Transformations of Darboux Integrable Systems

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Abstract This article reviews some recent theoretical results about the structure of Darboux integrable differential systems and their relationship with symmetry reduction of exterior differential systems. The symmetry reduction representation of Darboux integrable equations is then used to derive some new and unusual transformations.

1 Introduction

Broadly speaking, a system of partial differential equations $\Delta_1 = 0$ is said to be ***Darboux integrable*** if there exists an auxiliary system of compatible PDE $\Delta_2 = 0$ [20] such that

1. The combined system $\{\, \Delta_1 = 0,\ \Delta_2 = 0 \,\}$ is a system of total partial differential equations, that is, one which can be integrated by ODE methods, and
2. The auxiliary system $\Delta_2 = 0$ is parameterized by a number of arbitrary functions, sufficient in number so as to insure that every (local) solution to $\Delta_1 = 0$ can be realized as a solution to $\{\, \Delta_1 = 0,\ \Delta_2 = 0 \,\}$.

Partial differential equations which admit closed-form general solutions can be shown to be Darboux integrable ([16], p. 225) but the general definition of Darboux integrability extends well beyond this special case.

The auxiliary equations $\Delta_2 = 0$ are classically referred to as ***intermediate integrals*** for the given system $\Delta_1 = 0$ and, apart from E. Vessiot's remarkable papers [26], [27], the analysis of the method of Darboux has focused exclusively on the existence of these integrals. Vessiot observed that inherent in the integration of Darboux integrable equations are certain ODE systems known as equations of Lie type and this led, for the first time, to a group theoretical formulation of the method

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of Darboux. In a recent article [2], the authors (with P. J. Vassiliou) re-interpreted Vessiot's approach within the more general setting of symmetry reduction of differential systems and used this general setting to develop a new, algorithmic approach for the explicit integration of Darboux integrable systems. We also introduced the concept of a ***non-linear superposition formula*** for differential systems and showed how Darboux integrable systems always admit such a formula. This concept of a non-linear superposition formula provides a new way of looking at Darboux integrable systems which has proven to be quiet useful.

From the outset, it should be emphasized that the group-theoretic tools which arise in this new approach to Darboux integrability are rather different from the more familiar methods due to Sophus Lie – the relevant Lie groups arise, not as symmetry groups for the given equations, but rather from certain canonical normalizations of the structure equations for the exterior differential systems associated to $\Delta_1 = 0$. We cite the theory of equations of Lie type [21](Chap. 3), [24] (Chap. 10) and papers by Cartan [8] and Vessiot [25] as other instances where Lie groups, which do not arise as symmetry groups, have lead to new integration methods for various classes of differential equations.

The best-known example of a Darboux integrable equation is the Liouville equation

$$u_{xy} = e^u. \tag{1}$$

The intermediate integrals for this equation are

$$u_{xx} = \frac{1}{2}u_x^2 + f(x) \quad \text{and} \quad u_{yy} = \frac{1}{2}u_y^2 + g(y). \tag{2}$$

It is a simple but nevertheless instructive exercise, to check that the compatibility conditions for (1) and (2) are satisfied. The method by which one derives (2) from (1) is well-established (see, for example [7], [3], [4], [23]). Vessiot's critical observation is that these intermediate integrals may be viewed as a Ricatti equation and these are equations of Lie type for the standard fractional linear action of the Lie group $SL(2)$ on the line.

The method of Darboux is one of the cornerstones of the classical geometric theory of differential equations developed in the nineteenth century by Monge, Ampere, Laplace, Goursat and Darboux. For the following reasons, we believe it remains an important topic:

1. Darboux integrable systems always have infinitely many generalized symmetries and conservation laws ([4], [23]) and consequently are always present in any classification of PDE with these properties.
2. One of the principle goals of the geometric theory of PDE is to relate properties of geometric invariants of PDE to solution techniques and properties of the solutions. Surely Darboux integrable equations afford the simplest situation where such relationships can be developed.
3. The mathematical physics literature contains many ad hoc methods for finding closed-form general solutions to reductions of various fundamental field theories. The method of Darboux, as generalized in [2], provides a completely systematic and algorithmic approach to the derivation of these solutions.

4. The integration of PDE by computer algebra systems is a very active area of current research and classical geometric methods, such as the method of Darboux, provide a powerful complement to differential elimination methods.

The goal of the present article is to summarize the key results of [2] and, as an application, establish some new and rather surprising transformations between various Darboux integrable differential systems.

2 Symmetry Reduction of Exterior Differential Systems and the Method of Darboux

We begin this section with the general definition of symmetric reduction of an EDS ([1]). We illustrate this definition within the familiar context of ODE reduction and make a few comments regarding the general theory. A (non-technical) summary of the application of symmetry reduction to the study of Darboux integrablity, as developed in [2], is presented.

Definition 2.1 Let G be a Lie group acting regularly on a manifold M and let $\mathbf{q}\colon M \to M/G$ be the projection map to the quotient space M/G of M by the orbits of G. Let $\mathcal{I}$ be an EDS on M and suppose that G is a symmetry group of $\mathcal{I}$. Then the G-reduction of $\mathcal{I}$ is the EDS $\mathcal{I}/G$ on M/G defined by

$$\mathcal{I}/G = \{\, \omega \in \Omega^*(M/G) \mid \mathbf{q}^*(\omega) \in \mathcal{I} \,\}. \tag{3}$$

To calculate the reduced differential system $\mathcal{I}/G$, one first calculates the forms $\mathcal{I}_{\mathbf{sb}} \subset \mathcal{I}$ which are semi-basic for the action of G on M. Specifically, if Γ denotes the Lie algebra of infinitesimal generators for the action of G on M, then

$$\mathcal{I}_{\mathbf{sb}} = \{\, \omega \in \mathcal{I} \mid X \lrcorner\, \omega = 0 \text{ for all } X \in \Gamma \,\}. \tag{4}$$

One can interpret $\mathcal{I}_{\mathbf{sb}}$ as the largest differential sub-system of $\mathcal{I}$ for which each $X \in \Gamma$ is a Cauchy characteristic. At this point, a standard result on reduction by Cauchy characteristics (see, for example [5], p. 31) asserts that one can construct a basis for $\mathcal{I}_{\mathbf{sb}}$ in terms of the G-invariant functions on M and their differentials. This basis naturally projects under $\mathbf{q}$ to give a basis for $\mathcal{I}/G$.

As a simple example, let us consider the problem of integrating the 4th order ODE ([19],(7.16))

$$3y''y^{(iv)} - 5(y''')^2 = 0. \tag{5}$$

In terms of standard jet coordinates $\{\, x,\ y,\ y_1,\ y_2,\ y_3 \,\}$, the differential system for this ODE is the rank 4 Pfaffian system

$$\mathcal{I} = \{\, \theta^1 = dy - y_1dx,\ \theta^2 = dy_1 - y_2dx,\ \theta^3 = dy_2 - y_3dx,\ \theta^4 = dy_3 - \frac{5y_3^2}{3y_2}dx \} \tag{6}$$

and we take as our symmetry group (with group parameters $(a, b, c) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^*$) the transformation group

$$x = a + cx,\ \ y = b + cy,\ \ y_1 = y_1,\ \ y_2 = \frac{y_2}{c}, y_3 = \frac{y_3}{c^2}. \tag{7}$$

The infinitesimal generators for this action are

$$\Gamma = \{\ \partial_x,\ \partial_y,\ x\partial_x + y\partial_y - y_2\partial_{y_2} - 2y_3\partial_{y_3}\ \}. \tag{8}$$

The orbits of G are two dimensional so that the quotient space M/G has dimension two with coordinates (say) $\{u,\ v\}$. The invariants for action of G on M are y_1 and y_3/y_2^2 so that we may write the projection $\mathbf{q}$ as

$$u = y_1, \quad v = \frac{y_3}{y_2^2}\ . \tag{9}$$

By solving the linear system

$$X \lrcorner\ (a_1\theta^1 + a_2\theta^2 + a_3\theta^3 + a_4\theta^4) = 0 \quad \text{for all } X \in \Gamma \tag{10}$$

we determine that

$$\mathcal{I}_{\mathbf{sb}} = \{\,\theta^4 - \frac{2y_3}{y_2}\,\theta^3 + \frac{y_3^2}{3y_2^2}\,\theta^2\,\} = \{\,dy_3 - \frac{2y_3}{y_2}dy_2 + \frac{y_3^2}{3y_2^2}dy_1\,\}. \tag{11}$$

We substitute $y_3 = vy_2^2$ and $y_1 = u$ into this result to find that the reduced EDS is

$$\mathcal{I}/G = \{\,v^2du + 3dv\,\} \tag{12}$$

and the reduced ODE is

$$\frac{dv}{du} = -\frac{1}{3}v^2. \tag{13}$$

This is precisely the result one would obtain using a step-by-step reduction of (5) following the well-known algorithm as presented in [22], pp. 130–161. In [1] we show how the general solution to (5) can be obtained from the solution to (13) and the group action (7).

Thus, for ODE, the reduction procedure given by Definition 2.1 coincides, more or less, with the usual reduction by differential invariants although it does lead to a new approach for lifting solutions of the reduced system to the original system and for dealing with non-regular group actions and singular orbits [13]. What is of real importance for us here is that Definition 2.1 provides us with a simple, unambiguous approach to the symmetry reduction of partial differential equations within an EDS setting. We should emphasize, however, that if $\mathcal{I}$ is the canonical Pfaffian system for some system of PDE (obtained by the restriction of the contact ideal on the appropriate jet space), then the reduction $\mathcal{I}/G$ may not be the canonical Pfaffian system for a PDE with the same order and same number of dependent and independent variables – even when the Cartan character and Cartan integer remain the same. In this sense, the ODE reduction presented in the foregoing example, where we effortlessly passed from the reduced EDS (12) to the ODE (13) is not reflective of the general situation.

This last remark naturally leads to the ***PDE recognition problem*** for differential systems of the kind discussed by Vessiot (see also Stomark [24], page 274) and more recently by Yamaguchi [28]. The construction (3) also raises a wide range of interesting (and often very challenging) problems regarding the behavior of the various geometric properties for differential systems under reduction. As a simple illustration, we cite Theorem 5.1 in [1], where conditions under which the reduction of a Pfaffian system is Pfaffian are established.

We now frame the theory of Darboux integrablity within the context of symmetry reduction of differential systems. Roughly speaking, a differential system $\mathcal{I}$ is Darboux integrable if (1) it is algebraically generated by 1-forms and 2-forms; (2) if the structure equations for the 1-forms decompose, at the symbol level, into a certain block diagonal form; and (3) if the singular Pfaffian systems determined by this decomposition admit a sufficient number of intermediate integrals. See [2] for the precise definition of Darboux integrablity. This definition generalizes the classical definition of Darboux integrablity for scalar PDE in the plane. We remark that, as with the classical definition, it frequently happens that a differential system is not Darboux integrable but that its prolongation to some order is.

The main results of [2] can be summarized as follows.

Result 1. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be Pfaffian systems on manifolds M_1 and M_2, respectively. Then the differential system $\mathcal{W}_1 + \mathcal{W}_2$ on $M_1 \times M_2$ is Darboux integrable (Theorem 3.1).

Result 2. Let G be a Lie group acting freely on M_1 and M_2 and as symmetries of $\mathcal{W}_1$ and $\mathcal{W}_2$ Asssume that G is transverse to $\mathcal{W}_1$ and $\mathcal{W}_2$. Then the quotient differential system $(\mathcal{W}_1 + \mathcal{W}_2)/G$ is Darboux integrable (Corollary 3.4).

Result 3. Let $\mathcal{I}$ on M be a Darboux integrable differential system with associated singular Pfaffian systems $\widehat{\mathcal{V}}$ and $\widecheck{\mathcal{V}}$. Then there is a (local) Lie group G and free right and left actions $\hat{\mu}\colon G \times M \to M$ and $\check{\mu}\colon G \times M \to M$ such that $\hat{\mu}$ preserves $\widehat{\mathcal{V}}$, $\check{\mu}$ preserves $\widecheck{\mathcal{V}}$, and $\hat{\mu}$ commutes with $\check{\mu}$. (For the complete list of properties which characterize these actions, see Sect. 5.3 of [2].) The Lie group G is called the ***Vessiot group*** for the Darboux integrable differential system $\mathcal{I}$.

Result 4. Let $\mathcal{I}$ on M be a Darboux integrable differential system with associated singular Pfaffian systems $\widehat{\mathcal{V}}$ and $\widecheck{\mathcal{V}}$ and Vessiot group G. Let $\mathcal{W}_1$ be the restriction of $\widehat{\mathcal{V}}$ to a fixed, maximal integral manifold of $\widecheck{\mathcal{V}}^\infty$ and let $\mathcal{W}_2$ be the restriction of $\widecheck{\mathcal{V}}$ to a fixed, maximal integral manifold of $\widehat{\mathcal{V}}^\infty$. Then

$$\mathcal{I} \cong (\mathcal{W}_1 + \mathcal{W}_2)/G. \tag{14}$$

We call the quotient differential systems $(\mathcal{W}_1 + \mathcal{W}_2)/G$ the ***quotient representation*** of the Darboux integrable differential system $\mathcal{I}$. This is Theorem 5.1 in [2].

Result 5. Every Darboux integrable differential system is uniquely characterized by its restricted singular Pfaffian systems $\mathcal{W}_1$ and $\mathcal{W}_2$, the Lie group G, and the actions $\hat{\mu}$ and $\check{\mu}$.

Result 6. The integral manifolds of a Darboux integrable differential system $\mathcal{I}$ can be constructed from the integral manifolds of its restricted singular Pfaffian systems $\mathcal{W}_1$ and $\mathcal{W}_2$ (Corollary 5.12).

Result 2 is extremely important to the entire subject of Darboux integrablity. While, in the past, it has been quite difficult to construct new examples of Darboux integrable systems it is now possible, using Result 2, to create entire new classes of Darboux integrable EDS.

Result 4 shows that every Darboux integrable system can be realized as a reduction of the trivial type of Darboux integrable system consider in Result 1. Regrettably, the present proof of this result is rather difficult. This is primarily because there are many groups actions which satisfy the conclusions of Result 3 but only a very careful and lengthy analysis of the structure equations for the singular Pfaffian systems leads to the correct choice of group actions required for the validity of (14).

Result 5 emphasizes the fact that Darboux integrablity, instead of being studied from the viewpoint of compatibility theory, can now be studied entirely within the setting of group actions and symmetry groups of differential systems. For example, (differential) invariants for the action of the Lie group G on the manifold M_1 and M_2 project under $\mathbf{q}$ to give the intermediate integrals for the EDS $\mathcal{I}$.

Result 6 shows explicitly that the explicit integration of a Darboux integrable $\mathcal{I}$ depends upon the explicit integration of the its restricted singular Pfaffian systems $\widehat{\mathcal{W}}$ and $\check{\mathcal{W}}$. In particular, one is assured of algebraic, closed-form general solutions for $\mathcal{I}$ whenever $\widehat{\mathcal{W}}$ and $\check{\mathcal{W}}$ can be identified with the canonical contact structures on jet spaces. Result 6 is also the key to the symbolic implementation of the method of Darboux.

3 Transformations of Darboux Integrable Systems

The symmetry reduction approach to the method of Darboux described in the previous section allows us to develop a new, coherent transformation theory for Darboux integrable differential systems. The following three Principles summarize the key results obtained thus far. Module various technical transversality conditions these principles are indeed theorems. Precise statements and proofs of these theorems will appear elsewhere.

Principle A (Prolongation). **[i]** *Let $\mathcal{I}$ be a differential system with independence condition $\mathcal{J}$. Let G be a symmetry group of $(\mathcal{I}, \mathcal{J})$ and suppose that $\mathcal{I}/G$ is a (regular) differential system. Then the symmetry group G lifts to a symmetry group of any prolongation (or partial prolongation [7]) $\mathcal{I}^{[p]}$ of $\mathcal{I}$ and*

$$(\mathcal{I}/G)^{[\mathfrak{p}]} = (\mathcal{I}^{[\mathfrak{p}]})/G.$$

[ii] *Let $\mathcal{I}$ be a differential system with independence condition $\mathcal{J}$. If $\mathcal{I}$ is Darboux integrable, then every prolongation of $\mathcal{I}$ is Darboux integrable.*

[iii] *Let $\mathcal{I}$ be a differential system with independence condition $\mathcal{J}$. Let G be a symmetry group of $(\mathcal{I}, \mathcal{J})$ and suppose that $\mathcal{I}/G$ is a (regular)differential system. If $\mathcal{I}$ is Darboux integrable, then some prolongation of $\mathcal{I}/G$ is Darboux integrable.*

Principle B (Mappings). **[i]** *Let $\mathcal{I}$ be a differential system with symmetry group G. Let H be a normal subgroup of G and suppose that $\mathcal{I}/G$ and $\mathcal{I}/H$ are regular differential systems. Then the quotient group G/H is a symmetry group of $\mathcal{I}/H$ and*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathbf{q}_H} & \mathcal{I}/H \\ & \searrow^{\mathbf{q}_G} & \downarrow \mathbf{q}_{G/H} \\ & & \mathcal{I}/G \end{array}$$

is a commutative diagram of differential systems.

[ii] *Let $\mathcal{I}$ and $\mathcal{J}$ be two differential systems with a common symmetry group G. Suppose that $\mathcal{I}/G$ and $\mathcal{J}/G$ are regular differential systems and that $\mathcal{I}$ and $\mathcal{J}$ are equivalent by a G equivariant diffeomorphism. Then there is an induced equivalence*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\cong} & \mathcal{J} \\ \downarrow \mathbf{q}_G & & \downarrow \mathbf{q}_G \\ \mathcal{I}/G & \xrightarrow{\cong} & \mathcal{J}/G \end{array} .$$

Principle C (Extensions). **[i]** *If $\pi : \widetilde{\mathcal{I}} \to \mathcal{I}$ is an integrable extension (see [6]) of a Darboux integrable differential systems $\mathcal{I}$, then $\widetilde{\mathcal{I}}$ is Darboux integrable.*

[ii] *If $\pi : \widetilde{\mathcal{I}} \to \mathcal{I}$ is an integrable extension of a Darboux integrable differential systems $\mathcal{I}$, then there are integrable extensions $\pi_1 : \widetilde{\mathcal{W}}_1 \to \mathcal{W}_1$ and $\pi_2 : \widetilde{\mathcal{W}}_2 \to \mathcal{W}_2$ and symmetry groups G and $\widetilde{G}$ of $\mathcal{W}_i$ and $\widetilde{\mathcal{W}}_i$, $i = 1, 2$ such that the diagram of differential systems*

$$\begin{array}{ccc} \widetilde{\mathcal{W}}_1 + \widetilde{\mathcal{W}}_2 & \xrightarrow{\mathbf{q}_{\widetilde{G}}} & \widetilde{\mathcal{I}} \\ \pi_1 \times \pi_2 \downarrow & & \downarrow \pi \\ \mathcal{W}_1 + \mathcal{W}_2 & \xrightarrow{\mathbf{q}_G} & \mathcal{I} \end{array}$$

commutes.

[iii] *There is a normal subgroup H of $\widetilde{G}$ such that $G = \widetilde{G}/H$. The projection maps π, π_1 and π_2 are all $\widetilde{G}$ equivariant maps.*

4 Linear Equations

In this first example we consider Darboux integrable linear partial differential equations

$$u_{xy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = 0. \tag{15}$$

We describe the quotient representation for these equations and we show how Principle B leads naturally to the constructions of the classical Laplace transformations between linear Darboux integrable equations.

Let $J^m(\mathbf{R}, \mathbf{R}) \times J^n(\mathbf{R}, \mathbf{R})$ be the product of two jet spaces with standard jet coordinates

$$(x,\ \phi,\ \phi_1,\ \ldots,\ \phi_m,\ y,\ \psi,\ \psi_1\ , \ldots,\ \psi_n)$$

and let $\mathcal{C}^m + \mathcal{C}^n$ be the sum of the canonical contact systems

$$\begin{aligned} \mathcal{C}^m &= \{\theta_0 = \ d\phi - \phi_1\, dx,\ \theta_1 = d\phi_1 - \phi_2\, dx,\ \ldots, \theta_{m-1} = d\phi_{m-1} - \phi_m\, dx\} \\ \mathcal{C}^n &= \{\vartheta_0 = d\psi - \psi_1\, dy,\ \vartheta_1 = d\psi_1 - \psi_2\, dy,\ \ldots, \vartheta_{n-1} = d\psi_{n-1} - \psi_n\, dy\}. \end{aligned} \tag{16}$$

Result 1 states that this sum is trivially a Darboux integrable differential system. Principle A states that quotients of $\mathcal{C}^m + \mathcal{C}^n$ will be Darboux integrable. Here we establish precisely which group actions will lead to the Pfaffian systems for (15).

Let G_p be the p dimensional Abelian group acting on $J^m \times J^n$ with infinitesimal generators defined by the prolongation of the vector fields.

$$Z^i = f^i(x)\frac{\partial}{\partial\phi} + f^i(y)\frac{\partial}{\partial\psi}, \quad i = 1, \ldots, p. \tag{17}$$

The functions f^i are smooth and subject only to the condition stated below in Theorem 4.1. By definition, G_p is a symmetry group for the canonical contact system $\mathcal{C}^m + \mathcal{C}^n$.

Theorem 4.1 **[i]** *Let $p = m + n - 3$ and assume that the action of G_p on $J^{n-2} \times J^{m-2}$ is free. Then the quotient differential system*

$$\mathcal{I} = (\mathcal{C}^m + \mathcal{C}^n)/G_p \tag{18}$$

is the standard rank 3 Pfaffian system, defined on a seven manifold, for a linear PDE (15).

[ii] *The Pfaffian system for any Darboux integrable linear PDE* (15) *is a quotient differential system of the type* (18).

Proof. The prolonged infinitesimal actions for G_p are

$$\mathrm{pr}Z^i = \sum_{k=0}^{m}[\frac{d^k f^i}{d\,x^k}(x)]\frac{\partial}{\partial\phi_k} + \sum_{l=0}^{n}[\frac{d^l\, f^i}{d\,y^l}(y)]\frac{\partial}{\partial\psi_l},$$

so that a form

$$\omega = \sum_{k=0}^{n-1} A^k \theta_k + \sum_{l=0}^{n-1} B^l \vartheta_l$$

is semi-basic if and only if

$$\sum_{k=0}^{m-1} [\frac{d^k f^i}{d x^k}(x)] A^k \ + \sum_{l=0}^{n-1} [\frac{d^l f^i}{d y^l}(y)] C^l = 0.$$

With the functions f^i chosen so that action of G_p on $J^{n-2} \times J^{m-2}$ is free, the rank of this linear system of $m+n-3$ equations for $m+n$ unknowns A^k and B^l is maximal. This implies that G_p is transverse to the derived system $\mathcal{C}^{m-1} + \mathcal{C}^{n-1}$ and therefore the quotient EDS is a rank 3 Pfaffian system.

The functions x, y are obviously invariants for this action and there is precisely 1 additional differential invariant U on $J^{(m-2)} \times J^{(n-2)}$. As the solution to the equations $\mathrm{pr}Z^i(U) = 0$, the function U is linear in the variables ϕ_m and ψ_n, this is,

$$U = \sum_{k=0}^{m-2} C^k(x, y)\phi_k + \sum_{l=0}^{n-2} D^l(x, y)\psi_l. \tag{19}$$

We next note the six functions x, y, U, $D_x U$ $D_y U$ and $D_{xy} U$ on $J^{m-1} \times J^{n-1}$ are all G_p invariant. But there can only be five independent G_p invariant functions on $J^{n-1} \times J^{m-1}$ so that these six functions are necessary functionally dependent. In fact, the linearity of these invariants in the variables ϕ_k and ψ_l forces this dependence to be of the form

$$D_{xy}U + a(x, y)D_x U + b(x, y)D_y U + c(x, y)U = 0 \tag{20}$$

for some choice of functions a, b, c. These coefficients determine the PDE (15) for our quotient EDS.

The quotient of $J^m \times J^n$ by G_p is a seven dimensional manifold with coordinates $(x, y, u, u_x, u_u, u_{xx}, u_{yy})$, where the quotient map is defined by

$$x = x,\ \ y = y,\ \ u = U,\ \ u_x = D_x U,\ \ u_y = D_y U,$$
$$u_{xx} = D_{xx} U, \quad u_{yy} = D_{yy} U.$$

On account of (19) and (20), the forms

$$\mathcal{I} = \{\, du - u_x\, dx - u_y\, dy,\ du_x - u_{xx}\, du_x - u_{xy} dy,\ du_y - u_{xy}\, dx - u_{yy}\, dy \,\}, \tag{21}$$

where u_{xy} is given by (15), pullback under $\mathbf{q}$ into $\mathcal{C}^m + \mathcal{C}^n$ and therefore determine the quotient Pfaffian differential system. The system $\mathcal{I}$ is therefore the canonical differential system for PDE of the form (15).

We remark that the intermediate integrals for (15) are given by the projection of the differential invariants for the action of G_p on the individual jet spaces $J^{m+n-1}(x,\phi)$ and $J^{m+n-1}(y,\psi)$. □

This symmetry approach allows one to generate many new families of linear, Darboux integrable, scalar PDE. But, in order to arrive at some simple concrete examples for which we can give complete formulas, we consider the elementary action (17) determined by the functions $f^i(z) = z^i$ for $i = 0\dots 4$ and acting on the jet spaces

$$J^4(\mathbf{R},\mathbf{R})\times J^4(\mathbf{R},\mathbf{R}),\quad J^3(\mathbf{R},\mathbf{R})\times J^5(\mathbf{R},\mathbf{R})\quad \text{and}\quad J^2(\mathbf{R},\mathbf{R})\times J^6(\mathbf{R},\mathbf{R}). \tag{22}$$

In each case the canonical contacts systems on these jet spaces define rank 6 Pfaffian systems on 12 dimensional manifolds. The quotient differential systems

$$\mathbf{q}_1 : \mathcal{C}^4\times\mathcal{C}^4 \to \mathcal{I},\quad \mathbf{q}_2 : \mathcal{C}^3\times\mathcal{C}^5 \to \mathcal{J},\quad \mathbf{q}_3 : \mathcal{C}^2\times\mathcal{C}^6 \to \mathcal{K} \tag{23}$$

are found to be the canonical 3 rank Pfaffian systems for the equations

$$\mathcal{I} : u_{xy} + \frac{6u}{\zeta^2} = 0,\quad \mathcal{J} : v_{xy} - \frac{2v_x}{\zeta} + \frac{6v}{\zeta^2} = 0,\quad \mathcal{K} : w_{xy} - \frac{4w_x}{\zeta} + \frac{4w}{\zeta^2} = 0. \tag{24}$$

Here $\zeta = x - y$. The projection map $\mathbf{q}_1$ is

$$\begin{aligned}
u = U \quad &= \frac{12\phi}{\zeta^2} - \frac{6\phi_1}{\zeta} + \phi_2 - \frac{12\psi}{\zeta^2} - \frac{6\psi_1}{\zeta} - \phi_2,\\
u_x = D_xU \quad &= -\frac{24}{\zeta^3}\phi + \frac{18}{\zeta^2}\phi_1 - \frac{6}{\zeta}\phi_2 + \phi_3 + \frac{24}{\zeta^3}\psi + \frac{6}{\zeta^2}\psi_1,\\
u_y = D_yU \quad &= \frac{24}{\zeta^3}\phi - \frac{6}{\zeta^2}\phi_1 - \frac{24}{\zeta^3}\psi - \frac{18}{\zeta^2}\psi_1 - \frac{6}{\zeta}\psi_2 - \psi_3,\\
u_{xx} = D_{xx}U &= \frac{72}{\zeta^4}\phi - \frac{60}{\zeta^3}\phi_1 + \frac{24}{\zeta^2}\phi_2 - \frac{6}{\zeta}\phi_3 + \phi_4 - \frac{72}{\zeta^4}\phi - \frac{12}{\zeta^3}\phi_1,\\
u_{yy} = D_{yy}U &= \frac{72}{\zeta^4}\phi - \frac{12}{\zeta^3}\phi_1 - \frac{72}{\zeta^4}\psi - \frac{60}{\zeta^3}\psi_1 - \frac{24}{\zeta^2}\psi_2 - \frac{6}{\zeta}\psi_3 - \psi_4,
\end{aligned}$$

with partial prolongations $\mathbf{q}_1^{[0,1]} :\ \mathcal{C}^4\times\mathcal{C}^5 \to \mathcal{I}^{[0,1]}$ and $\mathbf{q}_1^{[0,2]} :\ \mathcal{C}^4\times\mathcal{C}^6 \to \mathcal{I}^{[0,2]}$ defined by

$$\begin{aligned}
u_{yyy} &= \frac{288}{\zeta^5}\phi - \frac{36}{\zeta^4}\phi_1 - \frac{288}{\zeta^5}\psi - \frac{252}{\zeta^4}\psi_1 - \frac{108}{\zeta^3}\psi_2 - \frac{30}{\zeta^2}\psi_3 - \frac{6}{\zeta}\psi_4 - \psi_5, \text{ and}\\
u_{yyyy} &= \frac{1440}{\zeta^6}\phi - \frac{144}{\zeta^5}\phi_1 - \frac{1440}{\zeta^6}\psi - \frac{1296}{\zeta^5}\psi - \frac{576}{\zeta^4}\psi_2 - \frac{168}{\zeta^3}\psi_3\\
&\quad - \frac{36}{\zeta^2}\psi_4 - \frac{6}{\zeta}\psi_5 - \psi_6.
\end{aligned}$$

The projection map $\mathbf{q}_2$ is

$$\begin{aligned}
v = V \quad &= +\frac{24}{\zeta^3}\phi - \frac{6}{\zeta^2}\phi_1 - \frac{24}{\zeta^3}\psi - \frac{18}{\zeta^2}\psi_1 - \frac{6}{\zeta}\psi_2 - \psi_3,\\
v_x = D_x V \quad &= -\frac{72}{\zeta^4}\phi + \frac{36}{\zeta^3}\phi_1 - \frac{6}{\zeta^2}\phi_2 + \frac{72}{\zeta^4}\psi + \frac{36}{\zeta^3}\psi_1 + \frac{6}{\zeta^2}\psi_2,\\
v_y = D_y V \quad &= +\frac{72}{\zeta^4}\phi - \frac{12}{\zeta^3}\phi_1 - \frac{72}{\zeta^4}\psi - \frac{60}{\zeta^3}\psi_1 - \frac{24}{\zeta^2}\psi_2 - \frac{6}{\zeta}\psi_3 - \psi_4,\\
v_{xx} = D_{xx} V &= +\frac{288}{\zeta^5}\phi - \frac{180}{\zeta^4}\phi_1 + \frac{48}{\zeta^3}\phi_2 - \frac{6}{\zeta^2}\phi_3 - \frac{288}{\zeta^5}\psi - \frac{108}{\zeta^4}\psi_1 - \frac{12}{\zeta^3}\psi_2,\\
v_{yy} = D_{yy} V &= +\frac{288}{\zeta^5}\phi - \frac{36}{\zeta^4}\phi_1 - \frac{288}{\zeta^5}\psi - \frac{252}{\zeta^4}\psi_1 - \frac{108}{\zeta^3}\psi_2 - \frac{30}{\zeta^2}\psi_3\\
&\quad - \frac{6}{\zeta}\psi_4 - \psi_5
\end{aligned}$$

with partial prolongations $\mathbf{q}_2^{[1,0]}\colon \mathcal{C}^4 \times \mathcal{C}^5 \to \mathcal{J}^{[1,0]}$ and $\mathbf{q}_2^{[0,1]}\colon \mathcal{C}^3 \times \mathcal{C}^6 \to \mathcal{J}^{[0,1]}$ defined by

$$\begin{aligned}
v_{xxx} &= -\frac{1440}{\zeta^6}\phi + \frac{1008}{\zeta^5}\phi_1 - \frac{324}{\zeta^4}\phi_2 + \frac{60}{\zeta^3}\phi_3 - \frac{6}{\zeta^2}\phi_4 + \frac{1440}{\zeta^6}\psi\\
&\quad + \frac{432}{\zeta^5}\psi_1 + \frac{36}{\zeta^4}\psi_2,\\
v_{yyy} &= +\frac{1440}{\zeta^6}\phi - \frac{144}{\zeta^5}\phi_1 - \frac{1440}{\zeta^6}\psi - \frac{1296}{\zeta^5}\psi_1 - \frac{576}{\zeta^4}\psi_2\\
&\quad - \frac{168}{\zeta^3}\psi_3 - \frac{36}{\zeta^2}\psi_4 - \frac{6}{\zeta}\psi_5 - \phi_6.
\end{aligned}$$

The projection map $\mathbf{q}_3$ is

$$\begin{aligned}
w = W \quad &= \frac{24}{\zeta^4}\phi - \frac{24}{\zeta^4}\psi - \frac{24}{\zeta^3}\psi_1 - \frac{12}{\zeta^2}\psi_2 - \frac{4}{\zeta}\psi_3 - \psi_4,\\
w_x = D_x W \quad &= -\frac{96}{\zeta^5}\phi + \frac{24}{\zeta^4}\phi_1 + \frac{96}{\zeta^5}\psi + \frac{72}{\zeta^4}\psi_1 + \frac{24}{\zeta^3}\psi_2 + \frac{4}{\zeta^2}\psi_3,\\
w_y = D_y W \quad &= \frac{96}{\zeta^5}\phi - \frac{96}{\zeta^5}\psi - \frac{96}{\zeta^4}\psi_1 - \frac{48}{\zeta^3}\psi_2 - \frac{16}{\zeta^2}\psi_3 - \frac{4}{\zeta}\psi_4 - \psi_5,\\
w_{xx} = D_{xx} W &= \frac{480}{\zeta^6}\phi - \frac{192}{\zeta^5}\phi_1 + \frac{24}{\zeta^4}\phi_2 - \frac{480}{\zeta^6}\psi - \frac{288}{\zeta^5}\psi_1 - \frac{72}{\zeta^4}\psi_2 - \frac{8}{\zeta^3}\psi_3,\\
w_{yy} = D_{yy} W &= \frac{480}{\zeta^6}\phi - \frac{480}{\zeta^6}\psi - \frac{480}{\zeta^5}\psi_1 - \frac{240}{\zeta^4}\psi_2 - \frac{80}{\zeta^3}\psi_3 - \frac{20}{\zeta^2}\psi_4\\
&\quad - \frac{4}{\zeta}\psi_5 - \psi_6,
\end{aligned}$$

with partial prolongations $\mathbf{q}_3^{[1,0]}\colon\ \mathcal{C}^3 \times \mathcal{C}^6 \to \mathcal{K}^{[1,0]}$ and $\mathbf{q}_3^{[2,0]}\colon\ \mathcal{C}^5 \times \mathcal{C}^6 \to \mathcal{K}^{[2,0]}$ defined by

$$
\begin{aligned}
w_{xxx} &= -\frac{2880}{\zeta^7}\phi + \frac{1440}{\zeta^6}\phi_1 - \frac{288}{\zeta^5}\phi_2 + \frac{24}{\zeta^4}\phi_3 + \frac{2880}{\zeta^7}\psi + \frac{1440}{\zeta^6}\psi_1 + \frac{288}{\zeta^5}\psi_2 \\
&\quad + \frac{24}{\zeta^4}\psi_3,\ w_{xxxx} = \frac{20160}{\zeta^8}\phi - \frac{11520}{\zeta^7}\phi_1 + \frac{2880}{\zeta^6}\phi_2 - \frac{384}{\zeta^5}\phi_3 + \frac{24}{\zeta^4}\phi_4 \\
&\quad - \frac{20160}{\zeta^8}\psi - \frac{8640}{\zeta^7}\psi_1 - \frac{1440}{\zeta^6}\psi_2 - \frac{96}{\zeta^5}\psi_3.
\end{aligned}
$$

Because the quotients (23) are all with respect to (the prolongations of) the same group action, we can use Principle B to calculate the internal equivalences

$$
\begin{array}{ccc}
\mathcal{C}^4 \times \mathcal{C}^5 & \xleftrightarrow{\ \mathrm{id}\ } & \mathcal{C}^4 \times \mathcal{C}^5 \\
\Sigma_1^{[0,1]} \Big\uparrow \Big\downarrow \mathbf{q}_1^{[0,1]} & & \mathbf{q}_2^{[1,0]} \Big\downarrow \Big\uparrow \Sigma_2^{[1,0]} \\
\mathcal{I}^{[0,1]} & \underset{\Psi_1}{\overset{\Phi_1}{\rightleftarrows}} & \mathcal{J}^{[1,0]}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}^3 \times \mathcal{C}^6 & \xleftrightarrow{\ \mathrm{id}\ } & \mathcal{C}^3 \times \mathcal{C}^6 \\
\Sigma_2^{[0,1]} \Big\uparrow \Big\downarrow \mathbf{q}_2^{[0,1]} & & \mathbf{q}_3^{[1,0]} \Big\downarrow \Big\uparrow \Sigma_3^{[1,0]} \\
\mathcal{J}^{[0,1]} & \underset{\Psi_2}{\overset{\Phi_2}{\rightleftarrows}} & \mathcal{K}^{[1,0]}
\end{array}
$$

$$
\text{and}\quad
\begin{array}{ccc}
\mathcal{C}^4 \times \mathcal{C}^6 & \xleftrightarrow{\ \mathrm{id}\ } & \mathcal{C}^4 \times \mathcal{C}^6 \\
\Sigma_1^{[0,2]} \Big\uparrow \Big\downarrow \mathbf{q}_1^{[0,2]} & & \mathbf{q}_3^{[2,0]} \Big\downarrow \Big\uparrow \Sigma_3^{[2,0]} \\
\mathcal{I}^{[0,2]} & \underset{\Psi_3}{\overset{\Phi_3}{\rightleftarrows}} & \mathcal{K}^{[2,0]}
\end{array}
$$

We choose cross-sections $\phi = \phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$ and, for $\mathbf{q}_1$

$$
\begin{aligned}
&\psi = -\frac{\zeta^4}{24}u_{xx} - \frac{\zeta^3}{12}u_x,\ \ \psi_1 = \frac{\zeta^3}{6}u_{xx} + \frac{\zeta^2}{2}u_x,\ \ \psi_2 = -\frac{\zeta^2}{2}u_{xx} - 2\zeta u_x - u, \\
&\psi_3 = \zeta u_{xx} + 5u_x - u_y + \frac{6u}{\zeta}, \quad \psi_4 = -u_{xx} - u_{yy} + \frac{6}{\zeta}(u_y - u_x) - \frac{12u}{\zeta^2}, \\
&\psi_5 = -u_{yyy} + \frac{6}{\zeta}u_{yy} - \frac{6}{\zeta^2}u_y, \quad \psi_6 = -u_{yyyy} + \frac{6}{\zeta}u_{yyy} - \frac{12}{\zeta^3}u_y;
\end{aligned}
$$

for $\mathbf{q}_2$

$$
\begin{aligned}
&\psi = +\frac{\zeta^4}{24}v_x + \frac{\zeta^5}{24}v_{xx} + \frac{\zeta^6}{144}v_{xxx}, \quad \psi_1 = -\frac{2\zeta^3}{9}v_x - \frac{7\zeta^4}{36}v_{xx} - \frac{\zeta^5}{36}v_{xxx}, \\
&\psi_2 = +\zeta^2 v_x + \frac{2\zeta^3}{3}v_{xx} + \frac{\zeta^4}{12}v_{xxx}, \quad \psi_3 = -v - 3\zeta v_x - \frac{3\zeta^2}{2}v_{xx} - \frac{\zeta^3}{6}v_{xxx},
\end{aligned}
$$

$$\psi_4 = +\frac{6}{\zeta}v + \frac{13}{3}v_x - v_y + \frac{5\zeta}{3}v_{xx} + \frac{\zeta^2}{6}v_{xxx},$$
$$\psi_5 = -\frac{6}{\zeta^2}v + \frac{6}{\zeta}v_y - v_{yy}, \quad \psi_6 = -\frac{12}{\zeta^3}v + \frac{6}{\zeta}v_{yy} - v_{yyy},$$

and for $\mathbf{q}_3$

$$\psi = -\frac{\zeta^5}{24}w_x - \frac{\zeta^6}{16}w_{xx} - \frac{\zeta^7}{48}w_{xxx} - \frac{\zeta^8}{576}w_{xxxx}, \quad \psi_5 = \frac{4}{\zeta}w - w_y,$$
$$\psi_1 = \frac{5\zeta^4}{24}w_x + \frac{7\zeta^5}{24}w_{xx} + \frac{13\zeta^6}{144}w_{xxx} + \frac{\zeta^7}{144}w_{xxxx}, \quad \psi_6 = \frac{4}{\zeta^2}w + \frac{4}{\zeta}w_y - w_{yy},$$
$$\psi_2 = -\frac{5\zeta^3}{6}w_x - \frac{25\zeta^4}{24}w_{xx} - \frac{7\zeta^5}{24}w_{xxx} - \frac{\zeta^6}{48}w_{xxxx},$$
$$\psi_3 = \frac{5\zeta^2}{2}w_x + \frac{5\zeta^3}{2}w_{xx} + \frac{5\zeta^4}{8}w_{xxx} + \frac{\zeta^5}{24}w_{xxxx},$$
$$\psi_4 = -w - 4\zeta w_x - 3\zeta^2 w_{xx} - \frac{2\zeta^3}{3}w_{xxx} - \frac{\zeta^4}{24}w_{xxxx}.$$

The internal equivalences Φ_1 Φ_2, Φ_3 and their inverses Ψ_1, Ψ_2, Ψ_3 can now be easily computed as the prolongations of the formulas

$$\Phi_1 = \mathbf{q}_2^{[1,0]} \circ \Sigma_1^{[0,1]} : \ v = u_y, \qquad \Psi_1 = \mathbf{q}_1^{[0,1]} \circ \Sigma_2^{[1,0]} : \ u = -\frac{\zeta^2}{6}v_x,$$
$$\Phi_2 = \mathbf{q}_3^{[1,0]} \circ \Sigma_2^{[0,1]} : \ w = v_y - \frac{2}{\zeta}v, \quad \Psi_2 = \mathbf{q}_2^{[0,1]} \circ \Sigma_3^{[1,0]} : \ v = -\frac{\zeta^2}{4}w_x,$$
$$\Phi_3 = \mathbf{q}_3^{[2,0]} \circ \Sigma_1^{[0,2]} : w = u_{yy} - \frac{2}{\zeta}u_y, \Psi_3 = \mathbf{q}_1^{[0,2]} \circ \Sigma_3^{[2,0]} : u = \frac{\zeta^4}{24}w_{xx} + \frac{\zeta^3}{12}w_x.$$

The maps Φ_1 and Φ_2 are precisely the classical Laplace transformations ([11], Vol 2, p. 23–53, [14], Vol 6, p. 39–104) for the equations defined by $\mathcal{I}$ and $\mathcal{J}$ so that, as promised, we have re-constructed these transformations from the symmetry reduction viewpoint. We emphasize that had these Laplace transformations been unknown to us, we would have discovered them by applying the algorithms of [2] to recognize $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ as the quotients (23) by the same group action (17) (with $f^i(z) = z^i$).

The differential invariants for the action (17) on the individual jet spaces are simply ϕ_5 and ψ_5 and these project under $\mathbf{q}_1^{[1,1]}$, $\mathbf{q}_2^{[2,0]}$ $\mathbf{q}_3^{[3,0]}$ to give the following intermediate integrals

$$I(\mathcal{I}) = u_{xxx} + \frac{6}{\zeta}u_{xx} + \frac{6}{\zeta^2}u_x, \quad J(\mathcal{I}) = -u_{yyy} + \frac{6}{\zeta}u_{yy} - \frac{6}{\zeta^2}u_y,$$

$$I(\mathcal{J}) = \frac{\zeta^2}{6} v_{xxxx} + 2\zeta v_{xxx} + 6 v_{xx} + \frac{4}{\zeta} v_x, \quad J(\mathcal{J}) = -\frac{6}{\zeta^2} v + \frac{6}{\zeta} v_y - v_{yy},$$
$$I(\mathcal{K}) = \frac{\zeta^4}{24} w_{xxxxx} + \frac{5\zeta^3}{6} w_{xxxx} + 5\zeta^2 w_{xxx} + 10\zeta w_{xx} + 5 w_x,$$
$$J(\mathcal{K}) = -w_y + \frac{4}{\zeta} w.$$

From the orders of these intermediate integrals we can deduce that the Laplace invariants for our three equations vanish at orders

$$h_2(\mathcal{I}) = h_3(\mathcal{J}) = h_4(\mathcal{K}) = 0 \quad \text{and} \quad k_2(\mathcal{I}) = k_1(\mathcal{J}) = k_0(\mathcal{J}) = 0.$$

The inferences of this last computation hold generally [14].

Theorem 4.2 (The Canonical Form for Darboux Integrable Linear PDE) *Let $\mathcal{I}$ be the Pfaffian system for a linear Darboux integrable equation* (15). *Then there is another linear Darboux integrable equation* (15) *with associated Pfaffian system $\mathcal{J}$ such that Laplace invariant $k_0(\mathcal{J}) = 0$ and the appropriate partial prolongations of $\mathcal{I}$ and $\mathcal{J}$ are internally equivalent.*

5 Internal Equivalences of Some Non-linear PDE

We have systematically calculated the quotient representations for all the examples in Goursat [16] and, in so doing, we have uncovered a number of new internal equivalences.

Example 5.1 *The canonical Pfaffian systems (on seven manifolds) for the two equations ([16] p. 124 and p. 134)*

$$\mathcal{I} : u_{xy} = e^u \quad \text{and} \quad \mathcal{J} : v_{xy} = v v_x \tag{25}$$

are quotients of the contact systems $\mathcal{C}^3 \times \mathcal{C}^3$ and $\mathcal{C}^2 \times \mathcal{C}^4$ by the diagonal action of $SL(2)$, acting by fractional linear transformations on the dependent variables ϕ and ψ. The two projection maps

$$\mathbf{q}_1 : \mathcal{C}^3 \times \mathcal{C}^3 \to \mathcal{I} \quad \text{and} \quad \mathbf{q}_2 : \mathcal{C}^2 \times \mathcal{C}^4 \to \mathcal{J}$$

are given by

$$u = \ln \frac{2\phi_1 \psi_1}{(\phi + \psi)^2} \quad \text{and} \quad v = \frac{\psi_2}{\psi_1} - \frac{2\psi_1}{\phi + \psi}, \tag{26}$$

and the prolongations of these equations to order 2. Then, just as in the previous section, we find that there is an internal equivalence

$$\Phi\colon \mathcal{I}^{[0,1]} \to \mathcal{J}^{[1,0]} \quad \textit{with inverse} \quad \Psi\colon \mathcal{J}^{[1,0]} \to \mathcal{I}^{[0,1]}$$

which is given by

$$v = u_y \quad \textit{and} \quad u = \log(v_x). \tag{27}$$

The differential invariants for the action of $SL(2)$ *on* $J^3 \times J^3$ *are*

$$x, \quad \frac{2\phi_1\phi_3 - 3\phi_2^2}{\phi_1^2}, \quad y, \quad \frac{2\psi_1\psi_3 - 3\psi_2^2}{\psi_1^2} \tag{28}$$

and these project under $\mathbf{q}_1$ *to the intermediate integrals*

$$I_1 = x, \quad I_2 = u_{xx} - \frac{1}{2}u_x^2, \quad J_1 = y, \quad J_2 = u_{yy} - \frac{1}{2}u_y^2$$

for $\mathcal{I}$. *These, in turn , are transformed by* (27) *to the intermediate integrals*

$$\tilde{I}_1 = x, \quad \tilde{I}_2 = \frac{v_{xxx}}{v_x} - \frac{3}{2}v_{xx}^2, \quad \tilde{J}_1 = y, \quad \tilde{J}_2 = v_y^2 - \frac{1}{2}v^2,$$

for $\mathcal{J}$.

Example 5.2 *A more complex example is given by the two systems ([16], p. 186 and p. 231)*

$$\mathcal{I} : z_{xy} = \frac{2}{x+y}\sqrt{z_x z_y} \quad \textit{and} \quad \mathcal{J} : w_{uv} + w^2 w_{vv} + 2ww_v^2 = 0. \tag{29}$$

Remarkably, the quotient representations for these two differential systems are

$$\mathbf{q}_1 : \mathcal{H}^{2,[1]} \times \mathcal{H}^{2,[1]} \to \mathcal{I} \quad \textit{and} \quad \mathbf{q}_2 : \mathcal{H}^{2,[2]} \times \mathcal{H}^2 \to \mathcal{J}, \tag{30}$$

where $\mathcal{H}^2$ *is the rank 2 Pfaffian system defined on a 5 manifold with coordinates* $\{t, \sigma, \phi, \phi_1, \phi_2\}$, *by* $\mathcal{H}^2 = \{d\phi - \phi_1\, dt, d\sigma - \phi_1^2\, dt\}$, *and where the prolongations are given by*

$$\mathcal{H}^{2,[1]} = \mathcal{H}^2 \cup \{\, d\phi_1 - \phi_2\, dt\,\} \quad \textit{and} \quad \mathcal{H}^{2,[2]} = \mathcal{H}^{2,[1]} \cup \{\, d\phi_2 - \phi_3\, dt\,\}. \tag{31}$$

(We write the second copy of $\mathcal{H}^2$ *in terms of the coordinates* $\{s, \tau, \psi, \phi_1, \psi_2\}$ *as* $\{d\psi - \psi_1 ds,\ d\tau - \psi_1^2 dt\}$.*) The symmetry group for the reductions* (30) *is the three dimensional Heisenberg group, acting with infinitesimal generators*

$$\{\partial_\phi - \partial_\psi,\ \partial_\sigma - \partial_\tau,\ t\partial_\phi + \partial_{\phi_1} + 2\phi\partial_\sigma + s\partial_\psi + \partial_{\psi_1} + 2\psi\partial_\tau\}. \tag{32}$$

We are in precisely the situation of Theorem A and therefore $\mathcal{I}^{[1,0]}$ *and* $\mathcal{J}^{[0,1]}$ *are internally equivalent. With* $\zeta = s + t$ *the (prolonged) projection* $\mathbf{q}_1^{[1,0]}$ *is given by*

$$x = t, \quad y = s, \quad z = \sigma + \tau - \frac{(\phi + \psi)^2}{\zeta}, \quad z_x = P^2, \quad z_y = Q^2,$$
$$z_{xx} = 2PP_t, \quad z_{yy} = 2QQ_s, \quad z_{xxx} = 2P_t^2 + 2PP_{tt}, \quad \textit{where}$$
$$P = \frac{\phi + \psi - \zeta\phi_1}{\zeta}, \quad Q = \frac{\phi + \psi - \zeta\psi_1}{\zeta}.$$

The projection $\mathbf{q}_2^{[0,1]}$ *is*

$$u = t, \quad v = \sigma + \tau + \zeta\psi_1^2 - 2(\phi + \psi)\psi_1, \quad w = \psi_1 - \phi_1,$$
$$w_u = -\phi_2 - w^2 w_v, \quad w_v = -\frac{1}{2(\phi + \psi - \zeta\psi)}, \quad w_{vv} = -2\zeta w_v^3,$$
$$w_{uu} = -2\zeta w^4 z_v^3 + 4w^4 w_v^2 + 2\phi_2 w w_v - \phi_3, \quad w_{vvv} = 12\zeta^2 w_v^5 - \frac{2w_v^4}{\psi_2}.$$

Again, we proceed as in Sect. 4 to arrive at the equivalence $\Phi : \mathcal{I}^{[1,0]} \to \mathcal{J}^{[0,1]}$, *given by*

$$\Phi : u = x, \quad v = (x + y)z_y + z, \quad w = \sqrt{z_x} + \sqrt{z_y}$$

with inverse

$$\Psi : x = u, \quad y = -u - \frac{w_{vv}}{2w_v^3}, \quad z = v + \frac{w_v}{2w_{vv}}.$$

Under the mapping Ψ *the intermediate integrals*

$$I_1 = x, \quad I_2 = \frac{z_{xx}}{2\sqrt{z_x}} + \frac{\sqrt{z_x}}{x + y}, \quad J_1 = y, \quad J_2 = \frac{z_{yy}}{2\sqrt{z_y}} + \frac{\sqrt{z_y}}{x + y}$$

for $\mathcal{I}$ *are transformed to the intermediate integrals*

$$\tilde{I}_1 = u, \quad \tilde{I}_2 = w^2 w_v + w_u, \quad \tilde{J}_1 = -\frac{w_{vv}}{2w_v^3} - u, \quad \tilde{J}_2 = \frac{2w_v^5}{w_{vvv}w_v - 3w_{vv}^2}$$

for $\mathcal{J}$. *Note that the invariants* $\tilde{J}_1$ *and* $\tilde{J}_2$ *are of lower order than that suggested by Goursat.*

6 Moutard Equations

Moutard equations are non-linear scalar PDE of the form

$$v_{xy} + \frac{\partial}{\partial x}(A_0 e^v) - \frac{\partial}{\partial y}(B_0 e^{-v}) + C_0 = 0, \tag{33}$$

where the coefficients A_0, B_0 and C_0 are functions of the independent variables x, y. Assuming that $B_0 > 0$, the change of dependent variables $v \to v + \log(B_0)$ transforms (33) to the standard form

$$v_{xy} + \frac{\partial}{\partial x}(Ae^v) - \frac{\partial}{\partial y}(e^{-v}) + C = 0, \tag{34}$$

where A and C are functions of x, y. Let $\mathcal{M}$ be the usual rank 3 Pfaffian system for (34) on the seven manifold N with coordinates $(x, y, v, v_x, v_y, v_{xx}, v_{yy})$. Goursat [16] (p. 249) establishes a close relationship between Darboux integrable Moutard equations and Darboux integrable linear equations. From our perspective of symmetry reduction, this relationship is given by

Theorem 6.1 **[i]** *Let $\mathcal{L}^{[1,0]}$ be the rank 4 Pfaffian system for the partially prolonged linear PDE* (15)*, and let S be the 1 dimensional scaling symmetry group of $\mathcal{L}^{[1,0]}$ with infinitesimal generator*

$$W = u\partial_u + u_x\partial_{u_x} + u_y\partial_{u_y} + u_{xx}\partial_{u_{xx}} + u_{yy}\partial_{u_{yy}} + u_{xxx}\partial_{u_{xxx}}. \tag{35}$$

Then the quotient system $\mathcal{L}^{[1,0]}/S$ is the standard Pfaffian system for a Moutard equation (34).

[ii] *Every Moutard system $\mathcal{M}$ is the quotient $\mathbf{q}_S \colon \mathcal{L}^{[1,0]} \to \mathcal{M}$ of a Pfaffian system for a linear equation. The projection map $\mathbf{q}_S$ defines $\mathcal{L}^{[1,0]}$ as a rank 1 integrable extension of $\mathcal{M}$.*

[iii] *The Moutard system $\mathcal{M}$ is Darboux integrable (at some order of prolongation) if and only if $\mathcal{L}^{[1,0]}$ is Darboux integrable.*

Proof. The differential system $\mathcal{L}^{[1,0]}$ is the rank 4 Pfaffian system with generators

$$\begin{aligned} \theta &= du - u_x\,dx - u_y\,dy, & \theta_x &= du_x - u_{xx}\,dx - u_{yy}\,dy, \\ \theta_y &= du - u_{xy}\,dx - u_{yy}\,dy, & \theta_{xx} &= du_{xx} - u_{xxx}\,dx - u_{xxy}\,dy, \end{aligned} \tag{36}$$

where u_{xy} and u_{xxy} are given by the PDE (15) and its x derivative.

A basis for the semi-basic forms $\mathcal{L}^{[1,0]}_{\mathbf{sb}}$, with respect to the symmetry group S, is easily determined to be

$$\vartheta_1 = \theta_x - \frac{u_x}{u}\theta, \quad \vartheta_2 = \theta_y - \frac{u_y}{u}\theta, \quad \vartheta_3 = \theta_{xx} - \frac{u_{xx}}{u}\theta. \tag{37}$$

The quotient map, for the scaling group S, from the eight manifold for $\mathcal{L}^{[1,0]}$ to the seven manifold N is defined by the prolongation of

$$x = x, \quad y = y, \quad v = \frac{u_x}{u}. \tag{38}$$

It is a simple matter to re-write the semi-basic forms (37) in terms of v and its derivatives to deduce that the quotient differential system on N is that of a PDE of the form

$$v_{xy} + \rho\, v_x v_y + \tilde{A}\, v_x + \tilde{B}\, v_y + \tilde{C} = 0, \quad \text{where} \quad \rho = -\frac{1}{(b+v)} \tag{39}$$

and where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are certain functions of x, y, and v. This is not in the form of a Moutard equation but the point transformation

$$\tilde{v} = -\log(b+v)$$

will eliminate the quadratic term $\rho\, v_x v_y$ in (39) and lead to the Moutard equation (34), with

$$A = \frac{\partial b}{\partial y} + ab - c, \quad \text{and} \quad C = \frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}. \tag{40}$$

In other words, instead of using the obvious projection map (38), it is better to use

$$x = x \quad y = y, \quad v = -\log(\frac{u_x}{u} + b). \tag{41}$$

To prove the first part of **[ii]**, we simply check that that for given functions $A(x, y)$ and $B(x, y)$, it is always possible find $a(x, y)$, $b(x, y)$, $c(x, y)$ satisfying equations (40). To prove the second part of **[ii]**, we need only observe that the form θ is a complement to the semi-basic forms (37), relative to (36), and that

$$d\,\theta \equiv 0 \mod \{\,\theta, \vartheta_1, \vartheta_2, \vartheta_3\,\}. \tag{42}$$

Part **[iii]** then follows directly from Principles A and C. □

The combination of Theorems 4.1 and 6.1 yield the following corollary.

Corollary 6.2 *For any Darboux integrable Moutard system $\mathcal{M}$, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}^{m+1} \times \mathcal{C}^{n} & \xrightarrow{\mathbf{q}_{G_p}} & \mathcal{L}^{[1,0]} \\ & \searrow^{\mathbf{q}_{K_{p+1}}} & \downarrow \mathbf{q}_S \\ & & \mathcal{M} \end{array} \;. \tag{43}$$

Here $\mathcal{C}^{m+1}$ and $\mathcal{C}^{n}$ are the contact systems (16)*, K_{p+1} is the $p+1$ dimensional symmetry group with generators* (17) *and $W = \phi\partial_\phi + \psi\partial_\psi$ (prolonged). The group G_p is defined as in the statement of Theorem* 4.1.

Example 6.1 *The quotient of the linear equation $\mathcal{J}$ (the second Pfaffian system in* (24)*) by the scaling action $v\partial_v$ is the Moutard equation*

$$V_{xy} - 6D_x(\frac{e^V}{\zeta^2}) - D_y(e^{-V}) - \frac{2}{\zeta^2} = 0. \tag{44}$$

The composition of the projections $\mathbf{q}_2$ *and* $\mathbf{q}_S$ *gives the projection map* $\mathbf{q} : \mathcal{C}^4 \times \mathcal{C}^5 \to \mathcal{M}$ *as*

$$V = \log\Big(\frac{\zeta(24\phi - 6\zeta\phi_1 - 24\psi - 18\zeta\psi_1 - 6\zeta^2\psi_2 - \zeta^3\psi_3)}{6(-12\phi + 6\zeta\phi_1 - \zeta^2\phi_2 + 12\psi + 6\zeta\psi_1 + \zeta^2\psi_2)}\Big). \tag{45}$$

7 First Order Linear Systems

In this example we consider the simple class of first order linear PDE

$$u_y = \alpha_0\, u + \beta_0\, v, \quad v_x = \gamma_0\, u + \delta_0\, v, \tag{46}$$

where the coefficients $\alpha_0, \beta_0, \gamma_0, \delta_0$ are functions of the independent variables x, y. A simple scaling of the dependent variables u and v transforms this system to the form

$$u_y = \alpha\, v, \quad v_x = \beta\, u. \tag{47}$$

We associate to each such system a rank 2 Pfaffian system $\mathcal{S}$ on a six manifold. The quotient representation for Darboux systems of the type (47) can be obtained directly using the arguments of Theorem 4.1 or indirectly using the integrable extensions approach of the previous section.

Let $\mathcal{C}^m$ and $\mathcal{C}^n$ be the contact systems (16) and let G_p be the p dimensional Abelian group acting on $J^m \times J^n$ with infinitesimal generators (17).

Theorem 7.1 *The quotient differential system* $\mathcal{I} = (\mathcal{C}^m + \mathcal{C}^n)/G_p$*, where* $p = m + n - 2$ *is the standard rank 2 Pfaffian system, defined on a six manifold, for a linear PDE system* (47)*.*

Proof. The detailed argument follows the same lines as given for the proof of Theorem 4.1. Here we simply note that on $J^{n-1} \times J^{m-2}$ there are three differential invariants x, y and

$$U = U_0(x, y, \phi_0, \phi_1, \dots, \phi_{n-2}, \psi_0, \psi_1, \dots, \psi_{m-2}) + \phi_{n-1}$$

while on $J^{n-2} \times J^{m-1}$ there are three differential invariants x, y and

$$V = V_0(x, y, \phi_0, \phi_1, \dots, \phi_{n-2}, \psi_0, \psi_1, \dots, \psi_{m-2}) + \psi_{m-1}.$$

The functions U_0 and V_0 are linear in the jet coordinates ψ_k and ψ_l. These invariants are necessarily related by identities of the form

$$D_y U = \alpha\, V \quad \text{and} \quad D_x V = \beta\, U$$

which determine the coefficients for the quotient system (47). □

Theorem 7.2 *For any Pfaffian system $\mathcal{S}$ defined by* (47) *there is an integrable extension to a system $\mathcal{L}$ defined by* (15). *For any Pfaffian system $\mathcal{L}$ defined by* (15) *there is an integrable extension to the prolonged system $\mathcal{S}^{[1]}$ defined by* (47).

Proof. We first remark that by change of a dependent variable $u \to \mu(x, y)\, u$ one can always transform (15) to an equivalent linear equation with either $b = 0$ or $c = 0$.

If u and v solve (47), then

$$z = e^{\lambda} u - v, \quad \lambda = \lambda(x, y) \tag{48}$$

satisfies a 2nd order linear PDE

$$z_{xy} + a z_x + b z_y + c z = 0 \tag{49}$$

precisely when λ satisfies the Moutard equation

$$\lambda_{xy} + D_x(\alpha e^{\lambda}) - D_y(\beta e^{-\lambda}) = 0. \tag{50}$$

The coefficients of (49) of are given by

$$a = -\lambda_y, \quad b = 0, \quad c = e^{\lambda}\alpha_x + e^{\lambda}\lambda_x \alpha - \alpha\beta. \tag{51}$$

Note that in the special case where $\alpha_y = \beta_x$, then $\lambda = 0$ is a solution to (50).

Conversely, functions $\mu(x, y)$ and $\nu(x, y)$ can be chosen so that for any solution u to the linear equation (49), with $c = 0$, the functions

$$u = \mu(x, y) z_x \quad \text{and} \quad v = \nu(x, y) z_y \tag{52}$$

satisfy a system of the form (47).

The required integrable extensions $\pi_1 \colon \mathcal{S}^{[1]} \to \mathcal{L}$ and $\pi_2 \colon \mathcal{L} \to \mathcal{S}$ are determined by (48) (and its derivatives) and (52). □

Theorem 7.3 *For the Pfaffian system $\mathcal{L}$ associated to a Darboux integrable 2nd order linear PDE* (15) *or for the Pfaffian system $\mathcal{S}$ associated to a Darboux integrable 1st order linear system* (47), *there are commutative diagrams*

$$\begin{array}{ccc} \mathcal{C}^m \times \mathcal{C}^n & \xrightarrow{\mathbf{q}_{\tilde{G}_{p-1}}} & \mathcal{L} \\ & \searrow^{\mathbf{q}_{G_p}} & \downarrow \mathbf{q}_H \\ & & \mathcal{S} \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{C}^{m+1} \times \mathcal{C}^{n+1} & \xrightarrow{\mathbf{q}_{\tilde{G}_{p-1}}} & \mathcal{S}^{[1]} \\ & \searrow^{\mathbf{q}_{G_p}} & \downarrow \mathbf{q}_H \\ & & \mathcal{L} \end{array}. \tag{53}$$

The infinitesimal generators for G_p are given by (17), *$\tilde{G}_{p-1}$ is a subgroup of G_p and $G_p = \tilde{G}_{p-1} \oplus H$. In the first diagram G_p is of dimension $p = n + m - 2$ while in the second diagram G_p has dimension $p = n + m - 1$.*

Example 7.1 *As a simple example, let G_i be the four dimensional Abelian subgroup of the 5 dimensional group* (17) *(with $f^i(z) = z^i$, $i = 0 \dots 4$) obtained by removing the vector Z^i. The quotients $S_i^{[1]}$ of $\mathcal{C}^4 \times \mathcal{C}^4$ by G_i give the first prolongations of the systems*

$$u_y = \alpha_i\, v, \quad v_x = \beta_i\, u,$$

with

$$\alpha_0 = \frac{2y}{x(x-y)}, \quad \beta_0 = -\frac{2x}{y(x-y)}, \quad \alpha_3 = \frac{x+3y}{x^2-y^2}, \quad \beta_3 = -\frac{3x+y}{x^2-y^2},$$

$$\alpha_1 = \frac{y(3x+y)}{x(x^2-y^2)}, \quad \beta_1 = -\frac{x(x+3y)}{y(x^2-y^2)}, \quad \alpha_4 = \frac{2}{x-y}, \quad \beta_4 = -\frac{2}{x-y},$$

$$\alpha_2 = \frac{6y(x+y)}{(x-y)(x^2+4xy+y^2)}, \quad \beta_2 = -\frac{6x(x+y)}{(x-y)(x^2+4xy+y^2)}.$$

These coefficients all satisfy $\alpha_y = \beta_x$ and therefore all the Pfaffian system $S_i^{[1]}$ quotient to same(!) Pfaffian systems $\mathcal{I}$, defined by (24) *(with u replaced by z) via the projection map $z = u - v$.*

8 Goursat's Equation

Goursat ([15], [18]) showed that the non-linear equation

$$u_{xy} = 2A(x, y)\sqrt{u_x u_y} \tag{54}$$

can be linearized to the first order system

$$P_x = AQ, \quad Q_y = AP \tag{55}$$

by setting

$$u_x = P^2 \qquad u_y = Q^2, \tag{56}$$

or, alternatively, to the second order linear equation

$$v_{xy} - \frac{A_x}{2A} v_y - Av = 0 \tag{57}$$

by setting $v_x = u^2$.

In this example, we shall use Principle C to determine the general form of the quotient representation for Darboux integrable systems of the type (54) and we shall study, in some detail, the special case $A = \dfrac{n}{x+y}$.

Let $\mathcal{G}_A$ be the canonical rank 3 Pfaffian system (on a seven dimensional manifold) for (54) and let $\mathcal{S}_A$ the canonical rank 2 Pfaffian system (on a six dimensional manifold) for (55). Then (56) defines a map $\pi : \mathcal{G}_A \to \mathcal{S}_A$ for which $\mathcal{G}_A$ is an integrable extension of $\mathcal{S}_A$. Principles A and C show that (54) is Darboux integrable at some prolonged order whenever (55) is Darboux integrable (prolonged to the same order).

We know from Sect. 7 that the linear system (55) is Darboux integrable if it is the quotient of jet spaces $\mathcal{C}^m \times \mathcal{C}^n$ by an Abelian Lie group G of dimension $m + n - 2$, acting freely. We then infer, again by Principle C, that (54) will be the quotient of Pfaffian systems $\mathcal{D}^{m+1} \times \mathcal{E}^{n+1}$ by a Lie group $\tilde{G}$, where $\mathcal{D}^{m+1}$ is a 1 dimensional integrable extension of $\mathcal{C}^m$ and $\mathcal{E}^{n+1}$ is a 1 dimensional integrable extension of $\mathcal{C}^n$. Moreover, there are projection maps $\pi_1 : \mathcal{D}^{m+1} \to \mathcal{C}^m$ and $\pi_2 : \mathcal{E}^{n+1} \to \mathcal{C}^n$ such that the diagram

$$\begin{array}{ccc} \mathcal{D}^{m+1} + \mathcal{E}^{n+1} & \xrightarrow{\mathbf{q}_{\tilde{G}}} & \mathcal{G}_A \\ {\scriptstyle \pi_1 \times \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{C}^m + \mathcal{C}^n & \xrightarrow{\mathbf{q}_G} & \mathcal{S}_A \end{array} \tag{58}$$

commutes.

Principle C also implies that $\tilde{G}$ is a 1-step solvable Lie group of dimension $n + m - 1$.

If we denote the fiber coordinate for the projection map π_1 by σ, then generators for the action of $\tilde{G}$ on $\mathcal{D}^{m+1}$ may be taken to be of the form

$$W = \frac{\partial}{\partial \sigma} \quad \text{and} \quad \tilde{X}_i = f_i(x)\frac{\partial}{\partial \phi} + f_i'(x)\frac{\partial}{\partial \phi} + \cdots + f_i^{(m)}(x)\frac{\partial}{\partial \phi_m} + \xi_i \frac{\partial}{\partial \sigma}, \tag{59}$$

where $\xi_i = \xi_i(x, \phi, \phi_1, \dots, \phi_{m-1}, \sigma)$. The integrable extension $\mathcal{D}^{m+1}$ can be written as

$$\mathcal{D}^{m+1} = \mathcal{C}^m \cup \{d\sigma - H(x, \phi, \phi_1, \dots, \phi_{m-1})\, dx\}.$$

Theorem 8.1 *The Goursat equation*

$$u_{xy} = 2A(x, y)\sqrt{u_x u_y}$$

is Darboux integrable if and only if it is the quotient of a pair of Monge equations

$$\frac{d\sigma}{dx} = H(x, \phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2} \dots)$$

by a product action of an Abelian or 1-step solvable group with infinitesimal generators (59).

Just as with the case of linear equations (Sect. 5), the functions $f_i(x)$ can be prescribed arbitrarily. The functions H and ξ_i can be determined directly from the symmetry condition

$$Z_i(H) = D_x(\xi_i) \quad \text{which implies that} \quad E(Z_i(H)) = Z_i(E(H)) = 0, \tag{60}$$

where E is the Euler–Lagrange operator for the variable ϕ. This over-determined system of equations can be used to first determine H and then the coefficients ξ_i (independent of the method contained in the proof of Principle C).

The special case $A = \dfrac{2n}{x+y}$, corresponding to the choice of functions $f_i = x^i$, is easily solved.

Theorem 8.2 *The standard differential system $\mathcal{G}_n$ for the Goursat system $u_{xy} = \dfrac{2n}{x+y}\sqrt{u_x u_y}$ is the quotient differential system*

$$\mathcal{G}_n = \left(\mathcal{H}^{n+1,[1]} + \mathcal{H}^{n+1,[1]}\right)/G_{2n+1}, \tag{61}$$

where $\mathcal{H}^{n+1}$ is the rank $n+1$ Pfaffian system for the generalized Hilbert-Cartan equation

$$\frac{d\sigma}{dx} = \left[\frac{d^n\phi}{dx^n}\right]^2, \tag{62}$$

$\mathcal{H}^{n+1,[1]}$ is the first prolongation of $\mathcal{H}^{n+1}$, and G_{2n+1} is the $2n+1$dimensional, 1 step nilpotent Lie group with infinitesimal generators (59) *for $f_i = x^i$, $i = 0 \dots 2n-1$.*

The Monge system (62) enjoys a number of remarkable properties which we describe in the Sect. 9.

As in Sect. 4, the explicit formulas for the quotient maps $\mathbf{q}_n : \mathcal{H}^{n+1,[1]} + \mathcal{H}^{n+1,[1]} \to \mathcal{G}_n$ are determined by the prolongation of the lowest order joint differential invariants for the diagonal actions of the group G_{2n+1}. Let $\gamma = x + y$, $\Phi_k = \gamma^k \phi_k$, and $\Psi_k = \gamma^k \psi_k$. Then, for $n = 1$, the infinitesimal generators are $\{\, \partial_\sigma, \partial_\phi, t\partial_\phi + \partial_{\phi_1} + 2\phi\partial_\sigma \,\}$ and the joint invariant is

$$u_1 = \sigma_1 + \tau_1 + \frac{1}{\gamma}\begin{bmatrix}\Phi_0\\ \Psi_0\end{bmatrix}^t \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}\Phi_0\\ \Psi_0\end{bmatrix}; \tag{63}$$

for $n = 2$ the infinitesimal generators are vector fields X_i^+, $i = 2, \dots, 6$ (see (74)) and the joint invariant is

$$u_2 = \sigma_2 + \tau_2 + \frac{1}{\gamma^3}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Psi_0\\ \Psi_1\end{bmatrix}^t \begin{bmatrix}-12 & 6 & -12 & 6\\ 6 & -4 & 6 & -2\\ -12 & 6 & -12 & 6\\ 6 & -2 & 6 & -4\end{bmatrix}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Psi_0\\ \Psi_1\end{bmatrix}; \tag{64}$$

for $n = 3$ the generators are $\{W, X_0, X_1, X_2, X_3, X_4, X_5\}$ (see (76)) and the joint invariant is

$$u_3 = \sigma_3 + \tau_3 + \frac{1}{\gamma^5}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Psi_0\\ \Psi_1\\ \Psi_2\end{bmatrix}^t \begin{bmatrix} -720 & 360 & -60 & -720 & 360 & -60\\ 360 & -192 & 36 & 360 & -168 & 24\\ -60 & 36 & -9 & -60 & 24 & -3\\ -720 & 360 & -60 & -720 & 360 & -60\\ 360 & -168 & 24 & 360 & -192 & 36\\ -60 & 24 & -3 & -60 & 36 & -9 \end{bmatrix} \begin{bmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Psi_0\\ \Psi_1\\ \Psi_2\end{bmatrix}. \tag{65}$$

We can use Theorem 8.2 to establish, in a rather novel fashion a connection between the systems $\mathcal{G}_{n+1}$ and $\mathcal{G}_n$. We start with the simple observation that the transformation $\phi = \Phi'$ defines the equation $\sigma' = [\phi^{(n)}]^2$ as the quotient of $\sigma' = [\Phi^{(n+1)}]^2$ by the 1 dimensional group L with generator ∂_{Φ_0}. This, together with Theorem 8.2, then yields

$$\begin{array}{ccc} & & \mathcal{H}^{n+1,[1]} + \mathcal{H}^{n+1,[1]} \\ & \mathbf{q}_L \times \mathbf{q}_L \nearrow & \Big\downarrow \mathbf{q}_{G_{2n+1}} \\ \mathcal{H}^{n+2,[1]} + \mathcal{H}^{n+2,[1]} & \xrightarrow[\pi_{n+3}]{} & \mathcal{G}_n \end{array} . \tag{66}$$

From this diagram and (61) we arrive at

$$\begin{array}{ccc} & \mathcal{H}^{n+2,[1]} + \mathcal{H}^{n+2,[1]} & \\ \mathbf{q}_{G_{2n+3}} \swarrow & & \searrow \pi_{n+3} \\ \mathcal{G}_{n+1} & & \mathcal{G}_n \end{array} . \tag{67}$$

Now, because G_{2n+3} is a solvable group, every solution or integral manifold for $\mathcal{G}_{n+1}$ determines, by quadratures, an integral manifold to $\mathcal{H}^{n+2,[1]} \times \mathcal{H}^{n+2,[1]}$ (see [1], Theorem 6.2) which then projects under π_{n+3} to an integral manifold for $\mathcal{G}_n$. Because the projection map π_{n+3} is invariant with respect to the flows of all the generators of G_{2n+3} except the last one, the explicit formulas for this construction turn out to be remarkably simple.

Theorem 8.3 *If $U(x, y)$ solves $U_{xy} = \dfrac{2(n+1)}{(x+y)}\sqrt{U_x U_y}$ and $\lambda(x, y)$ solves*

$$\lambda_x = \frac{(2n+1)\sqrt{U_x}}{(x+y)^{n+1}}, \quad \lambda_y = -\frac{(2n+1)\sqrt{U_y}}{(x+y)^{n+1}} \tag{68}$$

then

$$V(x, y) = \frac{(x+y)^{2n}}{2n+1}\lambda^2 + U(x, y) \tag{69}$$

solves the equation $V_{xy} = \dfrac{2n}{(x+y)}\sqrt{V_x V_y}$.

This final theorem suggests the very intriguing possibility of adapting these Lie group theoretic methods towards the construction of Bäcklund transformations between various Darboux integrable systems ([10], [29]).

9 The Monge Equations $\sigma' = [\phi^{(n)}]^2$

Here we review some of the basic properties for the rank $n+1$ Pfafian systems $\mathcal{H}^{n+1}$ defined by the Monge equation $\sigma' = [\phi^{(n)}]^2$. For $n = 1$, the Pfaffian system

$$\mathcal{H}^2 = \{\, d\phi - \phi_1 dx,\ d\sigma - \phi_1^2 dx \,\} \tag{70}$$

has derived flag dimensions [2, 1, 0] and is therefore contact equivalent, by Engel's theorem (see, for example, [5], p. 50), to

$$\mathcal{C}^2(\mathbf{R}, \mathbf{R}) = \{\, d\Phi - \Phi_1 dX,\ d\Phi_1 - \Phi_2 dX \,\}. \tag{71}$$

An explicit equivalence is given by

$$x = \Phi_2, \quad \phi = X\Phi_2 - \Phi_1, \quad \phi_1 = X, \quad \sigma = X^2\Phi_2 - 2X\Phi_1 + 2\Phi. \tag{72}$$

For $n = 2$, the Pfaffian system

$$\mathcal{H}^3 = \{\, d\phi - \phi_1 dx,\ d\phi_1 - \phi_2 dx,\ d\sigma - \phi_2^2 dx \,\} \tag{73}$$

has derived flag dimensions [3, 2, 0] (the generic flag dimensions) and, amongst all generic rank 3 Pfaffian systems in five variables, has the symmetry algebra of largest dimension, namely, the real split form of the exceptional Lie algebra $\mathfrak{g}_2$. The generators for this symmetry algebra are:

$$H_1 = 2x\partial_x + 3\phi\partial_\phi + \phi_1\partial_{\phi_1} - \phi_2\partial_{\phi_2}, \quad H_2 = -(\phi\partial_\phi + \phi_1\partial_{\phi_1} + \phi_2\partial_{\phi_2} + 2\sigma\partial_\sigma),$$
$$X_1^+ = \frac{1}{2}x^2\partial_x + \frac{3}{2}\phi x\partial_\phi + (\frac{3}{2}\phi + \frac{1}{2}x\phi_1)\partial_{\phi_1} + (2\phi_1 - \frac{1}{2}\phi_2 x)\partial_{\phi_2} + 2\phi_1^2\partial_\sigma,$$
$$X_2^+ = \partial_\phi, \quad X_3^+ = x\partial_\phi + \partial_{\phi_1}, \quad X_4^+ = \frac{1}{2}x^2\partial_\phi + x\partial_{\phi_1} + \partial_{\phi_2} + 2\phi_1\partial_\sigma,$$
$$X_5^+ = \frac{1}{6}x^3\partial_\phi + \frac{1}{2}x^2\partial_{\phi_1} + x\partial_{\phi_2} + (2x\phi_1 - 2\phi)\partial_\sigma, \quad X_6^+ = \partial_\sigma, \quad X_1^- = \partial_x,$$

$$
\begin{aligned}
X_2^- &= (\frac{4}{3}\phi_1 x^2 - 2\phi x - \frac{1}{3}\phi_2 x^3)\partial_x + (\frac{1}{6}x^3\sigma + \frac{2}{3}\phi_1^2 x^2 - 2\phi^2 - \frac{1}{3}x^3\phi_2\phi_1)\partial_\phi, \\
&+(\frac{1}{2}x^2\sigma + \frac{2}{3}\phi_1^2 x - 2\phi\phi_1 - \frac{1}{6}\phi_2^2 x^3)\partial_{\phi_1} + (\sigma x - \frac{4}{3}\phi_1^2 + \frac{2}{3}x\phi_1\phi_2 - \frac{1}{3}\phi_2^2 x^2)\partial_{\phi_2} \\
&+(2\sigma x\phi_1 - 2\sigma\phi - \frac{1}{9}x^3\phi_2^3 - \frac{8}{9}\phi_1^3)\partial_\sigma, \\
X_3^- &= (\frac{8}{3}x\phi_1 - 2\phi - \phi_2 x^2)\partial_x + (\frac{1}{2}x^2\sigma + \frac{4}{3}\phi_1^2 x - x^2\phi_2\phi_1)\partial_\phi \\
&+(\sigma x + \frac{2}{3}\phi_1^2 - \frac{1}{2}\phi_2^2 x^2)\partial_{\phi_1} + (\sigma + \frac{2}{3}\phi_2\phi_1 - \frac{2}{3}\phi_2^2 x)\partial_{\phi_2} + (2\phi_1\sigma - \frac{1}{3}\phi_2^3 x^2)\partial_\sigma, \\
X_4^- &= (\frac{8}{3}\phi_1 - 2\phi_2 x)\partial_x + (\sigma x + \frac{4}{3}\phi_1^2 - 2x\phi_1\phi_2)\partial_\phi + (\sigma - \phi_2^2 x)\partial_{\phi_1} \\
&-\frac{2}{3}\phi_2^2\partial_{\phi_2} - \frac{2}{3}x\phi_2^3\partial_\sigma, \\
X_5^- &= -2\phi_2\partial_x + (\sigma - 2\phi_2\phi_1)\partial_\phi - \phi_2^2\partial_{\phi_1} - \frac{2}{3}\phi_2^3\partial_\sigma, \\
X_6^- &= (\frac{2}{3}\phi_1^2 - \phi\phi_2)\partial_x + (\frac{1}{2}\sigma\phi + \frac{4}{9}\phi_1^3 - \phi_2\phi\phi_1)\partial_\phi + (\frac{1}{2}\phi_1\sigma - \frac{1}{2}\phi\phi_2^2)\partial_{\phi_1} \\
&+(-\frac{1}{3}\phi_1\phi_2^2 + \frac{1}{2}\sigma\phi_2)\partial_{\phi_2} + (\frac{1}{2}\sigma^2 - \frac{1}{3}\phi\phi_2^3)\partial_\sigma.
\end{aligned} \tag{74}
$$

The vector fields X_i^+ have positive weight, the vectors H_1, H_2 define a Cartan subalgebra for $\mathfrak{g}_2$, and the vectors X_i^- have negative weight.

For $n = 3$, the Pfaffian system

$$
\mathcal{H}^4 = \{\, d\phi - \phi_1\,dx,\ d\phi_1 - \phi_2\,dx,\ d\phi_2 - \phi_2\,dx,\ d\sigma - \phi_3^2\,dx\,\} \tag{75}
$$

has derived flag dimensions [4, 3, 1, 0] and, amongst all rank 4 Pfaffian systems in six variables with such derived flag dimensions, has the symmetry algebra of largest dimension. In this case the symmetry algebra has Levi decomposition $\mathfrak{sl}(2) \ltimes \mathfrak{r}$, where the radical $\mathfrak{r}$ is an eight dimensional solvable algebra. The explicit formulas for this algebra are:

$$
\begin{aligned}
&W = \partial_\sigma, \quad X_0 = \partial_\phi, \quad X_1 = x\,\partial_\phi + \partial_{\phi_1}, \quad X_2 = \frac{1}{2}x^2\,\partial_\phi + x\,\partial_{\phi_1} + \partial_{\phi_2}, \\
&X_3 = \frac{1}{6}x^3\,\partial_\phi + \frac{1}{2}x^2\,\partial_{\phi_1} + x\,\partial_{\phi_2} + \partial_{\phi_3} + 2\phi_2\,\partial_\sigma, \\
&X_4 = \frac{1}{24}x^4\,\partial_\phi + \frac{1}{6}x^3\,\partial_{\phi_1} + \frac{1}{2}x^2\,\partial_{\phi_2} + x\,\partial_{\phi_3} + (-2\phi_1 + 2x\phi_2)\,\partial_\sigma, \\
&X_5 = \frac{1}{120}x^5\partial_\phi + \frac{1}{24}x^4\partial_{\phi_1} + \frac{1}{6}x^3\,\partial_{\phi_2} + \frac{1}{2}x^2\,\partial_{\phi_3} + (2\phi - 2x\phi_1 + x^2\phi_2)\,\partial_\sigma \\
&R = \phi\,\partial_\phi + \phi_1\,\partial_{\phi_1} + \phi_2\,\partial_{\phi_2} + \phi_3\,\partial_{\phi_3} + 2\sigma\,\partial_\sigma, \quad S_0 = \partial_x, \\
&S_1 = 2x\,\partial_x + 5\phi\,\partial_\phi + 3\phi_1\,\partial_{\phi_1} + \phi_2\partial_{\phi_2} - \phi_3\,\partial_{\phi_3}, \quad S_2 = x^2\,\partial_x + 5x\phi\,\partial_\phi \\
&+(5\phi + 3x\phi_1)\,\partial_{\phi_1} + (8\phi_1 + x\phi_2)\partial_{\phi_2} + (9\phi_2 - x\phi_3)\,\partial_{\phi_3} + 9\phi_2^2\,\partial_\sigma.
\end{aligned} \tag{76}
$$

The vector fields S_0, S_1, S_2 define the semi-simple part, the nilradical is the seven dimensional 1 step nilpotent subalgebra given by $\{W, X_0, X_1, \ldots, X_5\}$.

For $n \geq 3$ this pattern persists. The derived flag for $\mathcal{H}^{n+1}$ is $[n, n-1, n-3, n-4, n-5, \ldots]$ and the symmetry algebra is $\mathfrak{sl}(2) \ltimes \mathfrak{r}$, where the radical has dimension $2n+2$. In all cases the nilradical is a 1 step nilpotent algebra of dimension $2n+1$.

The Pfaffian systems $\mathcal{H}^{n+1}$ are also the canonical (flat) models in the Tanaka theory associated to the unique $2n+1$ graded nilpotent Lie algebras with grading $[2, 1, 2, 1, 1, \ldots]$. See Doubrov and Zelenko [12], Theorem 3.

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References

1. I. M. Anderson and M. E. Fels, *Exterior Differential Systems with Symmetry*, Acta. Appl. Math. **87** (2005), 3–31.
2. I. M. Anderson, M. E. Fels, and P. J. Vassiliou, *Superposition Formulas for Exterior Differential Systems*, available at `arXiv:math/0708.0678`.
3. I. M. Anderson and M. Juráš, *Generalized Laplace Invariants and the Method of Darboux*, Duke J. Math **89** (1997), 351–375.
4. I. M. Anderson and N. Kamran, *The Variational Bicomplex for Second Order Partial Differential Equations in the Plane*, Duke J. Math. **87** (1997), 265–319.
5. R. L. Bryant, S. S. Chern, R. B. Gardner, H. Goldschmidt, and P. A Griffiths, *Essays on Exterior Differential Systems*, MSRI Publications, Vol. 18, Springer-Verlag, New York, 1991.
6. R. L. Bryant and P. A. Griffiths, *Characteristic cohomology of differential systems, II: Conservation laws for a class of parabolic equations*, Duke Math. J. **78** (1995), no. 3, 531–676.
7. R. L. Bryant, P. A. Griffiths, and L. Hsu, *Hyperbolic exterior differential systems and their conservation laws, Parts I and II*, Selecta Math., New series **1** (1995), 21–122 and 265–323.
8. É. Cartan, *Sur intégration des systèmes différentiels complètement intégrables*, C. R. Acad. Sc, **134** (1902), 1415–1418 and 1564–1566.
9. É Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. **3** (1910), no. 27, 109–192.
10. J. N. Clelland and T. A. Ivey, *Bäcklund transformatiosn and Darboux integrability for nonlinear wave equations*, to appear, Asian J. Math.
11. G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Gauthier-Villars, Paris, 1896.
12. B. Doubrov and I. Zelenko, *On local geometry of nonholonomic rank 2 distributions* (March 22, 2007), available at `arXiv:math/073662v1`.
13. M. E. Fels, *Integrating scalar ordinary differential equations with symmetry revisited*, Foundations of Comp. Math. **7** (2007), 417–454.
14. A. Forsyth, *Theory of Differential Equations, Vol 6*, Dover Press, New York, 1959.
15. E. Goursat, *Sur une équations aux dérivées partielle*, Bulletin de Société Mathématique de France **25** (1897), 36–48.
16. ___, *Lecon sur l'intégration des équations aux dériées partielles du second ordre á deux variables indépendantes, Tome 1, Tome 2*, Hermann, Paris, 1897.
17. ___, *Recherches sur quelques équations aux dériées partielles du second ordre*, Ann. Fac. Sci. Toulouse **1** (1899), 31–78 and 439–464.

18. —, *Sur une transformation de léquation* $s^2 = 4\lambda(x, y)pq$, Bulletin de Société Mathématique de France **28** (1900), 1–6.
19. E. Kamke, *Differentialgleichungen*, 3rd ed., Chelsea, New York, 1947.
20. B. Kruglikov and V. Lychagin, *A compatibility criteria for systems of PDEs and generalized Lagrange-Charpit method*, AIP Conference Proceedings, Global Analysis and Applied Mathematics: International Workshop on Global Analysis **729** (2004), no. 1, 39–53.
21. A. Kushner, V. Lychagin, and V. Rubtsov, *Contact Geometry and Nonlinear Differential Equations*, Encylodedia of Mathematics and its Applications, vol. 101, Cambridge Univ. Press, 2007.
22. P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed., Springer, New York, 1993.
23. V. V. Sokolov and A. V. Ziber, *On the Darboux integrable hyperbolic equations*, Phys Lett. A **208**, 303–308.
24. O. Stormark, *Lie's structural approach to PDE systems*, Encyclopedia of Mathematics and its Applications, vol. 80, Cambridge University Press, Cambridge, UK, 2000.
25. E. Vessiot, *Sur une classsse de faisceaux complets de degré 2*, Bull. Soc. Math. France **45** (1937), 149–167.
26. —, *Sur les équations aux dérivées partielles du second ordre,* $F(x, y, z, p, q, r, s, t) = 0$*, intégrables par la méthode de Darboux*, J. Math. Pure Appl. **18** (1939), 1–61.
27. —, *Sur les équations aux dérivées partielles du second ordre,* $F(x, y, z, p, q, r, s, t) = 0$*, intégrables par la méthode de Darboux*, J. Math. Pure Appl. **21** (1942), 1–66.
28. K. Yamaguchi, *Contact Geometry of Second Order I.* these proceedings.
29. M. Yu. Zvyagin, *Classification of Bäcklund transformations of second order partial differential equations*, Soviet Math. Dokl. **43** (1991), 422–429.

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