Stochastic Geometry of Classical and Quantum Ising Models

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Contents

1 Introduction

These lecture notes are based on a mini-course which I taught at Prague school in September 2006. The idea was to try to develop and explain to probabilistically minded students a unified approach to the

R. Kotecký (ed.), *Methods of Contemporary Mathematical Statistical Physics*,
Lecture Notes in Mathematics 1970, DOI: 10.1007/978-3-540-92796-9_2,

Fortuin-Kasteleyn (FK) and to the random current (RC) representation of classical and quantum Ising models via path integrals. No background in quantum statistical mechanics was assumed.

In Section 1 familiar classical Ising models are rewritten in the quantum language. In this way usual FK and RC representations emerge as different instances of Lie-Trotter product formula. Then I am following [4] and set up a general notation for the Poisson limits.

In Section 2 both FK and the RC representations are generalized to quantum Ising models in transverse field. The FK representation was originally derived in [8] and [3]. The observation regarding the RC representation seems to be new. Both representations are used to derive formulas for one and two point functions and for the matrix and reduced density matrix elements.

Section 3 is devoted to the quantum Curie-Weiss model in transverse field. In the quantum mean field case the FK representation is built upon a generalization of the classical random graph model. I briefly explain recent results of [15], where the critical curve for quantum random graphs was explicitly computed. The critical curve for the quantum Curie-Weiss model itself is computed in the concluding Subsection 3.3 via partial Trotterization and a large deviation approach.

Of course, stochastic geometric methods apply for a large variety of other models, see the seminal [4] as well as [18, 20] and references therein. I did not try to provide a complete bibliography on the subject - the emphasis was rather on trying to advertise probabilistic aspects of quantum spin systems to a reader who is (like me) not very well familiar with the latter. I, therefore, apologize for many excellent and relevant papers which were not mentioned.

2 Classical Ising Model

We use the following notation for the classical Ising model:

- $(\Lambda, \mathcal{E})$ is a finite graph with *unoriented* edges $e = \{i,j\} = \{j,i\} \in \mathcal{E}$.
- $\mathbf{J} = \{J_{ij} \geq 0\}$ are coupling constants. By definition $J_{ij} > 0 \Leftrightarrow \{i,j\} \in \mathcal{E}$.
- $h \in \mathbb{R}$ is a magnetic field.
- $\nu \in \Omega_\Lambda \stackrel{\Delta}{=} \{-1,1\}^\Lambda$ is a spin configuration on Λ.

The Hamiltonian $\mathbf{H}_\Lambda$ is a function on Ω_Λ,

$$-\mathbf{H}_\Lambda(\nu) = \sum_{(i,j)\in\mathcal{E}} J_{ij}\nu_i\nu_j + h\sum_{i\in\Lambda}\nu_i .$$

Given $\beta \geq 0$ (inverse temperature) define the classical Ising-Gibbs probability distribution $\mu_\Lambda^{\beta,h}$ on Ω_Λ as

$$\mu_\Lambda^{\beta,h}(\nu) = \frac{1}{\mathcal{Z}_\Lambda(\beta,h)} \mathrm{e}^{-\beta \mathbf{H}_\Lambda(\nu)},$$

where the normalizing constant (partition function) is given by

$$\mathcal{Z}_\Lambda(\beta,h) = \sum_{\nu \in \Omega_\Lambda} \mathrm{e}^{-\beta \mathbf{H}_\Lambda(\nu)}. \tag{2.1}$$

In the sequel we shall use $\mu_\Lambda^{\beta,h}(\bullet)$ for the expectation under $\mu_\Lambda^{\beta,h}$. In particular, the mean value of the spin at i is

$$\mu_\Lambda^{\beta,h}(\nu_i) = \frac{1}{\mathcal{Z}_\Lambda(\beta,h)} \sum_{\nu \in \Omega_\Lambda} \nu_i \mathrm{e}^{-\beta \mathbf{H}_\Lambda(\nu)}, \tag{2.2}$$

and the two-point function $\mu_\Lambda^{\beta,h}(\nu_i \nu_j)$ is

$$\mu_\Lambda^{\beta,h}(\nu_i \nu_j) = \frac{1}{\mathcal{Z}_\Lambda(\beta,h)} \sum_{\nu \in \Omega_\Lambda} \nu_i \nu_j \mathrm{e}^{-\beta \mathbf{H}_\Lambda(\nu)}. \tag{2.3}$$

Two examples we shall consider in this paper are:

1. Curie-Weiss model: $\Lambda = \{1, 2 \ldots, N\}$ and $J_{ij} \equiv 1/N$.
2. Finite range Ising model: $\Lambda \subset \mathbb{Z}^d$ and $J_{ij} = 0$ for $\|i - j\| \geq R$.

2.1 Classical Ising Model Dressed as Quantum

Let us re-derive formulas (2.1), (2.2) and (2.3) in the quantum language. In this way spin values ± 1 are understood as eigenvalues of Pauli matrix

$$\hat{\sigma}^{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

Let us define the corresponding eigenfunctions

$$\psi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.5}$$

Of course, $\hat{\sigma}^{\mathsf{z}} \psi_\nu = \nu \psi_\nu$ for $\nu = \pm 1$. Throughout these lectures we shall work only with real numbers. Using eigenfunctions $\psi_{\pm 1}$ one constructs the following "lifting" of classical configurations $\nu \in \Omega_\Lambda$: Define

$$\mathbb{X}_\Lambda = \bigotimes_{i \in \Lambda} \mathbb{R}^2.$$

$\mathbb{X}_\Lambda$ is a $2^{|\Lambda|}$-dimensional vector space over the field of reals. Classical configurations $\nu \in \Omega_\Lambda$ are encoded in $\mathbb{X}_\Lambda$ as tensor products,

$$\Psi_\nu \stackrel{\Delta}{=} \otimes_{i\in\Lambda}\psi_{\nu_i}. \tag{2.6}$$

The collection $\{\Psi_\nu\}_{\nu\in\Omega_\Lambda}$ is a complete orthonormal basis of $\mathbb{X}_\Lambda$ with respect to the scalar product

$$\langle\Psi_\nu|\Psi_{\nu'}\rangle \stackrel{\Delta}{=} \prod_{i\in\Lambda}\langle\psi_{\nu_i},\psi_{\nu_i'}\rangle_2,$$

where $\langle\bullet,\bullet\rangle_2$ is the usual scalar product of $\mathbb{R}^2$. With each $i\in\Lambda$ we associate a linear self-adjoint operator (symmetric matrix) $\hat\sigma_i^{\mathsf{z}}$ which acts on i-th coordinate of Ψ as a copy of Pauli matrix $\hat\sigma^{\mathsf{z}}$ defined in (2.4). Namely, for each $\nu\in\Omega_\Lambda$,

$$\hat\sigma_i^{\mathsf{z}}\Psi_\nu \stackrel{\Delta}{=} \psi_{\nu_1}\otimes\cdots\otimes\hat\sigma^{\mathsf{z}}\psi_{\nu_i}\otimes\ldots = \nu_i\Psi_\nu. \tag{2.7}$$

Obviously, $\hat\sigma_i^{\mathsf{z}}$ and $\hat\sigma_j^{\mathsf{z}}$ commute, and, moreover,

$$\hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}}\Psi_\nu = \nu_i\nu_j\Psi_\nu. \tag{2.8}$$

Define now the quantum Hamiltonian $\mathcal{H}_\Lambda$ as a linear self-adjoint operator on $\mathbb{X}_\Lambda$,

$$-\mathcal{H}_\Lambda = \sum_{(i,j)\in\mathcal{E}} J_{ij}\hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}} + h\sum_{i\in\Lambda}\hat\sigma_i^{\mathsf{z}}. \tag{2.9}$$

Then, (2.7) and (2.8) imply,

$$\mathcal{H}_\Lambda\Psi\nu = \mathbf{H}_\Lambda(\nu)\Psi_\nu.$$

In other words, $\mathcal{H}_\Lambda$ is diagonal in the $\{\Psi_\nu\}$ basis, and with the corresponding eigenvalues being equal to values of the classical Ising Hamiltonian on configurations ν.

It is possible now to rewrite classical formulas (2.1)-(2.3) in terms of the quantum Hamiltonian $\mathcal{H}_\Lambda$. First of all,

$$\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right) = \sum_{\nu\in\Omega_\Lambda}\langle\Psi_\nu|\mathrm{e}^{-\beta\mathcal{H}_\Lambda}|\Psi_\nu\rangle = \sum_{\nu\in\Omega_\Lambda}\mathrm{e}^{-\beta\mathbf{H}_\Lambda(\nu)} = \mathcal{Z}_\Lambda(\beta,h). \tag{2.10}$$

Similarly,

$$\mu_\Lambda^{\beta,h}(\nu_i) = \frac{\mathsf{Tr}\left(\hat\sigma_i^{\mathsf{z}}\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)} \quad\text{and}\quad \mu_\Lambda^{\beta,h}(\nu_i\nu_j) = \frac{\mathsf{Tr}\left(\hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}}\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}. \tag{2.11}$$

2.2 Path Integral Representation and Poisson Limits

Since all the operators $\{\hat{\sigma}_i^{\mathsf{z}}\}$ commute,

$$\mathrm{e}^{-\beta\mathcal{H}_\Lambda} = \left(\prod_{(i,j)} \mathrm{e}^{\Delta J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}} \prod_i \mathrm{e}^{\Delta h\hat{\sigma}_i^{\mathsf{z}}}\right)^{\beta/\Delta}. \tag{2.12}$$

To facilitate the exposition we shall focus now on the case of zero magnetic field $h = 0$, the full Hamiltonian with both non-zero h will be considered in Subsection 2.3 and in Subsection 2.4, furthermore, an additional positive field in the traverse direction will be considered in Section 3.

For small Δ we shall linearize $\mathrm{e}^{\Delta J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}}$ in (2.12) in two different ways:

1) Write

$$J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}} = J_{ij}\mathrm{I} - J_{ij}\mathrm{I} + J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}.$$

Then,

$$\mathrm{e}^{-\beta\mathcal{H}_\Lambda} = \mathrm{e}^{\beta\sum_{(i,j)} J_{ij}} \lim_{\Delta\to 0}\left(\prod_{(i,j)} \{(1-\Delta J_{ij})\mathrm{I} + \Delta J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}\}\right)^{\beta/\Delta}. \tag{2.13}$$

This will lead to the random current representation of the model.

2) Write

$$J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}} = J_{ij}\mathrm{I} - 2J_{ij}\mathrm{I} + 2J_{ij}\frac{\mathrm{I} + \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}}{2}.$$

In the latter case,

$$\mathrm{e}^{-\beta\mathcal{H}_\Lambda} = \mathrm{e}^{\beta\sum_{(i,j)} J_{ij}} \lim_{\Delta\to 0}\left(\prod_{(i,j)} \left\{(1-2\Delta J_{ij})\mathrm{I} + 2\Delta J_{ij}\frac{\mathrm{I} + \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}}{2}\right\}\right)^{\beta/\Delta}. \tag{2.14}$$

As we shall see below such linearization leads to the FK (Fortuin-Kasteleyn) representation of the model. Thus both the FK and the random current representations are instances of path integral representation via Poisson limits which, following [4], we proceed to discuss in a somewhat general context.

General Setup for Poisson Limits

The fact that the operators $\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}$ in (2.13) or operators $(\mathrm{I} + \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}})/2$ in (2.14) commute *is not* essential for the path integral representation via Poisson limits. For the rest of this Subsection we shall work in the following general context:

1. $\mathbb{X}$ is an M-dimensional vector space (over $\mathbb{R}$) with a scalar product $\langle \bullet | \bullet \rangle$ and an orthonormal basis $\{\Psi_i\}$
2. $K_1, \dots, K_m$ are self-adjoint operators (matrices) on $\mathbb{X}$, in general non-commuting.
3. $\lambda_1, \dots, \lambda_m$ are positive numbers.

Given $\beta > 0$, we would like to find a probabilistic representation for

$$\exp\Big\{ \beta \sum_1^m \lambda_l K_l \Big\} \tag{2.15}$$

The linearization relies on two basic facts from theory of matrices:

Lie-Trotter Formula

Let A and B be two matrices. Then

$$\mathrm{e}^{A+B} \;=\; \lim_{n\to\infty} \left(\mathrm{e}^{A/n}\mathrm{e}^{B/n} \right)^n . \tag{2.16}$$

Proof (following [19]). Set

$$T_n = \mathrm{e}^{(A+B)/n} \qquad \text{and} \qquad S_n = \mathrm{e}^{A/n}\mathrm{e}^{B/n}.$$

Then,

$$T_n^n - S_n^n \;=\; \sum_{l=0}^{n-1} \left(T_n^{n-l} S_n^l - T_n^{n-l-1} S_n^{l+1} \right) \;=\; \sum_{l=0}^{n-1} T_n^{n-l-1} \left(T_n - S_n \right) S_n^l .$$

Now,

$$T_n - S_n = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{A+B}{n} \right)^k - \Big\{ \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{A}{n} \right)^k \Big\} \Big\{ \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{B}{n} \right)^k \Big\} = O\big(\frac{1}{n^2}\big).$$

On the other hand, $\|T_n^{n-l-1}\| \cdot \|S_n^l\| \leq \mathrm{e}^{\|A\|+\|B\|}$ for all $l = 1, \dots, n-1$.
$\square$

Product Expansion Formula

Let $A_1, A_2, \dots, A_n$ be self-adjoint matrices and Ψ, Ψ' two vectors in $\mathbb{X}$. Then,

$$\langle \Psi | A_1 \dots A_n | \Psi' \rangle = \sum_{\Psi_{i_1}, \dots \Psi_{i_n}} \langle \Psi | A_1 | \Psi_{i_1} \rangle \langle \Psi_{i_1} | A_2 | \Psi_{i_2} \rangle \dots \langle \Psi_{i_{n-1}} | A_n | \Psi' \rangle, \tag{2.17}$$

where Ψ_{i_l}-s run through the elements of the orthonormal basis $\{\Psi_i\}$ for all $l = 1, \dots, n-1$.

Proof. In the case $n = 2$ (2.17) follows from expansion of $A_1 \Psi$ in the basis $\{\Psi_i\}$,

$$A_1 \Psi = \sum_{i_1 = 1}^{M} \langle \Psi | A_1 | \Psi_{i_1} \rangle \Psi_{i_1}.$$

The general case follows by induction. □

Path Integral Representation

Let us go back to (2.15). By Lie-Trotter formula (2.16),

$$\exp\left\{ \beta \sum_1^m \lambda_l K_l \right\} = \mathrm{e}^{\beta \sum \lambda_l} \lim_{\Delta \to 0} \left(\prod_{l=1}^m \{(1 - \Delta\lambda_l)\mathrm{I} + \Delta\lambda_l K_l\} \right)^{\beta/\Delta}. \tag{2.18}$$

In the sequel we shall tacitly assume that $\beta/\Delta \in \mathbb{N}$. For each $l = 1, \dots, m$ consider a sequence of iid Bernoulli random variables

$$\underline{\xi}_l = \{\xi_l(1), \xi_l(2) \dots, \xi_l(\beta/\Delta)\},$$

with the probability of success being equal to $\Delta\lambda_l$. We assume that the sequences $\underline{\xi}_l$ are independent and let $\mathbb{P}^{\underline{\lambda}}_{\beta,\Delta}$ be the corresponding probability measure on

$$\{0,1\}^{\beta/\Delta} \times \dots \times \{0,1\}^{\beta/\Delta}$$

Above $\underline{\lambda}$ is a shorthand notation for the vector of success rates $\{\lambda_1, \dots, \lambda_m\}$. Then we can expand the expression on the right-hand side of (2.18) as follows,

$$\left(\prod_{l=1}^m \{(1 - \Delta\lambda_l)\mathrm{I} + \Delta\lambda_l K_l\} \right)^{\beta/\Delta} = \sum_{\underline{a}_1, \dots \underline{a}_m} \mathbb{P}^{\underline{\lambda}}_{\beta,\Delta} \left(\bigcap_{l=1}^m \left\{ \underline{\xi}_l = \underline{a}_l \right\} \right) \mathcal{K}_{\underline{a}}, \tag{2.19}$$

where the matrix $\mathcal{K}_{\underline{a}}$ is defined by

$$\mathcal{K}_{\underline{a}} \stackrel{\Delta}{=} \mathcal{K}_{\underline{a}_1,\dots,\underline{a}_m} = \prod_{j=1}^{\beta/\Delta}\left\{\prod_{l=1}^{m}\left((1-a_l(j))\mathrm{I} + a_l(j)K_l\right)\right\}. \tag{2.20}$$

Our next step is to associate with each sequence $\underline{\xi}_l$ of Bernoulli trials a point process of arrivals of operators K_l on the interval $[0,\beta]$. Define,

$$\xi_l^{\Delta} = \sum_{j=1}^{\beta/\delta} \xi_l(j)\delta_{i\Delta}. \tag{2.21}$$

Let Ψ and Ψ' be two elements of the basis $\{\Psi_i\}$. In order to derive a path integral representation of $\langle \Psi | K_{\underline{a}_1},\dots,K_{\underline{a}_m}|\Psi'\rangle$ notice first of all that up to probabilities of order $O(\Delta)$ we may restrict attention to sequences $\underline{a}_1,\dots,\underline{a}_m$ with disjoint occurrence of successes, that is $\sum_l a_l(j) = 0$ or 1 for every $j = 1,\dots,\beta/\Delta$. In the language of (2.21) this means that the realizations of $\xi_1^{\Delta},\dots,\xi_m^{\Delta}$ are pairwise disjoint and hence for each arrival time

$$t \in \xi^{\Delta} \stackrel{\Delta}{=} \cup \xi_l^{\Delta} = \left\{ j\Delta \,:\, \sum_{l=1}^{m} a_l(j) = 1 \right\},$$

there is a well defined arrival type $\mathfrak{l}^{\Delta}(t) \in \{1,\dots,m\}$. Accordingly, one can rewrite

$$\mathcal{K}_{\underline{a}_1,\dots,\underline{a}_m} = \prod_{j=1}^{\beta/\Delta}\left\{\delta_{\{j\Delta\notin\xi^{\Delta}\}}\mathrm{I} + \delta_{\{j\Delta\in\xi^{\Delta}\}}K_{\mathfrak{l}^{\Delta}(j\Delta)}\right\} \stackrel{\Delta}{=} \prod_{j=1}^{\beta/\Delta}\tilde{K}_j^{\Delta}.$$

By the product expansion formula (2.17),

$$\langle\Psi|\mathcal{K}_{\underline{a}}|\Psi'\rangle = \sum_{\Psi_{i_1},\dots\Psi_{i_{\beta/\Delta-1}}} \langle\Psi|\tilde{K}_1^{\Delta}|\Psi_{i_1}\rangle \prod_{j=2}^{\beta/\Delta-1}\langle\Psi_{i_{j-1}}|\tilde{K}_j^{\Delta}|\Psi_{i_j}\rangle\langle\Psi_{i_{\beta/\Delta-1}}|\tilde{K}_{\beta/\Delta}^{\Delta}|\Psi'\rangle. \tag{2.22}$$

Of course,

$$\langle\Psi_l|\tilde{K}_j^{\Delta}|\Psi_k\rangle = \begin{cases} \delta_{\{\Psi_l=\Psi_k\}} & \text{if } j\Delta\notin\xi^{\Delta} \\ \langle\Psi_l|K_{\mathfrak{l}^{\Delta}(j\Delta)}|\Psi_k\rangle & \text{if } j\Delta\in\xi^{\Delta} \end{cases} \tag{2.23}$$

We can now put this into the continuous time context as follows: To a given sequence $\Psi,\Psi_{i_1},\dots,\Psi_{i_{\beta/\Delta-1}},\Psi'$ associate a piecewise constant function $\Psi^{\Delta} : [0, \beta] \to \{\Psi_j\}$, such that $\Psi^{\Delta} = \Psi$ on $[0, \Delta)$, $\Psi^{\Delta}(\beta) = \Psi'$, and,

$$\Psi^{\Delta} = \Psi_{i_j} \quad \text{on} \quad [j\Delta, (j+1)\Delta) \quad \text{for } j = 1, \ldots \beta/\Delta - 1.$$

Given a realization ξ^{Δ} let us say that a piecewise constant function Ψ^{Δ} as above is compatible with ξ^{Δ}, $\Psi^{\Delta} \sim \xi^{\Delta}$ if all the jumps of Ψ^{Δ} occur only at arrival times of ξ^{Δ}. By (2.23) only compatible functions contribute to (2.22). In fact, in the notation just introduced the latter expansion reads as,

$$\langle \Psi | \mathcal{K}_{\underline{a}_1, \ldots, \underline{a}_m} | \Psi' \rangle = \sum_{\Psi^{\Delta} \sim \xi^{\Delta}} \prod_{t \in \xi^{\Delta}} \langle \Psi^{\Delta}(t-) | K_{\mathfrak{l}^{\Delta}(t)} | \Psi^{\Delta}(t) \rangle. \tag{2.24}$$

Poisson Limits

A basic result on Poisson approximation implies that

$$\left(\xi_1^{\Delta}, \ldots, \xi_m^{\Delta}, \mathfrak{l}^{\Delta}\right) \Rightarrow \left(\xi_1, \ldots, \xi_m, \mathfrak{l}\right)$$

where $(\xi_1, \ldots, \xi_m)$ are independent Poisson point processes on $[0, \beta]$ with intensities $(\lambda_1, \ldots, \lambda_m)$ respectively. Let us use $\mathbb{P}_{\beta}^{\underline{\lambda}}$ for the distribution of the latter. By independence there are no simultaneous arrivals, that is the type $\mathfrak{l}(t) \in \{1, \ldots, m\}$ of an arrival is well defined for each $t \in \xi \stackrel{\Delta}{=} \cup \xi_l$. Furthermore, conditioned on the realization of ξ the arrival types $\mathfrak{l}(t)$ are independent and

$$\mathbb{P}_{\beta}^{\underline{\lambda}}\left(\mathfrak{l}(t) = l \,\middle|\, t \in \xi\right) = \frac{\lambda_l}{\lambda_1 + \cdots + \lambda_m}.$$

Passing to the limit $\Delta \to 0$ in (2.24) and (2.19), we arrive to the following representation of matrix elements of $\exp\{\beta \sum \lambda_l K_l\}$: For every two elements of the basis $\Psi, \Psi' \in \{\Psi_i\}$,

$$\frac{\langle \Psi | \mathrm{e}^{\beta \sum \lambda_l K_l} | \Psi' \rangle}{\exp\{\beta \sum_l \lambda_l\}} = \int \mathbb{P}_{\beta}^{\underline{\lambda}}(\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_m) \sum_{\Psi \sim \xi} \prod_{t \in \xi} \langle \Psi(t-) | K_{\mathfrak{l}(t)} | \Psi(t) \rangle, \tag{2.25}$$

where, given a realization of ξ the summation is over all ξ-compatible (having jumps only at arrival times of ξ) piecewise constant right-continuous functions $\Psi : [0, \beta] \mapsto \{\Psi_i\}$, which, in addition, satisfy boundary conditions $\Psi(0) = \Psi$ and $\Psi(\beta) = \Psi'$. Clearly, since $\mathbb{X}$ is finite dimensional, and since, there are $\mathbb{P}_{\beta}^{\underline{\lambda}}$-a.s. finite number of arrivals of ξ, there are $\mathbb{P}_{\beta}^{\underline{\lambda}}$-a.s. finitely many such compatible functions.

Formula (2.25) enables a re-interpretation of various quantities related to the Hamiltonians $\mathcal{H}$ in terms of stochastic geometry of the family of Poisson processes ξ. For example,

$$\frac{\mathsf{Tr}\left(e^{\beta\sum\lambda_l K_l}\right)}{\exp\left\{\beta\sum_l\lambda_l\right\}} = \int\mathbb{P}_\beta^\lambda(\mathrm{d}\xi)\sum_{\Psi\sim\xi}\langle\Psi(0)|\Psi(\beta)\rangle\prod_{t\in\xi}\langle\Psi(t-)|K_{\mathrm{l}(t)}|\Psi(t)\rangle. \tag{2.26}$$

In general, given a self-adjoint matrix A,

$$\frac{\mathsf{Tr}\left(Ae^{\beta\sum\lambda_l K_l}\right)}{\exp\left\{\beta\sum_l\lambda_l\right\}} = \int\mathbb{P}_\beta^\lambda(\mathrm{d}\xi)\sum_{\Psi\sim\xi}\langle\Psi(0)|A|\Psi(\beta)\rangle\prod_{t\in\xi}\langle\Psi(t-)|K_{\mathrm{l}(t)}|\Psi(t)\rangle. \tag{2.27}$$

The approach has two degrees of freedom to play with:

1. There are different ways to decompose $\mathcal{H}$ as $\mathcal{H} = -\sum\lambda_l K_l$.
2. There are different choices of orthonormal bases $\{\Psi_i\}$ of $\mathbb{X}$.

In the following two subsections we shall consider the FK and the RC (random current) representation of classical Ising systems (2.9) as different instances of the path integral representation (2.25). Then in Section 3 we shall develop the FK and the RC representation for genuine quantum systems in traverse magnetic field.

2.3 FK Representation

Classical FK representation corresponds to the decomposition of the Hamiltonian $\mathcal{H}_\Lambda$ in (2.9) as,

$$-\mathcal{H}_\Lambda = -\left(\sum_{(i,j)}J_{ij}+\sum_i h\right)\mathrm{I} + \sum_{(i,j)}2J_{ij}\frac{\mathrm{I}+\hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}}}{2} + \sum_i 2h\frac{\mathrm{I}+\hat\sigma_i^{\mathsf{z}}}{2},$$

with matrix elements of $e^{-\beta\mathcal{H}_\Lambda}$ being computed in the z-basis (2.6).

In the language of the preceeding Subsection, we are dealing with independent Poisson processes ξ_{ij} of arrivals of operators $K_{ij} \stackrel{\Delta}{=} \frac{\mathrm{I}+\hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}}}{2}$ with intensities $2J_{ij}$ and with independent Poisson processes ξ_i of arrivals of operators $K_i \stackrel{\Delta}{=} \frac{\mathrm{I}+\hat\sigma_i^{\mathsf{z}}}{2}$ with intensities $2h$ each. Let $\nu,\nu'\in\Omega_\Lambda$ be two classical configurations and let, as before, Ψ_ν and $\Psi_{\nu'}$ be the corresponding elements of the basis of $\mathbb{X}_\Lambda$. Then,

$$\langle\Psi_\nu|K_{ij}|\Psi_{\nu'}\rangle = \delta_{\{\nu=\nu'\}}\delta_{\{\nu_i=\nu_j\}}. \tag{2.28}$$

Similarly,

$$\langle \Psi_\nu | K_i | \Psi_{\nu'} \rangle = \delta_{\{\nu=\nu'\}} \delta_{\{\nu_i=1\}}. \tag{2.29}$$

Due to our choice of the orthonormal basis, any piecewise constant function $\Psi : [0,\beta] \mapsto \{\Psi_\nu\}$ is of the form $\Psi_{\nu(\bullet)}$, where $\nu : [0,\beta] \mapsto \Omega_\Lambda$ is a piecewise constant classical spin configuration valued function. In fact relations (2.28) and (2.29) imply that, whatever are the realizations of Poisson processes $\xi = \{\xi_{ij}, \xi_i\}$ the only compatible $\nu \sim \xi$ are constant configurations $\nu(\bullet) \equiv$ *const*. Furthermore, an arrival of K_{ij} at time t imposes and additional constraint $\nu_i(t) = \nu_j(t)$, whereas an arrival of K_i imposes an additional constraint $\nu_i(t) = 1$. It is convenient to explore (2.25) in terms of the following graphical representation (see Figure 1 below):

To each site $i \in \Lambda$ we attach a time interval $\mathbb{S}_\beta \stackrel{\Delta}{=} [0,\beta]$. In order to distinguish between intervals attached to different sites we use notation $\mathbb{S}^i_\beta$. Points on $\mathbb{S}^i_\beta$ labeled as (i,t). An arrival of ξ_{ij} at time t is visualized as a link between (i,t) and (j,t). An arrival of ξ_i at time t puts a $*$-mark at (i,t). It is also convenient to think about all $*$-marks being linked (wired) to some ghost site $\mathfrak{g}$. Two intervals $\mathbb{S}^i_\beta$ and $\mathbb{S}^j_\beta$ are said to be connected if $\xi_{ij} \neq \emptyset$. Thus, any realization of $\{\xi_{ij}\}$ splits

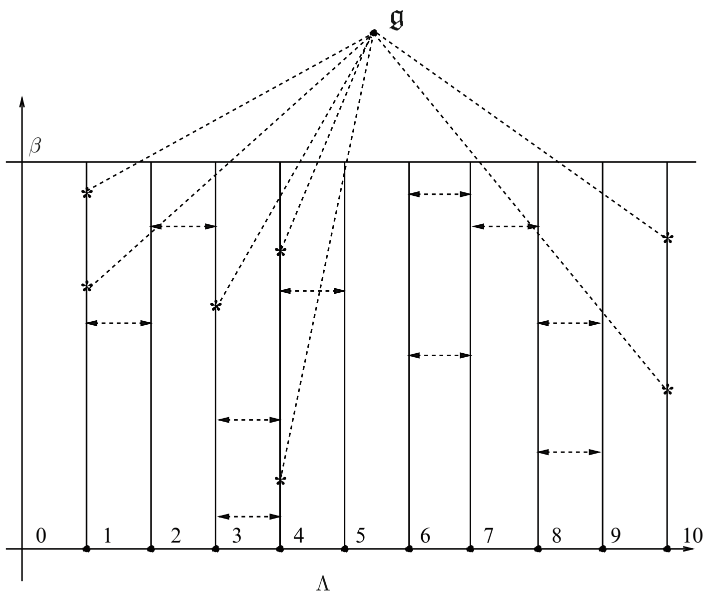

Fig. 1. The box Λ is split into three connected components, $\mathcal{C}_1 = \{1,2,3,4,5\} \times \mathbb{S}_\beta$, $\mathcal{C}_2 = \{6,7,8,9\} \times \mathbb{S}_\beta$ and $\mathcal{C}_3 = \{10\} \times \mathbb{S}_\beta$. Components $\mathcal{C}_1$ and $\mathcal{C}_3$ are wired, whereas $\mathcal{C}_2$ is free

$$\Lambda \times \mathbb{S}_\beta = \bigcup_{i\in\Lambda} \mathbb{S}_\beta^i = \cup \mathcal{C}_l$$

into the union of maximal connected components. Of course, each $\mathcal{C}_l$ above corresponds to a subset $\mathcal{A}_l$ of Λ,

$$\mathcal{C}_l = \bigcup_{i\in\mathcal{A}_l} \mathbb{S}_\beta^i .$$

A component $\mathcal{C}_l$ is said to be wired if $\xi_i \neq \emptyset$ for some $i \in \mathcal{A}_l$. It is convenient to link all wired components into one connected component. Given a realization $\xi = \{\xi_{ij}, \xi_i\}$ of all Poisson processes of arrivals of operators K_{ij} and K_i let $\#_{\mathsf{w}}(\xi)$ be the number of all maximal connected components $\mathcal{C}_l$ which are *not* wired to the ghost site $\mathfrak{g}$. Then the number of (constant) classical trajectories which satisfy (2.28) and (2.29) is precisely $2^{\#_{\mathsf{w}}(\xi)}$. For each such trajectory $\nu(\bullet) \equiv \nu$,

$$\prod_{t\in\xi} \langle \Psi_\nu | K_{\mathfrak{l}(t)} | \Psi_\nu \rangle = 1.$$

Consequently, let $\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}$ be the (Poisson) distribution of ξ. Then, (2.25) implies,

$$\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right) = \mathrm{e}^{\beta(\sum_{(i,j)} J_{ij} + \sum_i h)} \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(2^{\#_{\mathsf{w}}(\xi)}\right). \tag{2.30}$$

Define a new measure $\widetilde{\mathbb{P}}^{\mathbf{J},h}_{\beta,\Lambda}$ on trajectories of point processes ξ,

$$\widetilde{\mathbb{P}}^{\mathbf{J},h}_{\beta,\Lambda}(\mathrm{d}\xi) = \frac{2^{\#_{\mathsf{w}}(\xi)} \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}(\mathrm{d}\xi)}{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(2^{\#_{\mathsf{w}}(\xi)}\right)}. \tag{2.31}$$

$\widetilde{\mathbb{P}}^{\mathbf{J},h}_{\beta,\Lambda}$ is called FK or random cluster measure. Using (2.11) and (2.27) we arrive to the following stochastic geometric representation of classical expectations,

$$\mu^{\beta,h}_\Lambda(\nu_i) = \widetilde{\mathbb{P}}^{\mathbf{J},h}_{\beta,\Lambda}(i \longleftrightarrow \mathfrak{g}) \qquad \text{and} \qquad \mu^{\beta,h}_\Lambda(\nu_i\nu_j) = \widetilde{\mathbb{P}}^{\mathbf{J},h}_{\beta,\Lambda}(i \longleftrightarrow j), \tag{2.32}$$

where the event $\{i \longleftrightarrow \mathfrak{g}\}$ means that the connected component of $\mathbb{S}_\beta^i$ is wired, whereas $\{i \longleftrightarrow j\}$ means that $\mathbb{S}_\beta^i$ and $\mathbb{S}_\beta^j$ belong to the same connected component (including the case when $\{i \longleftrightarrow \mathfrak{g}\} \cap \{j \longleftrightarrow \mathfrak{g}\}$).

2.4 Random Current Representation

In its turn classical RC representation corresponds to the decomposition of the Hamiltonian $\mathcal{H}_\Lambda$ in (2.9) as,

$$-\mathcal{H}_\Lambda = \sum_{(i,j)} J_{ij}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}} + \sum_i h\hat{\sigma}_i^{\mathsf{z}}.$$

The trick is to compute matrix elements of $\mathrm{e}^{-\beta\mathcal{H}_\Lambda}$ in the x-basis, which is defined as follows: With $\psi_{\pm 1}$ being defined as in (2.5), set

$$\phi_{\pm 1} = \frac{1}{\sqrt{2}}\left(\psi_1 \pm \psi_{-1}\right). \tag{2.33}$$

Clearly $\{\phi_{-1}, \phi_1\}$ is an orthonormal basis of $\mathbb{R}^2$. To a given classical x-configuration $\vartheta \in \Omega_\Lambda$ one corresponds the vector,

$$\Phi_\vartheta = \otimes_{i\in\Lambda}\phi_{\vartheta_i}. \tag{2.34}$$

The collection $\{\Phi_\vartheta\}$ is an orthonormal basis of $\mathbb{X}_\Lambda$. In the x-basis Pauli matrix $\hat{\sigma}^{\mathsf{z}}$ looks like

$$\hat{\sigma}^{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{or} \quad \hat{\sigma}^{\mathsf{z}}\phi_{\pm 1} = \phi_{\mp 1}. \tag{2.35}$$

Thus in the x-basis $\hat{\sigma}^{\mathsf{z}}$ is just a spin-flip operator. As in (2.7) the action of $\hat{\sigma}_i^{\mathsf{z}}$ on Φ_ϑ is given by

$$\hat{\sigma}_i^{\mathsf{z}}\Phi_\vartheta = \phi_{\vartheta_1} \otimes \cdots \otimes \hat{\sigma}^{\mathsf{z}}\phi_{\vartheta_i} \otimes \dots .$$

In other words, $\hat{\sigma}_i^{\mathsf{z}}$ flips i-th component of Φ_ϑ.

In the language of Subsection 2.2, we are dealing with independent Poisson processes ξ_{ij} of arrivals of operators $K_{ij} \stackrel{\Delta}{=} \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}$ with intensities J_{ij} and with independent Poisson processes ξ_i of arrivals of operators $K_i \stackrel{\Delta}{=} \hat{\sigma}_i^{\mathsf{z}}$ with intensities h each. We continue to employ the running notation $\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},h}$ for the distribution of ξ.

Let $\vartheta, \vartheta' \in \Omega_\Lambda$ be two classical x-configurations and let, as before, Φ_ϑ and $\Phi_{\vartheta'}$ be the corresponding elements of the x-basis of $\mathbb{X}_\Lambda$. Then,

$$\langle\Psi_\vartheta|K_{ij}|\Psi_{\nu'}\rangle = \delta_{\left\{\vartheta'=\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}\vartheta\right\}} \quad \text{and} \quad \langle\Psi_\vartheta|K_i|\Psi_{\nu'}\rangle = \delta_{\left\{\vartheta'=\hat{\sigma}_i^{\mathsf{z}}\vartheta\right\}}. \tag{2.36}$$

In other words, each arrival of operator K_{ij} enforces a simultaneous flip of i-th and j-th coordinate of Φ, and each arrival of operator K_i enforces a flip of i-th coordinate of Φ. Therefore, given a realization of

ξ, compatible space-time configurations $\Phi(\cdot) \sim \xi$ are deterministically recovered from the initial value $\Phi(0)$. Therefore, there are exactly $2^{|\Lambda|}$ compatible configurations for each realization of ξ.

Consider now the representation of the trace in (2.26). Clearly a space-time configuration $\Phi(\cdot)$ contributes only if $\Phi(0) = \Phi(\beta)$. In view of the above description of action of operators K_{ij} and K_i, this obviously imposes a restriction on admissible realization of ξ: Namely, there are trajectories $\Phi \sim \xi$ with $\Phi(0) = \Phi(\beta)$ if and only if ξ flips each coordinate $i \in \Lambda$ *even* number of times.

With a slight abuse of notation let ξ_{ij} and ξ_i also denote the number of arrivals of K_{ij}, respectively K_i on the interval $[0, \beta]$. In this way ξ will be called random currents. The total current through $i \in \Lambda$ is $\xi[i] = \sum_j \xi_{ij} + \xi_i$ and the total current through the ghost site $\mathfrak{g}$ is $\xi[\mathfrak{g}] = \sum_i \xi_i$. The boundary of a current is,

$$\partial\xi \overset{\Delta}{=} \{u \in \Lambda \cup \mathfrak{g} \,:\, \xi[u] \text{ is odd}\}\,. \tag{2.37}$$

If $\partial\xi = \emptyset$, then all of $2^{|\Lambda|}$ compatible configurations $\Psi(\bullet) \sim \xi$ satisfy $\Phi(0) = \Phi(\beta)$, otherwise (if $\partial\xi \neq \emptyset$) none of them is periodic. Consequently, (2.26) implies,

$$\frac{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathrm{e}^{\beta(\sum_{(i,j)} J_{ij} + \sum_i h)}} = 2^{|\Lambda|}\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi = \emptyset\right). \tag{2.38}$$

The following representation of one and two point functions is now almost straightforward,

$$\mu^{\beta,h}_\Lambda(\nu_i) = \frac{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi = \{i, \mathfrak{g}\}\right)}{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi = \emptyset\right)} \text{ and } \mu^{\beta,h}_\Lambda(\nu_i\nu_j) = \frac{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi = \{i, j\}\right)}{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi = \emptyset\right)} \tag{2.39}$$

Switching Lemma

Let ξ and η be two independent random currents distributed according to the product Poisson measure $\mathbb{P}^{\mathbf{J},h}_\beta$ each. We continue to $\mathbb{P}^{\mathbf{J},h}_\beta$ to denote the product measure. Then, for every i, j and for every subset $A \subseteq \Lambda \cup \mathfrak{g}$,

$$\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi = \{i, j\}\right)\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\eta = A\right) = \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\big(\partial\xi = \emptyset; \partial\eta = A\Delta\{i, j\}\,; i \xleftrightarrow{\xi+\eta} j\big), \tag{2.40}$$

where the event $\left\{i \xleftrightarrow{\xi+\eta} j\right\}$ means that there exists a path of bonds $b \in \mathcal{E}$ from i to j with $\xi(b) + \eta(b) > 0$.

We refer to [1] for a proof of (2.40). In view of (2.39) an immediate consequence is the following representation of the truncated two-point function:

$$\mu_{\Lambda}^{\beta,h}(\nu_i\nu_j) - \mu_{\Lambda}^{\beta,h}(\nu_i)\,\mu_{\Lambda}^{\beta,h}(\nu_j) = \frac{\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},h}\Big(\partial\xi = \emptyset; \partial\nu = \{i,j\}\,; i \overset{\xi+\nu}{\not\longleftrightarrow} \mathfrak{g}\Big)}{\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},h}(\partial\xi = \emptyset; \partial\eta = \emptyset)}. \tag{2.41}$$

Exponential Decay of Two-point Functions at Non-zero Magnetic Fields

Representation (2.41) and similar formulas pave the way for a stochastic geometric interpretation of semi-invariants and give rise to a useful intuition. As an example let us show how (2.41) implies that classical Ising truncated two-point functions always have non-zero exponential rate of decay once $h \neq 0$. The argument below was developed together with Roberto Fernandez and Yvan Velenik some time ago. As it was pointed out by Yvan, a conventional proof could be found in [12].

Let $\kappa = \xi + \eta$ be the combined current. Any realization of κ splits Λ into a disjoint union of maximal connected components: as before we say that i and j are connected if there exists a chain of bonds b leading from i to j with $\kappa(b) > 0$ on each bond. Clearly $\{\partial\xi = \emptyset; \partial\eta = \{i,j\}\}$ implies that $\partial\kappa = \{i,j\}$ and, in particular that i and j are connected in κ or, in other words, that i and j belong to the same connected component C of κ. If R is the range of interaction, then $|C| \geq |i-j|/R$, as soon as we impose an additional constraint $\big\{C \overset{\kappa}{\not\longleftrightarrow} \mathfrak{g}\big\}$. It is almost obvious now why (2.41) implies exponential decay: one should pay a fixed price to disconnect each site $l \in C$ from the ghost site $\mathfrak{g}$, see Figure 2.

It remains to make the last remark precise. For any connected set $C \subset \Lambda$ define $\#(\mathcal{E}(C, \Lambda \setminus C))$ as the number of edges in $\mathcal{E}(C, \Lambda \setminus C)$, where the latter is the set of edges b with $J_b > 0$, which have one end-point in C and another in $\Lambda \setminus C$. The probability $p(C, \Lambda \setminus C)$ that none of the processes κ_b; $b \in \mathcal{E}(C, \Lambda \setminus C)$ arrives on the interval $[0, \beta]$ is

$$p(C, \Lambda \setminus C) = \exp\Big\{-2\beta \sum_{b \in \mathcal{E}(C,\Lambda\setminus C)} J_b\Big\}.$$

Given a connected set C and $i, j \in C$, define the following event

$$\mathcal{A}_{ij}(C) = \Big\{\partial\xi_C = \emptyset; \partial\eta_C = \{i,j\}\,;\ C \text{ is connected in } \kappa_C;\ C \overset{\kappa_C}{\not\longleftrightarrow} \mathfrak{g}\Big\},$$

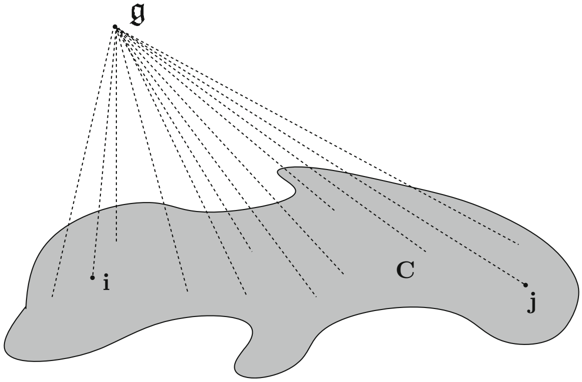

Fig. 2. Each site l inside the connected component C has a chance to be connected by a direct non-zero current to the ghost site $\mathfrak{g}$

where ξ_C, η_C (respectively $\xi_{\Lambda\setminus C}, \eta_{\Lambda\setminus C}$) and κ_C (respectively $\kappa_{\Lambda\setminus C}$) are the restrictions of the corresponding processes to the bonds with either both end-points at C ($\Lambda \setminus C$) or with one end-point at C ($\Lambda \setminus C$) and another end-point being $\mathfrak{g}$. In this notation the expression in the numerator in (2.41)

$$\sum_{C \text{ connected}} \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi_{\Lambda\setminus C} = \emptyset; \partial\eta_{\Lambda\setminus C} = \emptyset\right) \cdot p(C, \Lambda \setminus C) \cdot \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\mathcal{A}_{ij}(C)\right). \tag{2.42}$$

On the other hand, the denominator in (2.41) is certainly bounded below by

$$\sum_{C \text{ connected}} \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\partial\xi_{\Lambda\setminus C} = \emptyset; \partial\eta_{\Lambda\setminus C} = \emptyset\right) \cdot p(C, \Lambda \setminus C) \cdot \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\mathcal{A}^{\mathrm{e}}_{ij}(C)\right), \tag{2.43}$$

where the event

$$\mathcal{A}^{\mathrm{e}}_{ij}(C) = \{\partial\xi_C = \emptyset; \partial\eta_C = \emptyset;\ C \text{ is connected in } \kappa_C\}.$$

We claim that there exist two positive constants c_1 and c_2 which depend on β, h (but not on the range R of the interaction, the dimension of the lattice, connected C and $\{i,j\} \subseteq C$), such that,

$$\frac{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\mathcal{A}_{ij}(C)\right)}{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\mathcal{A}^{\mathrm{e}}_{ij}(C)\right)} \leq c_1 \mathrm{e}^{-c_2|i-j|/R}. \tag{2.44}$$

Indeed, for each current η_C with $\partial\eta = \{i, j\}$ and $\mathfrak{g} \overset{\eta_C}{\not\longleftrightarrow} C$, we may construct a family of currents

$$\{\eta_C\}^{\rm e} = \left\{\eta + (2r_i+1)\delta_{(i,\mathfrak{g})} + (2r_j+1)\delta_{(j,\mathfrak{g})} + \sum_{k\in C\setminus\{i,j\}} 2r_k\delta_{(k,\mathfrak{g})}\right\}_{r_l=0,1,\dots \text{for } l\in C}.$$

Thus, the family $\{\eta_C\}^{\rm e}$ is generated by tuples $\underline{r} = \{r_l\}_{l\in\Lambda}$ of non-negative integers. Evidently,

$$\{\eta_C\}^{\rm e} \cap \{\eta'_C\}^{\rm e} = \emptyset,$$

whenever $\eta_C \neq \eta'_C$. Furthermore, $(\xi_C, \eta_C) \in \mathcal{A}_{ij}(C) \Rightarrow \xi_C \times \{\eta_C\}^{\rm e} \subseteq \mathcal{A}^{\rm e}_{ij}(C)$. However, for such η_C,

$$\frac{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\{\eta_C\}^{\rm e}\right)}{\mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\left(\eta_C\right)} = (\sinh(\beta h))^2 \cdot (\cosh(\beta h))^{|C|-2},$$

and (2.44) follows.

3 Quantum Ising Models in Transverse Field

Quantum Ising Hamiltonian in transverse field λ is given by

$$-\mathcal{H}_\Lambda = \sum_{(i,j)} J_{ij}\hat{\sigma}^{\mathsf{z}}_i\hat{\sigma}^{\mathsf{x}}_j + h\sum_i \hat{\sigma}^{\mathsf{z}}_i + \lambda\sum_i \hat{\sigma}^{\mathsf{x}}_i, \tag{3.1}$$

where $\lambda \geq 0$, and (in the z-basis),

$$\hat{\sigma}^{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{and} \quad \hat{\sigma}^{\mathsf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.2}$$

Since matrices $\hat{\sigma}^{\mathsf{x}}$ and $\hat{\sigma}^{\mathsf{z}}$ do not commute, as soon as the strength of the transverse field $\lambda > 0$, the operator $\mathcal{H}_\Lambda$ does not have diagonal form neither in z-basis (2.6), nor in the x-basis (2.34). Nevertheless, the analog of Lie-Trotter product formula still holds,

$$\mathrm{e}^{-\beta\mathcal{H}_\Lambda} = \lim_{\Delta\to 0}\left(\prod_{(i,j)} \mathrm{e}^{\Delta J_{ij}\hat{\sigma}^{\mathsf{z}}_i\hat{\sigma}^{\mathsf{z}}_j} \prod_i \mathrm{e}^{\Delta h\hat{\sigma}^{\mathsf{z}}_i} \prod_i \mathrm{e}^{\Delta\lambda\hat{\sigma}^{\mathsf{x}}_i}\right)^{\beta/\Delta}. \tag{3.3}$$

As in the classical case various choices of bases and of decomposition of $\mathcal{H}_\Lambda$ lead to different stochastic geometric representations of the model.

3.1 FK Representation

As in the classical case the traces are computed in the z-basis. As for the decomposition represent $-\mathcal{H}_\Lambda$ as

$$-\Big(\sum_{(i,j)} J_{ij} + \sum_i h + \sum_i \lambda\Big)\mathrm{I} + \sum_{(i,j)} 2J_{ij}\frac{\mathrm{I} + \hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}}}{2} + \sum_i 2h\frac{\mathrm{I} + \hat\sigma_i^{\mathsf{z}}}{2} + \sum_i \lambda(\hat\sigma_i^{\mathsf{x}} + \mathrm{I}).$$

In the language of Subsection 2.2 we are dealing with Poisson process ξ of arrivals on the interval $[0,\beta]$ of the following type of operators:

- Operators $K_{ij} = \left(\hat\sigma_i^{\mathsf{z}}\hat\sigma_j^{\mathsf{z}} + \mathrm{I}\right)/2$ which arrive with intensities $2J_{ij}$. We shall call these processes *links* and denote them as ξ_{ij}.
- Operators $K_i^h = \left(\hat\sigma_i^{\mathsf{z}} + \mathrm{I}\right)/2$ which arrive with intensities $2h$. We shall call these processes *links to* $\mathfrak{g}$ and denote them as ξ_i^h.
- Operators $K_i^\lambda = \hat\sigma_i^{\mathsf{x}} + \mathrm{I}$ which arrive with intensities λ. We shall call these processes *holes* and denote them as ξ_i^λ.

As in Subsection 2.3 piece-wise constant functions $\Psi : [0,\beta] \mapsto \{\Psi_\nu\}$ are labeled by piece-wise constant classical trajectories $\nu : [0,\beta] \mapsto \Omega_\Lambda$. Given a realization $\xi = \left\{\xi_{ij}, \xi_i^h, \xi_i^\lambda\right\}$ let us try to describe the family of compatible trajectories $\nu \sim \xi$.

1. Since $\langle\Psi_\nu|K_{ij}|\Psi_{\nu'}\rangle = \delta_{\{\nu=\nu'\}}\delta_{\{\nu_i=\nu_j\}}$, an arrival of an (i,j)-link at time t imposes the constraint $\nu(t,i) = \nu(t,j)$.
2. Since $\langle\Psi_\nu|K_i^h|\Psi_{\nu'}\rangle = \delta_{\{\nu=\nu'\}}\delta_{\{\nu_i=1\}}$, an arrival of an $(i,\mathfrak{g})$-link at time t imposes the constraint $\nu(t,i) = 1$.
3. Since
$$\langle\Psi_\nu|K_i^\lambda|\Psi_{\nu'}\rangle = \delta_{\left\{\nu_j=\nu_j' \quad \text{for all } j\neq i\right\}},$$
an arrival of an i-hole at time t *enables* a flip of i-th coordinate of $\nu(t,\cdot)$.

Thus, contrary to the classical situation considered in Subsection 2.3, compatible configurations $\nu \sim \xi$ are permitted to have jumps at arrival times of ξ^λ. It is convenient to visualize compatible *periodic* $\nu(\cdot)$ as follows (see Figure 3): For each $i \in \Lambda$ the process of holes ξ_i^λ splits the circle $\mathbb{S}_\beta^i$ (which is the interval $i \times [0,\beta]$ with the end-points $(i,0)$ and (i,β) identified) into a disjoint union of connected intervals. Two such intervals $i \times I \subseteq \mathbb{S}_\beta^i$ and $j \times J \subseteq \mathbb{S}_\beta^j$ are said to be connected in ξ if there is an arrival of ξ_{ij} at a time $t \in I \cap J$. A maximal connected cluster $\cup_l \{i_l \times I_l\}$ (with i_l-s being not necessarily different, but with $\{i_l \times I_l\} \cap \{i_m \times I_m\} = \emptyset$ whenever $l \neq m$) is said to be connected to the ghost site $\mathfrak{g}$ if for some for some i_l a process $\xi_{i_l}^h$ arrives at $t \in I_l$. Otherwise

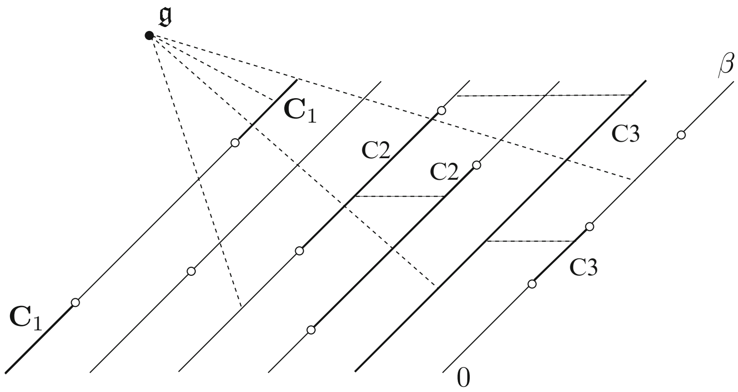

Fig. 3. Configurations with periodic boundary conditions $\nu(i,0) = \nu(i,\beta)$. Connected components $\mathbf{C}_1$ and $\mathbf{C}_3$ are linked to $\mathfrak{g}$ and hence $\nu \equiv 1$ on them. Connected component $\mathbf{C}_2$ is "free" and hence one can colour it in either of ± 1 colours.

such maximal connected cluster is called free. Define $\#_{\mathsf{w}}(\xi)$ to be the number of maximal free connected clusters of ξ. Then, using $\mathbb{P}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}$ for the reference product distribution of independent Poisson processes ξ, we arrive to the following quantum version of the FK representation (2.30) of the trace,

$$\mathsf{Tr}\big(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\big) = \mathrm{e}^{\beta(\sum_{(i,j)} J_{ij} + \sum_i h + \sum_i \lambda)} \mathbb{P}^{\mathbf{J},h}_{\beta,\Lambda}\big(2^{\#_{\mathsf{w}}(\xi)}\big). \tag{3.4}$$

As in the classical case define a new measure $\widetilde{\mathbb{P}}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}$ on trajectories of point processes ξ,

$$\widetilde{\mathbb{P}}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}(\mathrm{d}\xi) = \frac{2^{\#_{\mathsf{w}}(\xi)} \mathbb{P}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}(\mathrm{d}\xi)}{\mathbb{P}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}\big(2^{\#_{\mathsf{w}}(\xi)}\big)}. \tag{3.5}$$

Once again, using (2.11) and (2.27) we arrive to the following stochastic geometric representation of expectations,

$$\frac{\mathsf{Tr}\left(\hat{\sigma}^{\mathsf{z}}_i \mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)} = \widetilde{\mathbb{P}}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}\left((i,0) \longleftrightarrow \mathfrak{g}\right) \tag{3.6}$$

where the event $\{(i,0) \longleftrightarrow \mathfrak{g}\}$ means that the $\mathbb{S}^i_\beta$ interval containing $(i,0)$ belongs to a cluster which is connected to $\mathfrak{g}$. Similarly,

$$\frac{\mathsf{Tr}\big(\hat{\sigma}^{\mathsf{z}}_i \hat{\sigma}^{\mathsf{z}}_j \mathrm{e}^{-\beta\mathcal{H}_\Lambda}\big)}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)} = \widetilde{\mathbb{P}}^{\mathbf{J},h,\lambda}_{\beta,\Lambda}\left((i,0) \longleftrightarrow (j,0)\right), \tag{3.7}$$

where the event $\{(i,0) \longleftrightarrow (j,0)\}$ means that the corresponding $\mathbb{S}_\beta^i$ and $\mathbb{S}_\beta^j$ intervals belong to the same connected cluster.

Ground States, Matrix and Reduced Density Matrix Elements Let us fix a finite graph $(\Lambda, \mathcal{E})$, coupling constants $\mathbf{J}$ and $\lambda \geq 0$. In order to facilitate the notation we shall set magnetic filed in z-direction to zero, $h = 0$. For each $\beta \in \mathbb{R}$, z-matrix elements $\rho_\beta^{\mathsf{z}}(\nu, \nu')$ are defined via,

$$\rho_\beta^{\mathsf{z}}(\nu, \nu') \;=\; \frac{\langle \Psi_\nu | \mathrm{e}^{-\beta \mathcal{H}_\Lambda} | \Psi_{\nu'} \rangle}{\mathsf{Tr}\left(\mathrm{e}^{-\beta \mathcal{H}_\Lambda}\right)}. \tag{3.8}$$

In order to derive an appropriate expression in terms of Poisson arrival measures $\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},\lambda}$ or in terms of the FK measures $\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},\lambda}$ we should introduce a modification of the notion of connected components of ξ. Originally, those were defined as unions of sub-intervals of $\mathbb{S}_\beta$. However, in the computation of matrix elements we, obviously, do not impose periodicity conditions. In the sequel, given a subset $A \subset \Lambda$ and a configuration ξ let ξ_A be obtained from ξ via adding holes at all the points $(i,0) = (i,\beta)$ with $i \in A$. One can think about ξ_A in terms of slitting the A-part of ξ along $t = 0$.

Any piece-wise trajectory $\nu : [0,\beta] \to \Omega_\Lambda$ which contributes to the numerator in (3.8) satisfies boundary conditions,

$$\nu(i,0) \;=\; \nu_i \quad \text{and} \quad \nu(i,\beta) \;=\; \nu_i' \quad \forall\, i \in \Lambda.$$

As a result, realizations of ξ which place points (i,T) and (j,S) (with $i,j \in \Lambda$ and $T,S = 0 \,\text{or}\, \beta$) with $\nu(i,T) \neq \nu(j,S)$ into same connected components of the slit configuration ξ_Λ do not have compatible trajectories at all. Let us say that $\xi_\Lambda \sim \{\nu,\nu'\}$, if the latter does not happen. If $\xi_\Lambda \sim \{\nu,\nu'\}$, then the set of all ξ-compatible trajectories, which contribute to the denominator in (3.8) is constructed in the following way: Each connected cluster of ξ_Λ whose closure hits either $t = 0$ or $t = \beta$ layers inherits the z-spin value from ν or ν'. On the other hand, each interior cluster of ξ_Λ or, alternatively each cluster of ξ which does not contain points with $0 = \beta$ time coordinates, could be still coloured into ± 1. Clusters of ξ which are not interior are called boundary. Thus, if we use $\#_0(\xi)$ and $\#_\partial(\xi) = \#(\xi) - \#_0(\xi)$ for the number of interior (respectively boundary) clusters of ξ,

$$\rho_\beta^{\mathsf{z}}(\nu,\nu') = \frac{\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},\lambda}\left(\xi_\Lambda \sim \{\nu,\nu'\}\ ;\ 2^{\#_0(\xi)}\right)}{\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},\lambda}\left(2^{\#(\xi)}\right)} = \widetilde{\mathbb{P}}_{\beta,\Lambda}^{\mathbf{J},\lambda}\bigl(\xi_\Lambda \sim \{\nu,\nu'\}\ ;\ 2^{-\#_\partial(\xi)}\bigr). \tag{3.9}$$

For each $\lambda > 0$ there exist non-trivial limits $\mathbb{P}^{\mathbf{J},\lambda}_{\infty,\Lambda}$ and $\widetilde{\mathbb{P}}^{\mathbf{J},\lambda}_{\infty,\Lambda}$ as $\beta \to \infty$. These measures could be constructed directly: $\mathbb{P}^{\mathbf{J},\lambda}_{\infty,\Lambda}$ is just the distribution of Poisson processes of arrival ξ on $\mathbb{R}$. Connected components of ξ are understood now as linked sub-intervals of $\mathbb{R}$ over various spatial coordinates $i \in \Lambda$. The FK measure $\widetilde{\mathbb{P}}^{\mathbf{J},\lambda}_{\infty,\Lambda}$ is then constructed via modification of $\mathbb{P}^{\mathbf{J},\lambda}_{\infty,\Lambda}$ by the $2^{\#(\xi)}$ factor (as a limiting procedure, of course). Boundary clusters of the slit configuration ξ_Λ are coloured in this way according to ν just above the $t = 0$ layer and according to ν' just below it. If we slit along all of Λ, then the compatibility condition $\xi_\Lambda \sim \{\nu, \nu'\}$ decouples into $\{\xi_+ \sim \nu\} \cap \{\xi_- \sim \nu'\}$ for the upper and lower halves ξ_+ and ξ_- of configuration ξ. At this point it makes sense to introduce Poisson $\mathbb{P}^{\mathbf{J},\lambda,+}_{\infty,\Lambda}$ and, accordingly, FK $\widetilde{\mathbb{P}}^{\mathbf{J},\lambda,+}_{\infty,\Lambda}$ measures for arrival processes on $\mathbb{R}_+$. It is straightforward now to check that matrix elements $\rho^{\mathsf{z}}_{\infty}(\nu, \nu') = \langle \Psi_\nu | \Psi \rangle \langle \Psi | \Psi_{\nu'} \rangle$, which are generated by projections of the ground state Ψ of $\mathcal{H}_\Lambda$ are given by,

$$\rho^{\mathsf{z}}_{\infty}(\nu, \nu') \;=\; \widetilde{\mathbb{P}}^{\mathbf{J},\lambda}_{\infty,\Lambda}\bigl(\xi_\Lambda \sim \bigl\{\nu, \nu'\bigr\}\;;\; 2^{-\#_\partial(\xi)}\bigr). \qquad (3.10)$$

In the notation just introduced above the latter expression equals to

$$\langle \Psi_\nu | \Psi \rangle \langle \Psi | \Psi_{\nu'} \rangle \;=\; \widetilde{\mathbb{P}}^{\mathbf{J},\lambda,+}_{\infty,\Lambda}\bigl(\xi \sim \nu;\, 2^{-\#_\partial(\xi)}\bigr)\widetilde{\mathbb{P}}^{\mathbf{J},\lambda,+}_{\infty,\Lambda}\bigl(\xi \sim \nu';\, 2^{-\#_\partial(\xi)}\bigr).$$

Similarly, for $A \subseteq \Lambda$ and $\theta, \theta' \in \{\pm 1\}^A$, the reduced density matrix entry $\rho^{\mathsf{z}}_{\infty,A}(\theta, \theta')$ is given by

$$\rho^{\mathsf{z}}_{\infty,A}(\theta, \theta') \;=\; \widetilde{\mathbb{P}}^{\mathbf{J},\lambda}_{\infty,\Lambda}\bigl(\xi_A \sim \bigl\{\theta, \theta'\bigr\}\;;\; 2^{-\#_{\partial,A}(\xi)}\bigr), \qquad (3.11)$$

where the compatibility condition $\xi_A \sim \{\nu, \nu'\}$ for the slit configuration ξ_A is defined in the obvious way, and $\#_{\partial,A}(\xi)$ stands for the number of connected clusters of ξ which contain points $(0, i)$ with $i \in A$.

3.2 Random Current Representation

In order to derive an appropriate version of random current representation let us rewrite the Hamiltonian (3.1) as

$$-\Bigl(\sum_i \lambda\Bigr)\mathrm{I} + \sum_{(i,j)} J_{ij}\hat{\sigma}^{\mathsf{z}}_i\hat{\sigma}^{\mathsf{z}}_j + \sum_i h\hat{\sigma}^{\mathsf{z}}_i + \sum_i 2\lambda\frac{\hat{\sigma}^{\mathsf{x}}_i + \mathrm{I}}{2}.$$

As in the classical case the traces are going to be computed in the x-basis (2.34). Thus, in the language of Subsection 2.2 we are dealing with Poisson process ξ of independent arrivals on $[0,\beta]$ of the following type of operators:

- Operators of simultaneous (ij)-flips $K_{ij} = \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}$ which arrive with intensities J_{ij}. We shall denote the corresponding Poisson process ξ_{ij}.
- Operators of i-flips $K_i^h = \hat{\sigma}_i^{\mathsf{z}}$ which arrive with intensity h each. The corresponding Poisson processes are denoted as ξ_i^h.
- Operators $K_i^\lambda = (\hat{\sigma}^{\mathsf{x}} + \mathrm{I})/2$ which arrive with intensity 2λ each. The corresponding Poisson process is denoted ξ_i^λ. Since,

$$\langle \Phi_\vartheta | K_i^\lambda | \Phi_{\vartheta'} \rangle \;=\; \delta_{\{\vartheta = \vartheta'\}}\delta_{\{\vartheta_i = 1\}},$$

  an arrival of ξ_i^λ at time t imposes the constraint $\vartheta(i,t) = 1$ for every ξ-compatible classical piece-wise constant x-trajectory $\vartheta : [0,\beta] \mapsto \Omega_\Lambda$. We shall refer to ξ^λ as to processes of *marks*.

Accordingly, for a given realization of ξ compatible *periodic* piece-wise constant trajectories $\vartheta(\cdot)$ are characterized as follows:

1. Arrivals of ξ_{ij} and of ξ_i^h enforce simultaneous flips of i-th and j-th coordinates of ϑ, respectively of i-th coordinate of ϑ. These are the only jumps of $\vartheta(\cdot)$.
2. For each $i \in \Lambda$, $\vartheta(i,t) = 1$ at all arrival times of ξ_i^λ.

Let us try to compute the number of ξ-compatible trajectories ϑ for a given realization ξ. It is natural to modify the notion of the boundary $\partial\xi$ as follows: For every $i \in \Lambda$ the process of marks ξ_i^λ splits the circle $\mathbb{S}_\beta^i$ into the disjoint union of intervals,

$$\mathbb{S}_\beta^i \setminus \xi_i^\lambda \;=\; \cup_{l=1}^{m(i)} J_l^{(i)} \;\stackrel{\Delta}{=}\; \cup_{l=1}^{m(i)} i_l \times I_l^{(i)}. \tag{3.12}$$

The number $m(i)$ of such disjoint intervals equals to 1 if $\xi_i^\lambda = 0$ and to ξ_i^λ otherwise. Let us say that an interval $J_l^{(i)}$ in the decomposition (3.12) belongs to the boundary $\partial\xi$ if (see Figure 4) the total current through $J_l^{(i)}$

$$\xi[J_l^{(i)}] \stackrel{\Delta}{=} \sum_{j \in \Lambda\setminus i} \xi_{ij}(J_l^{(i)}) \,+\, \xi_i^h(J_l^{(i)}),$$

is odd. Evidently, there are periodic compatible $\vartheta \sim \xi$ iff $\partial\xi = \emptyset$. In the later case, there is a unique compatible trajectory $\nu(i,\cdot)$ for every *marked* $i \in \Lambda$ such that $\xi_i^\lambda > 0$ and, accordingly, there are precisely two compatible trajectories for every *unmarked* i with $\xi_i^\lambda = 0$. Let $\#_{\mathsf{m}}(\xi) \;=\; \#\left\{i : \xi_i^\lambda = 0\right\}$ be the total number of unmarked intervals $[0,\beta]$. By the general trace formula (2.26),

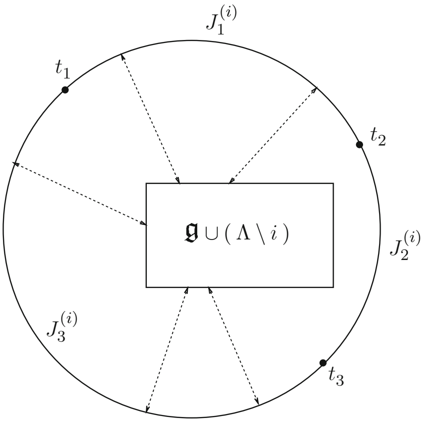

Fig. 4. The arrival times of the process of marks ξ_i^λ are t_1, t_2 and t_3. Accordingly, $\mathbb{S}_\beta^i$ is split into three marked intervals $J_1^{(i)}, J_2^{(i)}$ and $J_3^{(i)}$. The total number of arrivals of flips on $J_3^{(i)}$ equals to three, hence $J_3^{(i)} \subseteq \partial\xi$.

$$\frac{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathrm{e}^{\beta(\sum_{(i,j)} J_{ij} + \sum_i h + \sum_i \lambda)}} = \mathbb{P}_\beta^{\mathbf{J},h,\lambda}\left(2^{\#_{\mathsf{m}}(\xi)}\,;\,\partial\xi = \emptyset\right). \tag{3.13}$$

Thus, contrary to what happened in the the classical case, one should modify the reference (Poisson) measure. Define,

$$\widetilde{\mathbb{P}}_{\beta,\Lambda}^{\mathbf{J},\lambda,h}\left(\mathrm{d}\xi\right) = \frac{2^{\#_{\mathsf{m}}(\xi)}\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},\lambda,h}\left(\mathrm{d}\xi\right)}{\mathbb{P}_{\beta,\Lambda}^{\mathbf{J},\lambda,h}\left(2^{\#_{\mathsf{m}}(\xi)}\right)}.$$

Then, as in the classical case, the following random current representation of one and two point functions hold: Let $J(i,t)$ be the marked interval containing (i,t). Then,

$$\frac{\mathsf{Tr}\left(\hat{\sigma}_i^{\mathsf{z}}\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)} = \frac{\widetilde{\mathbb{P}}_{\beta,\Lambda}^{\mathbf{J},\lambda,h}\left(\partial\xi = J(i,0) \cup \mathfrak{g}\right)}{\widetilde{\mathbb{P}}_{\beta,\Lambda}^{\mathbf{J},\lambda,h}\left(\partial\xi = \emptyset\right)} \tag{3.14}$$

and, similarly,

$$\frac{\mathsf{Tr}\left(\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)} = \frac{\widetilde{\mathbb{P}}_{\beta,\Lambda}^{\mathbf{J},h,\lambda}\left(\partial\xi = J(i,0) \cup J(j,0)\right)}{\widetilde{\mathbb{P}}_{\beta,\Lambda}^{\mathbf{J},h,\lambda}\left(\partial\xi = \emptyset\right)} \tag{3.15}$$

It is, of course, a very natural question what should be a correct analog of the switching lemma in the quantum case. A closed form answer is still missing, but some aspects of this issue are discussed in [9].[1]

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[1] Appropriate versions of switching lemma were recently derived by Crawford and Ioffe [10] and by Björnberg and Grimmett [5].

Ground States, Matrix and Reduced Density Matrix Elements Let us briefly sketch how matrix and reduced density matrix elements in the x-basis could be written using the RC representation. Again, in order to simplify the notation we shall consider only the case of $h = 0$, and, exactly as in the end of Subsection 3.1, we shall directly pass to the ground state limit $\beta \to \infty$. In the ground state we are dealing with processes of arrivals ξ on the whole real line $\mathbb{R}$. We use $\mathbb{P}^{\mathbf{J},\lambda}_{\infty,\Lambda}$ to denote the corresponding product measure. Evidently, $\xi_i^\lambda \neq \emptyset \, \forall i \; \mathbb{P}^{\mathbf{J},\lambda}_{\infty,\Lambda}$-a.s. In other words, for each $i \in \Lambda$ the copy of the real line associated with i contains marks. As in the FK case, given ξ and a subset $A \subseteq \Lambda$, we use ξ_A to denote the slit configuration: except that now we view ξ_A as ξ with additional marks placed at time zero for each $i \in A$.

With such notation in mind we classify all marked intervals of ξ^λ and ξ_A^λ as follows:

1. Marked interval $i \times I$ of ξ^λ belong to $\mathcal{M}_0(\xi^\lambda)$ if $0 \in I$. Otherwise it belongs to $\mathcal{M}_{\rm ext}(\xi^\lambda)$.
2. Marked intervals of the type $i \times (0, t)$ of ξ_A belong to $\mathcal{M}_0^+(\xi_A)$. Similarly, marked intervals of the type $i \times (-t, 0)$ of ξ_A belong to $\mathcal{M}_0^-(\xi_A^\lambda)$.
3. All other marked intervals are ξ_A are also marked intervals of ξ and we classify them as $\mathcal{M}_0(\xi_A^\lambda)$ and $\mathcal{M}_{\rm ext}(\xi_A^\lambda)$.

Accordingly, we define the boundaries $\partial_0\xi$, $\partial_{\rm ext}\xi$, $\partial_0^+\xi_A$, $\partial_0^-\xi_A$, $\partial_0\xi_A$ and $\partial_{\rm ext}\xi_A$ as e.g.,

$$\partial_0\xi \;=\; \left\{ i \times I \in \mathcal{M}_0(\xi^\lambda) \,:\, \xi[i \times I] \text{ is odd} \right\}.$$

Let us introduce the following conditional measure

$$\mathbb{M}^{\mathbf{J},\lambda}_{\infty,\Lambda} \;=\; \mathbb{P}^{\mathbf{J},\lambda}_{\infty,\Lambda}\left(\cdot \,|\, \partial_{\rm ext}\xi = \emptyset\right).$$

Since Λ is finite the above definition can be easily turned into a meaningful one via an appropriate limiting procedure.

Let $\vartheta, \vartheta' \in \{\pm 1\}^\Lambda$ be two classical x-configurations, and let $\rho^{\mathsf{x}}_\infty(\vartheta, \vartheta')$ be the corresponding matrix element. From our interpretation of a mark in terms of a $+1$-spin enforcement at the corresponding space-time arrival point, it is apparent that that ξ contributes to $\rho^{\mathsf{x}}_\infty(\vartheta, \vartheta')$ iff the following event $\mathcal{E}_\pm(\vartheta, \vartheta') = \mathcal{E}_+(\vartheta) \cap \mathcal{E}_-(\vartheta')$ occurs:

1. Event $\mathcal{E}_+(\vartheta)$: For every $i \times I \in \mathcal{M}_0^+(\xi_\Lambda^\lambda)$, $i \times I \in \partial_0^+\xi_A$ iff ϑ_i is -1.
2. Event $\mathcal{E}_-(\vartheta')$: For every $i \times I \in \mathcal{M}_0^-(\xi_\Lambda^\lambda)$, $i \times I \in \partial_0^-\xi_A$ iff ϑ'_i is -1.

Then,

$$\rho^{\mathsf{x}}_{\infty}\left(\vartheta,\vartheta'\right) = \frac{\mathbb{M}^{\mathbf{J},\lambda}_{\infty,\Lambda}\left(\mathcal{E}_{\pm}(\vartheta,\vartheta')\right)}{\mathbb{M}^{\mathbf{J}}_{\infty,\Lambda},\lambda\left(\partial_0\xi=\emptyset\right)}.$$

In a similar fashion for $A \subseteq \Lambda$ and two classical x-configurations $\theta,\theta' \in \{\pm 1\}^A$ define the event $\mathcal{E}^A_{\pm}(\theta,\theta')$ exactly as above, except that even/odd conditions on currents are restricted to intervals $i \times I$ from $\mathcal{M}^{\pm}_0(\xi^{\lambda}_A)$. Then, the (θ,θ') entry of the reduced density matrix is given by,

$$\rho^{\mathsf{x}}_{\infty,A}\left(\theta,\theta'\right) = \frac{\mathbb{M}^{\mathbf{J},\lambda}_{\infty,\Lambda}\left(\mathcal{E}^A_{\pm}(\theta,\theta');\partial_0\xi_A=\emptyset\right)}{\mathbb{M}^{\mathbf{J},\lambda}_{\infty,\Lambda}\left(\partial_0\xi=\emptyset\right)}.$$

4 Curie-Weiss Model and Erdős-Rényi Random Graphs

Classical Curie-Weiss mean-field Hamiltonian $\mathbf{H}^{\mathsf{CW}}_N$ is a function on $\Omega_N = \{\pm 1\}^N$,

$$-\mathbf{H}^{\mathsf{CW}}_N(\nu) = \frac{1}{N}\sum_{(i,j)}\nu_i\nu_j, \tag{4.1}$$

where, as before, the summation is over all unordered pairs of $i \neq j$. In the language of Subsection 2.1, $\left\{\mathbf{H}^{\mathsf{CW}}_N(\nu)\right\}$ are eigenvalues of the quantum Hamiltonian $\mathcal{H}^{\mathsf{CW}}_N$,

$$\mathcal{H}^{\mathsf{CW}}_N\Psi_\nu = \mathbf{H}^{\mathsf{CW}}_N(\nu)\Psi_\nu, \quad \text{where} \quad -\mathcal{H}^{\mathsf{CW}}_N = \frac{1}{N}\sum_{(i,j)}\hat{\sigma}^{\mathsf{z}}_i\hat{\sigma}^{\mathsf{z}}_j.$$

Accordingly, for a given value of the inverse temperature β, the distribution of ν is,

$$\mu^{\beta}_N(\nu) = \frac{1}{\mathcal{Z}_N}\mathrm{e}^{-\beta\mathbf{H}^{\mathsf{CW}}_N(\nu)} = \frac{\langle\Psi_\nu|\mathrm{e}^{-\beta\mathcal{H}^{\mathsf{CW}}_N(\nu)}|\Psi_\nu\rangle}{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}^{\mathsf{CW}}_N(\nu)}\right)}. \tag{4.2}$$

One way to pin down phase transition in the CW model is to study statistical properties of the mean magnetization

$$\bar{\nu}_N \triangleq \frac{1}{N}\sum_i \nu_i,$$

under μ^{β}_N. As it is well known, for $\beta \leq 1$, the distribution of $\bar{\nu}_N$ is sharply concentrated around $\pm m^*$, where the spontaneous magnetization $m^* = m^*(\beta)$ equals to zero for $\beta \leq 1$ and is positive (and hence there are coexisting $\pm$ phases) for $\beta > 1$. This could be verified in two different ways, which correspond to two equality signs in (4.2): either directly through large deviation computations for Bernoulli random variables, or using the geometric FK representation as described in Subsection 2.3. In the latter case phase transition in the CW model is related to emergence of the giant component in the classical Erdős-Rényi random graph. Both methods are briefly recalled in Subsection 4.1

The main objective of this Section, however, is to explain that a very similar story happens with the quantum CW model in transverse field,

$$-\mathcal{H}_N^{\mathsf{CW}} = \frac{1}{N}\sum_{(i,j)} \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}} + \lambda \sum_i \hat{\sigma}_i^{\mathsf{x}}.$$

In particular, there is a natural inclusion of (one parameter) Erdős-Rényi random graph models into a two-parameter family of space-time random graphs. In this way classical Erdős-Rényi critical point $\beta = 1$ is just the limiting point on the whole critical curve in the (β, λ) plane. It is somewhat amusing that, apparently, such quantum version of Erdős-Rényi random graphs was overlooked for a long time, and the corresponding critical curve was originally computed only in [15].

Contrary to what happens in the classical case, however, for the moment it is not clear how recover the critical curve for the quantum CW model in the transverse field from the critical curve for the quantum Erdős-Rényi random graph, although a conjecture has appeared in [14]. In principle, the quantum CW critical curve could be derived from the results of [17], where limiting states were classified for essentially all mean field type models. Alternatively, one can use infinite dimensional theory of large deviations, see [11] and references therein. In the concluding Subsection 4.3 we shall briefly report on recent results of [9]. As in [11] the approach relies on a partial Trotterization of the mean-field Hamiltonian under, however, a different choice of arrival operators associated to transversal field: Ours corresponds to the FK setup of Subsection 3.1. Such FK point of view leads to certain advantages and, as a result, we go beyond just computing the critical curve itself. In particular, we are able to derive sharp asymptotics of the spontaneous magnetization $m^*(\beta, \lambda)$ in the vicinity of the critical curve, and for (β, λ) away from the critical curve we are able to derive quadratic stability bounds for maximizers of the corresponding infinite dimensional mean-field variational problem.

4.1 Classical Case

The probability measure ν_N^β in (4.2) could be described in the following way: Let $\mathbb{Q}$ be the uniform (1/2) distribution on $\{\pm 1\}$ and let $\otimes\mathbb{Q}$ be the corresponding product measure on $\Omega_N = \{\pm 1\}^N$. Then,

$$\mu_N^\beta(\nu) \;=\; \frac{\otimes\mathbb{Q}\big(\mathrm{e}^{N\beta(\bar{\nu}_N)^2/2};\nu\big)}{\otimes\mathbb{Q}\left(\mathrm{e}^{N\beta(\bar{\nu}_N)^2/2}\right)}. \tag{4.3}$$

Then, elementary one-dimensional theory of large deviations implies that μ_N^β exponentially concentrates around

$$\Big\{\nu\,:\,\bar{\nu}_N \text{ is close to } \mathrm{argmax}\Big(\frac{\beta}{2}m^2 - I(m)\Big)\Big\},$$

where I is the large deviation rate function for $\bar{\nu}_N$ under $\otimes\mathbb{Q}$,

$$I(m) \;=\; \sup_h \{hm - \Lambda(h)\} \quad \text{and} \quad \Lambda(h) = \log \mathbb{Q}\big(\mathrm{e}^{h\nu}\big) = \log \frac{\mathrm{e}^h + \mathrm{e}^{-h}}{2}.$$

It is easy to see that I is strictly convex and differentiable on $(-1,1)$ with $I'(m) \to \pm\infty$ as $m \to \pm 1$. In particular, the supremum of $\beta m^2/2 - I(m)$ is actually attained inside $(-1,1)$ for any $\beta \in \mathbb{R}_+$. Furthermore, since $I(\cdot)/\beta$ is the convex conjugate of $\Lambda(\beta\cdot)/\beta$,

$$\mathrm{argmax}\left\{\frac{m^2}{2} - \frac{1}{\beta}I(m)\right\} \;=\; \mathrm{argmax}\left\{\frac{1}{\beta}\Lambda(\beta h) - \frac{h^2}{2}\right\}. \tag{4.4}$$

But $\Lambda(\beta\cdot)$ is the log-moment generating function of the $\pm\beta$ Bernoulli random variable. If we use $\mathbb{Q}_\beta$ for the corresponding distribution, then it is straightforward to check that the maximizers in (4.4) are of the form $\pm m^*(\beta)$, where $m^*(\beta) > 0$ iff,

$$1 \;<\; \frac{1}{\beta}\mathbb{V}\mathrm{ar}\;(\beta)\,(\nu) \;=\; \frac{\beta^2}{\beta}, \tag{4.5}$$

and we, thereby, recover the critical value $\beta = 1$ of the classical CW model.

Relation to Random Graphs. Let us go back to the definition of the classical FK measure in (2.31), and let us use the shorthand notation $\widetilde{\mathbb{P}}_{\beta,N}$ for the CW case at zero magnetic field, $J \equiv 1/N$ and $h = 0$. By the second equality in (4.2), the distribution μ_N^β can be constructed from $\widetilde{\mathbb{P}}_{\beta,N}$ as follows:

First sample arrival processes $\xi = \{\xi_{ij}\}$ from $\widetilde{\mathbb{P}}_{\beta,N}$. Two sites i and j (or, equivalently, two circles $\mathbb{S}^i_\beta$ and $\mathbb{S}^j_\beta$) are said to be connected in ξ if $\xi_{ij} \neq \emptyset$. Thus, any realization of ξ splits $\{1, \dots, N\}$ into maximal connected components. At the second step paint those connected components into ± 1 independently and with probability $1/2$ each. In fact we have just constructed a joint measure $\mathbb{M}_{\beta,N}(\mathrm{d}\xi, \nu)$ with marginals $\widetilde{\mathbb{P}}_{\beta,N}$ and μ^β_N.

In view of such two-step construction of μ^β_N, the critical point $\beta = 1$ and the value of the spontaneous magnetization $m^*(\beta)$ could be recovered now from the following facts about the FK measures $\widetilde{\mathbb{P}}_{\beta,N}$ on complete graph: With $\widetilde{\mathbb{P}}_{\beta,N}$-probabilities tending to 1, as N tends to ∞,

1. For $\beta < 1$ all connected components of ξ have sizes $\mathrm{O}(\log N)$ at most.
2. For $\beta > 1$, there is exactly one giant connected component of size $\sim m^*(\beta)N$, whereas the remaining connected components of ξ have sizes $\mathrm{O}(\log N)$ at most.

Above statements are similar to classical results on the emergence of giant component in random complete graphs. Indeed, by construction,

$$\widetilde{\mathbb{P}}_{\beta,N}(\mathrm{d}\xi) = \frac{2^{\#(\xi)}\mathbb{P}_{\beta,N}(\mathrm{d}\xi)}{\mathbb{P}_{\beta,N}\left(2^{\#(\xi)}\right)}, \tag{4.6}$$

where $\#(\xi)$ is the number of connected components of ξ (recall that since we take $h = 0$ there are no wired components as in the general formula (2.31)). We can think about $\mathbb{P}_{\beta,N}$ in terms of Erdős-Rényi random graph on $\{1, \dots, N\}$ where bonds between different sites i, j are placed independently and with probability $2\beta/N$ each. Indeed, $1 - \mathrm{e}^{-2\beta/N}$ is the probability that $\xi_{ij} \neq \emptyset$. Furthermore, as it was observed by Edwards and Sokal [13], the conditional ξ-marginal of

$$\mathbb{M}_{\beta,N}(\cdot | \nu_1 = 1, \dots, \nu_M = 1, \nu_{M+1} = -1, \dots, \nu_N = -1)$$

is exactly $\mathbb{P}_{\beta,M} \otimes \mathbb{P}_{\beta,N-M}$. Since $\max\{M, N-M\} \geq N/2$, the inequality $\beta_c \leq 1$ for the critical FK value of β is immediately implied by classical Erdős-Rényi results, see e.g. [6]: Let $\{1, \dots, K\}$ be the complete graph of K sites. Assume that an (un-oriented) edge (i, j) is open with probability ϵ/K independently from all other edges. Then $\epsilon_c = 1$ is the threshold for the emergence of the giant component. Moreover, in the case of $\epsilon > 1$ the density $\rho(\epsilon)$ of the giant component is asymptotically close to the positive solution of

$$1 - \rho = \mathrm{e}^{-\epsilon\rho}. \tag{4.7}$$

In over case, $K = \max\{M, N - M\} \geq N/2$, and hence $2\beta/N > 1/K$ whenever $\beta > 1$.

The reverse inequality $\beta_c \geq 1$ is not much harder: Assume that $\beta < 1$. Without loss of generality we can consider only the case when the total number of $+$ spins $M \leq N/2$. Then, under $\mathbb{P}_{\beta,M}$ all the connected components of $\{1, \dots, M\}$ are small. A-priori, a giant connected component still could appear under $\mathbb{P}_{\beta,N-M}$. Let ρ be the density of this component. Then $(1 - \rho)(N - M)$ of the remaining $-$ spins live on small components of sizes $\mathrm{O}(\log N)$ at most. Since in the original coupled measure $\mathbb{M}_{\beta,N}$ all the small connected components were coloured independently, we infer that $M \sim (1-\rho)(N-M)$. Accordingly, $K \stackrel{\Delta}{=} N - M \sim N/(2 - \rho)$ and hence $2\beta/N \sim \epsilon/K$ with $\epsilon = 2\beta/(2 - \rho)$. Thus, by (4.7), the relative density ρ should satisfy

$$1 - \rho = \mathrm{e}^{-2\beta\rho/(2-\rho)}.$$

But the latter equation does not have a positive solution. Indeed, set $\theta = \rho/(2 - \rho)$ or $\rho = (1 - \theta)/(1 + \theta)$. Then θ is positive as soon as ρ is positive, and

$$\frac{1 - \theta}{1 + \theta} = \mathrm{e}^{-2\beta\theta}$$

Taking logs and expanding,

$$2\theta + \frac{2}{3}\theta^3 + \dots = 2\beta\theta,$$

which is impossible unless $\theta =$ or $\beta > 1$.

A general class of FK models on complete graphs is examined in [7].

4.2 Curie-Weiss Model in Transverse Field and Quantum Random Graphs

Quantum Curie-Weiss Hamiltonian in transverse field $\lambda \geq 0$ is given by,

$$-\mathcal{H}_N^{\mathsf{CW}} = \frac{1}{N}\sum_{(i,j)} \hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}} + \lambda \sum_i \hat{\sigma}_i^{\mathsf{x}}.$$

Following the approach of Subsection 3.1 we associate to $\mathcal{H}_N^{\mathsf{CW}}$ the following family ξ of independent Poisson processes of arrivals on the circle $\mathbb{S}_\beta$: For each un-oriented couple (i, j) operators $\left(\mathrm{I} + \hat{\sigma}_i^{\mathsf{z}} \hat{\sigma}_j^{\mathsf{z}}\right)$ arrive with intensity $2/N$, whereas operators $(\mathrm{I} + \hat{\sigma}_i^{\mathsf{x}})$ arrive with intensity λ for every $i = 1, \dots, N$. Connected components of $\{1, \dots, N\} \times \mathbb{S}_\beta$ induced by ξ are defined precisely as in Subsection 3.1. Recall that each such connected component $\mathcal{C}$ is represented as a union,

$$\mathcal{C} = \bigcup_l \{i_l \times \; I_l\} ,$$

of disjoint space-time intervals. The size of $\mathcal{C}$ could be measured in several ways: For example we can compute number of different spatial coordinates (out of $\{1, \dots, N\}$) which contribute to $\mathcal{C}$. The most natural definition of the size, however, is

$$|\mathcal{C}| = \sum_l |I_l|, \tag{4.8}$$

that is the total length of all time intervals of $\mathcal{C}$.

Since we consider the case of zero z-field, all connected components of ξ are free. Consequently, the FK modification $\widetilde{\mathbb{P}}_{\beta,N}^{\lambda}$ of the reference product Poisson measure $\mathbb{P}_{\beta,N}^{\lambda}$ is given by

$$\widetilde{\mathbb{P}}_{\beta,N}^{\lambda}(\mathrm{d}\xi) = \frac{2^{\#(\xi)} \mathbb{P}_{\beta,N}^{\lambda}(\mathrm{d}\xi)}{\mathbb{P}_{\beta,N}^{\lambda}\left(2^{\#(\xi)}\right)}, \tag{4.9}$$

In view of (3.7) it is suggestive to try to study the question of phase co-existence in terms of emergence of giant components under $\widetilde{\mathbb{P}}_{\beta,N}^{\lambda}$. Note that in a genuine quantum case of $\lambda > 0$, this is a non-trivial question even in the ground state limit when $\beta \to \infty$. In fact, instead of one critical value of β one should face here a whole critical curve in the (λ, β) positive quarter plane. For the moment we do not know how to derive this curve via direct analysis of random space-time graphs induced by the family of quantum FK measures (4.9). This, however, is a meaningful question even for the reference family of measures $\mathbb{P}_{\beta,N}^{\lambda}$.

Quantum Random Graphs As it is apparent from a comparison between (4.9) and (4.6) the measures $\mathbb{P}_{\beta,N}^{\lambda}$ play the same role for the quantum Curie-Weiss model in transverse field as Erdős-Rényi random graphs $\mathbb{P}_{\beta,N}$ play for the classical CW model. Accordingly, we shall refer to the collection of independent Poisson processes of holes and links induced by $\mathbb{P}_{\beta,N}^{\lambda}$ as to quantum random graphs. In order to be compatible with the usual random graph notation let us modify the arrival rates under $\mathbb{P}_{\beta,N}^{\lambda}$ in the following way: The holes still arrive with intensity λ,

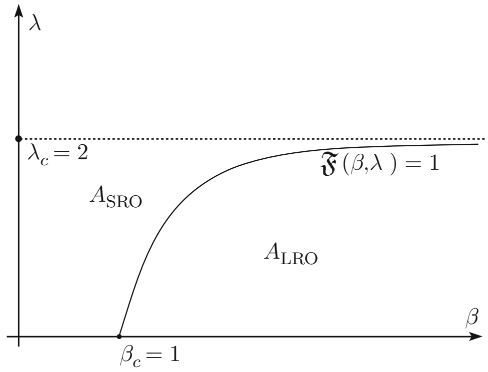

Fig. 5. Decomposition of the (β, λ) quarter plane into the short range and long range regions.

however the links between an unordered pair of sites $i \neq j$ arrive now with intensity $1/N$. In this way $\beta_c = 1$ is the classical critical value which corresponds to $\lambda = 0$. The main result of [15] asserts that the full critical curve for the family of quantum random graphs is implicitly given by,

$$\mathfrak{F}(\beta, \lambda) \stackrel{\Delta}{=} \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \beta\mathrm{e}^{-\lambda\beta} = 1. \tag{4.10}$$

The curve is depicted on Figure 5. Note that the classical critical value $\beta_c = 1$ is just the end-point of the curve on β-axis. Notice also that the critical value of λ in the ground state model $\beta = \infty$ equals to $\lambda_c = 2$. Let us be more specific about the nature of phase transition for quantum random graphs: The critical curve (4.10) splits the positive quarter-plane into

$$A_{\mathrm{LRO}} \stackrel{\Delta}{=} \{(\beta, \lambda) : \mathfrak{F}(\beta, \lambda) > 1\} \text{ and } A_{\mathrm{SRO}} \stackrel{\Delta}{=} \{(\beta, \lambda) : \mathfrak{F}(\beta, \lambda) < 1\},$$

where LRO (respectively SRO) stands for long (respectively short) range order. Here is a justification for such a terminology: By definition, two points $(i, t), (j, s)$ are connected in ξ, if the intervals containing these points belong to the same connected component $\mathcal{C}$ in the ξ-induced decomposition of $\{1, \ldots, N\} \times \mathbb{S}_\beta$. We shall denote the latter event as $\{(i, t) \longleftrightarrow (j, s)\}$. Then,

1. If $(\beta, \lambda) \in A_{\mathrm{SRO}}$, then

$$\mathbb{P}^{\lambda}_{\beta,N}\left((i, t) \longleftrightarrow (j, s)\right) = O\left(\frac{\log N}{N}\right) \tag{4.11}$$

   uniformly in $t, s \in \mathbb{S}_\beta$ and $i \neq j$.

2. On the other hand, if $\beta < \infty$ and $(\beta, \lambda) \in A_{\mathrm{LRO}}$, then there exists $\rho = \rho(\beta, \lambda) \in (0,1)$, such that

$$\mathbb{P}^{\lambda}_{\beta,N}\left((i,t) \longleftrightarrow (j,s)\right) \;=\; \rho(\beta,\lambda)^2\left(1 + o(1)\right), \tag{4.12}$$

also uniformly in $t, s \in \mathbb{S}_\beta$ and $i \neq j$.

As in the classical Erdős-Rényi case the short/long range order transition for $\beta < \infty$ is related to an emergence of a unique giant connected component. In fact, the number $\rho(\beta,\lambda)$ in (4.12) is precisely the limiting space-time density of the latter. More precisely, let us use (4.8) to measure sizes of random connected components of $\{1,\dots,N\} \times \mathbb{S}_\beta$. Let $\mathcal{M}$ and $\mathcal{M}^{\mathrm{next}}$ be the largest and the next to the largest sizes of these connected components (of course, these definitions make sense only for $\beta < \infty$). Then,

1. If $(\beta, \lambda) \in A_{\mathrm{SRO}}$, then for every $\kappa > 0$ there exists $c = c(\beta,\lambda,\kappa) < \infty$, such that

$$\mathbb{P}^{\lambda}_{\beta,N}\left(\left|\mathcal{C}\left((i,t)\right)\right| > c\log N\right) \;=\; o\Big(\frac{1}{N^{\kappa}}\Big), \tag{4.13}$$

where $\mathcal{C}((i,t))$ is the connected component containing (i,t). Clearly, the distribution of $\left|\mathcal{C}\left((i,t)\right)\right|$ is the same for all $i \in \{1,\dots,N\}$ and $t \in \mathbb{S}_\beta$ (by definition $\mathbb{S}_\infty = \mathbb{R}$). Furthermore, if $\beta < \infty$, then

$$\mathbb{P}^{\lambda}_{\beta,N}\left(\mathcal{M} > c\log N\right) \;=\; o\Big(\frac{1}{N^{\kappa-1}}\Big) \tag{4.14}$$

2. If, however, $\beta < \infty$ and $(\beta,\lambda) \in A_{\mathrm{LRO}}$ then there exists a sequence of positive numbers $\epsilon_N(\beta,\lambda) \to 0$ such that,

$$\mathbb{P}^{\lambda}_{\beta,N}\left(\left|\frac{|\mathcal{C}((i,t))|}{N\beta} - \rho\right| < \epsilon_N\right) \;=\; \rho(\beta,\lambda)(1 - o(1)), \tag{4.15}$$

where $\rho(\beta,\lambda)$ is the same probability as in (4.12). Furthermore, in the $\beta < \infty$ case, there exists a constant $c = c(\beta,\lambda) < \infty$ such that

$$\mathbb{P}^{\lambda}_{\beta,N}\left(\mathcal{E}(\rho,\epsilon_N,c)\right) = 1 - o(1), \tag{4.16}$$

where the event $\mathcal{E}(\rho,\epsilon_N,c)$ is defined via

$$\mathcal{E}(\rho,\epsilon_N,c) \;=\; \left\{\left|\frac{\mathcal{M}}{\beta N} - \rho\right| < \epsilon_N\right\} \cap \left\{\mathcal{M}^{\mathrm{next}} < c\log N\right\}. \tag{4.17}$$

The original proof of the above results appeared in [15]. Afterwards, the statements related to the $\beta < \infty$ case were re-proven using somewhat different methods in [16].

We finish this Subsection by indicating how the expression (4.10) comes into play. As in the classical case one couples a construction of a single connected component with a Galton-Watson process. In the quantum case descendant of a point $(i, t) \in \{1, \ldots, N\}$ are generated in the following fashion:

1. First generate a random interval $I \subseteq \mathbb{S}_\beta$ around (i, t),so that the end-points of I would imitate two successive holes. Since the holes arrive with intensity λ the length $|I|$ should be distributed as $\min\{\Gamma(2, \lambda), \beta\}$.
2. Given a realization of $I \ni t$, the number of all links to i which arrive during I is distributed Poisson$(\frac{N-1}{N}|I|)$. In the Galton-Watson approximation we take it to be exactly Poisson$(|I|)$.

Accordingly, if we denote the number of descendants in the Galton-Watson approximation by X, then $\mathbb{E}(X|I) = |I|$. Let $V \sim \Gamma(2, \lambda)$ Then,

$$\mathbb{E}(|I|) = \mathbb{E}(V; V < \beta) + \beta\mathbb{P}(V \geq \beta),$$

Now,

$$\mathbb{P}(V \geq \beta) = \int_\beta^\infty \lambda^2 t \mathrm{e}^{-\lambda t} \mathrm{d}t = (\lambda\beta + 1)\mathrm{e}^{-\lambda\beta}.$$

In the same fashion,

$$\mathbb{E}(V; V \leq \beta) = \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \left(\beta^2\lambda + 2\beta\right)\mathrm{e}^{-\lambda\beta}.$$

Consequently,

$$\mathbb{E}(|I|) = \frac{2}{\lambda}\left(1 - \mathrm{e}^{-\lambda\beta}\right) - \beta\mathrm{e}^{-\lambda\beta},$$

which is precisely the expression in (4.10).

4.3 Critical Curve for Quantum Curie-Weiss Model via Large Deviations

Large deviation representation of the CW model in transverse field is obtained via partial linearization in the Lie-Trotter product formula (or partial Poissonization of the CW Hamiltonian). Namely,

$$\frac{\mathrm{e}^{-\beta\mathcal{H}_N^{\mathsf{CW}}}}{\mathrm{e}^{\lambda N}} = \lim_{\Delta\to 0}\left(\prod_{(i,j)} \mathrm{e}^{\frac{\Delta}{N}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}} \prod_i \{(1 - \Delta\lambda)\mathrm{I} + \Delta\lambda(\hat{\sigma}_i^{\mathsf{x}} + \mathrm{I})\}\right)^{\beta/\Delta}. \quad (4.18)$$

Note that matrices $\mathrm{e}^{\frac{\Delta}{N}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}}$ are diagonal in the z-basis,

$$\langle \Psi_\nu | \mathrm{e}^{\frac{\Delta}{N}\hat{\sigma}_i^{\mathsf{z}}\hat{\sigma}_j^{\mathsf{z}}} | \Psi_{\nu'} \rangle = \mathrm{e}^{\frac{\Delta}{N}\nu_i\nu_j}. \tag{4.19}$$

Let $\mathbb{P}_\beta^\lambda$ be the distribution of the Poisson point process (of holes) on the circle $\mathbb{S}_\beta$ with arrival intensity λ. We shall use $\otimes\mathbb{P}_\beta^\lambda$ for the product distribution of N independent copies $\xi = (\xi_1, \dots, \xi_N)$. Given a realization of ξ let us say that a classical piece-wise constant trajectory $\nu : \mathbb{S}^\beta \mapsto \{\pm 1\}^N$ is compatible with ξ; $\nu \sim \xi$, if for every $i = 1, \dots, N$ jumps of $\nu_i(\cdot)$ occur only at arrival times of ξ_i. Passing to the limit in (4.18) we, in view of (4.19), infer

$$\frac{\mathsf{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_N^{\mathsf{CW}}}\right)}{\mathrm{e}^{\lambda N}} = \int \otimes\mathbb{P}_\beta^\lambda(\mathrm{d}\xi) \sum_{\nu\sim\xi} \exp\left\{\int_0^\beta \frac{1}{N}\sum_{(i,j)} \nu_i(t)\nu_j(t)\mathrm{d}t\right\}. \tag{4.20}$$

For every i let $\#(\xi_i)$ be the number of connected components of $\mathbb{S}_\beta \setminus \xi_i$. Evidently, the number of all compatible $\nu \sim \xi$ equals to $2^{\sum_i \#(\xi_i)}$. Define

$$\widetilde{\mathbb{P}}_\beta^\lambda(\mathrm{d}\xi) = \frac{2^{\#(\xi)}\mathbb{P}_\beta^\lambda(\mathrm{d}\xi)}{\mathbb{P}_\beta^\lambda\left(2^{\#(\xi)}\right)}$$

This is just the one-circle FK measure. Consider probability distribution $\mathbb{Q}_\beta^\lambda$ on piece-wise constant classical one-circle spin trajectories $\nu : \mathbb{S}_\beta \mapsto \{\pm 1\}$ which is generated by the following two step procedure: First sample ξ from $\widetilde{\mathbb{P}}_\beta^\lambda$, and then paint connected components of $\mathbb{S}_\beta \setminus \xi$ into ± 1, independently and with probability $1/2$ each. Let $\otimes\mathbb{Q}_\beta^\lambda$ be the corresponding product measure. It is straightforward to check that the righthand side of (4.20) equals to

$$\left[\widetilde{\mathbb{P}}_\beta^\lambda\left(\mathrm{e}^{\#(\xi)}\right)\right]^N \otimes \mathbb{Q}_\beta^\lambda\left(\exp\left\{\int_0^\beta \frac{1}{N}\sum_{(i,j)} \nu_i(t)\nu_j(t)\mathrm{d}t\right\}\right).$$

Consequently, an analysis of phase diagram of the CW model in transverse filed boils down to an investigation of asymptotic properties for weighted measures

$$\otimes\widetilde{\mathbb{Q}}_\beta^\lambda(\mathrm{d}\nu) \stackrel{\Delta}{=} \frac{\otimes\mathbb{Q}_\beta^\lambda\left(\exp\left\{\frac{N}{2}\int_0^\beta \left(\bar{\nu}_N(t)\right)^2\mathrm{d}t\right\}\ ;\ \mathrm{d}\nu\right)}{\otimes\mathbb{Q}_\beta^\lambda\left(\exp\left\{\frac{N}{2}\int_0^\beta \left(\bar{\nu}_N(t)\right)^2\mathrm{d}t\right\}\right)}, \tag{4.21}$$

where,

$$\bar{\nu}_N(t) \;=\; \frac{1}{N}\sum_i \nu_i(t).$$

This problem belongs to the realm of theory of large deviations. Formally, the measures (4.21) are asymptotically concentrated around solutions of

$$\sup_m \Big\{ \frac{1}{2}\int_0^\beta m^2(t)\mathrm{d}t \;-\; I(m) \Big\} \;\stackrel{\Delta}{=}\; \sup_m \mathfrak{G}(m), \tag{4.22}$$

where I is the large deviation rate function for the average $\bar{\nu}_N$ under the product measures $\otimes\mathbb{Q}_\beta^\lambda$. If we formulate the large deviation principle in $\mathbb{L}_2(\mathbb{S}_\beta)$, then, using $(\cdot,\cdot)_\beta$ for the corresponding scalar product,

$$I(m) \;=\; \sup_h \left\{(h,m)_\beta - \Lambda(h)\right\} \quad \text{where} \quad \Lambda(h) \;=\; \log \mathbb{Q}_\beta^\lambda\big(\mathrm{e}^{(h,\nu)_\beta}\big). \tag{4.23}$$

A detailed analysis of the variational problem (4.22) and of the weighted measures $\widetilde{\mathbb{Q}}_{\beta,N}^\lambda$ will appear in the forthcoming [9]. Here we shall try to give a brief sketch of the results and techniques, in particular, we shall explain how the critical curve of the CW model in the transverse field could be read from (4.22).

The critical curve is implicitly given by

$$\mathfrak{f}(\lambda,\beta) \;\stackrel{\Delta}{=}\; \frac{1}{\beta}\mathbb{V}\mathrm{ar}_\lambda\,(\beta)\left((\nu,\mathbb{1})_\beta\right) \;=\; \frac{1}{\lambda}\tanh(\lambda\beta) \;=\; 1, \tag{4.24}$$

where $\mathbb{V}\mathrm{ar}_\lambda\,(\beta)$ is the variance under the one-circle spin measure $\mathbb{Q}_\beta^\lambda$. As we show in [9], the variational problem (4.22) has constant maximizers $\pm m^*(\lambda,\beta)$, where the spontaneous z-magnetization m^* satisfies:

1. If $\mathfrak{f}(\lambda,\beta) \le 1$, then $m^* = 0$.
2. If $\mathfrak{f}(\lambda,\beta) > 1$, then $m^* > 0$, and, consequently there are two distinct solutions to (4.22).

Furthermore, away from the critical curve the solutions $\pm m^*\mathbb{1}$ are stable in the following sense: There exists $c = c(\lambda,\beta) > 0$ and a strictly convex symmetric function U with $U(r) \sim r\log r$ growth at infinity such that

$$\mathfrak{G}(\pm m^*\mathbb{1}) - \mathfrak{G}(m) \ge c\min\left\{\|m - m^*\mathbb{1}\|_\beta^2, \|m + m^*\mathbb{1}\|_\beta^2\right\} + \int_0^\beta U(m'(t))\mathrm{d}t. \tag{4.25}$$

The second term above is important in the super-critical regime ($\mathfrak{f}(\lambda,\beta) > 1$) since it rules out trajectories of $\bar{\nu}_N(\cdot)$ with rapid transitions between the optimal values $\pm m^*$.

Properties of One-circle Spin Measures

The following properties of $\mathbb{Q}_\beta^\lambda$ are crucial for the analysis of (4.22):

1. $\mathbb{Q}_\beta^\lambda$ possesses the FKG property.
2. $\mathbb{Q}_\beta^\lambda$ satisfies the following qualitative version of the GHS inequality: Given $h \in \mathbb{R}_+$ define the tilted measure
$$\mathbb{Q}_\beta^{\lambda,h}(\mathrm{d}\nu) = \frac{\mathbb{Q}_\beta^\lambda\left(\mathrm{e}^{h(\nu,\mathbb{1})_\beta}\,;\,\mathrm{d}\nu\right)}{\mathbb{Q}_\beta^\lambda\left(\mathrm{e}^{h(\nu,\mathbb{1})_\beta}\right)}.$$
Then, there exists $c_1 = c_1(\lambda,\beta) > 0$, such that
$$\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{V}\mathrm{ar}_{\lambda,h}(\beta)\left((\nu,\mathbb{1})_\beta\right) \le -c_1 h \mathrm{e}^{-2\beta h}. \tag{4.26}$$
3. $\mathbb{Q}_\beta^\lambda$ is reflection positive: Let $n \in \mathbb{N}$, $0 < t_1 < \cdots < t_n < \beta/2$ and let $f : \{\pm 1\}^n \to \mathbb{C}$. Set $s_k = \beta - t_k$. Then,
$$\mathbb{Q}_\beta^\lambda\left(f(\nu_{t_1},\dots,\nu_{t_n})\bar{f}(\nu_{s_1},\dots,\nu_{s_n})\right) \ge 0. \tag{4.27}$$

Properties 1. and 3. are more or less immediate since $\mathbb{Q}_\beta^\lambda$ could be viewed in terms of an approximation by ferromagnetic nearest neighbour one-dimensional Ising models. Namely, let us approximate ξ by Bernoulli point process of arrivals ξ^Δ, exactly as in (2.21). Modify Bernoulli weights by $2^{\#(\xi^\Delta)}$ and paint connected components of $\mathbb{S}\setminus\xi^\Delta$ into ± 1, independently and with probability $1/2$ each. Then, the resulting measure $\mathbb{Q}_{\beta,\Delta}^\lambda$ approximates $\mathbb{Q}_\beta^\lambda$. Of course $\mathbb{Q}_{\beta,\Delta}^\lambda$ charges only trajectories ν which jump at times $j\Delta$. For such trajectories,
$$\mathbb{Q}_{\beta,\Delta}^\lambda(\nu) \sim \prod_{i=0}^{\beta/\Delta-1}\left(\delta_{\{\nu(i\Delta)=\nu((i+1)\Delta)\}} + \Delta\lambda\delta_{\{\nu(i\Delta)=\nu(i+1)\Delta\}}\right).$$

Set $J = J(\Delta,\lambda) = -\log\sqrt{\Delta\lambda}$. Since
$$\delta_{\{\nu(i\Delta)=\nu((i+1)\Delta)\}} + \Delta\lambda\delta_{\{\nu(i\Delta)=\nu((i+1)\Delta)\}} = \frac{\mathrm{e}^{J\nu(i\Delta)\nu((i+1)\Delta)}}{\mathrm{e}^J},$$
we recognize $\mathbb{Q}_{\beta,\Delta}^\lambda$ as a scaling of the nearest neighbour Ising model on discrete circle $\mathbb{S}_\beta/\Delta$ at unit temperature and with interaction strength $J(\Delta,\lambda)$.

Inequality (4.26) is proved in [9] using the same approximation (by 1D Ising models) with an additional care being paid to limits of random current representation of third semi-invariants (based on [2]).

Dual Variational Problem

In order to explain the implications of the properties of $\mathbb{Q}_\beta^\lambda$ listed above, it is convenient to consider the dual variational problem,

$$\sup_h \left\{ \Lambda(h) - \frac{1}{2}\int_0^\beta h^2(t)\mathrm{d}t \right\} \stackrel{\Delta}{=} \sup_h \mathfrak{G}^*(h). \tag{4.28}$$

Any solution $\tilde{h}$ of (4.28) is also a solution to (4.22). This is a general fact from convex analysis: Let F and G be two proper lower-semicontinuous convex functionals (on say $\mathbb{L}_2(\mathbb{S}_\beta)$) and let F^* and G^* be their convex conjugates. Assume that

$$F^*(\tilde{h}) - G^*(\tilde{h}) \;=\; \max_h \left\{ F^*(h) - G^*(h) \right\},$$

and assume that both F^* and G^* are Gateaux differentiable (in fact sub-differentiability would be enough) at $\tilde{h}$. Let $\tilde{m} = \nabla F^*(\tilde{h}) = \nabla G^*(\tilde{h})$. Then,

$$F^*(\tilde{h}) - G^*(\tilde{h}) = G(\tilde{m}) - F(\tilde{m}).$$

Consequently, for each couple of functions m and h,

$$\left\{ (m,h)_\beta - G^*(h) \right\} - \left\{ (m,h)_\beta - F^*(h) \right\} \le G(\tilde{m}) - F(\tilde{m}).$$

It follows that for every m, $G(m) - F(m) \le G(\tilde{m}) - F(\tilde{m})$. Furthermore, assume that we can quantify stability property of the dual variational problem in the following way: There exists a non-negative functional D, such that $D = 0$ only on the solutions of the dual problem, and for any function h,

$$F^*(h) - G^*(h) + D(h) \;\le\; F^*(\tilde{h}) - G^*(\tilde{h}). \tag{4.29}$$

Then such stability bound is transferable to the direct problem: Assume that $h = \nabla G(m)$. Then,

$$G(m) - F(m) + D(h) + \left\{ F(m) + F^*(h) - (m,h)_\beta \right\} \;\le\; G(\tilde{m}) - F(\tilde{m}). \tag{4.30}$$

In particular, $G(m) - F(m) < G(\tilde{m}) - F(\tilde{m})$, whenever $\nabla G(m)$ *is not* a solution of the dual problem or whenever $h \notin \partial F(m)$.

Let us now go back to (4.22) and (4.28). In the above notation: $F(m) = I(m)$ and $G(m) = \|m\|_\beta^2/2$. Accordingly, $F^*(h) = \Lambda(h)$ and $G^*(h) = \|h\|_\beta^2/2$. In particular, G, G^* and F^* are everywhere Gateaux differentiable. Of course, $\nabla G(m) = m$. Consequently, once we derive a stability bound of the type (4.29) for the dual problem, we immediately recover a stability bound

$$\frac{1}{2}\int_0^\beta m^2(t)\mathrm{d}t - I(m) + D(m) + \left\{I(m) + \Lambda(m) - \|m\|_\beta^2\right\} \leq \mathfrak{G}(\tilde{m}) \tag{4.31}$$

for the original problem (4.22). In particular, any solution of (4.22) is a solution of (4.28).

We, therefore, proceed to study the dual variational problem (4.28).

Reduction to a One-dimensional Problem

Reflection positivity property (4.27) implies that for any $h \in \mathbb{L}_2(\mathbb{S}_\beta)$,

$$\Lambda(h) \leq \frac{1}{\beta}\int_0^\beta \Lambda\left(h(t)\mathbb{I}\right)\mathrm{d}t. \tag{4.32}$$

Note that (4.32) has been originally proved in a somewhat more general context in [11]. As a result,

$$\mathfrak{G}^*(h) \leq \int_0^\beta \left\{\frac{1}{\beta}\Lambda\left(h(t)\mathbb{I}\right) - \frac{1}{2}h^2(t)\right\}\mathrm{d}t \leq \beta \sup_{h\in\mathbb{R}}\left\{\frac{1}{\beta}\Lambda\left(h\mathbb{I}\right) - \frac{1}{2}h^2\right\}.$$

We claim that the maximizers of the one-dimensional variational problem

$$\max_{h\in\mathbb{R}}\left\{\frac{1}{\beta}\Lambda\left(h\mathbb{I}\right) - \frac{1}{2}h^2\right\}, \tag{4.33}$$

are of the form $\pm h^*$, where $h^* > 0$ if and only if $\mathfrak{f}(\lambda, \beta) > 1$.

The critical curve (4.24). Compute,

$$\frac{\mathrm{d}}{\mathrm{d}h}\left\{\frac{1}{\beta}\Lambda(h\mathbb{I}) - \frac{1}{2}h^2\right\} = \frac{1}{\beta}\mathbb{Q}_\beta^{\lambda,h}\left((\nu,1)_\beta\right) - h.$$

The latter expression is evidently negative for h large enough, hence the maximum in (4.33) is attained at a critical point. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{Q}_\beta^{\lambda,h}\left((\nu,1)_\beta\right) = \mathbb{V}\mathrm{ar}_\lambda\left(\beta\right)\left((\nu,1)_\beta\right).$$

Since by symmetry at $h = 0$ the expectation $\mathbb{Q}_\beta^\lambda\left((\nu,1)_\beta\right) = 0$, and since by (4.26) the function $h \to \mathbb{Q}_\beta^{\lambda,h}\left((\nu,1)_\beta\right)$ is strictly concave on $[0,\infty)$, we infer that:

Either $\mathbb{V}\mathrm{ar}_\lambda\left(\beta\right)\left((\nu,1)_\beta\right) \leq \beta$, and then $h = 0$ is the only critical point of the function in (4.33). Or, $\mathbb{V}\mathrm{ar}_\lambda\left(\beta\right)\left((\nu,1)_\beta\right) > \beta$, and then there are exactly three critical points; 0 and $\pm h^*$, the latter inevitably being the global maxima.

Stability of the one-dimensional problem. We claim, furthermore, that whenever (λ, β) is away from the critical curve, the problem (4.33) is stable,

$$\left\{ \frac{1}{\beta} \Lambda\left(h\mathbb{I}\right) - \frac{1}{2}h^2 \right\} + d(h) \leq \frac{1}{\beta}\Lambda\left(\pm h^*\mathbb{I}\right) - \frac{1}{2}(h^*)^2, \tag{4.34}$$

where d satisfies the following bound: There exists $c_1 = c_1(\lambda, \beta) > 0$, such that,

$$d(h) \geq c_2 \mathrm{e}^{-2\beta|h|} \min\left\{(h-h^*)^2, (h+h^*)^2\right\}. \tag{4.35}$$

Proof of (4.34). Follows from (4.26).

Stability of the original variational problem. It follows that the dual variational problem (4.28) (recall that in our case $F^*(\cdot) = \Lambda(\cdot)$ and $G^*(\cdot) = 1/2\|\cdot\|_\beta^2$) satisfies (4.29) with

$$D(h) = \frac{1}{\beta}\int_0^\beta d(h(t))\mathrm{d}t.$$

Of course, the bound (4.34) could be improved for large values of $|h|$, however since we are primarily interested in transferring stability to the direct variational problem (4.22), the values of $|h| > 1$ are, in view of (4.31), irrelevant. In particular $D(m)$ clearly dominates (with $h^* = m^*$ and c chosen appropriately small) the first term on the right hand side of (4.25).

The second term $\int_0^\beta U(m'(t))\mathrm{d}t$ on the right hand side of (4.25) is related to a more careful analysis of $\left\{I(m) + \Lambda(m) - \|m\|_\beta^2\right\}$ term in (4.31), which is unfortunately beyond the scope of these lectures. We, therefore, refer the reader to [9].

Behaviour Near the Critical Curve

The GHS-type bound (4.26) implies that the 4-th semi-invariant

$$-\mathsf{s}_4(\lambda, \beta) \triangleq \left.\frac{\mathrm{d}^4\Lambda(h\mathbb{I})}{\mathrm{d}h^4}\right|_{h=0},$$

is locally uniformly negative. Let $\mathfrak{f}(\lambda, \beta) > 1$ and assume that (λ, β) is close to the critical curve, in particular that $h^*(\lambda, \beta)$ is small. Then,

$$h^*(\lambda,\beta) = \frac{1}{\beta}\int_0^{h^*} \mathbb{V}\mathrm{ar}_{\lambda,\tau}(\beta)\left((\nu,\mathbb{I})\right)\mathrm{d}\tau$$
$$= h^*\mathfrak{f}(\lambda,\beta) - \frac{\mathsf{s}_4(\lambda,\beta)(h^*)^3}{6\beta}\left(1+\mathrm{O}(h^*)\right). \quad (4.36)$$

It follows that in the vicinity of the critical curve spontaneous magnetization $m^*(\lambda,\beta) = h^*(\lambda,\beta)$ scales like

$$\frac{m^*(\lambda,\beta)}{\sqrt{6\beta(\mathfrak{f}(\lambda,\beta)-1)/\mathsf{s}_4(\lambda,\beta)}} = 1 + o(1).$$

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