2

Representations of Electromagnetic Fields

I Introduction

The aim of this chapter is to *introduce the relevant equations* for the computation of electromagnetic fields and their network representations in a unified and systematic format. As is well known, *Maxwell's equations* provide the basic equations governing electromagnetic fields when complemented by *constitutive relations* pertaining to the media under consideration and by their relevant *boundary conditions*. These equations are suitable for initiating the numerical/analytical solution of the given problem.

When dealing with Maxwell's equations we shall emphasize the *Laplace ($\bar{s}$–domain) formulation* which has several advantages. First, $\bar{s}$–domain solutions are numerically efficient because once the solution is computed, frequency sweeps and transient analysis are also feasible with modest numerical effort. Second, $\bar{s}$–domain solutions are well suited to conversion into equivalent networks; these equivalent networks can be combined with external voltage and current sources, and the entire system can be modeled by using circuit simulators. Third, the electromagnetic analysis may be performed by using either differential- or integral-equation methods. In addition, there is the advantage of expressing the set of equations in a format that is common in the theory of linear systems. The format is such as to allow us to identify the state variables of the system, the sources, the observable quantities and all corresponding transfer functions. This approach also highlights issues concerning the uniqueness of the solution, the possibility of expressing the state of the system with a minimal amount of data, and the strategy for the applications of reduced–order models.

In this chapter we shall deal with *abstract representations* where the electromagnetic fields vary on a spatial and temporal continuum, i.e. with systems of infinite dimensions. This formalism can be adapted in later chapters to discretization and truncation processes in finite dimensions, making these systems suitable for numerical computations.

II Maxwell's Equations

Equations linking electromagnetic field quantities have been introduced by James Clerk Maxwell in an elegant treatise first published in 1873 and then inserted into [1] (see also [2] for more historical information). We shall assume that a student reader is familiar with these equations, which are usually introduced in preliminary courses, and that he/she has a general knowledge of the relevant experimental facts and their theoretical interpretation. In what follows, we summarize Maxwell's equations in the time, frequency and Laplace ($\bar{s}$) domains.

II.1 Maxwell's Equations in Time–Dependent Form

It is customary to write Maxwell's equations in either local or in global form; we shall first consider their local form. We also note that, unfortunately, it is customary to designate the local form as differential form and this generates some confusion with the general meaning that differential forms have. In the following of this book, since differential forms are not used, the ambiguity is resolved.

Local Form of Maxwell's Equations

In three–dimensional vector notation, with vector $\boldsymbol{r}$ indicating a position in space and t the time variable, Maxwell's equations are

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}\,, \qquad \text{Faraday's law} \tag{2.1a}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{J}(\boldsymbol{r},t)\,, \qquad \text{Ampère's law} \tag{2.1b}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho_e(\boldsymbol{r},t)\,, \qquad \text{Gauss' law} \tag{2.1c}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0\,, \qquad \text{Magnetic flux continuity} \tag{2.1d}$$

where bold face symbols denote vector quantities. The quantities are defined as

$\boldsymbol{E}(\boldsymbol{r},t)$	electric field strength
$\boldsymbol{D}(\boldsymbol{r},t)$	electric displacement
$\boldsymbol{B}(\boldsymbol{r},t)$	magnetic flux density
$\boldsymbol{H}(\boldsymbol{r},t)$	magnetic field strength
$\boldsymbol{J}(\boldsymbol{r},t)$	electric current density
$\rho_e(\boldsymbol{r},t)$	electric charge density

Equations (2.1a)–(2.1d) are not independent since, for example, we may derive (2.1d) by taking the divergence of (2.1a). Another fundamental relationship can be derived by introducing (2.1c) into the divergence of (2.1b)

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}, t) = -\frac{\partial \rho_e(\boldsymbol{r}, t)}{\partial t} \tag{2.2}$$

which provides the conservation law for electric charge and current densities. Actually, the set of three equations (2.1a), (2.1b) and (2.2) may be considered as the independent equations describing macroscopic electromagnetic fields, since the two Gauss equations (2.1c) and (2.1d) can be derived from this set.
Note that in the static case $\frac{\partial}{\partial t} = 0$ the electric and magnetic fields are not any more interdependent and the equations (2.1a) – (2.1d) become

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0\,, \tag{2.3a}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r})\,, \tag{2.3b}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_e(\boldsymbol{r})\,, \tag{2.3c}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0\,. \tag{2.3d}$$

Finally also note that, if we assign the electric current density $\boldsymbol{J}(\boldsymbol{r})$ and the electric charge density $\rho_e(\boldsymbol{r})$, we have, from (2.1a) and (2.1b), two vector equations (i.e. six scalar equations) while we have four unknown vectors (i.e. twelve scalar quantities). To complete the number of equations we have to account for the media properties expressed by the constitutive relations.

Integral (global) Form of Maxwell's Equations

The properties of an electromagnetic field may also be expressed globally by an equivalent system of integral relations through use of the two fundamental theorems of vector analysis: the divergence theorem and Stokes' theorem [3].

Divergence or Gauss' Theorem

Let $\boldsymbol{U}(\boldsymbol{r})$ be any vector function of position, continuous together with its first derivative throughout a volume V bounded by a surface S. The divergence theorem states that

$$\oint_S \boldsymbol{U}(\boldsymbol{r}) \cdot \boldsymbol{n}\, dS = \int_V \nabla \cdot \boldsymbol{U}(\boldsymbol{r})\, dV\,, \tag{2.4}$$

where $\mathbf{n}$ is the outward unit vector normal to S. In fact, Gauss's theorem may also be used to *define* the divergence.

Stokes' Theorem

Let $\boldsymbol{U}(\boldsymbol{r})$ be any vector function of position, continuous together with its first derivatives throughout an arbitrary surface S bounded by a contour C, and assumed to be resolvable into a finite number of regular arcs. Stokes' theorem (also called curl theorem) states that

$$\oint_C \boldsymbol{U}(\boldsymbol{r}) \cdot d\ell = \int_S [\nabla \times \boldsymbol{U}(\boldsymbol{r})] \cdot \boldsymbol{n}\, dS\,, \tag{2.5}$$

where $d\ell$ is an element of length along C, and $\boldsymbol{n}$ is a unit vector normal to the positive side of the element area dS as defined by the right–hand thumb rule. This relationship may also be considered as an equation defining the *curl* or *circulation*.

By applying the curl theorem to (2.1a) and (2.1b), and the divergence theorem to (2.1c) and (2.1d), we get

$$\oint_C \boldsymbol{E}(\boldsymbol{r},t) \cdot d\ell = -\int_S \frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \cdot \boldsymbol{n}\, dS\,, \tag{2.6a}$$

$$\oint_C \boldsymbol{H}(\boldsymbol{r},t) \cdot d\ell = \int_S \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \cdot \boldsymbol{n}\, dS + \int_S \boldsymbol{J}(\boldsymbol{r},t) \cdot \boldsymbol{n}\, dS\,, \tag{2.6b}$$

$$\int_V \nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) dV = \oint_S \boldsymbol{D}(\boldsymbol{r},t) \cdot \boldsymbol{n}\, dS = \int_V \rho_e(\boldsymbol{r},t)\, dV\,, \tag{2.6c}$$

$$\int_V \nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) dV = \oint_S \boldsymbol{B}(\boldsymbol{r},t) \cdot \boldsymbol{n}\, dS = 0\,. \tag{2.6d}$$

By defining the current $I(t)$ as

$$I(t) = \int_S \boldsymbol{J}(\boldsymbol{r},t) \cdot \boldsymbol{n}\, dS, \tag{2.7}$$

the charge $Q(t)$ as

$$Q(t) = \int_V \rho_e\, dV, \tag{2.8}$$

and the flux of the magnetic induction as

$$\Phi_m(t) = \int_S \boldsymbol{B}(\boldsymbol{r},t) \cdot \boldsymbol{n}\, dS, \tag{2.9}$$

we may write the previous equations as

$$\oint_C \boldsymbol{E}(\boldsymbol{r},t) \cdot d\ell = -\frac{\partial \Phi_m(t)}{\partial t}\,, \tag{2.10a}$$

$$\oint_C \boldsymbol{H}(\boldsymbol{r},t) \cdot d\ell = \int_S \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \cdot \boldsymbol{n}\, dS + I(t)\,, \tag{2.10b}$$

$$\int_V \nabla \cdot \boldsymbol{D}(\boldsymbol{r},t)\, dV = Q(t)\,. \tag{2.10c}$$

II.2 Maxwell's Equations in the Frequency Domain

Electromagnetic fields operating at a particular frequency are known as time–harmonic steady–state or monochromatic fields. By adopting the time dependence $e^{j\omega t}$ to denote a time–harmonic field with angular frequency ω, we write

$$\boldsymbol{E}(\boldsymbol{r},t) = \Re\left\{\boldsymbol{E}(\boldsymbol{r})e^{j\omega t}\right\}\,, \tag{2.11}$$

where $\Re$ denotes the mathematical operator which selects the real part of a complex quantity. The complex quantity is $\boldsymbol{E}(\boldsymbol{r})$ is called a *vector phasor*. In (2.11) we have used the same symbol to denote both the real quantity in the time domain, $\boldsymbol{E}(\boldsymbol{r},t)$, and the complex quantity, $\boldsymbol{E}(\boldsymbol{r})$, in the frequency domain. In what follows we shall generally refer to complex quantities unless otherwise explicitly stated.
By applying (2.11) to the field quantities appearing in (2.1a), (2.1b), (2.1c) and (2.1d) we obtain Maxwell's equations in the frequency domain. As an example, let us consider (2.1a) for which we have

$$\Re\left\{\left[\nabla\times\boldsymbol{E}(\boldsymbol{r}) + j\omega\boldsymbol{B}(\boldsymbol{r})\right]e^{j\omega t}\right\} = 0\,. \tag{2.12}$$

Since this equation is valid for *all times* t, we may make use of the above lemma and state that the quantity inside the square bracket must be equal to zero. By applying the same reasoning also to the other equations (2.1b), (2.1c) and (2.1d) we get

$$\nabla\times\boldsymbol{E}(\boldsymbol{r}) = -j\omega\boldsymbol{B}(\boldsymbol{r})\,, \tag{2.13a}$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r}) = j\omega\boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r})\,, \tag{2.13b}$$

$$\nabla\cdot\boldsymbol{D}(\boldsymbol{r}) = \rho_e(\boldsymbol{r})\,, \tag{2.13c}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r}) = 0\,. \tag{2.13d}$$

In the following, we make use of equivalence theorems which introduce magnetic current density, $\boldsymbol{M}(\boldsymbol{r})$, and magnetic charge distributions, $\rho_m(\boldsymbol{r})$. These quantities, although not physically present, help in the solution of several boundary value problems. When considering also magnetic currents and charges, the frequency–domain Maxwell's equations become

$$\nabla\times\boldsymbol{E}(\boldsymbol{r}) = -j\omega\boldsymbol{B}(\boldsymbol{r}) - \boldsymbol{M}(\boldsymbol{r})\,, \tag{2.14a}$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r}) = j\omega\boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r})\,, \tag{2.14b}$$

$$\nabla\cdot\boldsymbol{D}(\boldsymbol{r}) = \rho_e(\boldsymbol{r})\,, \tag{2.14c}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r}) = -\rho_m(\boldsymbol{r})\,. \tag{2.14d}$$

II.3 Maxwell's Equations in the $\bar{s}$–Domain

By introducing the complex variable $\bar{s} = \sigma + j\omega$, the Laplace transform is defined conventionally as

$$\boldsymbol{E}(\boldsymbol{r}, \bar{s}) = \int_0^\infty \boldsymbol{E}(\boldsymbol{r}, t) e^{-\bar{s}t} dt \,. \tag{2.15}$$

In (2.15) we have used the same symbol to denote both the quantity in the time domain, $\boldsymbol{E}(\boldsymbol{r}, t)$, and that in the $\bar{s}$-domain, $\boldsymbol{E}(\boldsymbol{r}, \bar{s})$. In what follows we shall generally refer to these quantities without explicitly exhibiting the $\bar{s}$ or t dependence, the latter being clear from the context.
Applying (2.15) to the field quantities appearing in (2.1a), (2.1b), (2.1c) and (2.1d) yields Maxwell's equations in the $\bar{s}$–domain,

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = -\bar{s}\boldsymbol{B}(\boldsymbol{r}) - \boldsymbol{M}(\boldsymbol{r}) \,, \tag{2.16a}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = \bar{s}\boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r}) \,, \tag{2.16b}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_e(\boldsymbol{r}) \,, \tag{2.16c}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = -\rho_m(\boldsymbol{r}) \,. \tag{2.16d}$$

II.4 Constitutive Relations

As already pointed out Maxwell's equations cannot be solved unless the relationships between the field vectors $\boldsymbol{D}$ and $\boldsymbol{B}$ with $\boldsymbol{E}$ and $\boldsymbol{H}$ are specified. The type of field generated by given sources depends on the medium characteristics, which are accounted for by *constitutive relations*; they may be written as

$$\boldsymbol{D} = \mathcal{F}_d(\boldsymbol{E}, \boldsymbol{H}) \,, \tag{2.17a}$$

$$\boldsymbol{B} = \mathcal{F}_b(\boldsymbol{E}, \boldsymbol{H}) \,. \tag{2.17b}$$

Here, $\mathcal{F}_d$ and $\mathcal{F}_d$ are suitable functionals dependent on the medium considered; they may be classified as:

- *nonlinear*, when functionals depend on the electromagnetic field;
- *inhomogeneous*, when functionals depend on *space coordinates*; they are called *spatially–dispersive* when functionals also depend on *spatial derivatives*;
- *nonstationary*, if functionals depend on *time* or *temporally–dispersive* when functionals depend on *time derivatives*.

We shall deal only with linear, stationary media; however, inhomogeneous media are included because of their practical importance.
Another classification of media is provided by the vector form of the constitutive relations. The simplest possibility arises when considering *isotropic media*, where the constitutive relations are given by

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}\,, \tag{2.18a}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}\,. \tag{2.18b}$$

with ε denoting permittivity and μ permeability. In this case $\boldsymbol{E}$ is parallel to $\boldsymbol{D}$ and $\boldsymbol{B}$ is parallel to $\boldsymbol{H}$. In particular, in free space, the above equations are rewritten by using the vacuum constitutive parameters, i.e. permittivity ε_0 and permeability μ_0, as

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}\,, \tag{2.19a}$$
$$\boldsymbol{B} = \mu_0 \boldsymbol{H}\,, \tag{2.19b}$$

with

$$\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{Fm}^{-1} \cong \frac{1}{36\pi} 10^{-9}\,\mathrm{Fm}^{-1}\,, \tag{2.20a}$$
$$\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{Hm}^{-1}\,. \tag{2.20b}$$

Anisotropic media are characterized by constitutive relations of the type

$$\boldsymbol{D} = \underline{\underline{\varepsilon}}\, \boldsymbol{E}\,, \tag{2.21a}$$
$$\boldsymbol{B} = \underline{\underline{\mu}}\, \boldsymbol{H}\,, \tag{2.21b}$$

where $\underline{\underline{\mu}}$ is the permeability tensor and $\underline{\underline{\varepsilon}}$ is the permittivity tensor. The medium is called *electrically anisotropic* if it is described by the permittivity tensor $\underline{\underline{\varepsilon}}$, and *magnetically anisotropic* when it is described by the permeability tensor $\underline{\underline{\mu}}$. A medium can be both electrically and magnetically anisotropic. An interesting particular case is that of *biaxial* crystals, which may be described, by choosing a suitable particular coordinate system, the so–called principal system, in terms of a tensor of the type:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}\,. \tag{2.22}$$

Cubic crystals, where $\varepsilon_x = \varepsilon_y = \varepsilon_z$, are isotropic; tetragonal, hexagonal and rhombohedral crystals have two parameters equal and the medium is called *uniaxial*. The *principal axis* that exhibits this anisotropy is also referred to as the *optic axis*. When all the three parameters are different, as in orthorhombic crystals, the medium is referred to as *biaxial*.

When the medium has elements possessing permanent electric and magnetic dipoles parallel or antiparallel to each other, an applied electric field simultaneously aligns electric *and* magnetic dipoles; analogously, an applied magnetic field that aligns the magnetic dipoles simultaneously aligns the electric dipoles [4, p.8]. In order to describe such media Tellegen, in 1948, introduced a new element, the *gyrator*, in addition to the resistor, the capacitor, the inductor and,

the ideal transformer. These media, when placed in an electric field or a magnetic field, become both polarized and magnetized, and they are referred to as *bianisotropic*, being characterized by constitutive relations of the type

$$\boldsymbol{D} = \underline{\underline{\varepsilon}}\, \boldsymbol{E} + \underline{\underline{\xi}}\, \boldsymbol{H}\,, \tag{2.23a}$$

$$\boldsymbol{B} = \underline{\underline{\zeta}}\, \boldsymbol{E} + \underline{\underline{\mu}}\, \boldsymbol{H}\,. \tag{2.23b}$$

Examples of hypothetic materials which directly relate electric and magnetic fields are the perfect electromagnetic conductors (PEMCs) as discussed by Sihvola and Lindell [5]. In a PEMC electric and magnetic fields on a material response level both cause electric and magnetic polarizations, however the medium response is not sensitive to the vector orientation of the electric and magnetic fields. P. Russer has introduced the field theoretical analogon to the gyrator circuit of network theory by boundary surfaces with gyrator properties [6].

II.5 Boundary Conditions

In order to obtain a unique solution of the Maxwell field equations, one must impose appropriate boundary, radiation, and edge conditions. Radiation and edge conditions formalize, respectively, the outgoing wave requirement on fields in an infinite region excited by sources in a bounded domain and by conservation of energy in the possibly singular fields induced in the vicinity of edges and corners (tips) on obstacle scatterers. These conditions are discussed customarily using field solutions of the wave equation in an appropriate coordinate system, and they are treated later on in Section VII. We shall deal here only with the boundary conditions arising at the interface between two different media.
Consider a regular surface S of a medium discontinuity, as shown in Figure 2.1, where the subscripts 1 and 2 distinguish quantities in regions 1 and 2, respectively. From (2.6a) and (2.6b), as a consequence of a limiting process, one obtains the following conditions:

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}\,, \tag{2.24a}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = -\boldsymbol{M}\,, \tag{2.24b}$$

where $\boldsymbol{J}$ and $\boldsymbol{M}$ are, respectively, the electric and magnetic surface current density distributions at the interface. Similarly, from (2.6c) and (2.6d), for a small volume at the interface, a limiting process yields,

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = -\rho_m\,, \tag{2.25a}$$

$$\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \rho_e\,, \tag{2.25b}$$

where ρ_e and ρ_m are, respectively, the electric and magnetic surface charge density distributions on the interface.

If neither medium is perfectly conducting, the tangential component of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous while their normal components undergo a jump due to the discontinuity in the permittivity and permeability.
When medium 1 is a perfect electric conductor, the field inside the medium vanishes everywhere and induced electric charges and currents exist on the surface. In this case we have:

$$\boldsymbol{n} \times \boldsymbol{H}_2 = \boldsymbol{J}\,, \tag{2.26a}$$

$$\boldsymbol{n} \times \boldsymbol{E}_2 = 0\,, \tag{2.26b}$$

$$\boldsymbol{n} \cdot \boldsymbol{B}_2 = 0\,, \tag{2.26c}$$

$$\boldsymbol{n} \cdot \boldsymbol{D}_2 = \rho_e\,, \tag{2.26d}$$

which states the vanishing, at the metal surface, of the tangential components of $\boldsymbol{E}$ and of the normal component of $\boldsymbol{H}$.
In certain cases, it is convenient to include fields generated from equivalent magnetic currents. Accordingly, the field generated by a magnetic current distribution in the immediate vicinity of a perfectly (electrically) conducting surface is given by

$$\boldsymbol{n} \times \boldsymbol{E}_2 = -\boldsymbol{M}\,. \tag{2.27}$$

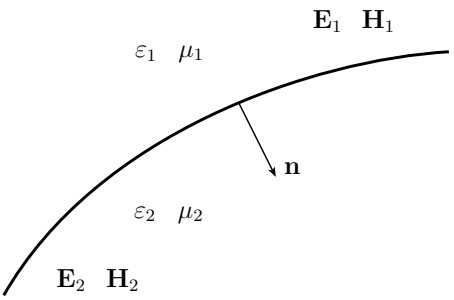


Fig. 2.1. Interface between two media.

III Theorems and Concepts for Electromagnetic Field Computation

Certain theorems and concepts of electromagnetic theory are of fundamental importance for efficient and systematic electromagnetic field computation. Their short description follows.

III.1 Energy and Power

The field concept is based upon the hypothesis that the electromagnetic energy is distributed over the space. We introduce the electric energy density

$$w_e(\boldsymbol{r},t) = \frac{\varepsilon}{2}\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t) \tag{2.28}$$

and the magnetic energy density

$$w_m(\boldsymbol{r},t) = \frac{\mu}{2}\boldsymbol{H}(\boldsymbol{r},t)\cdot\boldsymbol{H}(\boldsymbol{r},t). \tag{2.29}$$

In order to investigate energy storage and power flow in the electromagnetic field, we start again with Maxwell's equations. By scalar multiplication of Ampére's law with $-\boldsymbol{E}$ and Faraday's law with $\boldsymbol{H}$, we obtain

$$\begin{aligned} \nabla\times\boldsymbol{H} &= \tfrac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{J} \quad | \quad \cdot(-\boldsymbol{E})\,, \\ \nabla\times\boldsymbol{E} &= \quad -\tfrac{\partial\boldsymbol{B}}{\partial t} \quad | \quad \cdot\boldsymbol{H}\,. \end{aligned} \tag{2.30}$$

After inserting (2.18a) and (2.18b) into equation (2.30), we obtain

$$\boldsymbol{H}\cdot\nabla\times\boldsymbol{E} - \boldsymbol{E}\cdot\nabla\times\boldsymbol{H} = -\mu\boldsymbol{H}\cdot\frac{\partial\boldsymbol{H}}{\partial t} - \varepsilon\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t} - \boldsymbol{E}\cdot\boldsymbol{J}. \tag{2.31}$$

Using the relation

$$\nabla\cdot(\boldsymbol{U}\times\boldsymbol{V}) = \boldsymbol{V}\cdot\nabla\times\boldsymbol{U} - \boldsymbol{U}\cdot\nabla\times\boldsymbol{V}, \tag{2.32}$$

we transform the left side of (2.31) and obtain the differential form of *Poynting's theorem*

$$-\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \frac{\partial}{\partial t}\left(\frac{\mu}{2}\boldsymbol{H}\cdot\boldsymbol{H} + \frac{\varepsilon}{2}\boldsymbol{E}\cdot\boldsymbol{E}\right) + \sigma\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{J}_0\,. \tag{2.33}$$

On the right side of (2.33), we have the time derivative of the electric and magnetic energy densities corresponding to (2.28) and (2.29). The third term is the power loss density

$$p_v(\boldsymbol{r},t) = \sigma(\boldsymbol{r})\boldsymbol{E}\cdot\boldsymbol{E}. \tag{2.34}$$

Due to the impressed current density $\boldsymbol{J}_0$, a power

$$p_0(\boldsymbol{r},t) = -\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{J}_0(\boldsymbol{r},t) \tag{2.35}$$

is added to the electromagnetic field per unit of volume. Introducing the *Poynting* vector

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) \tag{2.36}$$

allows to write down Poynting's theorem in the following form:

$$\nabla \cdot \boldsymbol{S} = -\frac{\partial w_m}{\partial t} - \frac{\partial w_e}{\partial t} - p_v + p_0. \tag{2.37}$$

Integrating (2.37) over a volume V and transforming the integral over S into a surface integral over the boundary ∂V, we obtain the integral form of Poynting's theorem:

$$\oint_{\partial V} \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{A} = \int_V p_0 \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t}\int_V w_m \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t}\int_V w_e \mathrm{d}V - \int_V p_v \mathrm{d}V\,. \tag{2.38}$$

The first term on the right side of equation (2.38) describes the power added into the volume V via impressed currents. The second and the third term, respectively, describe time variation of the magnetic and electric energy stored in the volume. The last term describes the conductive losses occurring inside the volume V. The right side of the equation comprises the total electromagnetic power generated within the volume V minus the power losses in the volume minus the increase of electric and magnetic power stored in the volume. This net power must be equal to the power, which is flowing out from the volume V through the boundary ∂V. Therefore we may interpret the surface integral over the pointing vector on the left side of (2.38) as the total power flowing from inside the volume V to the outside. Since this is valid for an arbitrary choice of volume V, it follows that the Poynting vector describes the energy flowing by unit of time through an unit area oriented perpendicular to S.
For harmonic electromagnetic fields, the introduction of a complex Poynting vector is useful. For this we construct

$$\begin{aligned} \nabla \times \boldsymbol{H}^* &= -j\omega\varepsilon^* \boldsymbol{E}^* + \boldsymbol{J}_0^* \quad | \quad \cdot(-\boldsymbol{E})\,, \\ \nabla \times \boldsymbol{E} &= \quad -j\omega\mu \boldsymbol{H} \quad | \quad \cdot \boldsymbol{H}^*\,. \end{aligned} \tag{2.39}$$

Summing both equations, we obtain

$$\boldsymbol{H}^* \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H}^* = -j\omega(\mu \mid \boldsymbol{H} \mid)^2 - \varepsilon^* \mid \boldsymbol{E} \mid^2) - \boldsymbol{E} \cdot \boldsymbol{J}_0^*\,. \tag{2.40}$$

With the relation

$$\nabla \cdot (\boldsymbol{U} \times \boldsymbol{V}) = \boldsymbol{V} \cdot \nabla \times \boldsymbol{U} - \boldsymbol{U} \cdot \nabla \times \boldsymbol{V}\,, \tag{2.41}$$

we can transform (2.40) into the differential form of the *complex Poynting's theorem*

$$\boldsymbol{\nabla} \cdot \frac{1}{2}(\boldsymbol{E} \times \boldsymbol{H}^*) = -2j\omega \left(\frac{\mu}{4} \mid \boldsymbol{H} \mid^2 - \frac{\varepsilon^*}{4} \mid \boldsymbol{E} \mid^2 \right) - \frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{J}_0^*\,. \tag{2.42}$$

We now introduce the *complex Poynting vector* $\boldsymbol{T}$:

$$\boldsymbol{T}(\boldsymbol{r}) = \frac{1}{2}\left(\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})\right). \tag{2.43}$$

We have to note that $\boldsymbol{T}$ is not the phasor corresponding to $\boldsymbol{S}$. Therefore we have used a different character to distinguish between the complex Poynting vector and the real Poynting vector. In order to give an interpretation of the complex Poynting vector $\boldsymbol{T}$, we compute first the time-dependent Poynting vector $\boldsymbol{S}$ for a harmonic electromagnetic field

$$\boldsymbol{E}(\boldsymbol{r},t) = \Re\left\{\boldsymbol{E}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}\omega\mathrm{t}}\right\} = \frac{1}{2}\left(\boldsymbol{E}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}\omega\mathrm{t}} + \boldsymbol{E}^{*}(\boldsymbol{r})\mathrm{e}^{-\mathrm{j}\omega\mathrm{t}}\right), \tag{2.44a}$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \Re\left\{\boldsymbol{H}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}\omega\mathrm{t}}\right\} = \frac{1}{2}\left(\boldsymbol{H}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}\omega\mathrm{t}} + \boldsymbol{H}^{*}(\boldsymbol{r})\mathrm{e}^{-\mathrm{j}\omega\mathrm{t}}\right) \tag{2.44b}$$

we obtain

$$\boldsymbol{S}(\boldsymbol{r},t) = \frac{1}{2}\Re\left\{\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^{*}(\boldsymbol{r})\right\} + \frac{1}{2}\Re\left\{\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r})\mathrm{e}^{2\mathrm{j}\omega\mathrm{t}}\right\}. \tag{2.45}$$

The first term on the right side of (2.45) is equal to the real part of the complex Poynting vector $\boldsymbol{T}$ after equation (2.43). This term is independent of time. The second on the right-hand side of (2.45) oscillates with twice the frequency of the alternating electromagnetic field. The time average of this part vanishes. Therefore the real part of the complex Poynting vector $\boldsymbol{T}$ is the time average of the Poynting vector $\boldsymbol{S}$.

$$\overline{\boldsymbol{S}(\boldsymbol{r},t)} = \Re\{\boldsymbol{T}(\boldsymbol{r})\}. \tag{2.46}$$

The real part of the complex Poynting vector $\boldsymbol{T}$ denotes the power flowing through an unit area oriented perpendicular to $\boldsymbol{T}$. We write the time averages of the electric and magnetic energy densities $\overline{w}_e$ and $\overline{w}_m$ as

$$\overline{w}_e = \frac{\varepsilon}{2}\overline{\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)} = \frac{\varepsilon'}{4} \mid \boldsymbol{E}(\boldsymbol{r}) \mid^2, \tag{2.47}$$

$$\overline{w}_m = \frac{\mu}{2}\overline{\boldsymbol{H}(\boldsymbol{r},t)\cdot\boldsymbol{H}(\boldsymbol{r},t)} = \frac{\mu'}{4} \mid \boldsymbol{H}(\boldsymbol{r}) \mid^2 . \tag{2.48}$$

We have to consider that the quantities ε' and μ' in the complex representation correspond to the quantities ε and μ in the time-dependent formulation. From equations (2.29), (2.43) and (2.34) we obtain the average electric power dissipation density

$$\overline{p}_{ve} = \frac{1}{2}\sigma \mid \boldsymbol{E}(\boldsymbol{r}) \mid^2 = \frac{1}{2}\omega\varepsilon'' \mid \boldsymbol{E}(\boldsymbol{r}) \mid^2 . \tag{2.49}$$

The introduction of the complex permittivity μ allows also to consider the magnetic losses. The average power dissipation density is given by

$$\overline{p}_{v} = \frac{w}{2}(\varepsilon'' \mid \boldsymbol{E}(\boldsymbol{r}) \mid^2 + \mu'' \mid \boldsymbol{H}(\boldsymbol{r}) \mid^2 . \tag{2.50}$$

The complex power, which is added to the field due to the impressed current density $\boldsymbol{J}_0$ is given by

$$p_{s0} = -\frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{J}_0^*. \tag{2.51}$$

The real part of p_{s0} equals the time average $\overline{p}_{s0}$ according to equation (2.37).

$$\overline{p}_0 \;=\; \Re\{p_{s0}\}. \tag{2.52}$$

The proof is similar to the one of (2.46). After inserting of (2.43), (2.47), (2.48), (2.50) and (2.51) into (2.42), we can write down the complex Poynting's theorem in the following form

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} \;=\; -2j\omega(\overline{w}_m - \overline{w}_e) - \overline{p}_v + p_{s0}\,. \tag{2.53}$$

By integration over a volume V, we obtain the integral form of the complex Poynting's theorem

$$\oint_{\partial V} \boldsymbol{T} \cdot \mathrm{d}\boldsymbol{A} \;=\; \int_V p_{s0}\mathrm{d}V - 2j\omega \int_V (\overline{w}_m - \overline{w}_e)\mathrm{d}V - \int_V \overline{p}_v \mathrm{d}V\,. \tag{2.54}$$

We consider first the real part of (2.54).

$$\Re\left\{\oint_{\partial V} \boldsymbol{T} \cdot \mathrm{d}\boldsymbol{A}\right\} \;=\; \Re\left\{\int_V p_{s0}\mathrm{d}V\right\} - \int_V \overline{p}_v \mathrm{d}V\,. \tag{2.55}$$

The left side of (2.55) equals the active power radiated from inside the volume V through the boundary ∂V. On the right side of this equation, the first term denotes the power added via the impressed current density $\boldsymbol{J}_0$; the second term describes the conductive losses, the dielectric losses and the magnetic losses inside the volume V.
The imaginary part of (2.54) is

$$\Im\left\{\oint_{\partial V} \boldsymbol{T} \cdot \mathrm{d}\boldsymbol{A}\right\} \;=\; \Im\left\{\int_V p_{s0}\mathrm{d}V\right\} - 2\omega \int_V (\overline{w}_m - \overline{w}_e)\mathrm{d}V\,. \tag{2.56}$$

The first term on the right side gives the reactive power inserted into the volume V via the impressed current density $\boldsymbol{J}_0$. Let us first consider the case where the second term on the right side is vanishing. In this case we see that the left side of (2.56) denotes the power radiated from volume V. Since the volume V can be chosen arbitrarily, it follows that the imaginary part of the complex Poynting vector $\boldsymbol{T}$ describes the reactive power radiated through an unit area normally oriented to the vector $\boldsymbol{T}$.

The second term on the right side of (2.56) contains the product of the double angular frequency with the difference of the average stored magnetic and electric energies. This term yields no contribution, if the magnetic energy stored in the volume V equals the average electric energy stored in V. The magnetic energy as well as electric energy oscillates with an angular frequency 2ω. The energy is permanently converted between electric energy and magnetic energy. If the averages $\overline{w}_e$ and $\overline{w}_m$ are equal, electric and magnetic energies may be mutually converted completely. In this case the energy oscillates between electric and magnetic field inside the volume V. If the average electric and magnetic energies are not equal, energy as well oscillates between volume V and the space outside V. In this case there is a power flow between V and the outer region. For $\overline{w}_m > \overline{w}_e$ the reactive power flowing into volume V is positive, whereas for $\overline{w}_m < \overline{w}_e$ the reactive power flowing into V is negative.

III.2 Field Theoretic Formulation of Tellegen's Theorem

Tellegen's theorem states fundamental relations between voltages and currents in a network, and is of considerable versatility and generality in network theory [7–9]. A notable property of this theorem is that it is only based on Kirchhoff's current and voltage laws, i.e. on topological relationships, and that it is independent of the constitutive laws of the network. The same reasoning that leads from Kirchhoff's laws to Tellegen's theorem permits direct derivation of a field form of Tellegen's theorem from Maxwell's equations [9–11].
In order to derive Tellegen's theorem for partitioned electromagnetic structures, let us consider two cases based on the same partitioning but filled with different materials. The connection network is established via relating the tangential field components on both sides of the boundaries; since the connection network has zero volume, no field energy is stored therein. An important point for the following discussion is that the materials filling the subdomains may be completely different. Starting directly from Maxwell's equations we may derive for a closed volume V with boundary surface ∂V and normal unit vector $\boldsymbol{n}$ the following relation

$$\int_{\partial V} \boldsymbol{E}'(\boldsymbol{\rho}, t') \times \boldsymbol{H}''(\boldsymbol{\rho}, t'') \cdot \boldsymbol{n}\, d\mathcal{A} = -\int_V \mathbf{E}'(\mathbf{r}, t') \cdot \mathbf{J}''(\mathbf{r}, t'') dV - \int_V \mathbf{E}'(\mathbf{r}, t') \cdot \frac{\partial \mathbf{D}''(\mathbf{r}, t'')}{\partial t''}\, dV - \int_V \mathbf{H}'(\mathbf{r}, \mathbf{t}') \cdot \frac{\partial \mathbf{B}''(\mathbf{r}, \mathbf{t}'')}{\partial t''}\, dV\,. \quad (2.57)$$

The single and double primes relate to the case of a different choice of sources, different material parameters and also a different time reference. For volumes V of zero measure, we obtain the following equation

$$\int_{\partial V} \boldsymbol{E}'(\boldsymbol{\rho}, t') \times \boldsymbol{H}''(\boldsymbol{\rho}, t'') \cdot \boldsymbol{n} d\mathcal{A} = 0\,. \quad (2.58)$$

The above equation may be considered as the field form of Tellegen's theorem. Since it applies to a volume of zero measure, it is independent of the domain equations.

III.3 Uniqueness Theorem

The uniqueness theorem indicates how a problem should be properly formulated in order to provide one and only one solution. Uniqueness of the solution is a consequence of the proper imposition of the boundary conditions, since overdetermination, i.e. too many boundary conditions, may lead to no solution for a given problem, while a lack of boundary conditions may lead to multiple solutions. For time–harmonic electromagnetic fields, the uniqueness theorem states that when the sources and the tangential components of the electric *or* magnetic field are specified over the *whole* boundary surface of a given region, then the solution within this region is unique. This is actually true only if the medium is slightly lossy; otherwise it is possible to have a multiplicity of solution as, for example, for a closed resonator.
The proof of the uniqueness theorem follows from considering two different solutions $\boldsymbol{E}_1$, $\boldsymbol{H}_1$ and $\boldsymbol{E}_2$, $\boldsymbol{H}_2$ in the volume V bounded by the surface S excited by the same system of sources. Let us define the difference fields $\delta\boldsymbol{E}$ and $\delta\boldsymbol{H}$ as

$$\delta\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2 \,, \tag{2.59a}$$

$$\delta\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2 \,. \tag{2.59b}$$

By linearity, and since the sources are the same, the difference fields satisfy the source–free Maxwell equations

$$\nabla \times \delta\boldsymbol{E} = -\, j\omega\mu\delta\boldsymbol{H} \,, \tag{2.60a}$$

$$\nabla \times \delta\boldsymbol{H} = j\omega\varepsilon\delta\boldsymbol{E} \,, \tag{2.60b}$$

where it has been assumed that the permittivity ε and the permeability μ are of the following form

$$\varepsilon = \varepsilon_r - j\varepsilon_i \,, \tag{2.61a}$$

$$\mu = \mu_r - j\mu_i \,, \tag{2.61b}$$

i.e. a small, positive, imaginary part is present. As noted in [4] the proof also holds when the imaginary parts are both negative. By scalar multiplication of (2.60a) by $\delta\boldsymbol{H}^*$ and of the complex conjugate of (2.60b) by $\delta\boldsymbol{E}$ we obtain

$$\nabla \cdot (\delta\boldsymbol{E} \times \delta\boldsymbol{H}^*) = j\omega\varepsilon^* |\delta\boldsymbol{E}|^2 - j\omega\mu |\delta\boldsymbol{H}|^2 \,. \tag{2.62}$$

The complex conjugate of (2.62) also holds, giving

$$\nabla \cdot (\delta \boldsymbol{E}^* \times \delta \boldsymbol{H}) = -j\omega\varepsilon |\delta \boldsymbol{E}|^2 + j\omega\mu^* |\delta \boldsymbol{H}|^2 \,. \tag{2.63}$$

By adding (2.62) and (2.63), integrating over the volume V and applying the divergence theorem, we recover

$$\oint_S (\delta \boldsymbol{E} \times \delta \boldsymbol{H}^* + \delta \boldsymbol{E}^* \times \delta \boldsymbol{H}) \cdot dS = -2\omega \int_V \left(\varepsilon_i |\delta \boldsymbol{E}|^2 + \mu_i |\delta \boldsymbol{H}|^2 \right) dV \,. \tag{2.64}$$

When the tangential components of $\boldsymbol{E}$ or $\boldsymbol{H}$ coincide on the boundary surface S, i.e. when either $\delta \boldsymbol{E}$ or $\delta \boldsymbol{H}$ are zero on S, we have

$$\oint_S (\delta \boldsymbol{E} \times \delta \boldsymbol{H}^* + \delta \boldsymbol{E}^* \times \delta \boldsymbol{H}) \cdot dS = 0 \,. \tag{2.65}$$

In this case, the right–hand side of (2.64) is zero only if $\delta \boldsymbol{E}$ and $\delta \boldsymbol{H}$ are identically zero in the region V. This proves the theorem.
As a last remark, observe that for lossless structures, when we look for modal spectra, we are seeking resonant solutions. In this case, the uniqueness theorem does not apply and an infinity of solutions is present.

III.4 Equivalence Theorem

There are several forms in which to state the *equivalence theorem* [4, 12] and, in view of its importance in the solution of electromagnetic field problems, it seems appropriate to examine relevant issues in detail.
Let us consider a volume V bounded by a surface S separating the internal region, labeled as region 1, from the external region, labeled as region 2. Our objective in applying the equivalence theorem is to maintain the field in region 2 even when modifying the field in region 1. By so doing, we obtain a modified field problem which, at least in region 2, and only in region 2, is equivalent to the original one. We denote by $\boldsymbol{E}_1$, $\boldsymbol{H}_1$ and $\boldsymbol{E}_2$, $\boldsymbol{H}_2$ the original fields in regions 1 and 2, respectively, as shown in Figure 2.2. Now suppose that the field in region 1 is altered, thus changing the field in this region from $\boldsymbol{E}_1, \boldsymbol{H}_1$ to $\boldsymbol{E}'_1, \boldsymbol{H}'_1$. In order to maintain the original field in region 2, we must insert equivalent magnetic and electric currents, $\boldsymbol{M}_s$ and $\boldsymbol{J}_s$, respectively, on the surface S such that

$$\boldsymbol{J}_s = \boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}'_1) \,, \tag{2.66a}$$

$$\boldsymbol{M}_s = -\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}'_1) \,, \tag{2.66b}$$

as shown in Figure 2.3.

Love Equivalence Theorem

A particular case is to set the field in region 1 equal to zero. Thus we have the case shown in Figure 2.4 where the surface currents are now given by

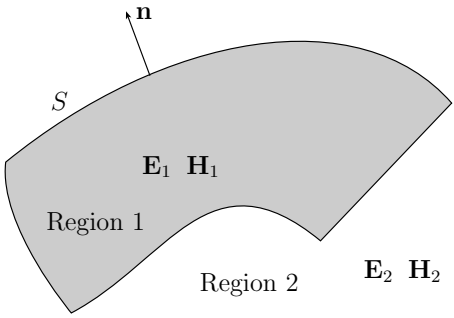


Fig. 2.2. Original field.

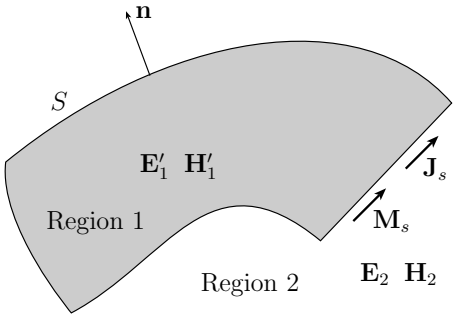


Fig. 2.3. The field in region 1 has been modified. By inserting the equivalent electric and magnetic currents on the surface S the field in region 2 is unchanged.

$$\boldsymbol{J}_s = \boldsymbol{n} \times \boldsymbol{H}_2 \,, \tag{2.67a}$$

$$\boldsymbol{M}_s = -\,\boldsymbol{n} \times \boldsymbol{E}_2 \,, \tag{2.67b}$$

The *Love equivalence theorem* states that the field in region 2 produced by the given sources in region 1 is the same as that produced by a system of *virtual* sources on the surface S.

Perfect Electric Conductor

The Love theorem only specifies a zero field in region 1. This may be obtained by filling region 1 with a perfect electric conductor (PEC) as considered here, or by filling region 1 with a perfect magnetic conductor (PMC) as considered below. It is easy to see that electric currents $\boldsymbol{J}_s$ on the PEC are short-circuited and therefore do not radiate any field. In fact, near the perfect conductor, the electric field is perpendicular to the surface S, while the magnetic field is parallel. The resulting Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ is thus parallel to the surface of the conductor and no

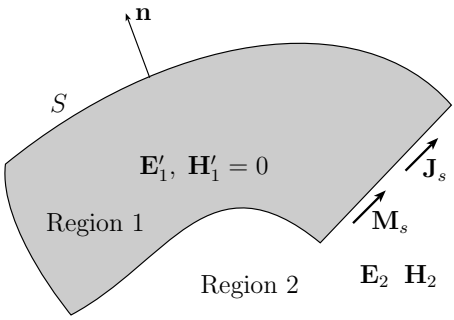


Fig. 2.4. The field in region 1 has been set to zero. Equivalent currents maintain the original field in region 2.

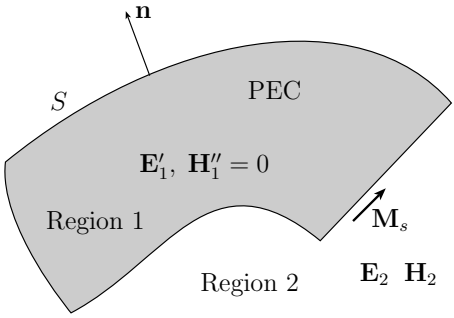


Fig. 2.5. Region 1 has been filled with a PEC. Only magnetic currents contribute since electric currents do not radiate.

power is radiated into space. A different proof of this fact may be obtained by using the Lorentz theorem. Thus, when region 1 is filled with a PEC, the resulting configuration is that shown in Figure 2.5. This form of equivalence theorem is used in practical applications where structures are bounded by metallic walls.

Perfect Magnetic Conductor

The other possibility of obtaining a null field in region 1, is to fill this region with a perfect magnetic conductor (PMC). In this case, since surface magnetic currents do not radiate, we are left with the case of Figure 2.6. Note that as in the previously, when calculating the field produced by sources in region 2, we must take into account the presence of the PMC, since the Green's function to be considered must satisfy the appropriate boundary conditions on the surface S. On the contrary, when applying the Love theorem without filling region 1

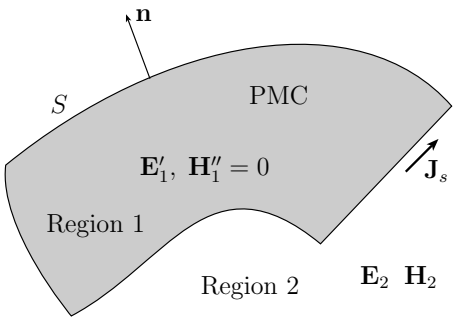


Fig. 2.6. Region 1 has been filled with a PMC Only electric currents contribute since magnetic currents do not radiate.

with either a PMC or a PEC the Green's function to be considered is that of free–space.

The Circuit Theory Analog

The Circuit Theory analog of the equivalence theorem provides a simple and effective way to illustrate its utility [13]. Let region 2 be without sources, represented by the passive network in Figure 2.7(a), while region 1 is represented by the source–excited (active) network. We can set up an equivalent problem by

- switching off the sources in the active network, leaving the source impedance connected;
- placing a shunt current generator I equal to the terminal current in the original problem;
- placing a series voltage generator V equal to the terminal voltages into the original problem.

This replaces the original sources in region 1, the active network, by the virtual sources at the interface as shown in Figure 2.7(b). From conventional circuit concepts, it is evident that there is no excitation of the source impedance from these equivalent sources whereas the excitation of the passive network is unchanged. This fact offers the possibility of replacing the source impedance by either a short or an open circuit. By considering a short circuit, we obtain the case of Figure 2.7(c), equivalent to considering a PEC when applying the Love theorem. When using an open circuit, we obtain the case of Figure 2.7(d), equivalent to considering a PMC in the Love theorem.

Duality

Returning to the Maxwell equations in (2.14a)–(2.14d) it is noted that performing the substitutions

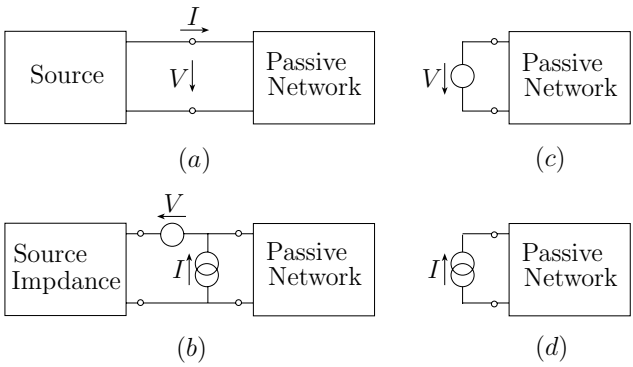


Fig. 2.7. Circuit analogue of the equivalence theorem: (a) original problem; (b) actual source deactivated, replaced by equivalent (virtual) sources; (c) source impedance replaced by a short circuit; (d) source impedance replaced by an open circuit.

$$\begin{aligned} &\boldsymbol{E} \to \boldsymbol{H} && \varepsilon \to \mu && \mu \to \varepsilon \\ &\boldsymbol{H} \to -\boldsymbol{E} && \boldsymbol{J} \to \boldsymbol{M} && \boldsymbol{M} \to -\boldsymbol{J}, \end{aligned} \tag{2.68}$$

equation (2.14a) becomes (2.14b), and vice versa. This is generally referred to as the "duality principle". However, the above substitutions imply a medium "dual" (or "adjoint") to free space, i.e. a medium with a permittivity of $4\pi \times 10^{-7}$ F/m and with permeability 8.854×10^{-12} H/m, which is undesirable.
A form of duality which is more suitable for antenna and radiation problems is established by the following equalities [4]

$$\begin{aligned} &\boldsymbol{E} \to \eta \boldsymbol{H}\,, && \boldsymbol{H} \to -\frac{1}{\eta}\boldsymbol{E}\,, \\ &\boldsymbol{J} \to \frac{1}{\eta}\boldsymbol{M}\,, && \boldsymbol{M} \to -\eta \boldsymbol{J}\,, \end{aligned} \tag{2.69}$$

with $\eta = \sqrt{\frac{\varepsilon}{\mu}}$ the free space impedance. With the substitutions in (2.69), equation (2.14a) becomes (2.14b) and vice versa, without need to replace free space with a different medium. Now, the form of duality in (2.69) does not apply to anisotropic or bianisotropic media, while (2.68) is more general.

IV Field Potentials

Auxiliary potentials are conventionally introduced to simplify the solution of the vector field equations [3, 14, 15]. When only *electric* sources are present in a homogeneous region, the two curl equations

$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}\,, \tag{2.70a}$$

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J} \tag{2.70b}$$

provide six scalar equations. The divergence equations

$$\nabla \cdot \boldsymbol{D} = \rho_e\,, \tag{2.71a}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{2.71b}$$

provide two additional scalar equations, which need to be complemented by the constitutive relations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E}\,, \tag{2.72a}$$

$$\boldsymbol{B} = \mu\boldsymbol{H}\,, \tag{2.72b}$$

and the relevant boundary conditions. The use of potential functions can systematize the solution of this large set of equations.

Magnetic Vector and Electric Scalar Potentials

The *vector* and *scalar potential* functions $\boldsymbol{A}$ and Φ, respectively, represent the electrodynamic extensions of the static magnetic vector potential and electrical scalar potential, respectively. While potential theory is generally developed for the time–dependent form of Maxwell's equations [3, 14, 15], we shall deal directly with the time–harmonic potentials (an $\exp(j\omega t)$ time–dependence is assumed and suppressed). By taking the divergence of (2.70a) we see that

$$\nabla \cdot \boldsymbol{H} = 0, \tag{2.73}$$

i.e. the divergence equation for $\boldsymbol{H}$ is automatically satisfied. This suggests expressing $\boldsymbol{H}$ as

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A}\,, \tag{2.74}$$

where $\boldsymbol{A}$ is referred to as the *magnetic vector potential*. By inserting (2.74) into (2.70a) we note that

$$\nabla \times (\boldsymbol{E} + j\omega\boldsymbol{A}) = 0 \tag{2.75}$$

and since

$$\nabla \times \nabla\Phi = 0 \tag{2.76}$$

the vector $\boldsymbol{E}$ can be expressed as

$$\boldsymbol{E} = -j\omega\boldsymbol{A} - \nabla\Phi \tag{2.77}$$

with Φ denoting the *electric scalar potential*. By substitution of (2.74), (2.77) into (2.70b) and recalling the vector identity

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla\nabla \cdot \boldsymbol{A} - \nabla^2\boldsymbol{A} \tag{2.78}$$

we obtain

$$\nabla\nabla \cdot \boldsymbol{A} - \nabla^2\boldsymbol{A} = k^2\boldsymbol{A} - j\omega\mu\varepsilon\nabla\Phi + \mu\boldsymbol{J}\,. \tag{2.79}$$

Lorentz Potentials

Equation (2.79) can be phrased in a different manner by selecting the as yet unspecified divergence (lamellar part) of $\boldsymbol{A}$. One possible choice is to satisfy the *Lorentz* or *gauge condition*,

$$\nabla \cdot \boldsymbol{A} = -j\omega\mu\varepsilon\Phi \tag{2.80}$$

which reduces (2.79) to the *vector Helmholtz equation*

$$\nabla^2\boldsymbol{A} + k^2\boldsymbol{A} = -\mu\boldsymbol{J}. \tag{2.81}$$

Taking the divergence of (2.77) and using (2.80) it follows that the scalar potential Φ satisfies the scalar Helmholtz equation

$$\nabla^2\Phi + k^2\Phi = -\frac{\rho_e}{\varepsilon}. \tag{2.82}$$

The electric and magnetic fields, when using the Lorentz condition (2.80), are

$$\boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\mu\varepsilon}\,, \tag{2.83a}$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla\times\boldsymbol{A}\,. \tag{2.83b}$$

Electric Vector and Magnetic Scalar Potentials

When only *magnetic* sources are present, the vector $\boldsymbol{E}$ has zero divergence. By duality, introducing an *electric vector potential* $\boldsymbol{F}$ and a *scalar potential* Ψ, we obtain [13, p.129]

$$\boldsymbol{E} = -\frac{1}{\varepsilon}\nabla\times\boldsymbol{F}\,, \tag{2.84a}$$

$$\boldsymbol{H} = -j\omega\boldsymbol{F} + \frac{\nabla\nabla\cdot\boldsymbol{F}}{j\omega\varepsilon\mu} \tag{2.84b}$$

with

$$\nabla^2\boldsymbol{F} + k^2\boldsymbol{F} = -\varepsilon\boldsymbol{M}\,, \tag{2.85a}$$

$$\nabla^2\Psi + k^2\Psi = -\frac{\rho_m}{\mu}\,, \tag{2.85b}$$

where $\boldsymbol{M}$ denotes magnetic currents and ρ_m magnetic charges. When *electric and magnetic* currents are present simultaneously in a linear system, we make use of superposition to obtain:

$$\boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\mu\varepsilon} - \frac{1}{\varepsilon}\nabla\times\boldsymbol{F}\,, \tag{2.86a}$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla\times\boldsymbol{A} - j\omega\boldsymbol{F} + \frac{\nabla\nabla\cdot\boldsymbol{F}}{j\omega\varepsilon\mu}\,. \tag{2.86b}$$

Hertz Potentials

Hertz vector potentials for electric and magnetic time-harmonic fields are simply related to the electric and magnetic vector potentials via

$$\boldsymbol{A} = j\omega\varepsilon\mu\boldsymbol{\Pi}_e\,, \tag{2.87a}$$

$$\boldsymbol{F} = j\omega\varepsilon\mu\boldsymbol{\Pi}_h\,. \tag{2.87b}$$

The general field expression in terms of the Hertz vector potentials is:

$$\boldsymbol{E} = k^2\boldsymbol{\Pi}_e + \nabla\nabla\cdot\boldsymbol{\Pi}_e - j\omega\mu\nabla\times\boldsymbol{\Pi}_h\,, \tag{2.88a}$$

$$\boldsymbol{H} = j\omega\varepsilon\nabla\times\boldsymbol{\Pi}_e + k^2\boldsymbol{\Pi}_h + \nabla\nabla\cdot\boldsymbol{\Pi}_h\,. \tag{2.88b}$$

For Hertz potentials related to time-dependent fields see [15].

V Separation of Variables: The Scalar Wave Equation

Explicit solution of wave problems is facilitated substantially in special configurations that render the relevant wave equations fully or partially separable. Much about the physics of wave phenomena is learned from such special *canonical problems*. This section introduces concepts and notation for the *scalar wave equation*

$$\nabla^2\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = f(\mathbf{r},t), \tag{2.89}$$

where c_0 is the ambient wave speed. If it is assumed that the time dependence of the source distribution $f(\mathbf{r},t)$ is sinusoidal with frequency ω, the scalar field $\varphi(\mathbf{r},t)$ can be written as

$$\varphi(\mathbf{r},t) = \Re\left[U(u,v,w)e^{j\omega t}\right], \tag{2.90}$$

where (u,v,w) have been introduced as spatial coordinates. Outside the source region, (2.89) then becomes the *homogeneous Helmholtz equation*

$$(\nabla^2 + k_0^2)U = 0, \tag{2.91}$$

for the complex function U, where the ambient wavenumber k_0 is

$$k_0 = \frac{\omega}{c_0}, \tag{2.92}$$

with c_0 denoting the propagation speed. This equation is to be solved subject to the prevailing boundary conditions.
In certain spatial coordinate systems (see [16] for a complete discussion), it is possible to apply the *separation of variables* technique to (2.91), by which the

field $U(u, v, w)$ is written as the product of functions which individually depend on only one spatial variable,

$$U(u, v, w) = U_u(u)U_v(v)U_w(w) \tag{2.93}$$

(see Table 2.1). Separability must hold for the operator ∇^2 *and* for the prevailing boundary conditions. Separability of the operator implies that the partial differential Laplacian ∇^2 can be arranged into three second-order one-dimensional ordinary differential operators ∇_τ^2, where τ stands for either u, v or w. On the boundaries $\tau = \tau_1$ and $\tau = \tau_2$ of each τ-domain, the boundary conditions are assumed to be of the linear homogeneous (impedance) type

$$U_\tau(\tau_{1,2}) + \gamma_{\tau_{1,2}} \left.\frac{\partial U_\tau(\tau)}{d\tau}\right|_{\tau_{1,2}} = 0 \quad \text{on} \quad B_{\tau_{1,2}} \tag{2.94}$$

Here, U_τ is one of the functions in (2.93) depending only on τ, B_τ defines the boundary surfaces $\tau = \text{const.}$ in the τ-domain, and $\gamma_{\tau 1,2}$ are constants. The method is best illustrated by example.

Table 2.1. Summary of boundary conditions for coordinate-separable solutions of the scalar wave equation.

Generic coordinates:	$\boldsymbol{r} = (u, v, w)$		
boundaries along:	$u = u_1, u_2,$	$v = v_1, v_2,$	$w = w_1, w_2,$
range of variables:	$u_1 \le u \le u_2,$	$v_1 \le v \le u_2,$	$w_1 \le w \le w_2$

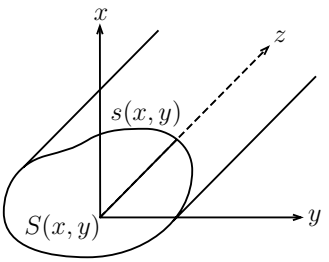
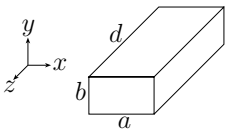
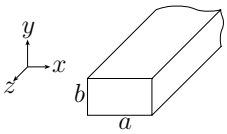
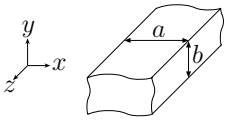


Fig. 2.8. Cross section $S(x, y)$ and boundary curve $s(x, y)$ for z-domain separation in Cartesian coordinates.

Table 2.2. Completely coordinate-separable configurations for rectangular Cartesian coordinates $(u, v, w) = (x, y, z)$. Note that, although only the cases for different conditions in the z–domain are shown, it is possible to change in a similar way also the x and y domains obtaining several other configurations.

Domain	Configuration
$x_1 = 0,\ x_2 = a,$ $y_1 = 0,\ y_2 = b,$ $z_1 = 0,\ z_2 = d,$	
$x_1 = 0, \quad x_2 = a,$ $y_1 = 0, \quad y_2 = b,$ $z_1 \to -\infty,\ z_2 = 0,$	
$x_1 = 0, \quad x_2 = a,$ $y_1 = 0, \quad y_2 = b,$ $z_1 \to -\infty,\ z_2 \to \infty,$	

V.1 The Scalar Wave Equation in Cartesian Coordinates

The simplest demonstration of separability is for Cartesian coordinates, where $(u, v, w) = (x, y, z)$ (see Table 2.2). For a problem separable with respect to one of the coordinates, to be designated by z, the Laplacian ∇^2 is decomposed as

$$\nabla^2 = \nabla^2_{xy} + \nabla^2_z\,, \tag{2.95}$$

where

$$\nabla^2_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.96}$$

and

$$\nabla^2_z = \frac{\partial^2}{\partial z^2}\,. \tag{2.97}$$

Accordingly, a solution for the field $U(u, v, w)$ is sought in the form

$$U(x, y, z) = U_z(z)U_{xy}(x, y), \quad z_1 \le z \le z_2, \tag{2.98}$$

wherein the z-variable has been explicitly separated, and its domain has been explicitly identified. Any boundaries in the (x, y) domain must be z-independent cylindrical surfaces with transverse-to-z cross sections $S = S(x, y)$ bounded by

the curve $s(x, y)$ (Figure 2.8). Separability of the z-domain boundary conditions implies that the boundary conditions on the planes $z = z_1$ and $z = z_2$ must be independent of x and y, as in (2.94), with $\tau = z$. Similarly, the boundary conditions on the surface $s(x, y)$ must be independent of z.
Substitution of (2.95) and (2.98) into (2.91) gives

$$-\frac{\nabla_z^2 U_z}{U_z} = \frac{\nabla_{xy}^2 U_{xy}}{U_{xy}} + k_0^2. \tag{2.99}$$

The left-hand side of (2.99) is a function of z only, while the right-hand side is a function of x and y only. Therefore both sides must be equal to a constant, because (2.99) must hold for arbitrary values (x, y, z) and k_0. If this constant is designated by k_z^2, then the partial field functions U_z and U_{xy} satisfy the reduced equations

$$(\frac{d^2}{dz^2} + k_z^2)U_z = 0, \tag{2.100}$$

subject to the z-domain boundary conditions, and

$$(\nabla_{xy}^2 + k_{xy}^2)U_{xy} = 0, \tag{2.101}$$

subject to the (x, y)-domain boundary conditions. The constants k_z and k_{xy} satisfy the relation

$$k_0^2 - k_z^2 - k_{xy}^2 = 0. \tag{2.102}$$

With respect to wave propagation, k_0 is the total wavenumber, and the separation constants k_z and k_{xy} are therefore the wavenumber components associated with the z and the (x, y) subdomains, respectively. (2.102) is the *spatial dispersion relation* that constrains the wavenumber components.
When boundary conditions as in (2.94) are imposed at the endpoints z_1 and z_2 of the domain $z_1 \leq z \leq z_2$, solutions of (2.100) can be found for special values $k_z = k_{z\alpha}$, with corresponding solutions $U_{z\alpha}$. The problem

$$\left(\frac{d^2}{dz^2} + k_{z\alpha}^2\right) U_{z\alpha}(z) = 0\,, \qquad z_1 \leq z \leq z_2 \tag{2.103}$$

is called an *eigenvalue problem*, the wavenumbers (separation constants) $k_{z\alpha}$ are called *eigenvalues*, and the solutions $U_{z\alpha}(z)$ are called *eigenfunctions*. Depending on the boundary conditions at z_1 and z_2, the eigenvalues $k_{z\alpha}$ may form an infinite set of discrete values or they may be continuously distributed. Eigenvalue problems are discussed in detail within the context of the *Sturm–Liouville problem* (see Section VI).
If the boundary conditions are also (x, y)-separable (i.e., the three-dimensional problem is completely separable), the field U_{xy} in (2.98) is written as $U_{xy}(x, y) = U_x(x)U_y(y)$, so that the total scalar field $U(x, y, z) = U_{xy}U_z$ becomes

$$U(x,y,z) = U_x(x)U_y(y)U_z(z). \tag{2.104}$$

The individual partial fields U_x, U_y, U_z then satisfy the equations

$$(\frac{d^2}{dx^2} + k_x^2)U_x = 0\,, \tag{2.105a}$$

$$(\frac{d^2}{dy^2} + k_y^2)U_y = 0\,, \tag{2.105b}$$

$$(\frac{d^2}{dz^2} + k_z^2)U_z = 0,\,, \tag{2.105c}$$

with separation constants (wavenumbers) k_x, k_y and k_z that satisfy the dispersion relation

$$k_0^2 - k_x^2 - k_y^2 - k_z^2 = 0. \tag{2.106}$$

Imposition of the boundary conditions of (2.94) in the separate x and y domains leads to two eigenvalue problems analogous to that in (2.103), with corresponding interpretations. The solutions for $U_{z\alpha}$ are the trigonometric functions

$$U_{z\alpha}(z) = \sin k_{z\alpha}z,\;\; \cos k_{z\alpha}z,\;\; e^{ik_{z\alpha}z},\;\; e^{-ik_{z\alpha}z}, \tag{2.107}$$

any two of which are linearly independent, and are chosen in configurations that satisfy (2.94) at $z = z_{1,2}$. Solutions for $U_y(y)$ and $U_z(z)$ are similar.

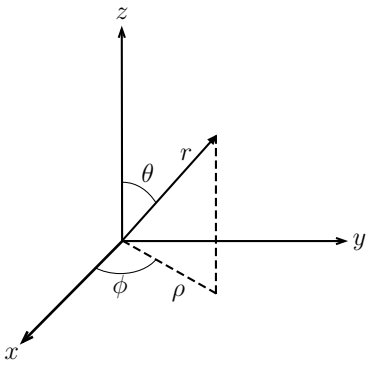


Fig. 2.9. Completely coordinate-separable configurations for spherical coordinates $(u,v,w) = (r,\theta,\phi)$. The domains $r = r_1, r = r_2$ correspond to spherical boundaries, $\theta = \theta_1, \theta = \theta_2$ to conical boundaries and $\phi = \phi_1, \phi = \phi_2$ to plane boundaries.

V.2 The Scalar Wave Equation in Spherical and Polar Coordinates

In separable curvilinear coordinates, the reduction process is similar but more subtle, because not all of the coordinates are direct measures of length. For example, in spherical polar coordinates $(u, v, w) = (r, \theta, \phi)$ (Figure 2.9), the Laplace operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right). \tag{2.108}$$

The azimuthal ϕ-coordinate is separated readily because the associated partial differential operator is $\partial^2/\partial\phi^2$ just as in the Cartesian case. Removing the coefficient $(r^2\sin^2\theta)^{-1}$ by cross multiplication, i.e., writing $U(r, \theta, \phi) = U_\phi(\phi)U_{r\theta}(r, \theta)$, (2.91) and (2.108) yields

$$-\frac{1}{U_\phi}\frac{\partial^2 U_\phi}{\partial\phi^2} = \frac{1}{U_{r\theta}}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]U_{r\theta} + k_0^2 r^2\sin^2\theta. \tag{2.109}$$

The left-hand side of (2.109) is a function of ϕ only, while the right-hand side is a function of r and θ only. Therefore both sides must be equal to a constant, and if this constant is denoted by k_ϕ^2, then the functions U_ϕ and $U_{r\theta}$ satisfy the reduced equations

$$\left(\frac{d^2}{d\phi^2} + k_\phi^2\right)U_\phi = 0, \tag{2.110}$$

subject to the ϕ-domain boundary conditions on $\phi =$ const. planes, or for 2π-periodic conditions, and

$$\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k_{r\theta}^2\sin^2\theta\right)U_{r\theta} = 0, \tag{2.111}$$

subject to the ϕ-independent (r, θ)-domain boundary conditions on surfaces of revolution $S(r, \theta) = 0$. The dispersion relation for the angular wavenumbers k_ϕ and $k_{r\theta}$ is

$$k_{r\theta}^2 = k_0^2 r^2 - \frac{k_\phi^2}{\sin^2\theta}. \tag{2.112}$$

Note that k_ϕ here is the dimensionless angular wavenumber associated with the dimensionless angular azimuthal coordinate ϕ. The separation parameter (wavenumber) $k_{r\theta}$ in (2.112) has been chosen so that the r and θ dependencies in $k_{r\theta}$ appear in separable form. If the problem conditions are also (r, θ) separable, then writing $U_{r\theta}(r, \theta) = U_r(r)U_\theta(\theta)$ in (2.111) gives

$$-\frac{1}{U_\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)U_\theta + \frac{k_\phi^2}{\sin^2\theta} = \frac{1}{U_r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)U_r + r^2k_0^2. \tag{2.113}$$

Both sides of (2.113) must again be constant, and if this constant is denoted by k_θ^2, the partial fields U_θ and U_r satisfy the reduced equations

$$\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{k_\phi^2}{\sin^2\theta} + k_\theta^2\right) U_\theta = 0, \tag{2.114a}$$

$$\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{k_\theta^2}{r^2} + k_0^2\right) U_r = 0. \tag{2.114b}$$

Here, k_θ plays the role of the dimensionless wavenumber associated with the dimensionless angular latitudinal coordinate θ.
Solutions $U_\phi(\phi)$ of (2.110) are the trigonometric functions (see (2.107))

$$U_\phi(\phi) = \sin\mu\phi, \quad \cos\mu\phi, \quad e^{i\mu\phi}, \quad e^{-i\mu\phi}, \quad \mu = k_\phi\,. \tag{2.115}$$

Solutions $U_\theta(\theta)$ of (2.114a) are the associated Legendre functions, with a linearly independent pair given by

$$\begin{aligned} U_\theta(\theta) &= P_\nu^{-\mu}(\cos\theta), \quad P_\nu^{-\mu}(-\cos\theta), \\ \nu &= \sqrt{k_\theta^2 + \frac{1}{4}} - \frac{1}{2}\,, \\ \text{i.e.,} \quad k_\theta^2 &= \nu(\nu+1) \end{aligned} \tag{2.116}$$

and μ defined in (2.115). Solutions $U_r(r)$ of (2.114a) are the spherical Bessel functions

$$U_r(r) = j_\nu(k_0 r)\,, \quad n_\nu(k_0 r)\,, \quad h_\nu^{(1)}(k_0 r)\,, \quad h_\nu^{(2)}(k_0 r)\,, \tag{2.117}$$

with the order ν defined in (2.116). Any two of these solutions are linearly independent. The spherical Bessel functions are related to the cylindrical Bessel functions by

$$z_\nu(k_0 r) = \sqrt{\frac{\pi}{2k_0 r}} Z_{\nu+1/2}(k_0 r)\,, \tag{2.118}$$

where z_ν stands for any of the spherical functions in (2.117) and $Z_{\nu+1/2}$ stands for the corresponding cylindrical Bessel function of argument $(k_0 r)$ and order $(\nu + 1/2)$ (see (2.124)).

V.3 The Scalar Wave Equation in Cylindrical Polar Coordinates

In cylindrical polar coordinates $(u, v, w) = (\rho, \phi, z)$ (see Figure 2.10), the Laplacian operator is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\,. \tag{2.119}$$

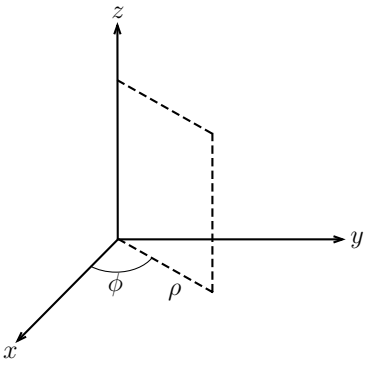
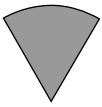
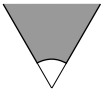
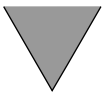
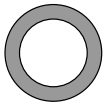
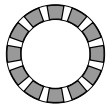


Fig. 2.10. Completely coordinate-separable configurations for cylindrical coordinates $(u, v, w) = (\rho, \phi, z)$. The domains $\rho = \rho_1, \rho = \rho_2$ correspond to cylindrical boundaries, $\phi = \phi_1, \phi = \phi_2$ and $z = z_1, z = z_2$ to plane boundaries.

Table 2.3. Some separable configurations in cylindrical coordinates. The figures show the ρ, ϕ plane.

$\rho_1 = 0$	$\rho_2 \to \infty$	$\rho_1 = 0,$ $\rho_2 \to \infty$	radial section	angular periodic
				

Writing $U(\rho, \phi, z) = U_z(z)U_{\rho\phi}(\rho, \phi)$ separates the z-dependence from the (ρ, ϕ)-dependence to yield, on substituting into (2.91) and proceeding as in (2.110) and (2.111), using k_z^2 as the separation parameter,

$$\left(\frac{d^2}{dz^2} + k_z^2\right) U_z(z) = 0\,, \tag{2.120}$$

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_0^2 - k_z^2\right) U_{\rho\phi}(\rho, \phi) = 0\,. \tag{2.121}$$

This decomposition applies to cylindrical boundaries of unchanged, but arbitrary, cross section along z, with non–separable boundary conditions (see Table 2.3). If the cross section is circular, the ρ- and ϕ-dependencies separate as well to yield, using k_ϕ^2 as the separation parameter,

$$\left(\frac{d^2}{d\phi^2} + k_\phi^2\right) U_\phi(\phi) = 0\,, \tag{2.122}$$

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}+k_\rho^2\right)U_\rho(\rho)=0\,, \tag{2.123}$$

where

$$k_\rho^2=k_0^2-k_z^2-\frac{k_\phi^2}{\rho^2} \tag{2.124}$$

is the corresponding dispersion relation. Solutions $U_z(z)$ of (2.120) and $U_\phi(\phi)$ of (2.122) are trigonometric functions as in (2.107) and (2.115), respectively. Solutions $U_\rho(\rho)$ of (2.123) are cylindrical Bessel functions

$$U_\rho(\rho)=J_\mu(\kappa\rho)\,,\quad N_\mu(\kappa\rho)\,,\quad H_\mu^{(1)}(\kappa\rho)\,,\quad H_\mu^{(2)}(\kappa\rho)\,,\qquad \kappa=\sqrt{k_0^2-k_z^2}\,, \tag{2.125}$$

any two of which are linearly independent. Here, $\mu=k_\phi$.
Table 2.3 summarizes the boundary configurations which allow a solution of the scalar wave equation by separation of variables. Figures 2.8, 2.9 and 2.10 show the corresponding domain configurations in Cartesian, spherical and cylindrical coordinate systems, respectively.

VI Sturm–Liouville Problems

VI.1 Source–Free Solutions: Eigenvalue Problem

Formulation

The reduced one-dimensional differential equations given by (2.105a)-(2.105c), (2.110), (2.114a), (2.114b), (2.120), (2.121), (2.122) and (2.123) all are special cases of the generic form

$$L_\alpha(u)f_\alpha(u)=0, \tag{2.126}$$

$$L_\alpha(u)\equiv\left[\frac{d}{du}\left(p(u)\frac{d}{du}\right)-q(u)+\lambda_\alpha w(u)\right]\,, \tag{2.127}$$

where p, q and the weight function w are positive real functions of u, $f_\alpha(u)$ is the wave function, and λ_α is the separation parameter. Equation (2.127) is a homogeneous (source–free) *Sturm–Liouville* (SL) problem, [17, p.719] and $L(u)$ is the *Sturm–Liouville operator*, defined, in general, for arbitrary λ, as

$$L(u,\lambda)\equiv\left[\frac{d}{du}\left(p(u)\frac{d}{du}\right)-q(u)+\lambda w(u)\right]\,. \tag{2.128}$$

(2.127) is to be solved on the interval $u_1\le u\le u_2$, subject to the linear homogeneous boundary conditions at the end points u_1 and u_2 (see (2.94)),

$$f_\alpha(u_{1,2})+\gamma_{1,2}\left.\frac{df_\alpha}{du}\right|_{u_{1,2}}=0, \tag{2.129}$$

where $\gamma_{1,2}$ are constants. As already noted in SectionV, a solution $f_\alpha(u)$ is called an *eigenfunction*, and the constant λ_α associated with $f_\alpha(u)$ is the corresponding *eigenvalue*. In general there will be a set of eigenfunction-eigenvalue pairs $\{(f_\alpha, \lambda_\alpha)\}$ which satisfy (2.127) and the boundary conditions in 2.129. Note that here and in the mathematical sections that follow, the *spectral* parameter λ in (2.128) plays a general role which, in the context of the wave equation, is equivalent to the squared wavenumber k^2. For the *eigenvalue problem* in (2.126), $\lambda \to \lambda_\alpha$ and $L(u, \lambda) \to L_\alpha(u)$.

Adjointness Properties

Before proceeding further, we demonstrate that the Sturm–Liouville (SL) operator $L(u)$ in (2.128) is *self-adjoint*; i.e., subject to the boundary conditions in (2.129), with f_α replaced by $F(u)$, $L(u)$ exhibits the adjointness property (suppressing the λ dependence)

$$\langle \bar{F}, LF \rangle \equiv \int_{u_1}^{u_2} \bar{F} LF \, du = \int_{u_1}^{u_2} F L\bar{F} \, du \equiv \langle F, L\bar{F} \rangle \,. \tag{2.130}$$

Equation (2.130) states that in the domain $u_1 \le u \le u_2$ of the operator $L(u)$ with the boundary conditions as in (2.129), the *inner product* $\langle \bar{F}, LF \rangle$ as defined on the left-hand side of (2.130) is equal to $\langle F, L\bar{F} \rangle$ on the right-hand side. The function $\bar{F}(a)$ is said to be adjoint to $F(a)$. Thus, the L-operation in the inner product is commutative. To prove (2.130), we construct

$$F L\bar{F} = F \left(\frac{d}{du} p \frac{d}{du} - q + \lambda w \right) \bar{F} \,, \tag{2.131}$$

$$\bar{F} LF = \bar{F} \left(\frac{d}{du} p \frac{d}{du} - q + \lambda w \right) F \,, \tag{2.132}$$

whence

$$\begin{aligned} F L\bar{F} - \bar{F} LF &= F \frac{d}{du} p \frac{d\bar{F}}{du} - \bar{F} \frac{d}{du} p \frac{dF}{du} \\ &= \frac{d}{du} \left[p \left(F\bar{F}' - \bar{F}F' \right) \right] \,. \end{aligned} \tag{2.133}$$

Here and hereafter, $F(u)$ and $\bar{F}(u)$ are two different twice–differentiable functions of u with a prime denoting the derivative with respect to u. Integrating both sides of (2.133) between the limits u_1 and u_2 yields

$$\int_{u_1}^{u_2} du \, (F L\bar{F} - \bar{F} LF) = \left[p \left(F\bar{F}' - \bar{F}F' \right) \right]_{u_1}^{u_2} \,. \tag{2.134}$$

The bracketed term on the right-hand side of (2.134),

$$W(\bar{F}, F) \equiv p\,(F\bar{F}' - \bar{F}F') = p\,\det\begin{vmatrix} F & F' \\ \bar{F} & \bar{F}' \end{vmatrix} \tag{2.135}$$

is the λ–dependent *Wronskian* which plays an important role in the theory that follows (see (2.186) - (2.196)). Subject to the boundary conditions in (2.129), the Wronskian vanishes, thereby establishing (2.130). Vanishing of the Wronskian at the boundary is confirmed by noting that, in view of (2.129),

$$F_1' = -\frac{F_1}{\gamma_1}\,, \qquad \bar{F}_1' = -\frac{\bar{F}_1}{\gamma_1}\,, \tag{2.136}$$

where $F_1 \equiv F(u_1)$, $\bar{F}_1 \equiv \bar{F}(u_1)$. The same holds for u_2.

Orthogonality, Completeness Relation, and Eigenfunction Expansions

In view of the adjointness property in (2.130), the eigenfunctions f_α satisfy an orthogonality property which can be derived as follows. Equation (2.127) is written for an eigenfunction-eigenvalue pair $(f_\alpha, \lambda_\alpha)$ and for a different eigenfunction-eigenvalue pair (f_β, λ_β). Proceeding as in (2.131) to (2.133), the f_α–equation is multiplied by f_β^*, where the asterisk denotes the complex conjugate, and the complex conjugate of the f_β–equation is multiplied by f_α. The resulting equations are subtracted to obtain

$$\frac{d}{du}W(f_\alpha, f_\beta^*) + (\lambda_\alpha - \lambda_\beta^*)wf_\alpha f_\beta^* = 0\,. \tag{2.137}$$

(2.137) is now integrated with respect to u between u_1 and u_2 to give

$$(\lambda_\alpha - \lambda_\beta^*)\int_{u_1}^{u_2} wf_\alpha f_\beta^*\,du = 0\,, \tag{2.138}$$

since the endpoint contribution vanishes via (2.136). Therefore it follows that

$$\int_{u_1}^{u_2} f_\alpha f_\beta^* w\,du = 0, \quad \alpha \neq \beta\,. \tag{2.139}$$

If $\lambda_\beta = \lambda_\alpha$, then from (2.138),

$$(\lambda_\alpha - \lambda_\alpha^*)\int_{u_1}^{u_2} |f_\alpha|^2 w\,du = 0\,. \tag{2.140}$$

Since w is positive and the trivial eigenfunction $f_\alpha = 0$ is not considered, the integral is nonzero. Thus,

$$\lambda_\alpha = \lambda_\alpha^*, \text{ i.e., } \textit{the eigenvalues are real}. \tag{2.141}$$

Returning to (2.139), these considerations imply that the integral vanishes for $\lambda_\alpha \neq \lambda_\beta$. Since the integral in (2.139) represents, in the function space, the

inner product of the functions f_α and f^*_β, (see the comments following (2.130)), vanishing of the integral implies that eigenfunctions corresponding to distinct eigenvalues are *orthogonal* with respect to the weighting function w. It can also be shown that the eigenvalues λ_α are non-negative when γ_1 is negative real and γ_2 is positive real. In (2.131), let $\bar{F} = f^*_\alpha$, $F = f_\alpha$ and $\lambda = \lambda_\alpha$, and equate the expression to zero in view of (2.127). Integrating over the interval from u_1 to u_2 yields

$$\int_{u_1}^{u_2} du\, f_\alpha \frac{d}{du} p \frac{df^*_\alpha}{du} - \int_{u_1}^{u_2} du\, q\, |f_\alpha|^2 + \lambda_\alpha \int_{u_1}^{u_2} du\, w\, |f_\alpha|^2 = 0\,. \tag{2.142}$$

Integrating by parts, the first integral in (2.142) becomes

$$\int_{u_1}^{u_2} du\, f_\alpha \frac{d}{du}\left(p f^{*\prime}_\alpha\right) = p f_\alpha f^{*\prime}_\alpha \Big|_{u_1}^{u_2} - \int_{u_1}^{u_2} du\, p\, \left|f^{*\prime}_\alpha\right|^2 . \tag{2.143}$$

For the boundary conditions in (2.129), with $\gamma_1 < 0$, $\gamma_2 > 0$ and real, the endpoint contributions at u_1 and u_2 can be written as $(-p|f_\alpha|^2|\gamma_1|^{-1})$ and $(-p|f_\alpha|^2\gamma_2^{-1})$, respectively. Thus, since $p > 0$, the left-hand side of (2.143), and therefore the first term in (2.142), is negative. The second term in (2.142) is also negative since $q > 0$, whereas the integral multiplying λ_α equals unity (see (2.144)). Thus, to satisfy (2.142), λ_α must be non-negative under the stated conditions.
It is convenient to normalize the eigenfunctions (multiply by an appropriate constant) so that

$$\int_{u_1}^{u_2} |f_\alpha|^2 w\, du = 1\,. \tag{2.144}$$

This renders the set $\{f_\alpha\}$ orthonormal. Equations (2.139) and (2.144) can then be written as the single expression

$$\int_{u_1}^{u_2} f_\alpha f^*_\beta w\, du = \delta_{\alpha\beta}\,, \tag{2.145}$$

with the Kronecker delta $\delta_{\alpha\beta}$ defined as $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$. To evaluate the normalizing integral on the left side of (2.144) (i.e., *before* the normalization implied by the right side), we return to (2.137) but replace the eigenfunction $f^*_\beta(u) \equiv f^*_\beta(u, \lambda_\beta)$ by a function $f^*(u, \lambda)$ satisfying $L(u)f(u, \lambda) = 0$ (see (2.128)) for any specified $\lambda \neq \lambda_\alpha$. Integrating the resulting modification of (2.137) between the limits u_1 and u_2 yields

$$(\lambda_\alpha - \lambda) \int_{u_1}^{u_2} du\, w f_\alpha f^* + W(f_\alpha, f^*)\Big|_{u_1}^{u_2} = 0\,, \tag{2.146}$$

which can be re-arranged as follows,

$$\int_{u_1}^{u_2} du\, w f_\alpha f^* = \frac{W(f_\alpha, f^*)\Big|_{u_1}^{u_2}}{\lambda - \lambda_\alpha}\,. \tag{2.147}$$

Now take the limit $\lambda \to \lambda_\alpha$, whence $f^* \to f^*_\alpha$. The limiting form of the Wronskian vanishes, and the resulting indeterminate right-hand side can be evaluated by L'Hospital's rule, i.e., taking $[(dW/d\lambda)/(d/d\lambda)(\lambda - \lambda_\alpha)]_{\lambda=\lambda_\alpha}$, to obtain

$$\int_{u_1}^{u_2} du\, w(u)|f_\alpha(u)|^2 = \left[p\left(\frac{d}{d\lambda}f^*(u,\lambda)\bigg|_{\lambda_\alpha}\frac{d}{du}f_\alpha(u,\lambda_\alpha) - f_\alpha(u,\lambda_\alpha)\frac{d^2}{d\lambda du}\, f^*(u,\lambda)|_{\lambda_\alpha}\right)\right]_{u_1}^{u_2} \quad (2.148)$$

with

$$f(u,\lambda) \equiv f(\sqrt{\lambda}u)\,, \qquad f_\alpha(u,\lambda_\alpha) \equiv f(\sqrt{\lambda_\alpha}u)\,. \quad (2.149)$$

In (2.149), the functional dependencies of f and f_α are shown explicitly. The normalized eigenfunctions f_α defined in (2.144) are now obtained by writing $f_\alpha = B^{-1/2}\overline{f}_\alpha$, where $\overline{f}_\alpha$ is the unnormalized form, and $B = \{\ldots\}^{1/2}$, with $\{\ldots\}$ representing the expression on the right-hand side of (2.148) with (2.149).
Assuming that the eigenfunction set $\{f_\alpha(u)\}$ is complete, any "representable" function $F(u)$ can be expanded formally as

$$F(u) = \sum_\alpha A_\alpha f_\alpha(u)\,. \quad (2.150)$$

Here, "representable" implies that the expansion converges. Multiplying both sides of (2.150) by $w(u)f^*_\beta(u)$, integrating over the (u_1, u_2) interval, invoking the orthonormality condition given by (2.145) and switching back to the index α, it follows that

$$A_\alpha = \int_{u_1}^{u_2} f^*_\alpha F w\, du\,. \quad (2.151)$$

Substitution of (2.151) into (2.150) gives, upon interchange of the orders of summation and integration,

$$F(u) = \int_{u_1}^{u_2} du' \,\{w(u')\sum_\alpha f_\alpha(u)f^*_\alpha(u')\}F(u')\,, \quad (2.152)$$

which implies that

$$w(u')\sum_\alpha f_\alpha(u)f^*_\alpha(u') = \delta(u-u') \quad (2.153)$$

or

$$\frac{\delta(u-u')}{w(u')} = \sum_\alpha f_\alpha(u)f^*_\alpha(u')\,. \quad (2.154)$$

Equation (2.154) expresses the *completeness statement* in compact symbolic form. The expansion of the weighted delta function in terms of the eigenfunctions implies that the set of eigenfunctions is complete, because any function $F(u)$ can be expressed by using the delta function property

$$F(u) = \int_{u_1}^{u_2} F(u')\delta(u - u')\, du' \,. \tag{2.155}$$

Thus, to *apply* (2.154), the previous steps are *reversed* as follows. Each side of (2.154) is multiplied by $w(u')F(u')$ and integrated with respect to the variable u' from u_1 to u_2, giving

$$F(u) = \sum_\alpha f_\alpha(u) \int_{u_1}^{u_2} du'\; w(u')F(u')f_\alpha^*(u') \,. \tag{2.156}$$

Equation (2.156) is of the form

$$F(u) = \sum_\alpha A_\alpha f_\alpha(u) \,, \tag{2.157}$$

with the coefficients A_α given by

$$A_\alpha = \int_{u_1}^{u_2} du'\; w(u')F(u')f_\alpha^*(u') \,. \tag{2.158}$$

The implied orthonormality of the eigenfunctions is verified by setting $F(u)$ in (2.156) equal to the eigenfunction $f_\beta(u)$, giving

$$f_\beta(u) = \sum_\alpha f_\alpha(u) \int_{u_1}^{u_2} du'\; w(u')f_\beta(u')f_\alpha^*(u') \,. \tag{2.159}$$

To satisfy (2.159) one is led to (2.145).

Large $|\lambda|$ Behavior of the Source-Free Solutions

The source-free solutions $f(u)$ of the SL equation (see (2.127)) reduce to trigonometric functions for large values of λ, and when $w = p$. To demonstrate this behavior, we reduce the $L(u)$ operator to its normal form (without the first derivative d/du) by the transformation

$$f(u) = p^{-1/2}\hat{f}(u) \,, \tag{2.160}$$

which changes $L(u)f(u) = 0$ to the normalized equation

$$\left[\frac{d^2}{du^2} + h(u)\right] \hat{f}(u) = 0 \,, \tag{2.161}$$

where

$$h(u) = \frac{\lambda w}{p} - \frac{q}{p} - p^{-1/2}\frac{d^2}{du^2}p^{1/2} \,. \tag{2.162}$$

For large λ, the $(\lambda w/p)$ term dominates, and when $w = p$, (2.161) reduces to

$$\left(\frac{d^2}{du^2}+\lambda\right)\hat{f}(u)\sim 0,\quad |\lambda|\gg 1,\quad w=p \tag{2.163}$$

Thus, the large-$|\lambda|$ solutions for $f(u)$ become

$$f(u)\sim p^{-1/2}\cdot\left(\sin\sqrt{\lambda}u\,,\quad \cos\sqrt{\lambda}u\,,\quad e^{\mp j\sqrt{\lambda}u}\right),\quad |\lambda|\gg 1\,,\quad w=p\,. \tag{2.164}$$

For the eigenvalue problem, (2.164) applies with $f(u)\to f_\alpha(u)$, $\lambda\to\lambda_\alpha$, $\lambda_\alpha\gg 1$. For the Green's function problem in Section VI.2, (2.164) applies to the synthesizing homogeneous solutions $\overleftarrow{f(u)}$ and $\overrightarrow{f(u)}$.

VI.2 Source-Driven Solutions: Green's Function Problem

Properties of the Green's Function

The eigenvalue problem defined by (2.126) describes a one-dimensional physical system which is free or unforced. Problems in which forcing functions or sources exist are solved through the introduction of a Green's function. The one-dimensional Green's function $g(u,u';\lambda)$ satisfies equation

$$L(u)g(u,u';\lambda)\equiv\left[\frac{d}{du}p(u)\frac{d}{du}-q(u)+\lambda w(u)\right]g(u,u';\lambda)=-\delta(u-u') \tag{2.165}$$

over the interval $u_1\le(u,u')\le u_2$, with boundary conditions at $u=u_{1,2}$ of the form (cf. VI.1)

$$g(u_{1,2})+\gamma_{1,2}\left.\frac{dg}{du}\right|_{u_{1,2}}=0\,. \tag{2.166}$$

The right-hand side of (2.165) represents a u-domain point source at location $u=u'$. Here, $L(u)$ is the general Sturm-Liouville (SL) operator in (2.128), which is self-adjoint subject to the boundary conditions in (2.129). The parameter λ is now unrestricted and may range over the entire complex λ-plane, provided that $\lambda\ne\lambda_\alpha$. All eigenvalues $\lambda=\lambda_\alpha$ must be avoided because the source-free (2.165) has the eigensolutions $f_\alpha(u)$. Any eigensolution can be added to g and still satisfy (2.165) and (2.166), thereby rendering the resulting g non unique.

Reciprocity

The Green's function $g(u,u';\lambda)$ is symmetric in its dependence on u and u'. This can be shown by referring to (2.134), with $F=g(u,u';\lambda)$ and $\bar{F}=g(u,u'';\lambda)$, where u' and u'' are source points in the interval $u_1<(u',u'')<u_2$. Thus (omitting the λ-dependence),

$$\int_{u_1}^{u_2}du\,[g(u,u')L(u)g(u,u'')-g(u,u'')L(u)g(u,u')]$$
$$=\{p(u)\,[g(u,u')g'(u,u'')-g(u,u'')g'(u,u')]\}_{u_1}^{u_2}\,. \tag{2.167}$$

Since $L(u)g(u,\overline{u}) = -\delta(u-\overline{u})$ and $g'(u_{1,2},\overline{u}) = -\gamma_{1,2}^{-1}g(u_{1,2},\overline{u})$ (see (2.165) and (2.166)), the endpoint contribution vanishes (self-adjointness property) and the integral is reduced via the delta functions, yielding the result

$$g(u'',u';\lambda) = g(u',u'';\lambda)\,. \tag{2.168}$$

Thus, the self-adjoint Sturm-Liouville (SL) Green's function $g(u,u';\lambda)$ is unchanged, i.e. *reciprocal* when u and u' are interchanged at any two locations in the interval (u_1,u_2).

Synthesis of the General Initial-Boundary Value Problem

The general SL initial-boundary value problem is of the form

$$L(u)F(u) = S(u)\,, \qquad u_1 \le u \le u_2\,, \tag{2.169}$$

subject to the initial-boundary condition

$$F(u_{1,2}) + \gamma_{1,2}F'(u_{1,2}) = \overline{S}(u_{1,2})\,, \tag{2.170}$$

where $S(u)$ are interior sources while $\overline{S}(u_{1,2})$ are sources impressed at the boundaries of the domain. The solution for $F(u)$ can be synthesized in terms of the Green's function $g(u,u';\lambda)$ defined in (2.165) together with (2.166). Returning to the adjointness relation in (2.133), let $\bar{F} = g(u,u';\lambda)$ and let $F(u)$ represent the solution of (2.169) and (2.170). Thus, omitting the λ-dependence,

$$\begin{aligned}\int_{u_1}^{u_2} du\,[F(u)L(u)g(u,u') - g(u,u')L(u)F(u)] \\ = p(u_2)[F(u_2)g'(u_2,u') - g(u_2,u')F'(u_2)] \\ - p(u_1)[F(u_1)g'(u_1,u') - g(u_1,u')F'(u_1)]\,.\end{aligned} \tag{2.171}$$

Inside the integral in (2.171), referring to (2.165) and (2.169), Lg and LF are replaced by $-\delta(u-u')$ and $S(u)$, respectively. On the right-hand side of (2.171), referring to (2.166) and (2.170), we use

$$g'(u_{1,2}) = -\frac{g(u_{1,2})}{\gamma_{1,2}}\,, \qquad F'(u_{1,2}) = \frac{\overline{S}(u_{1,2}) - F(u_{1,2})}{\gamma_{1,2}}\,. \tag{2.172}$$

This reduces (2.171) to the expression

$$\begin{aligned}F(u') = -\int_{u_1}^{u_2} du\; g(u,u')S(u) + p(u_2)g(u_2,u')\overline{S}(u_2)/\gamma_2 \\ - p(u_1)g(u_1,u')\overline{S}(u_1)/\gamma_1\end{aligned} \tag{2.173}$$

where u' is any point in the closed interval $u_1 \le u' \le u_2$.
Since u and u' in $g(u,u';\lambda)$ represent the field (observation) point and source point, respectively, it is customary to integrate the Green's function over the primed coordinates. The necessary interchange of u and u' can be implemented in view of the reciprocity property in (2.168) in the form (restoring the λ-dependence)

$$F(u,\lambda) = -\int_{u_1}^{u_2} du'\; g(u,u';\lambda)S(u') + p(u_2)g(u,u_2;\lambda)\overline{S}(u_2)/\gamma_2 \\ - p(u_1)g(u,u_1;\lambda)\overline{S}(u_1)/\gamma_1\,. \tag{2.174}$$

Solution for the Green's Function

The Green's function $g(u,u';\lambda)$ can be evaluated directly. When $u \neq u'$, the Green's function satisfies the homogeneous equation obtained by setting the right-hand side of (2.165) equal to zero. Let $\overleftarrow{f}$ be a solution of the homogeneous equation which satisfies the boundary condition given by (2.166) at $u = u_1$, and let $\overrightarrow{f}$ be a solution of the homogeneous equation which satisfies the boundary condition given by (2.166) at $u = u_2$. The functions $\overleftarrow{f}$ and $\overrightarrow{f}$ can be constructed by superposition of any two linearly independent solutions $f^{(1)}$ and $f^{(2)}$ of the homogeneous (2.126) using the expressions

$$\overrightarrow{f}\,(u) = f^{(1)}(u) + \overrightarrow{\Gamma}\, f^{(2)}(u)\,, \tag{2.175}$$

$$\overleftarrow{f}\,(u) = \overleftarrow{\Gamma}\, f^{(1)}(u) + f^{(2)}(u)\,, \tag{2.176}$$

where

$$\overrightarrow{\Gamma} = -\frac{\left[f^{(1)}(u_2) + \gamma_2\left(\dfrac{df^{(1)}}{du}\right)_{u=u_2}\right]}{\left[f^{(2)}(u_2) + \gamma_2\left(\dfrac{df^{(2)}}{du}\right)_{u=u_2}\right]}\,, \tag{2.177}$$

$$\overleftarrow{\Gamma} = -\frac{\left[f^{(2)}(u_1) + \gamma_1\left(\dfrac{df^{(2)}}{du}\right)_{u=u_1}\right]}{\left[f^{(1)}(u_1) + \gamma_1\left(\dfrac{df^{(1)}}{du}\right)_{u=u_1}\right]}\,. \tag{2.178}$$

To obtain the expression for $\overrightarrow{\Gamma}$, note from (2.166) that

$$\overrightarrow{f} = f^{(1)} + \overrightarrow{\Gamma}\, f^{(2)} = -\gamma_2 \frac{d\,\overrightarrow{f}}{du} = -\gamma_2\left[\frac{df^{(1)}}{du} + \overrightarrow{\Gamma}\,\frac{df^{(2)}}{du}\right]\,, \qquad u = u_2\,. \tag{2.179}$$

The second equality follows from (2.129) applied to $\vec{f}$, whereas the third equality implements $d\vec{f}/du$ via (2.175). Solving the first and third equalities for $\vec{\Gamma}$ yields (2.177). A similar calculation gives the expression for $\overleftarrow{\Gamma}$ in (2.178).
Next, it is noted that g is continuous at $u = u'$ but has a discontinuous slope (first derivative) at $u = u'$, consistent with the recognition that the delta function singularity at $u = u'$ in (2.165) is generated by the highest derivative, (d^2g/du^2). Implementing continuity at u', with discontinuous slope, suggests the expression

$$g(u,u';\lambda) = \begin{cases} \bar{C}\,\overleftarrow{f}\,(u)\,\vec{f}\,(u'), & u < u' \\ \bar{C}\,\overleftarrow{f}\,(u')\,\vec{f}\,(u), & u > u' \end{cases}, \tag{2.180}$$

which also satisfies both prescribed boundary conditions, as well as (2.165) for all $u \neq u'$. With the notation

$$u_> = \begin{cases} u, & u > u' \\ u', & u < u' \end{cases}, \tag{2.181}$$

$$u_< = \begin{cases} u, & u < u' \\ u', & u > u', \end{cases} \tag{2.182}$$

(2.180) can be written as

$$g(u,u';\lambda) = \bar{C}\,\vec{f}\,(u_>)\,\overleftarrow{f}\,(u_<)\,. \tag{2.183}$$

To determine the constant $\bar{C}$ we integrate (2.165) over the interval $u' - \epsilon < u < u' + \epsilon$, $\epsilon > 0$, and then allow $\epsilon \to 0$. Since g is bounded at u' and q, w and p have no singularities at $u = u'$, the contribution from the second and third terms in $L(u)$ vanishes in the limit. The result is

$$p\,\frac{dg}{du}\bigg|_{u'-\epsilon}^{u'+\epsilon} = -1\,, \tag{2.184}$$

which after using (2.183) gives

$$\bar{C} = -\frac{1}{W(\vec{f},\overleftarrow{f})}\,, \tag{2.185}$$

with the Wronskian $W(\vec{f},\overleftarrow{f})$ defined as in (2.135),

$$W(\vec{f},\overleftarrow{f}) = p(u')\left[\overleftarrow{f}\,\frac{d\,\vec{f}}{du} - \vec{f}\,\frac{d\,\overleftarrow{f}}{du}\right]_{u=u'}. \tag{2.186}$$

Using (2.183) and (2.185), the Green's function $g(u, u', \lambda)$ can now be written as

$$g(u, u'; \lambda) = -\frac{\overrightarrow{f}\,(u_>)\,\overleftarrow{f}\,(u_<)}{W(\overrightarrow{f}, \overleftarrow{f})}\,. \tag{2.187}$$

The Wronskian $W(\overrightarrow{f}, \overleftarrow{f})$ has the following properties (recall that the λ–dependence has been suppressed throughout):

- W is a λ–dependent constant, independent of u'.
- $W \neq 0$ if $\overrightarrow{f}$ and $\overleftarrow{f}$ are linearly independent functions over the interval $u_1 < u < u_2$.

To show that W is independent of u', the equation

$$\left[\frac{d}{du}p\frac{d}{du} - q + \lambda w\right]\overrightarrow{f} = 0 \tag{2.188}$$

is multiplied by $\overleftarrow{f}$, and the equation

$$\left[\frac{d}{du}p\frac{d}{du} - q + \lambda w\right]\overleftarrow{f} = 0 \tag{2.189}$$

is multiplied by $\overrightarrow{f}$. The resulting equations are subtracted to give

$$\overleftarrow{f}\,\frac{d}{du}p\frac{d\overrightarrow{f}}{du} - \overrightarrow{f}\,\frac{d}{du}p\frac{d\overleftarrow{f}}{du} = 0, \tag{2.190}$$

which is equivalent to

$$\frac{d}{du}\left[p\left(\overleftarrow{f}\,\frac{d\overrightarrow{f}}{du} - \overrightarrow{f}\,\frac{d\overleftarrow{f}}{du}\right)\right] = 0, \tag{2.191}$$

or

$$\frac{d}{du}\left(W(\overrightarrow{f}, \overleftarrow{f})\right) = 0. \tag{2.192}$$

Equation (2.192) states that W is independent of u, i.e., W equals a λ–dependent constant.

To show that W is nonzero if $\overrightarrow{f}$ and $\overleftarrow{f}$ are linearly independent, it will be shown conversely that $W = 0$ implies linear dependence, i.e., that $\overrightarrow{f}$ is then a constant multiple of $\overleftarrow{f}$. If $W = 0$, (2.186) gives

$$\overleftarrow{f}\,\frac{d\overrightarrow{f}}{du} = \overrightarrow{f}\,\frac{d\overleftarrow{f}}{du} \tag{2.193}$$

or

$$\frac{1}{\overleftarrow{f}}\frac{d\overleftarrow{f}}{du} = \frac{1}{\overrightarrow{f}}\frac{d\overleftarrow{f}}{du}. \tag{2.194}$$

Integration of (2.194) gives

$$\ln \frac{\overleftarrow{f}}{\overrightarrow{f}} = \bar{c} = \text{const.} \tag{2.195}$$

or

$$\overleftarrow{f} = \bar{c}' \overrightarrow{f} \tag{2.196}$$

which confirms that $W = 0$ implies linear dependence of $\overleftarrow{f}$ and $\overrightarrow{f}$. (2.196) implies furthermore that $\overrightarrow{f}$ or $\overleftarrow{f}$ satisfy *both* boundary conditions at u_1 and u_2, in addition to satisfying the source-free (2.165); i.e., $\overrightarrow{f}$ or $\overleftarrow{f}$ are eigensolutions $f_\alpha(u)$ with forbidden eigenvalues $\lambda = \lambda_\alpha$. This is in accord with the result in (2.136). Evidently, the solution for g in (2.187) becomes invalid when $W = 0$.

Large $|\lambda|$ Behavior of the Spectral Green's Function

In the investigation that follows, emphasis will be placed on the behavior of the Sturm–Liouville Green's function *throughout* the complex spectral $|\lambda|$–plane. For large values of λ, and when $w = p$, the synthesizing homogeneous solutions $\overrightarrow{f}$ and $\overleftarrow{f}$ in (2.187) reduce to trigonometric functions, as shown in Section VI.1, (2.164). The formal solution for $g(u, u'; \lambda)$ in (2.187) reduces accordingly in the large-λ range. Consider the case where $g = 0$ at $u_1 = 0$ (no loss of generality) and at u_2. The synthesizing solutions of (2.163) are $\overleftarrow{f}\ (u_<) = \sin\left(\sqrt{\lambda}u_<\right)$, $\overrightarrow{f}\ (u_>) = \sin\left[\sqrt{\lambda}(u_2 - u_>)\right]$, whereas the u–independent Wronskian is given by $w = \sqrt{\lambda}\sin(\sqrt{\lambda}u_2)$. For $|\ \lambda\ | \gg 1$, $|\ \Im\lambda\ | \neq 0$, retaining only the dominant (growing) exponentials, one obtains

$$g(u, u'; \lambda) \to \frac{e^{|\Im\sqrt{\lambda}|u_<} e^{|\Im\sqrt{\lambda}|(u_2 - u_>)}}{\sqrt{\lambda} e^{|\Im\sqrt{\lambda}|u_2}} \to \frac{e^{|\Im\sqrt{\lambda}|(u_< - u_>)}}{\sqrt{\lambda}}$$

$$\frac{e^{-|\Im\sqrt{\lambda}|\ |u-u'|}}{\sqrt{\lambda}}, \qquad |\Im\sqrt{\lambda}| \neq 0. \tag{2.197}$$

which decays exponentially at infinity in the complex λ–plane, and therefore yields no contribution when integrated over a circular contour at $|\ \lambda\ | \to \infty$.

VI.3 Relation Between the Spectral (Characteristic) Green's Function and the Eigenvalue Problems

In this section, it is shown how the complete orthonormal set of eigenfunctions in Section VI.1 can be constructed from knowledge of the spectral Green's function

in Section VI.2. In fact, it will become apparent that the Green's function route furnishes a far more general approach to the representation of wavefields.

To establish the Green's function–eigenfunction connection in qualitative physical terms, it is recalled that the Green's function represents the field due to a localized source, with the parameter λ in (2.165) proportional to the square of the spatial wavenumber (i.e, the squared frequency of the spatial oscillations). If the spatial frequency of the source is varied between 0 and ∞ in a lossless environment, the Green's function will exhibit amplitude singularities at each spatial frequency which corresponds to an eigenvalue λ_α; since λ_α identifies a *source-free* solution, the *driven* response at λ_α is unbounded. Therefore, the *totality* of singularities in the Green's function generates the *complete eigenfunction set*.

We begin by *assuming* that the set of eigenfunctions $\{f_\alpha\}$ is complete. The Green's function $g(u, u'; \lambda)$ may therefore be expanded in a series of eigenfunctions with coefficients g_α as

$$g(u, u'; \lambda) = \sum_\alpha g_\alpha(u'; \lambda) f_\alpha(u). \tag{2.198}$$

Applying the operator $L(u)$ in (2.128) to both sides of (2.198), and using (2.154) and (2.165), one obtains

$$-\sum_\alpha w(u) f_\alpha(u) f_\alpha^*(u') = \sum_\alpha g_\alpha(u'; \lambda) \left[\frac{d}{du} p(u) \frac{d}{du} - q(u) + \lambda w(u)\right] f_\alpha(u)\,. \tag{2.199}$$

Via (2.127), this can be written as

$$-\sum_\alpha w(u) f_\alpha(u) f_\alpha^*(u') = \sum_\alpha g_\alpha(u'; \lambda)(\lambda - \lambda_\alpha) w(u) f_\alpha(u)\,. \tag{2.200}$$

Equating the coefficients of the orthogonal functions $f_\alpha(u)$ on both sides of (2.200) yields

$$g_\alpha(u'; \lambda) = -\frac{f_\alpha^*(u')}{(\lambda - \lambda_\alpha)}. \tag{2.201}$$

Substitution of these coefficients into (2.199) gives

$$g(u, u'; \lambda) = -\sum_\alpha \frac{f_\alpha(u) f_\alpha^*(u')}{(\lambda - \lambda_\alpha)}, \tag{2.202}$$

which is an expression for the Green's function $g(u, u'; \lambda)$ in terms of the eigenfunctions of the homogeneous problem defined by (2.127).

The result in (2.202) can be used to derive a generalized completeness relation. Both sides of (2.202) are integrated in the complex-λ plane over a contour C which encloses in the counterclockwise sense all the pole singularities at the eigenvalues λ_α. The contour C is deformed into the contour C' consisting of small semicircles C_α centered at the poles λ_α and of line segments C'' which approach the real axis,

as shown in Figure 2.11. The contributions to the integral due to the oppositely directed C''-segments along the real axis cancel, and each pair of semicircular arcs C_α contributes a residue at the corresponding pole as the radius of the semicircles approaches zero. Therefore, by the residue theorem, the line integral of g is

$$\begin{aligned}\frac{1}{2\pi j}\oint_C g(u,u';\lambda)\,d\lambda &= \frac{1}{2\pi j}\oint_{C'} g(u,u';\lambda)\,d\lambda \\ &= -\sum_\alpha f_\alpha(u)f_\alpha^*(u')\left(\frac{1}{2\pi j}\oint_{C_\alpha}\frac{d\lambda}{\lambda-\lambda_\alpha}\right) \\ &= \sum_\alpha f_\alpha(u)f_\alpha^*(u') \\ &= \frac{\delta(u-u')}{w(u')}. \end{aligned} \tag{2.203}$$

Equation (2.203) establishes the Green's function-eigenfunction connection in the completeness relation, which now takes the form

$$\frac{\delta(u-u')}{w(u')} = \frac{1}{2\pi j}\oint_C g(u,u';\lambda)\,d\lambda. \tag{2.204}$$

The contour C in Figure 2.11 can be terminated *anywhere* at $|\lambda|\to\infty$ because, as shown in SectionVI.2, g converges exponentially at $|\lambda|\to\infty$, so that contour segments at infinity do not contribute to the integral. The contour C must, however, have all of the singularities of g on one side. Because of this resolving connection with the eigenvalue problem, the spectral Green's function is also referred to as the characteristic (resolvent) Green's function. Although demonstrated here only for discrete eigenspectra (poles λ_α in the complex λ–plane), the characteristic Green's function procedure in (2.204) remains valid for continuous eigenspectra (typically in unbounded regions) which give rise to branch points in the then multi–sheeted complex λ–plane.

The importance of (2.204) resides in the fact that $g(u,u';\lambda)$ can be evaluated *directly* as in (2.187) of Section VI.2. Thus, (2.204) furnishes a generalized completeness relation for representing an arbitrary function $F(u)$. Such a representation is obtained, as before, by multiplying both sides of (2.204) by $w(u')F(u')$ and integrating over u' between the limits u_1 and u_2, giving

$$F(u) = \frac{1}{2\pi j}\oint_C d\lambda\, g(u,u';\lambda)\left\{\int_{u_1}^{u_2} du'\, w(u')F(u')\right\}. \tag{2.205}$$

VII Radiation and Edge Condition

VII.1 Radiation Condition

For an unbounded region it is necessary to specify the field behavior on a surface at infinity. By assuming that all sources are contained in a finite region, only

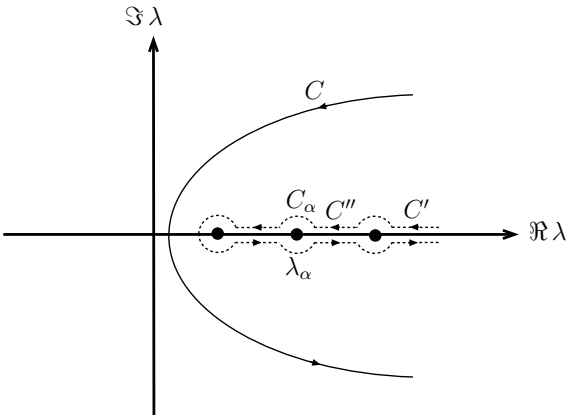


Fig. 2.11. Integration contours in the complex λ-plane.

outgoing waves can be present at large distances from the sources. In other words, the field behavior at large distances from the sources must meet the physical requirement that energy travel away from the source region. This requirement is the Sommerfeld "*radiation condition*" and constitutes a boundary condition on the surface at infinity. It assumes different expressions when dealing with 2D– or 3D–regions.

3D region.

Let A denote any field component transverse to the radial distance r. The transverse field of a spherically diverging wave in a homogeneous isotropic medium decays as $1/r$ at large distances r from the source region; locally the spherical wave behaves like a plane wave traveling in the *outward* r direction. As such (for an implied $e^{j\omega t}$ time dependence) each field component transverse to r must behave like $\exp(-jkr)/r$, where $k = \omega/c$ is the free–space wavenumber and c is the speed of light in vacuum. This requirement may be phrased mathematically as

$$\lim_{r\to\infty} r\left(\frac{\partial A}{\partial r} + jkA\right) = 0\,. \tag{2.206}$$

Observe that the above boundary condition is not self–adjoint in the Hermitian sense. The adjoint boundary condition would be

$$\lim_{r\to\infty} r\left(\frac{\partial A}{\partial r} - jkA\right) = 0\,. \tag{2.207}$$

corresponding to waves impinging from infinity.

2D region.

Let ρ denote the radial variable in the transverse plane, perpendicular to the direction of uniformity. The transverse to ρ field component A in a cylindrically diverging wave in a homogeneous isotropic medium decays as $1/\sqrt{\rho}$ at large distances ρ from the source region; locally A behaves like a plane wave travelling in the outward ρ direction. As such, each field component transverse to ρ must behave like $\exp(-jk\rho)/\sqrt{\rho}$ This requirement may be phrased mathematically as

$$\lim_{\rho\to\infty} \sqrt{\rho}\left(\frac{\partial A}{\partial \rho} + jkA\right) = 0\,. \tag{2.208}$$

The above equations apply to non–dissipative media. When the media are slightly lossy one may use the simpler requirement that all fields excited by sources in a finite region should vanish at infinity (i.e. k has a small *negative* imaginary part).

VII.2 Edge Condition in Two Dimensions

It is well recognized that, in many cases, boundary and radiation conditions alone are not sufficient to determine the solution uniquely [15, p.385], since it is possible to construct several different fields which satisfy these conditions [18]. As an example, let us consider a metallic wedge as shown in Figure 2.12, which we assume with no changes in the z–direction and separability in cylindrical coordinates. Assume a field E_0 which satisfies boundary and radiation conditions.

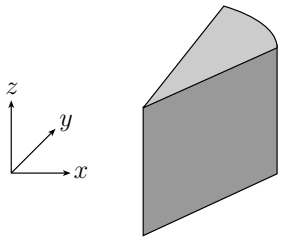


Fig. 2.12. Three-dimensional view of a perfectly conducting wedge extending from $\phi = 0$ to $\phi = \phi_2$ with no variations in the z–direction.

Now consider a field

$$E_z = E_0 + \bar{C}\, J_\nu(k\rho)\sin\left[\nu(\phi - \phi_2)\right] \tag{2.209}$$

which satisfies the Helmholtz equation (the scalar wave equation) for any value of $\bar{C}$, complies with the radiation condition and has the same boundary behavior

as E_0 since $E_z = 0$ when $\phi = 0, \phi_2$. However, an infinite set of solutions can be generated by giving different values to $\bar{C}$ [14, pp.531-532]. Therefore, it is necessary to apply an additional constraint in order to achieve a unique solution i.e. an edge condition [18–20]. We start by noting that *the electromagnetic energy density must be integrable over any finite domain* even if this domain contains singularities of the electromagnetic field. Differently stated, *the electromagnetic energy in any finite domain must be finite*. The sum of the electric and magnetic energies in a small volume V surrounding the edge is [12, p. 24]

$$\frac{1}{2}\int_V \left(\varepsilon \boldsymbol{E}\cdot\boldsymbol{E}^* + \mu \boldsymbol{H}\cdot\boldsymbol{H}^*\right)\rho\, d\phi\, d\rho\, dz \tag{2.210}$$

In the vicinity of the edge the fields can be expressed as a power series in ρ; this series will have a dominant term ρ^μ where μ may be negative. Therefore, as ρ approaches zero the dominant term of the field components of (E, H) appearing in (2.210) behaves like $\rho^{2\mu}$, and the entire integrand behaves like $\rho^{2\mu+1}$. Integration over ρ yields $\rho^{2(\mu+1)}$ which is bounded for $\mu > -1$. The actual degree of singularity that a field experiences near the edge is dependent on the wedge configuration. It is also noted that the field singularity does not depend on frequency since in the proximity of the edge, spatial derivatives of the fields are much larger than time derivatives, so that the latter can be neglected in Maxwell's equations (quasi–static regime).
The exact knowledge of the type of field singularity near the edge is of considerable importance for numerical applications. The reader may find more information on cases of practical importance in reference [21].

VIII Reciprocity and Field Equivalence Principles

VIII.1 Reaction in Electromagnetic Theory

The *reaction concept* in electromagnetic theory has been introduced in [22] in order to find a fundamental observable representing measurements which can be performed practically. For example, if we want to measure the field radiated by some source of electromagnetic energy, we may use an antenna probe and observe the signal received at terminals at the point of observation. However, the latter measurement does not provide the field *just* at the observation point, but it measures the effect of the field over a small, but finite, region. To take this fact into account, it is convenient to define the *reaction*, i.e. the coupling between the field that we want to measure and the antenna that we are using.
Consider a monochromatic source of electromagnetic field, denoted by a, consisting of electric and magnetic currents $\boldsymbol{J}_a$ and $\boldsymbol{M}_a$, respectively, and producing the field $\boldsymbol{E}_a$, $\boldsymbol{H}_a$. Similarly, consider also a source b of electric and magnetic currents $\boldsymbol{J}_b$ and $\boldsymbol{M}_b$, generating the field $\boldsymbol{E}_b$, $\boldsymbol{H}_b$. The interaction of source a with field b may be characterized by the complex number $\langle a, b\rangle$, defined as [4]

$$\langle a, b\rangle = \int_V (\boldsymbol{J}_a \cdot \boldsymbol{E}_b - \boldsymbol{M}_a \cdot \boldsymbol{H}_b)\, dV\,, \tag{2.211}$$

where the first entry, a, is associated with the source (or probe), and the second entry, b, is associated with the observed field. The integration is extended over the volume V, i.e. the region containing the source a, which may contain both volume current densities and surface current densities. Note that, for an ideal electric field probe, $\boldsymbol{J}_a$ is a delta function which measures the field just at the observation point. As noted in the previous paragraph, also for electromagnetic field quantities, the reaction is different from *complex power* since there is no complex–conjugate. Moreover, let Σ represent any scalar and Σa be the source a increased in strength by the factor Σ, then

$$\langle \Sigma a, b\rangle = \Sigma \langle a, b\rangle\,. \tag{2.212}$$

By considering another source c, radiating at the same frequency as a and b, we have

$$\langle a, (b + c)\rangle = \langle a, b\rangle + \langle a, c\rangle\,. \tag{2.213}$$

VIII.2 Lorentz Reciprocity Theorem

Having discussed the reaction concept we proceed to the Lorentz reciprocity theorem. A simple interpretation of this theorem is that, in isotropic media, the response of a system to a source is unchanged when source and detector are interchanged [13]. In order to establish this theorem let us consider the two monochromatic sources a, b and the field produced thereby. In each case Maxwell's equations are:

$$\nabla \times \boldsymbol{E}_a = -j\omega\mu \boldsymbol{H}_a - \boldsymbol{M}_a\,, \tag{2.214a}$$

$$\nabla \times \boldsymbol{H}_a = j\omega\varepsilon \boldsymbol{E}_a + \boldsymbol{J}_a \tag{2.214b}$$

and

$$\nabla \times \boldsymbol{E}_b = -j\omega\mu \boldsymbol{H}_b - \boldsymbol{M}_b\,, \tag{2.215a}$$

$$\nabla \times \boldsymbol{H}_b = j\omega\varepsilon \boldsymbol{E}_b + \boldsymbol{J}_b\,. \tag{2.215b}$$

Performing dot product multiplication of (2.214b) by $\boldsymbol{E}_b$ and of (2.215a) by $\boldsymbol{H}_a$, and subtracting one from the other we obtain

$$\nabla \cdot (\boldsymbol{E}_b \times \boldsymbol{H}_a) = -j\omega\varepsilon \boldsymbol{E}_a \cdot \boldsymbol{E}_b - \boldsymbol{J}_a \cdot \boldsymbol{E}_b - j\omega\mu \boldsymbol{H}_a \cdot \boldsymbol{H}_b - \boldsymbol{M}_b \cdot \boldsymbol{H}_b\,, \tag{2.216}$$

where use is made of the identity

$$\nabla \cdot (\boldsymbol{U} \times \boldsymbol{V}) = \boldsymbol{V} \cdot \nabla \times \boldsymbol{U} - \boldsymbol{U} \cdot \nabla \times \boldsymbol{V}\,. \tag{2.217}$$

Similarly, performing dot product multiplication of (2.215b) by $\boldsymbol{E}_a$, and of (2.214a) by $\boldsymbol{H}_b$, and subtracting one from the other, we obtain:

$$\nabla \cdot (\boldsymbol{E}_a \times \boldsymbol{H}_b) = -j\omega\varepsilon \boldsymbol{E}_a \cdot \boldsymbol{E}_b - \boldsymbol{J}_b \cdot \boldsymbol{E}_a - j\omega\mu \boldsymbol{H}_a \cdot \boldsymbol{H}_b - \boldsymbol{M}_a \cdot \boldsymbol{H}_b \,. \quad (2.218)$$

Finally, by subtracting (2.216) from (2.218), integrating throughout a source–free region, and applying the divergence theorem we arrive at

$$\oint_S (\boldsymbol{E}_a \times \boldsymbol{H}_b - \boldsymbol{E}_b \times \boldsymbol{H}_a)\, d\boldsymbol{S} = \langle a, b \rangle - \langle b, a \rangle \,. \quad (2.219)$$

By *definition*, isotropic media are reciprocal when

$$\oint_S (\boldsymbol{E}_a \times \boldsymbol{H}_b - \boldsymbol{E}_b \times \boldsymbol{H}_a)\, d\boldsymbol{S} = 0 \,. \quad (2.220)$$

In this case, the Lorentz reciprocity theorem can be stated as

$$\langle a, b \rangle - \langle b, a \rangle = 0. \quad (2.221)$$

The surface integral on the left side of (2.219) vanishes also when the surface S encloses all the sources. In fact, in this case we can consider the complementary source–free volume bounded by S and the surface S_∞ of a sphere with infinite radius. When the fields satisfy the radiation condition the integrand of the left side of (2.219) vanishes on S_∞ and (2.220) applies as well.

The Lorentz theorem has a variety of useful applications. It allows one to derive stationary formulas in variational problems in a direct manner. It is also suitable for proving simple assertions, such as the fact that an electric current sheet impressed on the surface of a perfect conductor does not radiate [4]. This is a trivial result when the surface of the conductor is planar, since image theory shows that no field is produced. In fact, by replacing the metallic plane by an image source, i.e. by an impressed current directed in the opposite direction, the two impressed currents annihilate, producing zero field. When the surface is not planar, application of reciprocity demonstrates the above assertion in the following way. With reference to Figure 2.13 let us consider source a on the perfect electric conductor. In order to measure the field $\boldsymbol{E}_a$, $\boldsymbol{H}_a$ produced by this source, let us place a probe (source b) at the observation point and evaluate the reaction of source b on the field a, i.e. $\langle b, a \rangle$. By the reciprocity theorem, the effect of source b on the field a is equal to the effect of source a on the field b, i.e.

$$\langle b, a \rangle = \langle a, b \rangle. \quad (2.222)$$

However, the tangential component of the electric field produced by b is zero on the metallic surface where $\boldsymbol{J}_a$ is present, thus

$$\langle a, b \rangle = 0 \,. \quad (2.223)$$

In view of the arbitrariness of source b it is proved that the impressed electric current sheets $\boldsymbol{J}_a$ on the surface of the perfect electric conductor do not produce any field.

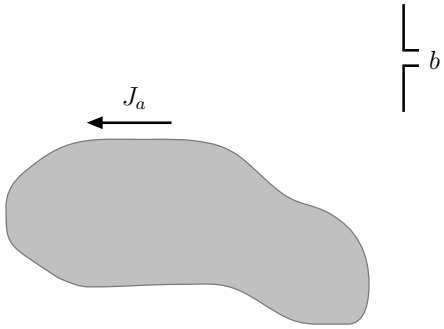


Fig. 2.13. The impressed electric current sheets $\boldsymbol{J}_a$ on the surface of a perfect electric conductor do not produce any field, as measured through probe b.

Huygens' Principle

The propagation of electromagnetic fields can be visualized according to Christian Huygens as wavefronts comprising a number of secondary sources or radiators, each generating new spherical wavelets. According to Huygens' principle the envelope of these wavelets forms a wavefront which in turn consists of new sources giving rise to a new generation of spherical wavelets. This in turn means that the field solution in a region is completely determined by the tangential fields specified over the surface enclosing the region. This principle can be rigorously stated in mathematical terms, as shown next. To this end we need to recall scalar and vector Green's theorems [3] which, as noted in [13, p.120], are mathematical statements of reciprocity (symmetrical in two functions). The difference between the Lorentz reciprocity theorem and Green's theorem is that no physical interpretation is ascribed to the latter.

Scalar Green's Theorem

Consider a closed regular surface S bounding a volume V where the two scalar functions $\bar{\phi}$ and $\bar{\psi}$, continuous together with their first and second derivatives throughout V and on the surface S, are defined. Applying the divergence theorem to the vector $\bar{\psi}\nabla\bar{\phi}$ yields

$$\int_V \nabla \cdot \left(\bar{\psi}\nabla\bar{\phi}\right) dV = \int_S \left(\bar{\psi}\nabla\bar{\phi}\right) \cdot \boldsymbol{n}\, dS\,. \tag{2.224}$$

The divergence on the left–hand side may be expanded as

$$\nabla \cdot \left(\bar{\psi}\nabla\bar{\phi}\right) = \nabla\bar{\psi} \cdot \nabla\bar{\phi} + \bar{\psi}\nabla \cdot \nabla\bar{\phi} = \nabla\bar{\psi} \cdot \nabla\bar{\phi} + \bar{\psi}\nabla^2\bar{\phi}\,. \tag{2.225}$$

while on the right–hand side, we may replace the normal component of the gradient by the normal derivative, i.e.

$$\nabla\bar{\phi}\cdot\boldsymbol{n}=\frac{\partial\bar{\phi}}{\partial n}\,. \tag{2.226}$$

Upon substituting (2.225) and (2.226) into (2.224) we obtain Green's *first* identity

$$\int_V \nabla\bar{\psi}\cdot\nabla\bar{\phi}\,dV+\int_V \bar{\psi}\nabla^2\bar{\phi}\,dV=\int_S \bar{\psi}\frac{\partial\bar{\phi}}{\partial n}\,dS\,. \tag{2.227}$$

This identity holds also when interchanging the roles of the functions $\bar{\phi},\bar{\psi}$; by so doing we obtain

$$\int_V \nabla\bar{\psi}\cdot\nabla\bar{\phi}\,dV+\int_V \bar{\phi}\nabla^2\bar{\psi}\,dV=\int_S \bar{\phi}\frac{\partial\bar{\psi}}{\partial n}\,dS\,. \tag{2.228}$$

Subtracting (2.228) from (2.227) we get another important identity, Green's *second* identity, namely,

$$\int_V \left(\bar{\psi}\nabla^2\bar{\phi}-\bar{\phi}\nabla^2\bar{\psi}\right)dV=\int_S \bar{\psi}\frac{\partial\bar{\phi}}{\partial n}\,dS-\int_S \bar{\phi}\frac{\partial\bar{\psi}}{\partial n}\,dS\,, \tag{2.229}$$

which is frequently referred to as *Green's theorem*.

Vector Green's Theorem

Let us return to the surface S and volume V as defined in the previous paragraph, but consider two vector functions $\boldsymbol{U}$ and $\boldsymbol{V}$ which, together with their first and second derivatives, are continuous throughout V and on the surface S. Then, replacing the gradient by the curl, i.e. ∇ by $\nabla\times$, and ∇^2 by $\nabla\times\nabla\times$, we have the building blocks for the vector analogue of the scalar Green's theorem. Applying the divergence theorem to the vector $\boldsymbol{U}\times\nabla\times\boldsymbol{V}$,

$$\int_V \nabla\cdot(\boldsymbol{U}\times\nabla\times\boldsymbol{V})\,dV=\int_S(\boldsymbol{U}\times\nabla\times\boldsymbol{V})\cdot\boldsymbol{n}dS \tag{2.230}$$

and expanding the divergence on the left hand side we get

$$\begin{gathered}\nabla\cdot(\boldsymbol{U}\times\nabla\times\boldsymbol{V})=\\ \nabla_P\cdot(\boldsymbol{U}\times\nabla\times\boldsymbol{V})+\nabla_Q\cdot(\boldsymbol{U}\times\nabla\times\boldsymbol{V})=\\ \nabla\times\boldsymbol{U}\cdot\nabla\times\boldsymbol{V}-\boldsymbol{U}\cdot\nabla\times\nabla\times\boldsymbol{V}\end{gathered} \tag{2.231}$$

which, by substitution into (2.230), provides the vector analogue of Green's first identity,

$$\int_V(\nabla\times\boldsymbol{U}\cdot\nabla\times\boldsymbol{V})-(\boldsymbol{U}\cdot\nabla\times\nabla\times\boldsymbol{V})\,dV=\int_S(\boldsymbol{U}\times\nabla\times\boldsymbol{V})\cdot\boldsymbol{n}\,dS\,. \tag{2.232}$$

Another form of the vector first identity may be obtained by interchanging $\boldsymbol{U}$ and $\boldsymbol{V}$,

$$\int_V (\nabla \times \boldsymbol{V} \cdot \nabla \times \boldsymbol{U}) - (\boldsymbol{V} \cdot \nabla \times \nabla \times \boldsymbol{U})\, dV = \int_S (\boldsymbol{V} \times \nabla \times \boldsymbol{U}) \cdot \boldsymbol{n} dS\,. \quad (2.233)$$

By subtracting (2.233) from (2.232) we get the vector analogue of Green's *second* identity,

$$\int_V (\boldsymbol{V} \cdot \nabla \times \nabla \times \boldsymbol{U} - \boldsymbol{U} \cdot \nabla \times \nabla \times \boldsymbol{V})\, dV = \int_S (\boldsymbol{U} \times \nabla \times \boldsymbol{V} \ - \boldsymbol{V} \times \nabla \times \boldsymbol{U}) \cdot \boldsymbol{n} dS\,. \quad (2.234)$$

The dyadic form of *Huygens' principle* is obtained on replacing the vector $\boldsymbol{V}$ in (2.234) by the scalar product of Green's dyad $\mathscr{G}$ with a vector $\boldsymbol{U}$, i.e. $\boldsymbol{V} = \mathscr{G} \cdot \boldsymbol{U}$.

Mathematical Formulation of Huygens' Principle

The equivalence principle is rigorously proved by introducing the mathematical formulation of Huygens' principle [3, 12, 23].
Consider a volume V, containing all sources, bounded by a smooth surface S. The electric field in V is a solution of the source–free vector wave equation

$$\nabla \times \nabla \times \boldsymbol{E} - k^2 \boldsymbol{E} = 0. \quad (2.235)$$

Consider also the dyadic Green's function $\mathscr{G}_e$ which, in turn, is a solution of

$$\nabla \times \nabla \times \mathscr{G}_e - k^2 \mathscr{G}_e = \mathscr{I}\, \delta(\boldsymbol{r} - \boldsymbol{r'})\,. \quad (2.236)$$

Both $\boldsymbol{E}$ and $\boldsymbol{G}_e$ satisfy the electric field boundary conditions on S as well as the radiation condition at infinity. Here $\mathscr{I}$ is the identity dyadic and $\delta(\boldsymbol{r} - \boldsymbol{r'})$ is the Dirac delta function. Forming the scalar products

$$\boldsymbol{E} \cdot \nabla \times \nabla \times \mathscr{G}_e - \nabla \times \nabla \times \boldsymbol{E} \cdot \mathscr{G}_e \quad (2.237)$$

and then applying Green's vector second identity in (2.234), yields

$$\int_V (\boldsymbol{E} \cdot \nabla \times \nabla \times \mathscr{G}_e - \nabla \times \nabla \times \boldsymbol{E} \cdot \mathscr{G}_e)\, dV = \int_S (\mathscr{G}_e \times \nabla \times \boldsymbol{E} - \boldsymbol{E} \times \nabla \times \mathscr{G}_e) \cdot \boldsymbol{n} dS\,, \quad (2.238)$$

where the integral over the sphere at infinity has been set to zero because both $\boldsymbol{E}$ and $\mathscr{G}_e$ satisfy the radiation condition. Hence, using (2.236) we have

$$\int_S (\boldsymbol{n} \times \boldsymbol{E} \cdot \nabla \times \mathscr{G}_e + \boldsymbol{n} \times \nabla \times \boldsymbol{E} \cdot \mathscr{G}_e)\, dS = \begin{cases} \boldsymbol{E}(\boldsymbol{r'}) & \boldsymbol{r'} \text{ in } V \\ \boldsymbol{0} & \boldsymbol{r'} \text{ in } V_1 \end{cases} . \tag{2.239}$$

Using Maxwell's curl equation,

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}, \tag{2.240}$$

(2.239) may be written in terms of the currents flowing on S as

$$\boldsymbol{E}(\boldsymbol{r'}) = \int_S \boldsymbol{n} \times \boldsymbol{E} \cdot \nabla \times \mathscr{G}_e dS - j\omega\mu \int_S \boldsymbol{n} \times \boldsymbol{H} \cdot \mathscr{G}_e dS\,. \tag{2.241}$$

The formula in (2.241) provides the electric field at each point of V in terms of the boundary fields on S, and constitutes the mathematical version of Huygens' principle [12, p.135], [23]). By following the same steps, or by using duality, it is possible to derive a formula analogous to (2.241) for the magnetic field, i.e.

$$\boldsymbol{H}(\boldsymbol{r'}) = \int_S \boldsymbol{n} \times \boldsymbol{H} \cdot \nabla \times \mathscr{G}_m dS + j\omega\varepsilon \int_S \boldsymbol{n} \times \boldsymbol{E} \cdot \mathscr{G}_m dS\,, \tag{2.242}$$

where the magnetic field dyadic Green's function $\mathscr{G}_m$ satisfies (2.236) with magnetic field boundary conditions and the radiation condition.
By recalling the equivalence theorem, it follows that specification of the tangential components of the $\boldsymbol{E}, \boldsymbol{H}$ fields on S is the same as the specification of equivalent electric and magnetic currents $\boldsymbol{J}$ and $\boldsymbol{M}$. It is useful to write (2.241) and (2.242) operationally in the following way

$$\boldsymbol{E} = \hat{Z}(\boldsymbol{J}) + \hat{T}_e(\boldsymbol{M})\,, \tag{2.243a}$$

$$\boldsymbol{H} = \hat{T}_m(\boldsymbol{J}) + \hat{Y}(\boldsymbol{M})\,, \tag{2.243b}$$

which express the electromagnetic field (as obtained from the field on S) in terms of operators identified from (2.241) and (2.242). It can be proved by inserting (2.243b) into (2.14a) and (2.14b), and noting the arbitrariness of $\boldsymbol{J}$ and $\boldsymbol{M}$, that the above operators also satisfy the following equations

$$\nabla \times \hat{Z} = -j\omega\mu \hat{T}_m\,, \tag{2.244a}$$

$$\nabla \times \hat{Y} = j\omega\varepsilon \hat{T}_e\,, \tag{2.244b}$$

$$\nabla \times \hat{T}_e = -j\omega\mu \hat{Y} - \boldsymbol{M}\,, \tag{2.244c}$$

$$\nabla \times \hat{T}_m = j\omega\varepsilon \hat{Z} + \boldsymbol{J}\,, \tag{2.244d}$$

from which one obtains

$$(\nabla \times \nabla \times -k^2)\hat{Z} = -j\omega\mu \boldsymbol{J}\,, \tag{2.245a}$$

$$(\nabla \times \nabla \times -k^2)\hat{Y} = -j\omega\varepsilon \boldsymbol{M}\,, \tag{2.245b}$$

$$(\nabla \times \nabla \times -k^2)\hat{T}_e = -\nabla \times \boldsymbol{M}\,, \tag{2.245c}$$

$$(\nabla \times \nabla \times -k^2)\hat{T}_m = \nabla \times \boldsymbol{J}\,, \tag{2.245d}$$

The operators $\hat{Z}, \hat{T}_e$ satisfy the same boundary condition as the electric field, while the operators $\hat{Y}, \hat{T}_m$ satisfy the same boundary condition as the magnetic field.

An interesting circuit analogy of (2.243b) can be obtained by considering the equivalent sources $\boldsymbol{J}$ and $\boldsymbol{M}$ on the surface S_1 and the observation point $\boldsymbol{r'}$ on the surface S_2. In this case, (2.243b) corresponds to an $ABCD$ representation of the region of space between the two surfaces. In order to describe this region, we could also have chosen other representations, such as the Z (impedance) or the Y (admittance) representation. As an example, a Z representation is obtained by considering the two surfaces S_1, S_2 as *magnetic* walls. Accordingly only the electric currents, i.e. the magnetic fields, produce radiation away from the surfaces. By letting $\boldsymbol{E}_1, \boldsymbol{H}_1$ be the electric and magnetic fields on the surface S_1, and $\boldsymbol{E}_2, \boldsymbol{H}_2$ the electric and magnetic fields on the surface S_2, we may express the (impedance) relationship between electric and magnetic fields on these surfaces as

$$\boldsymbol{E}_1 = \hat{Z}_{11}(\boldsymbol{H}_1) + \hat{Z}_{12}(\boldsymbol{H}_2)\,, \tag{2.246a}$$

$$\boldsymbol{E}_2 = \hat{Z}_{21}(\boldsymbol{H}_1) + \hat{Z}_{22}(\boldsymbol{H}_2)\,. \tag{2.246b}$$

A similar relationship may be written for the admittance representation.

Finally note that, when the operator is expressed in a diagonalized form, i.e. when the region we are dealing with is coordinate and vector separable, we can pass from one representation, say the admittance representation, to another representation.

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