_{j=0}^{\infty} \frac{(q; q^2)_j (-q)^j}{(-q^2; q^2)_j}
\]
\[
= \sum_{j \geq 0} \frac{(q; q^2)_j (q^2 q^2)_j}{(-q^2; q^2)_j} (-q^3)^j q^{2j^2 - 2j} (1 + q^{4j+2})
\]
(by [3, eq. (9.1.1), p. 223, $q \to q^2$, then $\alpha = q$, $\beta = -q^2$, $\tau = -q$])
We require two results from the literature:

\[
\sum_{n=0}^{\infty} q^{(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \quad [9, p. 6, eq. (7.321)] \quad (2.11)
\]

and

\[
2\phi_1 \left( \frac{a, b, q, t}{c} \right) = \frac{(q^2; q^2)_{\infty}}{(t; q)_{\infty}} 2\phi_1 \left( \frac{c, a, b; q, t}{c} \right) \quad [10, p. 10, eq. (1.4.6)] \quad (2.12)
\]

where

\[
2\phi_1 \left( \frac{a, b; q, t}{c} \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n t^n}{(c; q)_n (q; q)_n}. \quad (2.13)
\]

Thus starting from (2.9)

\[
W_2(-q) + 1 = \sum_{n=0}^{\infty} \frac{(-1; q^2)_n q^n}{(-q; q^2)_n} = 2\phi_1 \left( -1, q^2, q^2; q \right)
= \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(-q^{-1}; q^2)_n (q; q^2)_n q^{2n}}{(q^2; q^2)_n (-q; q^2)_n} \quad \text{by (2.12)}
\]

Consequently

\[
W_2(-q) + 1 = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \sum_{n=1}^{\infty} \frac{(1 + q^{-1})}{(1 + q^{2n-1})} q^{2n} (q^{2n+2}; q^2)_{\infty} \quad (2.14)
= \sum_{n=1}^{\infty} \frac{(1 + q^{2n-1} + q^{-1}(1 - q^{2n}))}{(1 + q^{2n-1})} q^{2n} (q^{2n+2}; q^2)_{\infty}
= \sum_{n=1}^{\infty} \frac{q^{2n} (q^{2n+2}; q^2)_{\infty}}{(q^{2n+1}; q^2)_{\infty}} + \sum_{n=1}^{\infty} \frac{q^{2n-1} (q^2; q^2)_{\infty}}{(q^{2n-1}; q^2)_{\infty}}.
\]
Now the first sum above counts DE-partitions with smallest part even and a weight of \( +1 \) if there are an odd number of exceptional parts and \(-1\) if there are an even number. The second sum counts DE-partitions with smallest part odd and a weight of \( +1 \) if there are an odd number of exceptional parts and \(-1\) if there are an even number. Thus the right-hand side of (2.14) is the generating function for \( E(n) \). Invoking (2.11), we see that

\[
\sum_{n=1}^{\infty} E(n)q^n = W_2(-q) - \sum_{n=1}^{\infty} q^{\binom{n+1}{2}}. 
\]

\[\Box\]

2.5 Conclusion

There are a number of natural questions that arise from this study. First, combinatorial proofs of Theorems 2.4 and 2.7 might be possible and are much to be desired.

In addition, the ordinary rank of Dyson has led both to explications of the Ramanujan congruence for \( p(n) \) (cf. [5] and [8]) and to surprising and appealing combinatorial theorems (cf. [9, eqs. (2.3.91) and (2.4.6)]. These aspects of the DE-rank and of exceptional parts of DE-partitions are completely unexplored.

References

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