2

QCD on the lattice – a first look

In Chap. 1 we derived the lattice path integral for a scalar field theory using the canonical approach for the quantization of the system. Subsequently, we changed our point of view and adopted the lattice path integral itself as a method of quantization. The steps involved are a lattice discretization of the classical Euclidean action and the construction of the measure for the integration over "all configurations" of the classical fields. In this chapter we begin with the implementation of these two steps for quantum chromodynamics.

Quantum chromodynamics (QCD) is the theory of strongly interacting particles and fields, i.e., the theory of quarks and gluons. In the first section of this chapter we will review the QCD action functional and its symmetries. This serves as a preparation for the subsequent discretization of QCD on the lattice. The discretization proceeds in several steps. We begin with the naive discretization of the fermionic part of the QCD action followed by discussing the lattice action for the gluons. In the end we write down a complete expression for the QCD lattice path integral. Already at this point we stress, however, that the expression for the path integral obtained in this chapter is not the final word. The nature of the quark fields in our path integral will have to be changed in order to incorporate Fermi statistics. This step will be taken in Chap. 5 where we reinterpret the quark fields as anti-commuting numbers, so-called *Grassmann numbers*.

2.1 The QCD action in the continuum

We begin our construction of the QCD lattice path integral with a review of the Euclidean continuum action. After introducing the quark and gluon fields, we develop the QCD continuum action starting with the discussion of the fermionic part of the action and its invariance under gauge transformations. Subsequently, we construct an action for the gluon fields that are invariant under these transformations.

Gattringer, C., Lang, C.B.: *QCD on the Lattice – A First Look*. Lect. Notes Phys. **788**, 25–41 (2010)
DOI 10.1007/978-3-642-01850-3_2

2.1.1 Quark and gluon fields

The quarks are massive fermions and as such are described by Dirac 4-spinors

$$\psi^{(f)}(x)_{\substack{\alpha \\ c}} \; , \quad \overline{\psi}^{(f)}(x)_{\substack{\alpha \\ c}} \; . \tag{2.1}$$

These quark fields carry several indices and arguments. The space–time position is denoted by x, the Dirac index by $\alpha = 1, 2, 3, 4$, and the color index by $c = 1, 2, 3$. In general we will use Greek letters for Dirac indices and letters $a, b, c, \ldots$ for color. Each field $\psi^{(f)}(x)$ thus has 12 independent components. In addition the quarks come in several flavors called up, down, strange, charm, bottom, and top, which we indicate by a flavor index $f = 1, 2, \ldots, 6$. In many calculations it is sufficient to include only the lightest two or three flavors of quarks. Thus, our flavor index will run from 1 to N_f, the number of flavors. We remark that often we omit the indices and use matrix/vector notation instead.

In the Minkowskian operator approach the fields ψ and $\overline{\psi}$ are related by $\overline{\psi} = \psi^\dagger \gamma_0$, where γ_0 is the γ-matrix related to time. In the Euclidean path integral one uses independent integration variables ψ and $\overline{\psi}$.

In addition to the quarks, QCD contains gauge fields describing the gluons,

$$A_\mu(x)_{cd} \; . \tag{2.2}$$

These fields also carry several indices. As for the quark fields we have a space–time argument denoted by x. In addition, the gauge fields constitute a vector field carrying a Lorentz index μ which labels the direction of the different components in space–time. Since we are interested in the Euclidean action, the Lorentz index μ is Euclidean, i.e., we do not distinguish between covariant and contravariant indices. There is no metric tensor involved and $\mu = 1, 2, 3, 4$ simply label the different components. Finally, the gluon field carries color indices $c, d = 1, 2, 3$. For given x and μ, the field $A_\mu(x)$ is a traceless, hermitian 3×3 matrix at each space–time point x. We will discuss the structure of these matrices and their physical motivation in more detail below.

It is convenient to split the QCD action into a fermionic part, which includes quark fields and an interaction term coupling them to the gluons, and a gluonic part, which describes propagation and interaction of only the gluons.

2.1.2 The fermionic part of the QCD action

The fermionic part $S_F[\psi, \overline{\psi}, A]$ of the QCD action is a bilinear functional in the fields ψ and $\overline{\psi}$. It is given by

$$\begin{aligned} S_F[\psi,\overline{\psi},A] &= \sum_{f=1}^{N_f} \int \mathrm{d}^4x \, \overline{\psi}^{(f)}(x) \left(\gamma_\mu \left(\partial_\mu + \mathrm{i}\, A_\mu(x) \right) + m^{(f)} \right) \psi^{(f)}(x) \\ &= \sum_{f=1}^{N_f} \int \mathrm{d}^4x \, \overline{\psi}^{(f)}(x)_{\substack{\alpha \\ c}} \left((\gamma_\mu)_{\alpha\beta} \left(\delta_{cd}\partial_\mu + \mathrm{i}\, A_\mu(x)_{cd} \right) \right. \\ &\quad \left. + \, m^{(f)} \delta_{\alpha\beta}\delta_{cd} \right) \psi^{(f)}(x)_{\substack{\beta \\ d}} \; . \end{aligned} \tag{2.3}$$

In the first line of this equation we have used matrix/vector notation for the color and Dirac indices, while in the second line we write all indices explicitly. Note that we use the Einstein summation convention.

Equation (2.3) makes it obvious that the action is a sum of the actions for the individual flavors $f = 1, 2, \ldots, N_f$. The quarks with different flavor all couple in exactly the same way to the gluon field A_μ and only differ in their masses $m^{(f)}$. Of course, different flavors also have different electric charge and thus couple differently to the electromagnetic field. However, here we only discuss the strong interaction.

The color indices c and d of the quark fields $\overline{\psi}$, ψ are summed over with the corresponding indices of the gauge field and, in this way, couple the quarks to the gluons. The coupling of the gauge field is different for each component μ since each component is multiplied with a different matrix γ_μ. The γ-matrices are 4×4 matrices in Dirac space, and in the QCD action they mix the different Dirac components of the quark fields. They are the Euclidean versions of the (Minkowski) γ-matrices familiar from the Dirac equation. The Euclidean γ- matrices γ_μ, $\mu = 1, 2, 3, 4$, obey the Euclidean anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\, \delta_{\mu\nu} \, \mathbb{1} \; . \tag{2.4}$$

In Appendix A.2 we discuss how to construct the Euclidean γ-matrices from their Minkowski counterparts and give an explicit representation. The different partial derivatives in the kinetic term of (2.3) mix the Dirac components in the same way as the gauge fields, i.e., the ∂_μ are also contracted with the matrices γ_μ. The kinetic term is, however, trivial in color space. The mass term, finally, is trivial in both color and Dirac space.

Having discussed our notation in detail, we still should verify that the action (2.3) indeed gives rise to the relativistic wave equation for fermions, the Dirac equation. For a single flavor, the contribution to the action is given by (we drop the flavor index for the subsequent discussion and use matrix/vector notation for the color and Dirac indices)

$$S_F[\psi,\overline{\psi},A] \; = \; \int \mathrm{d}^4x \, \overline{\psi}(x) \left(\gamma_\mu \left(\partial_\mu + \mathrm{i}\, A_\mu(x) \right) + m \right) \psi(x) \; . \tag{2.5}$$

The simplest way of applying the Euler Lagrange equations (1.43) in (2.5) is to differentiate the integrand of (2.5) with respect to $\overline{\psi}(x)$. This gives rise to

$$(\gamma_\mu (\partial_\mu + \mathrm{i}\, A_\mu(x)) + m)\, \psi(x) = 0 \,, \tag{2.6}$$

which is indeed the (Euclidean) Dirac equation in an external field A_μ. Thus, we have verified that the action (2.5) has the correct form.

2.1.3 Gauge invariance of the fermion action

So far we have only discussed the different building blocks of QCD and how they are assembled in the fermionic part of the QCD action. Let us now dive a little bit deeper into the underlying structures and symmetries.

Up to the additional color structure, the action (2.5) is exactly the action of electrodynamics – when using matrix/vector notation this difference is not even explicit. As a matter of fact, the QCD action is obtained by generalizing the gauge invariance of electrodynamics.

In electrodynamics the action is invariant under multiplication of the fermion fields with an arbitrary phase at each space–time point x, combined with a transformation of the gauge field. In QCD we require invariance under local rotations among the color indices of the quarks. At each space-time point x we choose an independent complex 3×3 matrix $\Omega(x)$. The matrices are required to be unitary, $\Omega(x)^\dagger = \Omega(x)^{-1}$, and to have $\det[\Omega(x)] = 1$. Such matrices are the defining representation of the *special unitary group*, denoted by SU(3) for the case of 3×3 matrices. It is easy to see that this set is closed under matrix multiplication. Furthermore the unit matrix is contained in this set and for every element there exists an inverse (the hermitian conjugate matrix). Thus the set of SU(3) matrices forms a group. We collect the basic equations showing these statements in Appendix A.1. Note, however, that the group operation – the matrix multiplication – is not commutative. Groups with a non-commutative group operation are called *non-abelian groups*. The idea of using non-abelian groups for a gauge theory was pursued by Yang and Mills [1], and such theories are often referred to as *Yang–Mills theories*.

Returning to our discussion of the QCD gauge invariance, we require that the action is invariant under the transformation

$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x) \,,\; \overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\Omega(x)^\dagger \tag{2.7}$$

for the fermion fields and a yet unspecified transformation $A_\mu(x) \to A'_\mu(x)$ for the gauge fields. Invariance of the action means that we require

$$S_F[\psi', \overline{\psi}', A'] \;=\; S_F[\psi, \overline{\psi}, A] \,. \tag{2.8}$$

With (2.5) and (2.7), this gives

$$S_F[\psi', \overline{\psi}', A'] \;=\; \int \mathrm{d}^4 x\, \overline{\psi}(x)\Omega(x)^\dagger \left(\gamma_\mu \left(\partial_\mu + \mathrm{i}\, A'_\mu(x)\right) + m\right) \Omega(x)\psi(x) \,. \tag{2.9}$$

Using $\Omega(x)^\dagger = \Omega(x)^{-1}$, we see immediately that for the mass term the gauge transformation matrices cancel. For the other terms the situation is a bit more involved. From comparing (2.5) with (2.9) we obtain the condition

$$\begin{aligned} \partial_\mu + \mathrm{i}\, A_\mu(x) &= \Omega(x)^\dagger \left(\partial_\mu + \mathrm{i}\, A'_\mu(x)\right) \Omega(x) \\ &= \partial_\mu + \Omega(x)^\dagger \left(\partial_\mu \Omega(x)\right) + \mathrm{i}\Omega(x)^\dagger A'_\mu(x) \Omega(x) \,. \end{aligned} \tag{2.10}$$

This is an equation for an operator acting on a function of x. Thus, due to the product rule, we find two terms with derivatives. We now can solve for $A'_\mu(x)$ (again we use $\Omega(x)^\dagger = \Omega(x)^{-1}$) and we arrive at the transformation property for the gauge field

$$A_\mu(x) \to A'_\mu(x) = \Omega(x) A_\mu(x) \Omega(x)^\dagger + \mathrm{i}\left(\partial_\mu \Omega(x)\right) \Omega(x)^\dagger \,. \tag{2.11}$$

Note that also $A'_\mu(x)$ is hermitian and traceless as required for the gauge fields. For the first term on the right-hand side of (2.11), this follows from the fact that $A_\mu(x)$ is traceless and $\Omega(x)^\dagger = \Omega(x)^{-1}$. For the second term this is shown in Appendix A.1 (see (A.11) and (A.15)).

The requirement that the fermion action (2.5) remains invariant under the gauge transformation (2.7) of the fermions necessarily implies the presence of gauge fields $A_\mu(x)$ with the transformation properties given by (2.11).

In the next section, when we discretize QCD on the lattice, we again require the invariance of the lattice action under the local gauge transformations (2.7) for the quark fields. Gauge fields have to be introduced to achieve gauge invariance of the action.

2.1.4 The gluon action

Let us now discuss the action for the gluon fields $A_\mu(x)$. The gluon action $S_G[A_\mu]$ is a functional of only the gauge fields and is required to be invariant under the transformation (2.11):

$$S_G[A'] = S_G[A] \,. \tag{2.12}$$

To construct an action with this property we define the *covariant derivative*

$$D_\mu(x) = \partial_\mu + \mathrm{i}\, A_\mu(x) \,. \tag{2.13}$$

From our intermediate result in the first line of (2.10) we read off the transformation property for the covariant derivative as

$$D_\mu(x) \to D'_\mu(x) = \partial_\mu + \mathrm{i}\, A'_\mu(x) = \Omega(x) D_\mu(x) \Omega(x)^\dagger \,. \tag{2.14}$$

These transformation properties ensure that $D_\mu(x)\psi(x)$ and $\psi(x)$ transform in exactly the same way – thus, the name "covariant derivative."

The covariant derivatives are now used to construct an action functional which is a generalization of the expression known from electrodynamics. We define the field strength tensor $F_{\mu\nu}(x)$ as the commutator

$$F_{\mu\nu}(x) = -\mathrm{i}[D_\mu(x), D_\nu(x)] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \mathrm{i}[A_\mu(x), A_\nu(x)] \;, \quad (2.15)$$

where the last term on the right-hand side does not vanish since $A_\mu(x)$ and $A_\nu(x)$ are matrices, i.e., objects where multiplication is a non-commutative operation. Up to this commutator, the field strength tensor $F_{\mu\nu}(x)$ has the same form as the field strength in electrodynamics.

The fact that the field strength tensor is the commutator of two covariant derivatives implies that it inherits the transformation properties (2.14), i.e., it transforms as

$$F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = \Omega(x) F_{\mu\nu}(x) \Omega(x)^\dagger \;. \quad (2.16)$$

As a candidate for the gauge action we now consider the expression

$$S_G[A] = \frac{1}{2\,g^2} \int \mathrm{d}^4 x \;\mathrm{tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right] \;. \quad (2.17)$$

Taking the trace over the color indices ensures that (2.17) is invariant under gauge transformations. One can use (2.16), the invariance of the trace under cyclic permutations, and $\Omega(x)^\dagger = \Omega(x)^{-1}$ to verify this property. Furthermore, the summation over the Lorentz indices μ, ν (summation convention) ensures that the action is a Lorentz scalar. We stress again that (2.17) is understood in Euclidean space.

From (2.15) and (2.17) it is obvious that our gauge action generalizes the action of electrodynamics. Up to the trace and the different overall factor, it exactly matches the action for the electromagnetic field. The trace is due to the fact that gluon fields are matrix valued. Below, we decompose the matrix-valued fields into components and in this way get rid of the trace, pushing the similarity to electrodynamics even further. The extra factor $1/g^2$ is just a convenient way to introduce the coupling. After rescaling the gauge fields

$$\frac{1}{g} A_\mu(x) \;\rightarrow\; A_\mu(x) \;, \quad (2.18)$$

the factor $1/g^2$ in (2.17) is gone and the gauge action assumes the more familiar form. Now, the gauge coupling shows up in the covariant derivative

$$D_\mu(x) \;\rightarrow\; \partial_\mu + \mathrm{i}\, g A_\mu(x) \;, \quad (2.19)$$

making obvious that g is the coupling strength of the gauge fields to the quarks. On the lattice it is more convenient to have the gauge coupling as an overall factor of the gauge action, i.e., we work with the form (2.17).

2.1.5 Color components of the gauge field

We have introduced the gauge fields $A_\mu(x)$ as hermitian, traceless matrices and have shown that the gauge transformation (2.11) preserves these properties. Thus the $A_\mu(x)$ are in the Lie algebra su(3) and we can write

$$A_\mu(x) = \sum_{i=1}^{8} A_\mu^{(i)}(x)\, T_i \; . \tag{2.20}$$

The components $A_\mu^{(i)}(x), i = 1, 2, \ldots 8$, are real-valued fields, the so-called color components, and the T_i are a basis for traceless hermitian 3×3 matrices (see Appendix A.1). We can use this representation (2.20) of the gauge field to write also the field strength tensor (2.15) in terms of its components. Inserting (2.20) into (2.15) we obtain

$$F_{\mu\nu}(x) = \sum_{i=1}^{8} \Big(\partial_\mu A_\nu^{(i)}(x) - \partial_\nu A_\mu^{(i)}(x) \Big)\, T_i \; + \mathrm{i} \sum_{j,k=1}^{8} A_\mu^{(j)}(x) A_\nu^{(k)}(x)\, [T_j, T_k] \; . \tag{2.21}$$

The commutator on the right-hand side can be simplified further with the commutation relations (A.4) and one ends up with

$$F_{\mu\nu}(x) = \sum_{i=1}^{8} F_{\mu\nu}^{(i)}(x)\, T_i \; , \tag{2.22}$$

$$F_{\mu\nu}^{(i)}(x) = \partial_\mu A_\nu^{(i)}(x) - \partial_\nu A_\mu^{(i)}(x) - f_{ijk} A_\mu^{(j)}(x) A_\nu^{(k)}(x) \; . \tag{2.23}$$

This representation of the field strength can now be inserted in the expression (2.17) for the gauge action and, using (A.3) to evaluate the trace, we obtain

$$S_G[A] = \frac{1}{4\, g^2} \sum_{i=1}^{8} \int \mathrm{d}^4 x\, F_{\mu\nu}^{(i)}(x) F_{\mu\nu}^{(i)}(x) \; . \tag{2.24}$$

From this equation we see that the gauge action is a sum over color components and each term has the form of the action of electrodynamics. However, there appears a qualitatively new feature: From the right-hand side of (2.23) we see that the field strength color components are not linear in the gauge field $A_\mu^{(i)}(x)$ but have a quadratic piece which mixes the different color components of the gluon field. Thus, in the action (2.24) we not only encounter the term quadratic in the gauge fields which is familiar from electrodynamics, but also find cubic and quartic terms. These terms give rise to self-interactions of the gluons, making QCD a highly nontrivial theory. The self-interactions are responsible for confinement of color, the most prominent feature of QCD.

In Fig. 2.1 we show a schematic picture (a so-called tree-level Feynman diagram) illustrating the cubic and quartic interaction terms. The curly lines represent the gluons and the dots are the interaction vertices.

These remarks conclude our review of the continuum action and we have all the concepts and notations necessary to begin the lattice discretization of quantum chromodynamics. We stress, however, that besides gauge invariance, there are other important symmetries of the QCD action. We discuss these symmetries later as we need them.

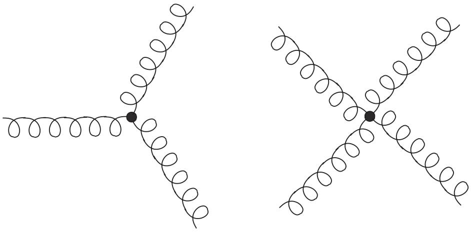

Fig. 2.1. Schematic picture of the cubic and quartic gluon self-interaction. The *wavy lines* represent the gluon propagators and the *dots* are the interaction vertices

2.2 Naive discretization of fermions

In this section we introduce the so-called *naive discretization* of the fermion action. Later, in Chap. 5, this discretization will be augmented with an additional term to remove lattice artifacts. Here, it serves to present the basic idea and, more importantly, to discuss the representation of the lattice gluon field which differs from the continuum form. We show that on the lattice the gluon fields must be introduced as elements of the gauge group and not as elements of the algebra, as is done in the continuum formulation.

2.2.1 Discretization of free fermions

As discussed in Sect. 1.4, the first step in the lattice formulation is the introduction of the 4D lattice Λ:

$$\Lambda = \{ n = (n_1, n_2, n_3, n_4) \mid n_1, n_2, n_3 = 0, 1, \ldots, N-1 \,;\; n_4 = 0, 1, \ldots, N_T - 1 \} \,. \tag{2.25}$$

The vectors $n \in \Lambda$ label points in space–time separated by a lattice constant a. In our lattice discretization of QCD we now place spinors at the lattice points only, i.e., our fermionic degrees of freedom are

$$\psi(n) \,,\, \overline{\psi}(n) \,, \quad n \in \Lambda \,, \tag{2.26}$$

where the spinors carry the same color, Dirac, and flavor indices as in the continuum (we suppress them in our notation). Note that for notational convenience we only use the integer-valued 4-coordinate n to label the lattice position of the quarks and not the actual physical space-time point $x = an$.

In the continuum the action S_F^0 for a free fermion is given by the expression (set $A_\mu = 0$ in (2.5))

$$S_F^0[\psi, \overline{\psi}] = \int \mathrm{d}^4 x \, \overline{\psi}(x) \left(\gamma_\mu \partial_\mu + m \right) \psi(x) \,. \tag{2.27}$$

When formulating this action on the lattice we have to discretize the integral over space–time as well as the partial derivative. The discretization is implemented as a sum over Λ, as we did for the scalar field theory in Chap. 1. The partial derivative is discretized with the symmetric expression

$$\partial_\mu \psi(x) \;\to\; \frac{1}{2a}\left(\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})\right) \,. \tag{2.28}$$

Thus, our lattice version of the free fermion action reads

$$S_F^0[\psi,\overline{\psi}] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \, \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} \;+\; m\,\psi(n) \right) \,. \tag{2.29}$$

This form is the starting point for the introduction of the gauge fields.

2.2.2 Introduction of the gauge fields as link variables

In the last section we showed that requiring the invariance of the action under the local rotation (2.7) of the color indices of the quark fields enforces the introduction of the gauge fields. On the lattice we implement the same transformation by choosing an element $\Omega(n)$ of SU(3) for each lattice site n and transforming the fermion fields according to

$$\psi(n) \to \psi'(n) = \Omega(n)\,\psi(n) \;,\; \overline{\psi}(n) \to \overline{\psi}'(n) = \overline{\psi}(n)\,\Omega(n)^\dagger \,. \tag{2.30}$$

As for the continuum case, we find that on the lattice the mass term of (2.29) is invariant under the transformation (2.30). For the discretized derivative terms in (2.29), this is not the case. Consider, e.g., the term

$$\overline{\psi}(n)\psi(n+\hat{\mu}) \to \overline{\psi}'(n)\,\psi'(n+\hat{\mu}) = \overline{\psi}(n)\,\Omega(n)^\dagger\,\Omega(n+\hat{\mu})\,\psi(n+\hat{\mu}) \,. \tag{2.31}$$

This is not gauge-invariant. If, however, we introduce a field $U_\mu(n)$ with a directional index μ, then

$$\overline{\psi}'(n)\,U'_\mu(n)\,\psi'(n+\hat{\mu}) \;=\; \overline{\psi}(n)\,\Omega(n)^\dagger\,U'_\mu(n)\,\Omega(n+\hat{\mu})\,\psi(n+\hat{\mu}) \tag{2.32}$$

is gauge-invariant if we define the gauge transformation of the new field by

$$U_\mu(n) \;\to\; U'_\mu(n) = \Omega(n)\,U_\mu(n)\,\Omega(n+\hat{\mu})^\dagger \,. \tag{2.33}$$

In order to make the fermionic action (2.29) gauge-invariant, we introduce the gauge fields $U_\mu(n)$ as elements of the gauge group SU(3) which transform as given in (2.33). These matrix-valued variables are oriented and are attached to the links of the lattice and thus are often referred to as *link variables*. $U_\mu(n)$ lives on the link which connects the sites n and $n+\hat{\mu}$.

Since the link variables are oriented, we can also define link variables that point in negative μ direction. Note that these are not independent link variables but are introduced only for notational convenience. In particular, $U_{-\mu}(n)$

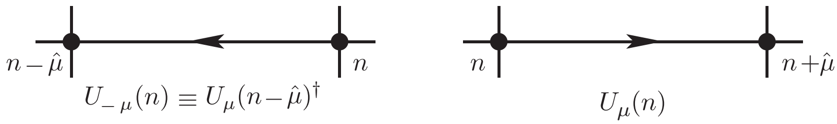

Fig. 2.2. The link variables $U_\mu(n)$ and $U_{-\mu}(n)$

points from n to $n - \hat{\mu}$ and is related to the positively oriented link variable $U_\mu(n - \hat{\mu})$ via the definition

$$U_{-\mu}(n) \equiv U_\mu(n - \hat{\mu})^\dagger \, . \tag{2.34}$$

In Fig. 2.2 we illustrate the geometrical setting of the link variables on the lattice. From the definitions (2.34) and (2.33) we obtain the transformation properties of the link in negative direction

$$U_{-\mu}(n) \;\rightarrow\; U'_{-\mu}(n) = \Omega(n)\, U_{-\mu}(n)\, \Omega(n - \hat{\mu})^\dagger \, . \tag{2.35}$$

Note that we have introduced the gluon fields $U_\mu(n)$ as elements of the gauge group SU(3), not as elements of the Lie algebra which were used in the continuum. According to the gauge transformations (2.33) and (2.35) also the transformed link variables are elements of the group SU(3).

Having introduced the link variables and their properties under gauge transformations, we can now generalize the free fermion action (2.29) to the so-called *naive fermion action* for fermions in an external gauge field U:

$$S_F[\psi, \overline{\psi}, U] = a^4 \sum_{n \in \Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a} + m\, \psi(n) \right) . \tag{2.36}$$

Using (2.30), (2.33), and (2.35) for the gauge transformation properties of fermions and link variables, one readily sees the gauge invariance of the fermion action (2.36), $S_F[\psi, \overline{\psi}, U] = S_F[\psi', \overline{\psi}', U']$.

2.2.3 Relating the link variables to the continuum gauge fields

Let us now discuss the link variables in more detail and see how they can be related to the algebra-valued gauge fields of the continuum formulation. We have introduced $U_\mu(n)$ as the link variable connecting the points n and $n + \hat{\mu}$. The gauge transformation properties (2.33) are consequently governed by the two transformation matrices $\Omega(n)$ and $\Omega(n + \hat{\mu})^\dagger$. Also in the continuum an object with such transformation properties is known: It is the path-ordered exponential integral of the gauge field A_μ along some curve $\mathcal{C}_{xy}$ connecting two points x and y, the so-called *gauge transporter*:

$$G(x,y) = P \exp\left(\mathrm{i}\int_{\mathcal{C}_{xy}} A\cdot \mathrm{d}s\right) . \tag{2.37}$$

We may assume that a lattice is embedded in the continuum, where smooth gauge fields live. For a detailed discussion of the path-ordered exponential in the continuum see, e.g., [2]. We do not need the precise definition of the continuum gauge transporters and only use that under a gauge transformation (2.11) they transform as

$$G(x,y) \;\rightarrow\; \Omega(x)\, G(x,y)\, \Omega(y)^\dagger . \tag{2.38}$$

These transformation properties are the same as for our link variables $U_\mu(n)$ when n and $n+\hat{\mu}$ are considered as end points of a path. Based on these transformation properties, we interpret the link variable $U_\mu(n)$ as a lattice version of the gauge transporter connecting the points n and $n+\hat{\mu}$, i.e., we wish to establish $U_\mu(n) = G(n, n+\hat{\mu}) + \mathcal{O}(a)$. For that purpose we introduce algebra-valued lattice gauge fields $A_\mu(n)$ and write

$$U_\mu(n) = \exp\left(\mathrm{i}\, a A_\mu(n)\right) . \tag{2.39}$$

When comparing (2.37) and (2.39) one sees that we have approximated the integral along the path from n to $n+\hat{\mu}$ by $aA_\mu(n)$, i.e., by the length a of the path times the value of the field $A_\mu(n)$ at the starting point.[1] This approximation is good to $\mathcal{O}(a)$ and no path ordering is necessary to that order. Since the link variables act as gauge transporters, we will often use this nomenclature instead of "link variable."

Based on the relation (2.39) we can now also connect the lattice fermion action (2.36) to its continuum counterpart (2.5). Since one of the guiding principles of our construction is the requirement that in the limit $a \rightarrow 0$ the lattice action approaches the continuum form, we expand (2.39) for small a,

$$U_\mu(n) = \mathbb{1} + \mathrm{i}\, a A_\mu(n) + \mathcal{O}(a^2) \;\;,\;\; U_{-\mu}(n) = \mathbb{1} - \mathrm{i}\, a A_\mu(n-\hat{\mu}) + \mathcal{O}(a^2) \;, \tag{2.40}$$

where we use (2.34) and $A_\mu = A_\mu^\dagger$ for the expansion of $U_{-\mu}(n)$. Inserting these expanded link variables into expression (2.36) we find

$$S_F[\psi,\overline{\psi},U] = S_F^0[\psi,\overline{\psi}] \;+\; S_F^I[\psi,\overline{\psi},A] \;, \tag{2.41}$$

where S_F^0 denotes the free part of the action and the interaction part reads

$$\begin{aligned} S_F^I[\psi,\overline{\psi},A] &= \mathrm{i}\, a^4 \sum_{n\in\Omega}\sum_{\mu=1}^{4} \overline{\psi}(n)\gamma_\mu \frac{1}{2}\left(A_\mu(n)\,\psi(n+\hat{\mu}) + A_\mu(n-\hat{\mu})\,\psi(n-\hat{\mu})\right) \\ &= \mathrm{i}\, a^4 \sum_{n\in\Omega}\sum_{\mu=1}^{4} \overline{\psi}(n)\gamma_\mu A_\mu(n)\psi(n) \;+\; \mathcal{O}(a) \;. \end{aligned} \tag{2.42}$$

[1] We remind the reader that for notational convenience we denote the lattice points only by their integer-valued 4-coordinates n. For the current discussion it should, however, be kept in mind that the physical space–time coordinates are $a\,n$, i.e., neighboring lattice points are separated by the distance a.

In the second step we have used $\psi(n \pm \hat{\mu}) = \psi(n) + \mathcal{O}(a)$ and $A_\mu(n - \hat{\mu}) = A_\mu(n) + \mathcal{O}(a)$. The two Eqs. (2.41) and (2.42) establish that when expanding the lattice version (2.36) of the fermionic action for $a \to 0$, we indeed recover the continuum form (2.5).

Before we continue with discretizing the gauge part of the QCD action we stress an important conceptual point: The group-valued link variables $U_\mu(n)$ are not merely an auxiliary construction to sneak the Lie algebra-valued fields $A_\mu(x)$ of the continuum into the lattice formulation. Instead, the group elements $U_\mu(n)$ are considered as the fundamental variables which are integrated over in the path integral (see Chap. 3). This change from algebra-valued to (compact) group-valued fields has important consequences. In particular, the role of gauge fixing changes considerably. We will discuss these issues in detail once we have completed the construction of QCD on the lattice.

2.3 The Wilson gauge action

We have introduced the link variables as the basic quantities for putting the gluon field on the lattice. Now we construct a lattice gauge action in terms of the link variables and show that in the limit $a \to 0$ it approaches its continuum counterpart (assuming that the lattice gauge fields are embedded in a continuous background). This is the *naive continuum limit* in contradistinction to the continuum limit of the full, integrated quantum theory.

2.3.1 Gauge-invariant objects built with link variables

As a preparation for the construction of the gluon action let us first discuss the transformation properties of a string of link variables along a path of links. Let $\mathcal{P}$ be such a path of k links on the lattice connecting the points n_0 and n_1. We define the ordered product

$$P[U] = U_{\mu_0}(n_0) U_{\mu_1}(n_0 + \hat{\mu}_0) \ldots U_{\mu_{k-1}}(n_1 - \hat{\mu}_{k-1}) \equiv \prod_{(n,\mu)\in\mathcal{P}} U_\mu(n) \; . \quad (2.43)$$

This object is the lattice version of the continuum gauge transporter (2.37). Note that the path $\mathcal{P}$ may contain link variables for both directions $\pm\mu$.

From the transformation properties for single link variables, (2.33) and (2.35), it follows that gauge rotations for all but the end points cancel: Consider the transformation of two subsequent link variables on the path, one ending at n the other one starting from this point. The two transformation matrices $\Omega(n)^\dagger$ and $\Omega(n)$ cancel each other at n. Only the matrices at the two end points of the path, n_0 and n_1, remain. Thus the product $P[U]$ transforms according to

$$P[U] \; \to \; P[U'] = \Omega(n_0)\, P[U]\, \Omega(n_1)^\dagger \; . \quad (2.44)$$

Like for the single link term, from such a product of link variables $P[U]$ a gauge-invariant quantity can be constructed by attaching quark fields at the starting point and at the end point,

$$\overline{\psi}(n_0)\, P[U]\, \psi(n_1)\ . \tag{2.45}$$

An alternative way of constructing a gauge-invariant product of link variables is to choose for the path $\mathcal{P}$ and a closed loop $\mathcal{L}$ and to take the trace,

$$L[U] = \operatorname{tr}\left[\prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\right]\ . \tag{2.46}$$

According to (2.44), under a gauge transformation only the matrices $\Omega(n_0)$ and $\Omega(n_0)^\dagger$ at the point n_0 where the loop is rooted remain. These matrices then cancel when taking the trace. We find

$$L[U'] = \operatorname{tr}\left[\Omega(n_0) \prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\, \Omega(n_0)^\dagger\right] = \operatorname{tr}\left[\prod_{(n,\mu)\in\mathcal{L}} U_\mu(n)\right] = L[U]\ . \tag{2.47}$$

Thus, we have established that the trace over a closed loop of link variables is a gauge-invariant object. Such loops of link variables are used for the construction of the gluon action and later will also serve as physical observables.

2.3.2 The gauge action

For the gluon action it is sufficient to use the shortest, nontrivial closed loop on the lattice, the so-called *plaquette*. The plaquette variable $U_{\mu\nu}(n)$ is a product of only four link variables defined as

$$\begin{aligned} U_{\mu\nu}(n) &= U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_{-\mu}(n+\hat{\mu}+\hat{\nu})\, U_{-\nu}(n+\hat{\nu}) \\ &= U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_\mu(n+\hat{\nu})^\dagger\, U_\nu(n)^\dagger\ . \end{aligned} \tag{2.48}$$

In the second formulation we have utilized the equivalence (2.34). We depict the plaquette in Fig. 2.3. As we have shown in the last paragraph, the trace of the plaquette variable is a gauge-invariant object.

We now present Wilson's form of the gauge action [3] – the first formulation of lattice gauge theory – and subsequently show that it indeed approaches the continuum form in the naive limit $a \to 0$. The *Wilson gauge action* is a sum over all plaquettes, with each plaquette counted with only one orientation. This sum can be realized by a sum over all lattice points n where the plaquettes are located, combined with a sum over the Lorentz indices $1 \leq \mu < \nu \leq 4$,

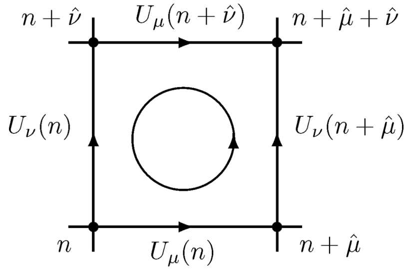

Fig. 2.3. The four link variables which build up the plaquette $U_{\mu\nu}(n)$. The *circle* indicates the order that the links are run through in the plaquette

$$S_G[U] = \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \text{Re tr} \left[\mathbb{1} - U_{\mu\nu}(n)\right] . \tag{2.49}$$

The individual contributions are the real parts of traces over the unit matrix minus the plaquette variable. The factor $2/g^2$ is included to match the form of the continuum action (2.17) in the limit $a \to 0$. Let us now discuss this limit.

For establishing the correct limit we need to expand the link variables in the form (2.39) for small a. In order to handle the products of the four link variables in the plaquette in an organized way, it is useful to invoke the Baker–Campbell–Hausdorff formula for the product of exponentials of matrices:

$$\exp(A)\, \exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + \cdots\right), \tag{2.50}$$

where A and B are arbitrary matrices and the dots indicate powers of the matrices larger than 2 which are omitted. The formula (2.50) can be proven easily by expanding both sides in powers of A and B. Inserting (2.39) into the definition (2.48) of the plaquette and applying (2.50) iteratively, we obtain

$$\begin{aligned} U_{\mu\nu}(n) = \; \exp\Big(& \mathrm{i}\, aA_\mu(n) + \mathrm{i}\, aA_\nu(n+\hat{\mu}) - \frac{a^2}{2}[A_\mu(n), A_\nu(n+\hat{\mu})] \\ & - \mathrm{i}\, aA_\mu(n+\hat{\nu}) - \mathrm{i}\, aA_\nu(n) - \frac{a^2}{2}[A_\mu(n+\hat{\nu}), A_\nu(n)] \\ & + \frac{a^2}{2}[A_\nu(n+\hat{\mu}), A_\mu(n+\hat{\nu})] + \frac{a^2}{2}[A_\mu(n), A_\nu(n)] \\ & + \frac{a^2}{2}[A_\mu(n), A_\mu(n+\hat{\nu})] + \frac{a^2}{2}[A_\nu(n+\hat{\mu}), A_\nu(n)] + \mathcal{O}(a^3)\Big) . \end{aligned} \tag{2.51}$$

In this expression we have gauge fields with shifted arguments such as $A_\nu(n+\hat{\mu})$. We now perform a Taylor expansion for these fields, i.e., we set

$$A_\nu(n+\hat{\mu}) = A_\nu(n) + a\,\partial_\mu A_\nu(n) + \mathcal{O}(a^2) , \tag{2.52}$$

in all these terms and take into account contributions up to $\mathcal{O}(a^2)$. With this expansion several terms cancel and we obtain

$$\begin{aligned} U_{\mu\nu}(n) &= \exp\left(\mathrm{i}\, a^2 \left(\partial_\mu A_\nu(n) - \partial_\nu A_\mu(n) + \mathrm{i}[A_\mu(n), A_\nu(n)]\right) + \mathcal{O}(a^3)\right) \\ &= \exp\left(\mathrm{i}\, a^2 F_{\mu\nu}(n) + \mathcal{O}(a^3)\right) \; . \end{aligned} \tag{2.53}$$

In the second step we use the continuum definition of the field strength given in (2.15). The form (2.53) can now be inserted in (2.49) for the Wilson gauge action. The exponential in (2.53) is expanded and we find

$$S_G[U] = \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\ \mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(n)\right] = \frac{a^4}{2\, g^2} \sum_{n\in\Lambda} \sum_{\mu,\nu} \mathrm{tr}[F_{\mu\nu}(n)^2] + \mathcal{O}(a^2) \; . \tag{2.54}$$

The terms of $\mathcal{O}(a^2)$ that appear in the expansion of the exponential in (2.53) cancel when taking the real part of $\mathrm{tr}[\mathbb{1} - U_{\mu\nu}(n)]$ (one may use $\mathrm{tr}[U_{\mu\nu}(n)]^* = \mathrm{tr}[U_{\mu\nu}(n)^\dagger] = \mathrm{tr}[U_{\nu\mu}(n)]$ to see this). In a similar way also the $\mathcal{O}(a^3)$ terms in the expansion of (2.53) cancel, such that the Wilson action approximates the continuum form up to $\mathcal{O}(a^2)$, as stated in (2.54). Note that the factor a^4 together with the sum over Λ is just the discretization of the space–time integral and thus $\lim_{a\to 0} S_G[U] = S_G[A]$. This completes our discussion of the naive continuum limit $a \to 0$ for the Wilson gauge action.

Concluding this section we remark that different lattice actions for the gauge fields have been proposed in order to reduce cutoff effects further. We return to this issue in Chap. 9.

2.4 Formal expression for the QCD lattice path integral

Having constructed the fermion and gauge field actions on the lattice we can now write down the complete expression for the lattice QCD path integral formula for Euclidean correlators. As already announced, the preliminary formulation presented here will be refined further in subsequent chapters. Nevertheless, we find such an intermediate summary helpful for understanding the line of the construction.

2.4.1 The QCD lattice path integral

Following the discussion of Sect. 1.4 we write Euclidean correlators as a lattice path integral in the form

$$\langle O_2(t)\, O_1(0)\rangle = \frac{1}{Z} \int \mathcal{D}\left[\psi, \overline{\psi}\right]\, \mathcal{D}[U]\; \mathrm{e}^{-S_F[\psi,\overline{\psi},U] - S_G[U]}\, O_2[\psi, \overline{\psi}, U]\, O_1[\psi, \overline{\psi}, U] \; , \tag{2.55}$$

where the partition function Z is given by

$$Z = \int \mathcal{D}\left[\psi, \overline{\psi}\right]\, \mathcal{D}[U]\; \mathrm{e}^{-S_F[\psi,\overline{\psi},U] - S_G[U]} \; . \tag{2.56}$$

The quantization of the system in the path integral formalism is implemented as an integral over all field configurations. On the lattice the corresponding path integral measures are products of measures of all quark field components and products of measures for all link variables:

$$\mathcal{D}\left[\psi,\overline{\psi}\right] = \prod_{n\in\Lambda}\prod_{f,\alpha,c} \mathrm{d}\psi^{(f)}(n)_{\substack{\alpha\\c}}\, \mathrm{d}\overline{\psi}^{(f)}(n)_{\substack{\alpha\\c}}\,, \quad \mathcal{D}[U] = \prod_{n\in\Lambda}\prod_{\mu=1}^{4} \mathrm{d}U_\mu(n)\,. \tag{2.57}$$

Both the fermion and gauge field measures shown here will be discussed in more detail in subsequent sections. For the fermion fields we have to include the Pauli's principle, turning the spinors ψ and $\overline{\psi}$ into anticommuting variables. These so-called *Grassmann numbers* and the corresponding rules of integration will be discussed in Chap. 5. For the gauge fields we have denoted in (2.57) the measure for a single link variable as $\mathrm{d}U_\mu(n)$, but not yet specified how we implement the integration over the group manifold of SU(3). This leads to the concept of *Haar measure* which we discuss in the next chapter. Despite these issues, the expressions in (2.57) already incorporate some of the essential features of lattice QCD, in particular the reduction of the original quantum fields to a countable number of classical variables.

As discussed in Sect. 1.4, in the path integral quantization (2.55) the configurations to be integrated over are weighted with the Boltzmann factor of the Euclidean action. The corresponding lattice versions of the fermion and gauge parts of the action have been derived as (compare (2.36) and (2.49))

$$S_F[\psi,\overline{\psi},U] = a^4 \sum_{f=1}^{N_f}\sum_{n\in\Lambda}\left(\overline{\psi}^{(f)}(n)\sum_{\mu=1}^{4}\gamma_\mu \frac{U_\mu(n)\psi^{(f)}(n+\hat{\mu}) - U_{-\mu}(n)\psi^{(f)}(n-\hat{\mu})}{2a} + m^{(f)}\,\overline{\psi}^{(f)}(n)\psi^{(f)}(n)\right) + \text{terms discussed in Chap. 5}, \tag{2.58}$$

where we now also sum over N_f flavors of quarks. We stress again that the fermion action has to be augmented with another term in order to remove lattice artifacts (see Chap. 5). The gauge action, however, is ready to go and is taken over unchanged from (2.49):

$$S_G[U] = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu} \mathrm{Re}\ \mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(n)\right]\,. \tag{2.59}$$

Let us remark that the functionals $O_1[\psi,\overline{\psi},U]$ and $O_2[\psi,\overline{\psi},U]$ are translations of the operators $\widehat{O}_1$ and $\widehat{O}_2$ acting in Hilbert space. The translation proceeds as expressed in (1.81) by evaluating the operators between field eigenstates. We stress that the functional O_2 depends only on the fields with time argument n_t related to Euclidean time t on the right-hand side of (2.55) via $t = a\,n_t$. The fields in the functional O_1 depend only on the fields with $t = 0$.

Equations (2.55), (2.56), (2.57), (2.58), and (2.59) comprise our current status in the construction of lattice QCD. So far we discussed the fundamental fields describing quarks and gluons and put them onto the lattice – quarks on the sites and the gauge fields on the links of the lattice. In order to allow for color rotations of the fermion fields in the same way as in the continuum, the gauge fields were introduced as group-valued link variables. Subsequently, we showed that products of link variables along closed paths are gauge-invariant and we have used the plaquette to construct Wilson's gauge action. Finally, we put together these ingredients according to the path integral quantization recipe of Sect. 1.4. Operators are translated into functionals of classical fields. These functionals are weighted with the Boltzmann factor and this product is then integrated over all possible field configurations. The precise definition of this integration – Grassmann integration for the fermions, Haar measure for the link variables – will be discussed in subsequent chapters.

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http://www.springer.com/978-3-642-01849-7

Quantum Chromodynamics on the Lattice
An Introductory Presentation
Gattringer, C.; Lang, C.B.
2010, XV, 343 p. 34 illus., Hardcover
ISBN: 978-3-642-01849-7