2.1 Introduction

A “rigid machine” is the simplest calculation model in machine dynamics. It can be defined as a constrained system of rigid bodies, the motion of which is uniquely determined for a given drive motion due to holonomic constraints. This calculation model can be used if the deformations that always exist in reality due to the acting forces are so minor that they have little influence on the motions. It is also assumed that the joints and bearings are ideally backlash-free.

The rigid machine model can be used as the basis for calculating “slow-running” machines, that is, when the lowest natural frequency of the real object under consideration is greater than the largest occurring excitation frequency. The calculation model of the rigid machine can be used for mechanisms with constant transmission ratio such as gearboxes, worm gear systems, belt and chain transmission systems, as well as mechanisms with variable transmission ratio such as linkages, cam drive mechanisms, and geared linkages.

The fundamentals of rigid machine theory go back to works by L. Euler (1707–1783) and J.L. Lagrange (1736–1813). When the steam engine was developed, these theories became interesting to mechanical engineers in the second half of the 19th century. Machine designers first used the method of kinetostatics, which considers the inertia forces of moving mechanisms according to d’Alembert’s principle as static forces and treats them with known the methods of statics (then primarily graphical methods). The book “Versuch einer grafischen Dynamik” (An Attempt at Graphical Dynamics) by Proell, which was published in Leipzig in 1874, is an example of the approach taken at that time.

The second volume of “Theoretische Maschinenlehre” (Machine Theory) by F. Grashof (1826–1893) that appeared in 1883 also contained fundamentals of machine dynamics. For example, the concept of reduced mass, which was to prove very useful later on, was introduced there. The arising questions about the balancing of masses were first addressed in the book by H. Lorenz (1865–1940) titled “Dynamik der Kurbelgetriebe” (Dynamics of crank mechanisms; Leipzig, 1901).
The works by Karl Heun (1859–1929), who stressed the mathematical aspects such as the integration of the differential equations, and R. von Mises (1883–1953) were summarized in 1907 in “Dynamische Probleme der Maschinenlehre” (Dynamic Problems of Machine Theory) so that the rigid machine theory had basically been worked out by the beginning of the 20th century.

For a long time, the authoritative book for mechanical engineers was the one by F. Wittenbauer (1857–1922) that presented graphical methods suitable for planar mechanisms. The extensions of these methods to spatial mechanisms were provided by K. Federhofer (1885–1960), who published his book “Grafische Kinematik und Kinetostatik des starren räumlichen Systems” (Graphical Kinematics and Kinetostatics of Rigid Spatial Systems) in Vienna in 1928.

These theories have since been included in textbooks on mechanism design, machine dynamics, and mechatronics. The monograph by Biezeno/Grammel [1] comprehensively discusses the mass balancing of machines. K. Magnus [23] wrote a fundamental book on gyroscopes. Rigid machine theory got a fresh impetus when computers emerged and industrial robots raised the question of useful algorithms for calculating rigid-body systems of any given topological structure.

Today, engineers can solve problems in this field using commercial software products, and the mathematical or numerical methods these solutions are based on do not have to be known in detail. However, a user of such programs has to get familiar with the basic ideas of model generation to understand what can be calculated using them and what cannot [3]. The Deutsche Forschungsgemeinschaft (German Research Foundation; DFG) has sponsored a program of key research projects titled “Multibody dynamics”, the results of which were summarized by W. Schiehlen in an anthology [28].

2.2 Kinematics of a Rigid Body

2.2.1 Coordinate Transformations

To describe the position and motions of a rigid body in space, it is useful to introduce a body-fixed, that is, a coordinate system \( \{O; \xi, \eta, \zeta\}\), that moves along with the body in addition to a fixed coordinate system \( \{O; x, y, z\}\), see Fig. 2.1. In kinematics and kinetics, there are geometrical and physical quantities that are defined by several components. These are vectors and tensors whose components in the body-fixed system differ from those in the fixed system. They can be converted when switching coordinate systems using specific rules (coordinate transformation).

It is advantageous, in conjunction with other problems of machine dynamics, to use matrix notation for representing the kinematic and dynamic relationships between vectors and tensors. Vectors are described by bold letters and column matrices and tensors by bold letters and quadratic \((3 \times 3)\) matrices. To apply matrix calculus for the vector product (or cross product), each vector is assigned a skew symmetric-
2.2 Kinematics of a Rigid Body

A matrix that is labeled by the letter of the vector and a superscript \textit{tilde} ($\tilde{\cdot}$). For example, the three coordinates of a vector are arranged as follows:

\[
\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix};
\]

\[
\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}; \quad \tilde{\mathbf{F}} = \begin{bmatrix} 0 & -F_z & F_y \\ F_z & 0 & -F_x \\ -F_y & F_x & 0 \end{bmatrix}.
\] (2.1)

The cross product of the position vector $\mathbf{r}$ and the force vector $\mathbf{F}$ yields the moment vector and can be expressed as a matrix product as follows:

\[
\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \cdot \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} -zF_y + yF_z \\ zF_x - xF_z \\ -yF_x + xF_y \end{bmatrix} = \begin{bmatrix} M^O_x \\ M^O_y \\ M^O_z \end{bmatrix}
\] (2.2)

The following applies, therefore, in matrix notation (using the tilde operator)

\[
\tilde{\mathbf{r}} \cdot \mathbf{F} = -\tilde{\mathbf{F}} \cdot \mathbf{r} = M^O
\] (2.3)
In an inertial system, point \( O \) is the origin of a Cartesian coordinate system comprising the fixed coordinate directions \( x, y, \) and \( z \), see Fig. 2.1. The position of an arbitrary point \( P \) of a rigid body is uniquely characterized by the three coordinates \( x_P, y_P, \) and \( z_P \) that are summarized in position vector \( r_P = (x_P, y_P, z_P)^T \). A body-fixed reference point \( \overline{O} \) is selected as the origin of a body-fixed \( \xi-\eta-\zeta \) coordinate system. It has the fixed coordinates \( r_{\overline{O}} = (x_{\overline{O}}, y_{\overline{O}}, z_{\overline{O}})^T \). With respect to the directions of the fixed coordinate system, the same point \( P \) that is viewed from this reference point \( \overline{O} \) has the components

\[
l_P = r_P - r_{\overline{O}} = (\Delta x, \Delta y, \Delta z)^T = (x_P - x_{\overline{O}}, y_P - y_{\overline{O}}, z_P - z_{\overline{O}})^T. \quad (2.4)
\]

In the body-fixed system, the position of that same point \( P \) can be given by the following components:

\[
\overline{l} = (\xi_P, \eta_P, \zeta_P)^T. \quad (2.5)
\]

The components of \( l_P \) and \( \overline{l}_P \) differ when the two coordinate systems do not have parallel axes. The index \( P \) that characterizes an arbitrary point in the body is omitted in other calculations, i.e. \( l_P \equiv \overline{l} = (\xi, \eta, \zeta)^T \). The coordinates \( x, y, \) and \( z \) as well as \( \xi, \eta, \) and \( \zeta \) then refer to all points that belong to the rigid body.

![Coordinate transformation for a planar rotation](image)

For motions in three-dimensional space, the rigid body has three rotational degrees of freedom in addition to the three translational degrees of freedom. These rotational ones can be described by three angles. First, the relationships between the
coordinates of a point for a planar rotation about the angle $q_1$ are established. The following relationships can be read for projections of the body-fixed coordinates onto the body-fixed axes (and vice versa) from Fig. 2.2:

$$\Delta x = 1 \cdot x^*, \quad x^* = 1 \cdot \Delta x,$$
$$\Delta y = \cos q_1 \cdot y^* - \sin q_1 \cdot z^*, \quad y^* = \cos q_1 \cdot \Delta y + \sin q_1 \cdot \Delta z,$$
$$\Delta z = \sin q_1 \cdot y^* + \cos q_1 \cdot z^*, \quad z^* = -\sin q_1 \cdot \Delta y + \cos q_1 \cdot \Delta z. \quad (2.6)$$

These two times three equations each correspond to a matrix equation if one introduces the vector $l^* = (x^*, y^*, z^*)^T$ and the rotational transformation matrix $A_1$:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_1 - \sin q_1 & 0 \\ 0 & \sin q_1 & \cos q_1 \end{bmatrix}; \quad A_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_1 & \sin q_1 \\ 0 & -\sin q_1 & \cos q_1 \end{bmatrix}. \quad (2.7)$$

The rotational transformation matrix is orthonormal, resulting in the following relationship with the unit matrix $E$

$$A_1 (A_1)^T = E; \quad (A_1)^T = (A_1)^{-1}. \quad (2.8)$$

Thus the relationships (2.6) are as follows:

$$l = A_1 l^*; \quad l^* = (A_1)^T l. \quad (2.9)$$

In a spatial rotation, the elements of the rotational transformation matrix $A$ depend on three angles to be defined specifically. The cardan angles that are designated $q_1$, $q_2$ and $q_3$ here are used to describe the position of the body, see Fig. 2.3.

The fixed $x$-$y$-$z$ system and the body-fixed $\xi$-$\eta$-$\zeta$ system coincide in the initial position. When rotating the outer frame about the angle of rotation $q_1$, the $x$ axis is retained ($x = x^*$), and the plane of the inner frame becomes the new $y^*$-$z^*$ plane. The angle of rotation $q_2$ describes the rotation of the inner frame about the positive $y^*$ axis that coincides with the $y^{**}$ axis, so that the $x^{**}$-$z^{**}$ plane, which is perpendicular to that, takes a new position. The angle of rotation $q_3$ finally relates to the $z^{**}$ axis that coincides with the $\zeta$ axis of the body-fixed coordinate system. The $\xi$-$\eta$ plane is perpendicular to $z^{**} = \zeta$. The body-fixed $\xi$-$\eta$-$\zeta$ system takes an arbitrary rotated position relative to the fixed $x$-$y$-$z$ system upon these three rotations.

Each of the three rotations in itself represents a planar rotation about another axis. According to Fig. 2.3, the following three elementary rotations apply:

$$l = A_1 l^*; \quad l^* = A_2 l^{**}; \quad l^{**} = A_3 l. \quad (2.10)$$

The rotational transformation matrices for rotations about the $y^*$ and $z^{**}$ axes can be obtained starting from the projections onto the other planes in analogy to Fig. 2.2. The following matrices implement the rotations about the angles $q_2$ and $q_3$ of the respective axes:
Fig. 2.3 Description of a spatial rotation

\[
A_2 = \begin{bmatrix}
\cos q_2 & 0 & \sin q_2 \\
0 & 1 & 0 \\
-\sin q_2 & 0 & \cos q_2
\end{bmatrix}; \quad A_3 = \begin{bmatrix}
\cos q_3 & -\sin q_3 & 0 \\
\sin q_3 & \cos q_3 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (2.11)

If these relationships are inserted into each other according to (2.10), one obtains:

\[
l = A_1 l^* = A_1 A_2 l^{**} = A_1 A_2 A_3 \tilde{l} = A \tilde{l}; \quad \tilde{l} = A^T l
\] (2.12)

and thus the transformation matrix for the spatial rotation

\[
A = A_1 A_2 A_3.
\] (2.13)

Performing the multiplication using the matrices known from (2.7) and (2.11) according to (2.13), one obtains:

\[
A = \begin{bmatrix}
\cos q_2 \cos q_3 & -\cos q_2 \sin q_3 & \sin q_2 \\
\sin q_1 \sin q_2 \cos q_3 + \cos q_1 \sin q_3 & -\sin q_1 \sin q_2 \sin q_3 + \cos q_1 \cos q_3 & -\sin q_1 \cos q_2 \\
-\cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3 & \cos q_1 \sin q_2 \sin q_3 + \sin q_1 \cos q_3 & \cos q_1 \cos q_2
\end{bmatrix}.
\] (2.14)

This matrix can not only be used to transform the position vectors, but all vectors, so that the following follows from (2.1) for the force and moment vectors:
\[ F = A F; \quad T = A^T F; \quad M^O = A M^O; \quad M^O = A^T M^O. \] (2.15)

The components of a vector are typically designated by the same letter as the vector itself but not printed in bold face, and given the indices \( x, y, z \) in the fixed system of reference. In the body-fixed system, a bar is added to the bold letter, and its components are given the indices \( \xi, \eta, \zeta \). For example, the same (physical) force vector has the components \( F = (F_x, F_y, F_z)^T \) or \( \bar{F} = (F_\xi, F_\eta, F_\zeta)^T \) depending on the reference system.

The elements of the rotational transformation matrix \( A \) are nonlinear functions of the three angles of rotation \( q_1, q_2 \) and \( q_3 \), cf. (2.14). The transformation matrices are orthonormal for spatial rotations as well, so the following applies in analogy to (2.8):

\[ A^T = (A)^{-1}; \quad A^T A = AA^T = E. \] (2.16)

The coordinates of a body point with respect to fixed directions relative to the origin \( O \) can be calculated in matrix notation from those of the reference point \( \bar{O} \) and the body-fixed coordinates:

\[ r = r_{\bar{O}} + \bar{l} = r_{\bar{O}} + A\bar{l}. \] (2.17)

### 2.2.2 Kinematic Parameters

The superordinate term kinematic parameters should be understood as to include velocity, acceleration, angular velocity, and angular acceleration. It is used in a similar way as "generalized force" that represents the superordinate term for force and moment/torque.

The three components of absolute velocity \( \mathbf{v} = \mathbf{\dot{r}} = (\dot{x}, \dot{y}, \dot{z})^T \) with respect to the fixed directions are derived using the time derivative of \( r \) from (2.17). Using (2.12), they are:

\[ \mathbf{v} = \mathbf{\dot{r}} = \mathbf{r}_{\bar{O}} + \mathbf{\dot{l}} = \mathbf{r}_{\bar{O}} + A\mathbf{\bar{l}} = \mathbf{\dot{l}} = A \mathbf{\bar{l}}. \] (2.18)

The time derivative of the relationship given on the right in (2.16) is:

\[ \frac{d(AA^T)}{dt} = \dot{A}A^T + A(A^T) = \dot{A}A^T + (\dot{A}A^T)^T = \ddot{\omega} + \bar{\omega}^T = \mathbf{o}. \] (2.19)

This equation gives rise to the following conclusion: since the sum of a matrix with its transpose is zero only if the matrix itself is skew symmetrical, the product of \( \dot{A}A^T = \ddot{\omega} \) in (2.19) must be a skew symmetrical matrix. One therefore may, according to (2.1), assign a \((3 \times 3)\) matrix \( \ddot{\omega} \) of the tensor of angular velocity to the vector of angular velocity \( \omega = (\omega_x, \omega_y, \omega_z)^T \):

\[ \dot{A}A^T = \ddot{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = -(\ddot{\omega})^T = -A(A^T)^T. \] (2.20)
The vectors and matrices of the angular velocity are transformed between the fixed and body-fixed coordinate systems in analogy to (2.12):

\[ \omega = A \bar{\omega}; \quad \bar{\omega} = A^T \omega, \]  
(2.21)

\[ \dot{\omega} = A \dot{\bar{\omega}} A^T; \quad \bar{\omega} = A^T \dot{\omega} A. \]  
(2.22)

It is useful for some applications to determine the body-fixed components of the angular velocity at once. The following applies as an alternative to (2.20)

\[ A^T \dot{A} = \bar{\omega} = \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} = \bar{\omega}. \]  
(2.23)

The \textbf{angular velocity} is the same at all points of the rigid body, it cannot be assigned to any one point of the rigid body and cannot be calculated by a time derivative of an angle for general spatial motion. Without going into more detail, note that each spatial motion of a rigid body can be described as a screw motion about an instantaneous axis, at which the vectors of velocity and angular velocity are proportional to each other \((v = k \omega)\).

The magnitude \(\omega\) of the angular velocity is derived from both (2.20) and (2.23):

\[ \omega = \sqrt{\omega^T \omega} = \sqrt{\omega_\zeta^2 + \omega_\eta^2 + \omega_\xi^2} = \sqrt{\bar{\omega}^T \bar{\omega}}. \]  
(2.24)

(2.18) can be written in two ways, taking into account (2.20):

\[ v = \dot{r} = \dot{r}_\Sigma + \bar{\omega} l = \dot{r}_\Sigma - \dot{\bar{\omega}} \bar{l}. \]  
(2.25)

Therefore, the components of the velocity vector are as follows:

\[ \begin{align*}
\dot{x} &= \dot{x}_\Sigma - \omega_\zeta \Delta y + \omega_\eta \Delta z \\
\dot{y} &= \dot{y}_\Sigma + \omega_\zeta \Delta x - \omega_\xi \Delta z \\
\dot{z} &= \dot{z}_\Sigma - \omega_\eta \Delta z + \omega_\xi \Delta y
\end{align*} \]  
(2.26)

The velocity can also be expressed as a function of the body-fixed components:

\[ v = \dot{r} = \dot{r}_\Sigma + \bar{\omega} A \bar{l} = \dot{r}_\Sigma + A \bar{\omega} \bar{l} = \dot{r}_\Sigma - A \dot{\bar{l}} \bar{\omega}. \]  
(2.27)

The components of the \(\omega\) vector can, in actual problems, not only be determined from (2.20) but also by projecting the angular velocity vector onto the directions of the respective system of reference.

Differentiation of the velocity in (2.25) and (2.27) finally provides the absolute \textbf{acceleration} of a point in the following form:

\[ \ddot{r} = \ddot{\bar{r}} = \frac{d(\dot{r}_\Sigma + \bar{\omega} l)}{dt} = \ddot{r}_\Sigma + \ddot{\bar{\omega}} \bar{l} + \dot{\bar{\omega}} \dot{l} \]  
(2.28)

\[ \ddot{r} = \ddot{\bar{r}} = \ddot{r}_\Sigma + A \dot{\bar{\omega}} \dot{\bar{l}} = \ddot{r}_\Sigma + (\dot{\bar{\omega}} + \ddot{\bar{\omega}}) \bar{l}, \]
2.2.3 Kinematics of the Gimbal-Mounted Gyroscope

Figure 2.3 shows a rigid body that can freely rotate (in the sketched massless apparatus) about three axes in space. A rigid body that can only perform three rotations is called a gyroscope. The position of the gyroscope can be uniquely described using cardan angles $q = (q_1, q_2, q_3)^T$, see. (2.14).

If one uses the matrix $A$ and its time derivative $\dot{A}$ to determine the product according to (2.20), one obtains the matrix of the tensor of angular velocity $\tilde{\omega}$, which contains the following components as matrix elements:

$$
\begin{align*}
\omega_x &= \dot{q}_1 + \dot{q}_3 \sin q_2 \\
\omega_y &= \dot{q}_2 \cos q_1 - \dot{q}_3 \sin q_1 \cos q_2 \\
\omega_z &= \dot{q}_2 \sin q_1 + \dot{q}_3 \cos q_1 \cos q_2.
\end{align*}
$$

(2.29)

For the body-fixed reference system, the components of the angular velocity are derived according to (2.21) using the $\omega = A^T \dot{\omega}$ transformation after performing matrix multiplication and some manipulations of the trigonometric functions:

$$
\begin{align*}
\omega_\xi &= \dot{q}_1 \cos q_2 \cos q_3 + \dot{q}_2 \sin q_3 \\
\omega_\eta &= -\dot{q}_1 \cos q_2 \sin q_3 + \dot{q}_2 \cos q_3 \\
\omega_\zeta &= \dot{q}_1 \sin q_2 + \dot{q}_3.
\end{align*}
$$

(2.30)

The magnitude $\omega$ of the angular velocity $\omega$ results from (2.24):

$$
\omega = \sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\dot{q}_1 \dot{q}_3 \sin q_2}.
$$

(2.31)

It follows from here that the magnitude of the angular velocity for gimbal-mounting according to Fig. 2.3, at constant angular velocities is not constant in general, but only if either ($\dot{q}_1$ or $\dot{q}_2$ or $\dot{q}_3$) is zero.

If the condition is met that the fixed $x$-$y$-$z$-system and the body-fixed $\xi$-$\eta$-$\zeta$-system coincide in their initial positions, the angle coordinates

$$
\varphi_x \approx q_1; \quad \varphi_y \approx q_2; \quad \varphi_z \approx q_3,
$$

(2.32)

can be introduced for small angles of rotation

$$
|q_1| \ll 1; \quad |q_2| \ll 1; \quad |q_3| \ll 1
$$

(2.33)

that describe “small motions”. Since $\sin q_k \approx q_k$ and $\cos q_k \approx 1$, it follows from (2.14) and (2.20):
\[ A \approx \begin{bmatrix} 1 & -\varphi_z & \varphi_y \\ \varphi_z & 1 & -\varphi_x \\ -\varphi_y & \varphi_x & 1 \end{bmatrix} \] (2.34)

and

\[ \dot{A}A^T = \ddot{\omega} \approx \begin{bmatrix} 0 & -\dot{\varphi}_z & \dot{\varphi}_y \\ \dot{\varphi}_z & 0 & -\dot{\varphi}_x \\ -\dot{\varphi}_y & \dot{\varphi}_x & 0 \end{bmatrix} \] (2.35)

These simple matrices are very popular for some applications. Whoever uses them should be aware of their scope, however.

### 2.2.4 Problems P2.1 and P2.2

**P2.1 Kinematics of a Pivoted Rotor**

In many engineering applications, rotating bodies are pivoted about an axis that is perpendicular to their longitudinal axis. Such motions occur during cornering of wheels (bicycle, motorbike, car), when pivoting a carousel, a drilling machine, or a running spin drier. Fig. 2.4 shows a model that describes such motions. A frame that can rotate about the \( x \) axis in a fixed \( x-y-z \) reference system carries a rotor that can be pivoted therein. Consider the motion of the rotor and one of its points.

**Given:**
- Frame dimensions \( l \) and \( h \)
- Distance of a point \( P \) in the rotor \( \eta_P \)
- Pivoting angle \( \alpha(t) \)
- Angle of rotation of the rotor \( \gamma(t) \)

**Find:**
1. Components of the angular velocity \( \omega \) and angular acceleration \( \ddot{\omega} \) of the rotor in the co-rotating \( \xi-\eta-\zeta \) coordinate system
2. Components of the absolute velocity \( v_P \) of point \( P \).
P2.2 Edge Mill

An edge mill is a machine for comminuting, grinding, or mixing, (e. g. ores, coal, clay, corn, etc.) in which rollers are guided along an angular path that compress and comminute the material to be ground.

Figure 2.5 shows the grindstone modeled as a homogeneous cylinder with a center of gravity that is guided at a distance \( \xi_S \) along a planar circular path around the fixed vertical \( z \) axis. The \( \xi \) axis of the grindstone is pivoted horizontally at the angular speed \( \dot{\varphi}(t) \). Pure rolling of the center plane of the roller at the grinding level is assumed.

**Given:**
- Roller radius \( R \)
- Distance to the center of gravity \( \xi_S \)
- Angular velocity of the axle \( \dot{\varphi}(t) \)

**Find:**
1. Rotational transformation matrix \( A \)
2. Angular velocity vector both in fixed \((\boldsymbol{\omega})\) and in body-fixed \((\overline{\boldsymbol{\omega}})\) coordinate directions
3. Velocity and acceleration distribution along \( \overline{AB} \)

**Fig. 2.5** Geometrical and kinematic quantities at a roller of the edge mill
2.2.5 Solutions S2.1 and S2.2

S2.1 The system in Fig. 2.4 is a special case of the gimbal-mounted gyroscope with regard to the rotational motion, see Fig. 2.3. The body-fixed components of the angular velocity of the body result from (2.30) with $\alpha = q_1$, $\beta = q_2 = 0$ and $\gamma = q_3$ as follows:

$$\mathbf{\omega} = [\omega_\xi, \omega_\eta, \omega_\zeta]^T = [\dot{\alpha} \cos \gamma, -\dot{\alpha} \sin \gamma, \dot{\gamma}]^T.$$  \hspace{1cm} (2.36)

The components of the angular acceleration $\mathbf{\dot{\omega}}$ are the derivatives with respect to time:

$$\dot{\omega}_\xi = \ddot{\alpha} \cos \gamma - \dot{\alpha} \dot{\gamma} \sin \gamma,$$

$$\dot{\omega}_\eta = -\dot{\alpha} \sin \gamma - \dot{\alpha} \dot{\gamma} \cos \gamma,$$

$$\dot{\omega}_\zeta = \ddot{\gamma}.$$ \hspace{1cm} (2.37)

The position of point $P$ is described, according to (2.17), using the body-fixed reference point $O$, the rotational transformation matrix $A$, and the coordinate in the body-fixed system:

$$r_P = r_\Sigma + l_P = r_\Sigma + A\tilde{l}_P.$$ \hspace{1cm} (2.38)

The matrix $A$ is either determined from the product of matrix $A_1$ in (2.7) and matrix $A_3$ in (2.11), or as a special case of (2.14) for $q_2 = 0$:

$$A = A_1 A_3 = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\cos \alpha \sin \gamma & \cos \alpha \cos \gamma - \sin \alpha & -\sin \alpha \\
\sin \alpha \sin \gamma & \sin \alpha \cos \gamma & \cos \alpha
\end{bmatrix}. \hspace{1cm} (2.40)$$

Using (2.18) or (2.27), the velocity of point $P$ is:

$$v_P = v_\Sigma + \tilde{A}\tilde{l}_P = v_\Sigma + A\dot{\mathbf{\omega}}\tilde{l}_P.$$ \hspace{1cm} (2.41)

After inserting (2.39) and a few calculation steps, the result is

$$\begin{bmatrix}
\dot{x}_P \\
\dot{y}_P \\
\dot{z}_P
\end{bmatrix} = \dot{\alpha} \begin{bmatrix}
0 \\
-l \sin \alpha - h \cos \alpha \\
l \cos \alpha - h \sin \alpha
\end{bmatrix}$$

$$+ \eta_P \begin{bmatrix}
-\dot{\gamma} \cos \gamma \\
-\dot{\gamma} \cos \alpha \sin \gamma - \dot{\alpha} \sin \alpha \cos \gamma \\
-\dot{\gamma} \sin \alpha \sin \gamma + \dot{\alpha} \cos \alpha \cos \gamma
\end{bmatrix}. \hspace{1cm} (2.42)$$

The first expression results from the differentiation of $r_\Sigma$ from (2.39). The second expression is either obtained by differentiation of $A$ and $\tilde{l}_P$ from (2.39), or by multiplying $A$ from (2.40) and $\ddot{\mathbf{\omega}}$ from (2.36) and $\tilde{l}_P$.

The acceleration $a_P$ could be determined by another differentiation of $v_P$ with respect to time. This involves terms with the factors $\ddot{\alpha}$, $\ddot{\gamma}$, $\dot{\alpha}^2$, $\dot{\gamma}^2$, and $\dot{\alpha} \dot{\gamma}$.

S2.2 A body-fixed $\xi$-$\eta$-$\zeta$ system with its origin in the bearing $\hat{O}$ was introduced in addition to the fixed $x$-$y$-$z$ reference system, see Fig. 2.5. In the initial position, the $\xi$ axis is parallel to the $x$ axis, the $\eta$ axis is parallel to the $y$ axis, and the $\zeta$ axis is parallel to the $z$ axis. Pure rolling of a circular cone would be possible if there were a conic support surface. However, a planar
2.2 Kinematics of a Rigid Body

support surface and a circular cylinder are used in the calculations here, assuming that the circle rolls off at the distance \( \xi_S \) on the \( z = 0 \) plane.

The stipulation \( \psi(\varphi = 0) = 0 \) means for the angles (when pure rolling takes place at the distance \( \xi_S \)) that the constraint

\[
\xi_S \varphi = R \psi; \quad \psi = \frac{\xi_S \varphi}{R}.
\]

(2.43)

applies when taking into account the positive directions of rotation as defined. Like in (2.10), the rotation matrix \( \mathbf{A} \) can be determined using the sequence of the two elementary rotations \( \varphi \) and \( \psi \):

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} = 
\begin{bmatrix}
    \cos \varphi & -\sin \varphi & 0 \\
    \sin \varphi & \cos \varphi & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix},
\]

(2.44)

\[
\begin{bmatrix}
    x' \\
    y' \\
    z'
\end{bmatrix} = 
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos \psi & \sin \psi \\
    0 & -\sin \psi & \cos \psi
\end{bmatrix}
\begin{bmatrix}
    \xi \\
    \eta \\
    \zeta
\end{bmatrix}
\]

If these equations are combined, the rotational transformation relations between the fixed \( x, y, z \) coordinates and the co-rotating \( \xi, \eta, \zeta \) coordinates are obtained:

\[
\mathbf{A} = \mathbf{A}_{\varphi} \cdot \mathbf{A}_{\psi} = 
\begin{bmatrix}
    \cos \varphi & -\sin \varphi \cos \psi & -\sin \varphi \sin \psi \\
    \sin \varphi \cos \psi & \cos \varphi & \cos \varphi \sin \psi \\
    -\sin \psi \cos \psi & 0 & \cos \psi
\end{bmatrix}
\]

(2.45)

The components of the angular velocity vector with respect to the body-fixed coordinate directions can be read from Fig. 2.5, keeping in mind that the angular velocity \( \dot{\psi} \) opposes the positive \( \xi \)-direction, and if the angular velocity \( \dot{\varphi} \) pointing in the \( z \)-direction is decomposed into its components in the directions of \( \eta \) and \( \zeta \) using the angle of rotation \( \psi \):

\[
\mathbf{\bar{\omega}} = 
\begin{bmatrix}
    \omega_\xi \\
    \omega_\eta \\
    \omega_\zeta
\end{bmatrix} = 
\begin{bmatrix}
    -\dot{\psi} \\
    -\dot{\varphi} \sin \psi \\
    \dot{\varphi} \cos \psi
\end{bmatrix} = \dot{\varphi}
\begin{bmatrix}
    -\xi_S / R \\
    -\sin \left( \frac{\xi_S \varphi}{R} \right) \\
    \cos \left( \frac{\xi_S \varphi}{R} \right)
\end{bmatrix}
\]

(2.46)

The result specified last in (2.46) was obtained by inserting the constraint (2.43) (also in differentiated form). Using the rotation matrix \( \mathbf{A} \), the following is obtained for the fixed directions:

\[
\mathbf{\dot{\omega}} = \begin{bmatrix} \omega_x, \omega_y, \omega_z \end{bmatrix}^T = \mathbf{A} \mathbf{\bar{\omega}} = \dot{\varphi}
\begin{bmatrix}
    -\xi_S / R \cos \varphi, \\
    -\xi_S / R \sin \varphi, \\
    1
\end{bmatrix}^T
\]

(2.47)

From this follows the skew symmetrical matrix \( \mathbf{\bar{\omega}} \), see (2.20), which is needed later:

\[
\mathbf{\bar{\omega}} = \dot{\varphi}
\begin{bmatrix}
    0 & -\xi_S \sin \varphi / R & \xi_S \cos \varphi / R \\
    -\xi_S \sin \varphi / R & 0 & \xi_S \cos \varphi / R \\
    \xi_S \sin \varphi / R & -\xi_S \cos \varphi / R & 0
\end{bmatrix}
\]

(2.48)

The components of the angular acceleration vector are obtained by differentiating (2.47) with respect to time:
\[ \begin{bmatrix} -\xi_S \cos \varphi \\ \xi_S \sin \varphi \\ 1 \end{bmatrix} + \dot{\varphi}^2 \frac{\xi_S}{R} \begin{bmatrix} \sin \varphi \\ -\cos \varphi \\ 0 \end{bmatrix} \]  

\[(2.49)\]

Differentiation of \( \ddot{x} \) from (2.47), due to the special property \( \dot{\omega} = \dot{\varphi} \), provides the components of the angular acceleration vector, with respect to the body-fixed directions:

\[ \ddot{x} = \ddot{\varphi} \begin{bmatrix} -\xi_S^2 \sin \varphi \\ 0 \\ 0 \end{bmatrix} \]

\[ + \xi_S^2 \frac{\xi_S}{R} \begin{bmatrix} \cos \left( \frac{\xi_S \varphi}{R} \right) \\ 0 \\ 0 \end{bmatrix} \]

\[(2.50)\]

For the velocity and acceleration distributions, which are most appropriately determined using Euler’s kinematic equations, the coordinates of the points located on the line segment \( \overline{AB} \) are required, to which

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[ \begin{bmatrix} \xi_S \cos \varphi \\ \xi_S \sin \varphi \\ z \end{bmatrix} \right]^T \]

applies according to Fig. 2.5. If the center of gravity \( S \) of the roller (and not \( \varnothing \)) with

\[ \begin{bmatrix} \xi_S \cos \varphi \\ \xi_S \sin \varphi \\ R \end{bmatrix} = \dot{r}_S \]

\[ \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} \cos \varphi \\ -\sin \varphi \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 0 \\ z - R \end{bmatrix} \]

\[ (2.52) \]

is selected as reference point, the following applies to the velocity distribution according to (2.25) with \( \varnothing \) according to (2.48):

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dot{r}_S + \varnothing (r - r_S) \]

\[ = \xi_S \varphi \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} + \xi_S \varphi \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ -\frac{\xi_S}{R} \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} \cos \varphi \\ -\sin \varphi \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 0 \\ z - R \end{bmatrix} \]

\[ (2.53) \]

Differentiation of (2.51) with respect to time would not have yielded the correct result (which can easily be verified by comparing with (2.53)) since the radius vector \( r \) in (2.50) describes the instantaneous position of points that are not body-fixed, that is, the points located on \( \overline{AB} \) are always different body points when the cylinder is rolling.
From (2.52) the linear velocity distribution already known from the rolling wheel can be seen: it equals zero at contact point $A$ ($z = 0$) and has its maximum at the upper point $B$ ($z = 2R$).

The following then applies to the acceleration distribution along $\mathcal{AB}$, see (2.28):

\[
\ddot{r} = \ddot{r}_S + (\dot{\omega} + \omega\ddot{\omega}) (r - r_S)
\]
\[
= \xi_S \ddot{\varphi} \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} - \xi_S \dot{\varphi}^2 \begin{bmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{bmatrix}
\]
\[
+ \left( \dot{\varphi} \xi_S \begin{bmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix} + \dot{\varphi}^2 \xi_S \begin{bmatrix} -2\cos \varphi \\ -2\sin \varphi \\ -\xi_S \frac{z}{R} \end{bmatrix} \right) (z - R)
\]

(2.54)

The special case $\dot{\varphi} = \Omega = \text{const.}$ (i.e. $\dot{\varphi} \equiv 0$) is shown for the two variants $\xi_S/R = 1$ and $\xi_S/R = 2$ in Fig. 2.6. It can be seen here that the distribution in conjunction with mass causes a moment effect that appears as gyroscopic moment in kinetics, see Sect. 2.3.3.

In Fig. 2.6, the instantaneous axis of rotation that passes through points $O$ and $A$ is shown as a dashed line.
2.3 Kinetics of the Rigid Body

2.3.1 Kinetic Energy and Moment of Inertia Tensor

The kinetic energy of a mass element $dm$ that moves at a velocity of $v = \dot{r}$ relative to a fixed reference system amounts to

$$dW_{\text{kin}} = \frac{1}{2} dm v^2 = \frac{1}{2} dm (\dot{r})^T \dot{r}. \quad (2.55)$$

The kinetic energy of a rigid body is determined by integrating over the entire body with a velocity distribution according to (2.25), resulting in

$$W_{\text{kin}} = \frac{1}{2} \int \omega^T J \omega \, dm. \quad (2.56)$$

Engineering mechanics proves that it is useful to select the center of gravity $S$ as the body-fixed reference point. Starting from (2.56), the kinetic energy of an arbitrarily moving rigid body can then be expressed as follows:

$$W_{\text{kin}} = \frac{1}{2} \frac{m v_S^T v_S}{2} + \frac{1}{2} \omega^T J_S \omega \quad (2.57)$$

The mass of the body is $m = \int dm$, and $v_S = (\dot{x}_S, \dot{y}_S, \dot{z}_S)^T$ is the absolute velocity of the center of gravity $S$. The translational kinetic energy can be obtained from it, together with the mass $m$ of the body. The vector $\overline{\omega} = (\omega_\xi, \omega_\eta, \omega_\zeta)^T$ of the angular velocity is related to the body-fixed $\xi$-$\eta$-$\zeta$ coordinate system, see Sects. 2.2.2 and 2.2.3. The rotational energy that corresponds to the other terms in (2.57) can be expressed using the moment of inertia $J_{kk}^S$ with respect to the instantaneous axis of rotation labeled with index $k$. The following applies:

$$J_{kk}^S \omega^2 = J_{\xi \xi}^S \omega_\xi^2 + J_{\eta \eta}^S \omega_\eta^2 + J_{\zeta \zeta}^S \omega_\zeta^2 + 2(J_{\xi \eta}^S \omega_\xi \omega_\eta + J_{\eta \zeta}^S \omega_\eta \omega_\zeta + J_{\zeta \xi}^S \omega_\zeta \omega_\xi). \quad (2.58)$$

Thus the kinetic energy is simply

$$W_{\text{kin}} = \frac{1}{2} m v_S^2 + \frac{1}{2} J_{kk}^S \omega^2. \quad (2.59)$$

The moment of inertia $J_{kk}^S$ refers to the direction of the instantaneous axis of rotation, see the application in Sect. 1.2.4. The direction of the instantaneous axis of rotation $k$ can be described with respect to the directions of the body-fixed reference system using the angles $\alpha_k$, $\beta_k$, and $\gamma_k$, see Fig. 2.7a. The components of angular velocity with respect to this direction are
\[
\omega_\xi = \omega \cos \alpha_k; \quad \omega_\eta = \omega \cos \beta_k; \quad \omega_\zeta = \omega \cos \gamma_k.
\] (2.60)

Fig. 2.7 Directional angles within the rigid body; a) Identification of a direction \( k \) (such as \( k = 1, \ II, \ III \)); b) Identification of the position of a \( \xi_k-\eta_k-\zeta_k \) system in the \( \xi-\eta-\zeta \) system

The dependence of the moment of inertia \( J^{S}_{kk} \) on these angles can be determined from (2.58) and (2.60):

\[
J^{S}_{kk} = J_{\xi\xi}^{S} \cos^2 \alpha_k + J_{\eta\eta}^{S} \cos^2 \beta_k + J_{\zeta\zeta}^{S} \cos^2 \gamma_k + 2(J_{\xi\eta}^{S} \cos \alpha_k \cos \beta_k + J_{\eta\zeta}^{S} \cos \beta_k \cos \gamma_k + J_{\zeta\xi}^{S} \cos \gamma_k \cos \alpha_k).
\] (2.61)

The matrix of the moment of inertia tensor with respect to the center of gravity is defined in the body-fixed system by:

\[
\mathbf{J}^S = \int (\mathbf{\tilde{l}} - \mathbf{\tilde{l}}_S)(\mathbf{\tilde{l}} - \mathbf{\tilde{l}}_S)^T dm.
\] (2.62)

The integration refers to the entire body volume and is theoretically performed by a triple integral that is hardly solved in closed form in practice because the bodies comprise so many shapes. In most cases, the moment of inertia tensor is calculated by subdividing a body into elementary bodies with small masses (or into bodies with a known moment of inertia tensor) from the CAD programs for any machine parts. When it comes to actually existing parts, it is recommended to determine the moment of inertia tensor from experimental results and to check the theoretical values, see in this context Sect. 1.2.4.

The mass \( m \) characterizes the body’s inertia during translational motions. Similarly, the moment of inertia tensor captures the respective properties of a rigid body with regard to rotational motions. If the center of gravity is the origin (\( S = \mathbf{O} \)), the matrix of the moment of inertia tensor is:
\( \mathbf{J}^S = \begin{bmatrix} J_{\xi\xi}^S & J_{\xi\eta}^S & J_{\xi\zeta}^S \\ J_{\eta\xi}^S & J_{\eta\eta}^S & J_{\eta\zeta}^S \\ J_{\zeta\xi}^S & J_{\zeta\eta}^S & J_{\zeta\zeta}^S \end{bmatrix} \)

\[
\mathbf{J}^S = \begin{bmatrix}
\int (\eta^2 + \zeta^2) dm \\
-\int \eta \xi dm \\
-\int \zeta \xi dm
\end{bmatrix}
- \begin{bmatrix}
\int \eta \xi dm \\
\int (\xi^2 + \zeta^2) dm \\
-\int \eta \zeta dm
\end{bmatrix}
- \begin{bmatrix}
-\int \eta \xi dm \\
\int (\xi^2 + \eta^2) dm \\
\int \eta \zeta dm
\end{bmatrix}.
\] (2.63)

This matrix is symmetrical. The elements on the principal diagonal are called moments of inertia (in short “rotating masses”), and the elements outside the principal diagonal are called products of inertia (also centrifugal moments). Unlike the moments of inertia, the products of inertia can be zero or negative. The moments of inertia are measures of the rotational inertia of a body and the products of inertia are measures of the body’s tendency to change its axis of rotation when rotating. They characterize the unsymmetrical mass distribution of the body, see also (2.75).

With respect to the fixed directions, the moment of inertia tensor according to transformation (2.22) results from

\[ \mathbf{J}^S = \mathbf{A} \mathbf{J}^S \mathbf{A}^T. \] (2.64)

In general, it is variable, that is, it depends on the angles of rotation in accordance with the rotational transformation matrix \( \mathbf{A} \). The matrix

\[ \mathbf{J}^S = \begin{bmatrix} J_{xx}^S & J_{xy}^S & J_{xz}^S \\ J_{yx}^S & J_{yy}^S & J_{yz}^S \\ J_{zx}^S & J_{zy}^S & J_{zz}^S \end{bmatrix}. \] (2.65)

corresponds to it. When the angles are small, as in (2.33), the linear approximation

\[ \mathbf{J}^S \approx \mathbf{J}^S - \mathbf{J}^S \tilde{q} + \tilde{q} \mathbf{J}^S. \] (2.66)

follows from (2.64) due to \( \mathbf{A} \approx \mathbf{E} + \tilde{q} \) (see (2.34)). The moment of inertia tensor is frequently used in the even more simplified form \( \mathbf{J}^S \approx \mathbf{J}^S \) (i.e. \( \mathbf{A} \approx \mathbf{E} \)) when calculating linear oscillations, see Sects. 1.2.4, 3.2.2 and 5.2.3.

The static moments and the moment of inertia tensor are dependent on the point of reference chosen. The center of gravity (or center of mass) \( S \) is a special body-fixed (reference) point. Its position is defined by the fact that the static moments with respect to it are zero. If it is the origin of the body-fixed \( \xi-\eta-\zeta \) coordinate system, the conditions

\[
\int \xi dm = \int \eta dm = \int \zeta dm = 0.
\]

must be satisfied. When switching from any reference point \( \mathbf{O} \) to the center of gravity \( S \) relative to parallel axes, the conversion of the matrix elements of the moment
of inertia tensor is governed by the parallel-axis theorem (Steiner’s theorem):
\[
\tilde{\mathbf{J}} = \mathbf{J}^S + m(I_S)^T I_S,
\]
with
\[
\tilde{l} = \begin{bmatrix}
0 & -\zeta_S & \eta_S \\
\zeta_S & 0 & -\xi_S \\
-\eta_S & \xi_S & 0
\end{bmatrix}.
\]
(2.68)

Thus the moments of inertia always have their smallest values with respect to axes of gravity because “Steiner terms” are added for other axes. The components of the moment of inertia tensor also change when switching to rotated body-fixed axes \(\xi_1-\eta_1-\zeta_1\). In analogy to (2.22), when it comes to the transformation between fixed and body-fixed directions, a transformation matrix can be used that is designated as \(A^*\). The directional cosines in \(A^*\) then refer to the nine angles, \(\alpha_{\xi k}\) to \(\gamma_{\zeta k}\), that are defined as in Fig. 2.7b between the \(\xi-\eta-\zeta\) system and the \(\xi_k-\eta_k-\zeta_k\) system that has the same point \(O\) as its body-fixed origin.

The moment of inertia tensor (here, exemplarily with respect to the center of gravity \(S\) – it applies in analogy to each body-fixed point) is transformed when rotating in the body-fixed reference system with the matrix
\[
A^* = \begin{bmatrix}
\cos \alpha_{\xi k} & \cos \beta_{\xi k} & \cos \gamma_{\xi k} \\
\cos \alpha_{\eta k} & \cos \beta_{\eta k} & \cos \gamma_{\eta k} \\
\cos \alpha_{\zeta k} & \cos \beta_{\zeta k} & \cos \gamma_{\zeta k}
\end{bmatrix}.
\]
(2.69)

by the following matrix multiplications:
\[
\mathbf{J}^S = A^* \mathbf{J}^S A^{*T}; \quad \mathbf{J}^* = A^{*T} \mathbf{J}^S A^*.
\]
(2.70)

The matrix \(\mathbf{J}^S\) contains the components known from (2.63) while the components in \(\mathbf{J}^*\) relate to the \(\xi_k-\eta_k-\zeta_k\) system that is rotated inside the rigid body.

For each reference point \(O\) there is a special coordinate system with three directions that are perpendicular to one another and for which the moment of inertia tensor becomes a diagonal matrix. These axes are called principal axes. The transformation onto the central principal axes, is of particular interest if the center of gravity is selected as the reference point \((O = S)\). The principal axes are identified by the Roman numerals I, II, and III. The principal moments of inertia \(J^S_I, J^S_{II}\) and \(J^S_{III}\) are the three eigenvalues of the eigenvalue problem
\[
(\mathbf{J}^S - J^S E)\mathbf{a} = \mathbf{0},
\]
(2.71)
which can be solved numerically using known software if parameter values are given. The three eigenvectors associated with the eigenvalues
\[
\mathbf{a}_k = [\cos \alpha_k, \cos \beta_k, \cos \gamma_k]^T; \quad k = I, II, III
\]
(2.72)
contain, as elements, the directional cosines, which define the orientation of the principal axes with spatial angles $\alpha_k$, $\beta_k$ and $\gamma_k$ relative to the original $\xi-\eta-\zeta$ system, see also Fig. 2.7a. They are normalized in such a way that

$$
(a_I)^T a_I = (a_{II})^T a_{II} = (a_{III})^T a_{III} = 1;
$$

$$
(a_I)^T a_{II} = (a_{II})^T a_{III} = (a_{III})^T a_I = 0
$$

and $\det(a_I, a_{II}, a_{III}) = 1$. The transformation matrix

$$
A^*_H = [a_I, a_{II}, a_{III}]
$$

is formed from these three eigenvectors, so that the moment of inertia tensor for the central principal axes can be expressed as follows:

$$
\hat{J}^S = A^*_H \hat{J}^S A^*_H = \begin{bmatrix}
J^S_I & 0 & 0 \\
0 & J^S_{II} & 0 \\
0 & 0 & J^S_{III}
\end{bmatrix}.
$$

The deviation moments with respect to the principal axes are zero. Symmetry axes of a homogeneous rigid body are principal axes. Table 5.2 specifies the moments of inertia with respect to the three principal axes for some bodies of revolution.

The components of the angular velocity with respect to the principal axes are derived from:

$$
\omega_H = A^*_H \overline{\omega} = [\omega_I, \omega_{II}, \omega_{III}]^T.
$$

The expression for the kinetic energy from (2.57) becomes simpler when reference can be made to the principal axes:

$$
W_{kin} = \frac{1}{2} m \dot{v}^T S \dot{v}^S + \frac{1}{2} (\omega_H)^T \hat{J}^S \omega_H
$$

$$
= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} (J^S_I \omega_I^2 + J^S_{II} \omega_{II}^2 + J^S_{III} \omega_{III}^2).
$$

If the motion is a rotation about a body point $\overline{O}$ that is fixed in space, the kinetic energy can simply be expressed using the moment of inertia tensor $\overline{J}$ with respect to this point, see (2.67):

$$
W_{kin} = \frac{1}{2} (\omega_H)^T \overline{J} \omega_H = \frac{1}{2} (J^O_1 \omega_I^2 + J^O_{II} \omega_{II}^2 + J^O_{III} \omega_{III}^2).
$$

The principal directions in (2.77) are generally different from those in (2.78), which is why the components of the angular velocity with respect to $S$ and $\overline{O}$ differ as well.
2.3.2 Principles of Linear Momentum and of Angular Momentum

The principle of linear momentum and the principle of conservation of angular momentum are fundamental laws that reveal the interconnection of force quantities and motion quantities of a rigid body.

The principle of linear momentum states that the center of gravity $S$ accelerates ($\ddot{r}_S$) as if the resultant $F$ of the external forces (both the applied forces and the reaction forces) acted on it and as if the mass $m$ was concentrated in $S$. With respect to a fixed reference system, it is:

$$m\ddot{r}_S = F \quad (2.79)$$

and for the components in the fixed reference system:

$$m\ddot{x}_S = F_x; \quad m\ddot{y}_S = F_y; \quad m\ddot{z}_S = F_z. \quad (2.80)$$

Newton’s second law can also be converted using (2.15) and (2.28) into

$$m\ddot{r}_S = m\left[ \ddot{r}_O + A(\ddot{\omega} + \omega \dot{\omega})l_S \right] = m\left[ \ddot{r}_O + (\dot{\omega} + \omega \dot{\omega})l_S \right] = F = AF \quad (2.81)$$

so that it takes the following form for fixed components, see (2.16):

$$m\left[ A^T \ddot{r}_O + (\ddot{\omega} + \omega \dot{\omega})l_S \right] = F. \quad (2.82)$$

The angular momentum of a mass element $dm$ with respect to a fixed reference point $O$ is the product of the components of its velocity and their perpendicular distances from the axes that pass through the reference point

$$dL^O = dm \ddot{r} \cdot r.$$

(2.83)

The angular momentum of a rigid body that is arbitrarily moving in space can be found by integration over the entire body:

$$L^O = \int \ddot{r} \cdot r dm = \int \ddot{r}(v_O - \ddot{l}\omega)dm. \quad (2.84)$$

The principle of conservation of angular momentum, formulated by L. Euler in 1750 takes the following form for the fixed reference point $O$ and the fixed directions

$$\frac{dL^O}{dt} \equiv \frac{d}{dt} \left[ m \left( \ddot{r}_O\ddot{r}_S + (\ddot{l}_S - \ddot{l}_O)\ddot{l}_O \right) + J^O\ddot{\omega} \right] = M^O. \quad (2.85)$$

The vector of the external moments, i.e. the sum of the applied moments $M^{O(e)}$ and the reaction moments $M^{O(z)}$, includes the components $M^O = [M^O_x, M^O_y, M^O_z]^T$ in the fixed reference system. While the principle of linear momentum is mostly used with respect to fixed coordinates, the principle of conservation of angular mo-
momentum is frequently applied with respect to body-fixed directions. Therefore, only the cases of most interest will be presented here in the form of Euler’s gyroscope equations: If the body-fixed point of reference $O$ is not accelerated ($\ddot{r}_O \equiv 0$), the principle of conservation of angular momentum is

$$M_{\text{kin}}^O \equiv \bar{\omega} J^O \dot{\omega} + J^O \ddot{\omega} = \bar{M}^O.$$  \hspace{1cm} (2.86)

The kinetic moment (or moment) due to rotational inertia is on the left-hand side. $M^O = [M^O_\xi, M^O_\eta, M^O_\zeta]^T$ is the vector of the resultant external moment in the body-fixed reference system with respect to $O$. For an arbitrarily moving center of gravity, the principle of conservation of angular momentum is similar to (2.86):

$$M_{\text{kin}}^S \equiv \bar{\omega} J^S \dot{\omega} + J^S \ddot{\omega} = \bar{M}^S.$$  \hspace{1cm} (2.87)

$M^S = [M^S_\xi, M^S_\eta, M^S_\zeta]^T$ is the vector of the resultant external moment in the body-fixed reference system with respect to $S$. The reader should state each of the three equations described in (2.87) in detail once. It will become evident that the following form results if the central principal axes are selected as the body-fixed reference system:

$$M_{\text{kin}}^I \equiv J^I \dot{\omega}_I - (J^I_{II} - J^I_{III}) \omega_{II} \omega_{III} = M^I$$

$$M_{\text{kin}}^{II} \equiv J^{II}_{III} \dot{\omega}_{III} - (J^{II}_{III} - J^{II}_{I}) \omega_{I} \omega_{III} = M^{II}$$

$$M_{\text{kin}}^{III} \equiv J^{III}_{I} \dot{\omega}_{I} - (J^{III}_{I} - J^{III}_{II}) \omega_{II} \omega_{I} = M^{III}.$$  \hspace{1cm} (2.88)

In addition to the term with the angular acceleration, the kinetic moment contains a term that occurs at constant angular velocities: the so-called gyroscopic moment. The following statement can be noted about the gyroscopic moment, e.g. due to the term $(J^I_{II} - J^I_{III}) \omega_{II} \omega_{III}$: as a result of inertia, the gyroscopic moment will occur about the respective third principal axis that is perpendicular to the two others. The right-hand rule applies to the direction of the gyroscopic moment: If the thumb and index finger of the right hand point in the direction of the vectors of $\omega_I$ and $\omega_{II}$, the middle finger points in direction $III$, about which the gyroscopic moment occurs. The body “wants” to turn in direction $III$. If it is prevented from this rotation, a reaction moment occurs that acts in opposite direction to direction $III$. If, for example, a wheel (rotation about horizontal component $I$) rolls around a bend (vertical component $II$), the gyroscopic moment acts about perpendicular horizontal axis $III$ in such a way that it exerts additional pressure towards the ground. This rule should be noted and checked in all examples, see, for example, problems P2.1, P2.3 and Sect. 2.3.3.

Like in (2.79) to (2.82), where external forces (applied force $F^{(e)}$ and reaction force $F^{(z)}$) are on the right and the inertia force is on the left side of the equations, the right sides of (2.85) to (2.88) always contain the external moments, and the left sides the kinetic moments $M_{\text{kin}}$ (or “inertia moments” in analogy to “inertia forces”). External moments can be both applied moments $M^{(e)}$ (e.g. input torques
or moments of friction) and reaction moments $M^{(z)}$, such as reaction forces that are absorbed by the bearings.

When solving problems, a free-body diagram for the rigid body is developed and all force quantities that act on it from outside are included. Due to inertia, inertia forces and moments that are also called kinetic forces $F_{\text{kin}}$ and kinetic moments $M_{\text{kin}}$, in accordance with the kinetic energy concept, come “from inside”. The inertia forces $F_{\text{kin}} \equiv m\ddot{r}$ are entered in the free-body diagram opposite to the positive coordinate direction of $r_S$, the kinetic moments $M_{\text{kin}} \equiv \dot{\omega} J_S \omega + J_S \ddot{\omega}$, in accordance with the kinetic energy concept, come “from inside”. The inertia forces $F_{\text{kin}} \equiv m\ddot{r}$ are entered in the free-body diagram opposite to the positive coordinate direction of the body-fixed $\xi-\eta-\zeta$ system.

The formal identity of (2.86) and (2.87) can also be transferred to (2.88) and the special forms (2.90), (2.92), and (2.93) of these equations discussed below. These will not be specified for the case of a fixed body point $\overrightarrow{O}$. As usual in engineering calculations, Newton’s second law and the principle of conservation of angular momentum can be stated as six conditions of equilibrium using the directions of the generalized forces shown in the free-body diagram:

$$F^{(e)} + F^{(z)} + (-F_{\text{kin}}) = \mathbf{0}; \quad M^{(e)} + M^{(z)} + (-M_{\text{kin}}) = \mathbf{0}. \quad (2.89)$$

Note their application when solving the problems in Sects. 2.3.3 to 2.3.5, see Figs. 2.8, 2.10, and 2.33.

If the body rotates about a single fixed axis only (which is the $\zeta$ axis here, such as for a rigid rotor in rigid bearings), the following equations of motion follow from (2.87) for $\omega_{\xi} \equiv \omega_{\eta} \equiv 0$:

$$M_{\text{kin}}^{S,\xi} \equiv J_{\xi}^{S} \omega_{\zeta} - J_{\eta}^{S} \omega_{\zeta}^2 = M_{\xi}^{S}$$
$$M_{\text{kin}}^{S,\eta} \equiv J_{\eta}^{S} \omega_{\zeta} + J_{\xi}^{S} \omega_{\zeta}^2 = M_{\eta}^{S}$$
$$M_{\text{kin}}^{S,\zeta} \equiv J_{\zeta}^{S} \omega_{\zeta} = M_{\zeta}^{S}. \quad (2.90)$$

This shows that kinetic moments occur about the $\xi$ and $\eta$ axes (that is, perpendicular to the axis of rotation $\zeta$) if the angular velocity is constant and the products of inertia are not zero. These kinetic moments must be absorbed by bearing forces perpendicular to the axis of rotation in order to force the fixed axis of rotation.

If the fixed $x$-$y$-$z$ system and the body-fixed $\xi-\eta-\zeta$ system coincide in their initial positions, small angles of rotation $\varphi_x$, $\varphi_y$ and $\varphi_z$ can be introduced for the fixed axes so that the following applies because of (2.32) and (2.35):

$$\omega_{\xi} \approx \dot{\varphi}_x; \quad \omega_{\eta} \approx \dot{\varphi}_y; \quad \omega_{\zeta} \approx \dot{\varphi}_z. \quad (2.91)$$

If the products of the angular velocities are neglected with respect to the angular accelerations, because they are small and of second order, the linearized form of the principle of conservation of angular momentum results from (2.87) taking into account (2.91) with a moment of inertia tensor that does change over time. Due to the small angles, the body-fixed and the fixed components approximately coincide if they were congruent in the initial position:
If a body rotates at the “large” angular velocity \( \omega \equiv \Omega = \text{const.} \), another form of the linearized gyroscope equations follows from (2.87) for \(|\omega| \ll \Omega \) and \(|\omega \eta| \ll \Omega \) when neglecting the products of the small components of the angular velocity:

\[
\begin{align*}
M_{\text{kin}}^\xi &\equiv J_{\xi \xi}^S \ddot{\phi}_x + J_{\xi \eta}^S \ddot{\phi}_y + J_{\xi \zeta}^S \ddot{\phi}_z = M_{\xi}^S \approx M_{\xi}^S, \\
M_{\text{kin}}^\eta &\equiv J_{\eta \xi}^S \ddot{\phi}_x + J_{\eta \eta}^S \ddot{\phi}_y + J_{\eta \zeta}^S \ddot{\phi}_z = M_{\eta}^S \approx M_{\eta}^S, \\
M_{\text{kin}}^\zeta &\equiv J_{\zeta \xi}^S \ddot{\phi}_x + J_{\eta \zeta}^S \ddot{\phi}_y + J_{\zeta \zeta}^S \ddot{\phi}_z = M_{\zeta}^S \approx M_{\zeta}^S. \\
\end{align*}
\]  

The principle of conservation of angular momentum is often used in the form of (2.92) or (2.93) if a rigid body is part of a vibration system, see also Sects. 3.2.2 and 5.2.3.

For more detailed information on the theory of gyroscopes and its applications, see [23].

### 2.3.3 Kinetics of Edge Mills

The kinematics of edge mills were discussed in Sect. 2.2.4 in the solution of problem P2.2 so that this discussion refers to the results obtained there.

The rotating body (grindstone) according to Fig. 2.5, which rolls off along a circular path, exerts a force in addition to its own weight on its base that is due to the gyroscopic effect. The problem is to calculate the required input torque, the normal force and the horizontal force on the grindstone for pure rolling motion for a given function of the pivoting angle \( \phi(t) \).

**Given:**
- Gravitational acceleration \( g \)
- Roller radius \( R \)
- Roller length \( L \)
- Distance to the center of gravity \( \xi_S \)
- Time function of the pivoting angle \( \phi(t) \)
- Mass of the roller (grindstone) \( m \)

Moments of inertia of the roller with respect to \( S \):

\[
J_{\xi \xi}^S = J_{\eta \eta}^S = \frac{m(3R^2 + L^2)}{12}, \quad J_{\zeta \zeta}^S = \frac{1}{2}mR^2.
\]

Since it is assumed here as in S2.2 that a pure rolling motion occurs at \( \xi = \xi_S \), a sliding motion will occur along the base at the other contact points between the grindstone and the plane, which may be a desired effect for grinding. The sliding velocity of the contact points in tangential direction between roller and grinding
plane is

\[ v_{\text{rel}} = (\xi_S - \xi)\dot{\phi}. \]  

(2.94)

To simplify, it is assumed in the calculation model that a vertical normal force \( F_N \) and a horizontal adhesive force \( F_H \) only act below the center of gravity of the roller. The frictional forces for \( \xi \neq \xi_S \) are not taken into account.

The components of the angular velocity of the roller with respect to the body-fixed and fixed directions are known from S2.2 (eqs. (2.47), (2.48)), and so is the angular acceleration ((2.49)).

The body-fixed \( \xi-\eta-\zeta \) system corresponds to the system of the principal axis of this symmetrical rigid body. Axis I can be assigned to the \( \xi \) coordinate, axis II to the \( \eta \) coordinate, and axis III to the \( \zeta \) coordinate. Euler’s gyroscope equations then result from (2.88) with respect to the fixed body point \( O \):

\[
M_{\text{kin}}^{\xi} \equiv J_{\text{O} \xi \xi} \dot{\omega}_{\xi} - (J_{\text{O} \eta \eta} - J_{\text{O} \zeta \zeta}) \omega_{\eta} \omega_{\zeta} = M_{\xi}^{O}
\]

$$M_{\text{kin}}^{\eta} \equiv J_{\text{O} \eta \eta} \dot{\omega}_{\eta} - (J_{\text{O} \zeta \zeta} - J_{\text{O} \xi \xi}) \omega_{\zeta} \omega_{\xi} = M_{\eta}^{O}$$

$$M_{\text{kin}}^{\zeta} \equiv J_{\text{O} \zeta \zeta} \dot{\omega}_{\zeta} - (J_{\text{O} \xi \xi} - J_{\text{O} \eta \eta}) \omega_{\xi} \omega_{\eta} = M_{\zeta}^{O}.$$  

(2.95)

These equations state that there is an equilibrium of the kinetic moments from the rotational inertia of the body with the external moments. The moments of inertia given with respect to the center of gravity have to be transformed to the fixed body point \( O \) using the parallel-axis theorem, see (2.72). They amount to:

\[
J_{\eta \eta}^{\text{O}} = J_{\xi \xi}^{\text{O}} = J_{\eta \eta}^{S} + m\xi_{S}^{2} = J_{a} = \frac{m(3R^{2} + L^{2} + 12\xi^{2}_{S})}{12}
\]

\[
J_{\xi \xi}^{\text{O}} = J_{\eta \eta}^{\text{O}} = J_{p} = \frac{1}{2}mR^{2}.
\]

(2.96)

If one takes into account (2.47) and (2.50), the kinetic moments can be calculated first from (2.95):

\[
M_{\text{kin}}^{\xi} \equiv J_{\text{O} \xi \xi} \dot{\omega}_{\xi} = -J_{p} \ddot{\psi}
\]

$$M_{\text{kin}}^{\eta} \equiv J_{\text{O} \eta \eta} \dot{\omega}_{\eta} - (J_{\text{O} \zeta \zeta} - J_{\text{O} \xi \xi}) \omega_{\eta} \omega_{\xi}$$

$$= J_{a} (\ddot{\varphi} \sin \psi + \dot{\varphi} \dot{\psi} \cos \psi) - (J_{a} - J_{p}) \dot{\varphi} \dot{\psi} \cos \psi
= J_{a} \ddot{\varphi} \sin \psi + J_{p} \dot{\varphi} \dot{\psi} \cos \psi$$

$$M_{\text{kin}}^{\zeta} \equiv J_{\text{O} \zeta \zeta} \dot{\omega}_{\zeta} - (J_{\text{O} \xi \xi} - J_{\text{O} \eta \eta}) \omega_{\xi} \omega_{\eta}$$

$$= J_{a} (\ddot{\varphi} \cos \psi - \dot{\varphi} \dot{\psi} \sin \psi) + (J_{p} - J_{a}) \dot{\varphi} \dot{\psi} \sin \psi
= J_{a} \ddot{\varphi} \cos \psi + J_{p} \dot{\varphi} \dot{\psi} \sin \psi.$$  

(2.97)

(2.98)

(2.99)

The components in (2.95) \( (M_{\xi}^{O}, M_{\eta}^{O} \text{ and } M_{\zeta}^{O}) \) of the resultant external moment \( M^{O} \) result from the external forces that act on the body, i.e. the input torque \( M_{\text{an}} \), the static weight \( mg \) and the reaction forces \( (F_{N} \text{ and } F_{H}) \) at the contact point. It
is more favorable to write the moment equilibrium about the \( y^* \) axis and about the \( z \) axis, rather than the moment equilibrium about the \( \eta \) and \( \zeta \) axes. The following can be derived for the \( \xi \) axis both formally from (2.97) to (2.99) and by inspection (Fig. 2.8b and c):

\[
M_{\text{kin} \xi} \equiv -J_p \ddot{\psi} = -F_H R
\]  

(2.100)

about the \( z \) axis (see Fig. 2.8a and b):

\[
M_{\text{kin} \zeta} \cos \psi - M_{\text{kin} \eta} \sin \psi \equiv J_a \ddot{\phi} = M_{\text{an}} - F_H \xi_S
\]  

(2.101)

and about the \( y^* \) axis, see Fig. 2.8a and c:

\[
M_{\text{kin} \zeta} \sin \psi + M_{\text{kin} \eta} \cos \psi \equiv J_p \dot{\phi} \dot{\psi} = (F_N - mg) \xi_S.
\]  

(2.102)

These are the equations for calculating the input torque as well as the reaction forces \( F_N \) and \( F_H \). One can express the results using the parameters given in the problem statement – the above form, however, is better suited to recognize the “origin” of each term, such as the gyroscopic effect of the rotor.

The horizontal force that ensures adhesion is

\[
F_H = \frac{J_p \ddot{\psi}}{R} = \frac{J_p}{R^2} \xi_S \ddot{\phi} = \frac{1}{2} m \xi_S \ddot{\phi}.
\]  

(2.103)
The input torque, that causes the given function $\varphi(t)$ is

$$M_{\text{an}} = \left( J_a + J_p \frac{\xi_S^2}{R^2} \right) \ddot{\varphi} = \frac{m\varphi(3R^2 + L^2 + 18\xi_S^2)}{12}$$ (2.104)

and the normal force results from (2.102):

$$F_N = mg + \frac{J_p \dot{\varphi} \dot{\psi}}{\xi_S} = mg \left( 1 + \frac{R \dot{\varphi}^2}{2g} \right).$$ (2.105)

It is proportional to the radius $R$ and the square of the angular velocity ($\dot{\varphi}^2$), but independent of the length $\xi_S$. It can be considerably larger than the (static) weight.

The influence of the roller radius appears to have been empirically known since ancient times, since one can find such edge mills mostly with great grindstone radii in old mills. The horizontal force $F_H$ that results from (2.100) is a reaction force that only occurs with angular accelerations. The individual forces assumed here are the resultants of the actually occurring line loads under the grindstone in both the vertical and horizontal directions. The mechanical behavior of the material to be milled has to be taken into account to calculate their distributions.

### 2.3.4 Problems P2.3 and P2.4

#### P2.3 Kinetics of a Pivoting Rotor

The bearing forces of rotating bodies that rotate about their bearing axis and at the same time about an axis perpendicular to this bearing axis are of interest in many engineering applications. Figure 2.4 shows a frame (considered massless) that is pivoted about the $x$ axis in the fixed $x$-$y$-$z$ reference system and in which a rotor is pivoted that can rotate about its $\zeta$ axis (principal axis III) inside the frame. The center of gravity of the rotor is at the origin of the body-fixed reference system ($\overline{O} = S$). Of interest are general formulae for calculating the moments with respect to the fixed system that occur when rotor and frame are rotated simultaneously.

**Given:**
- Frame dimensions $l$ and $h$
- Time functions of angles $\alpha(t)$ and $\gamma(t)$
- Rotor mass $m$
- Principal moments of inertia of the rotor $J_{\xi \xi}^S = J_\eta = J_\eta = J_{\zeta \zeta} = J_a^S$, $J_{\xi \zeta}^S = J_{\eta \zeta}^S = J_{\zeta \zeta}^S = J_p^S$

**Find:**
1. Components of center-of-gravity acceleration
2. Kinetic moments with respect to the center of gravity
3. Moment between rotor and frame ($M_x^S$, $M_y^S$, $M_z^S$)
4. Reaction forces and moments at the origin $O$ ($F_x$, $F_y$, $M_x^O$, $M_y^O$, $M_z^O$)
5. Input torques at the rotor ($M_{\text{an}}^a$) and frame ($M_{\text{an}}$)
P2.4 Bearing Forces of a Rotating Body

The bearing forces for the rigid rotor shown in Fig. 2.9 are to be determined. The body has an eccentric center of gravity $S$ with respect to the axis of rotation. A body-fixed $\xi-\eta-\zeta$ coordinate system is used, the origin of which coincides with that of the fixed coordinate system ($O = \bar{O}$) and the $\zeta$ axis of which is identical with the fixed $z$ axis.

Fig. 2.9 Nomenclature
for the rotating rigid body;
\begin{itemize}
\item a) general rotor,
\item b) inclined circular cylinder
\end{itemize}

Note: The topic of “balancing rigid rotors” is discussed in detail in Sect. 2.6.2. The principal purpose of this problem is to illustrate the relationships derived in the previous sections.
Given:

- Mass \( m \)
- Moment of inertia \( J^S \)
- Products of inertia \( J^S_{\xi \zeta}, J^S_{\eta \zeta} \)
- Body-fixed coordinates for the center of gravity \( \mathbf{I}_S = (\xi_S, \eta_S, \zeta_S)^T \)
- Angle of rotation \( \varphi(t) \)
- Distances of the bearings from the center of gravity \( a, b \)
- Circular cylinder with radius \( R \) and length \( L \)
- Inclination angle of \( \zeta_1 \) axis relative to the \( \zeta \) axis \( \gamma \)
- Polar moment of inertia \( J_p = \frac{1}{2} m R^2 \), see (2.96)
- Axial moment of inertia \( J_a = m(3R^2 + L^2)/12 \)

Find:

1. For any function \( \varphi(t) \) and a general rotor body
   - 1.1 Bearing forces \( \mathbf{F}_A \) and \( \mathbf{F}_B \) (body-fixed reference system)
   - 1.2 Input torque \( M_{an} \)
2. Moment of inertia tensor \( J^{*S} \) of the circular cylinder that is symmetrically positioned in the \( \xi_1-\eta_1-\zeta_1 \) system, inclined in the \( \xi-\zeta \) plane by the angle \( \gamma \) relative to the axis of rotation, see Fig. 2.9b.

### 2.3.5 Solutions S2.3 and S2.4

**S2.3** The acceleration of the center of gravity can be calculated from the velocities that were determined for a body point in S2.1. For \( \eta_P = 0, \ P = S \), and it follows from (2.42):

\[
\ddot{\mathbf{r}}_S = \begin{bmatrix} \ddot{x}_S \\ \ddot{y}_S \\ \ddot{z}_S \end{bmatrix} = \dot{\alpha} \begin{bmatrix} 0 \\ -l \sin \alpha - h \cos \alpha \\ l \cos \alpha - h \sin \alpha \end{bmatrix}.
\] (2.106)

The acceleration of the center of gravity, therefore, is

\[
\ddot{\mathbf{r}}_S = \begin{bmatrix} \ddot{x}_S \\ \ddot{y}_S \\ \ddot{z}_S \end{bmatrix} = \ddot{\alpha} \begin{bmatrix} 0 \\ -l \sin \alpha - h \cos \alpha \\ l \cos \alpha - h \sin \alpha \end{bmatrix} - \dot{\alpha}^2 \begin{bmatrix} 0 \\ l \cos \alpha - h \sin \alpha \\ l \sin \alpha + h \cos \alpha \end{bmatrix}.
\] (2.107)

The problem is solved here using a free-body diagram (it could also be solved using the method with \( \alpha \) and \( \gamma \) as independent drives as described in Sect. 2.4.1). The inertia forces \( (m\ddot{y}_S, m\ddot{z}_S) \), inertia moments (kinetic moments), the input torque \( M_{an} \) that acts on it, the constraint forces \( (F_y, F_z) \) and constraint moments \( (M^S_{x*}, M^S_{y*}, M^S_{z*}) \) of the frame are included in the free-body diagram of the rotor in Figs. 2.10a and c. The forces in \( x \) direction are zero.

The kinetic moments are defined by Euler’s gyroscope equations (2.95) and are obtained in conjunction with the angular velocities known from (2.36) and the angular accelerations known from (2.37):
\[
\begin{align*}
M_{\text{kin}}^S \xi & \equiv J_S^S \hat{\xi} \hat{\omega}_\xi - (J_S^S \eta - J_S^S \zeta) \omega_\eta \omega_\zeta \\
& = J_S^S (\hat{\alpha} \cos \gamma - \hat{\alpha} \hat{\gamma} \sin \gamma) + (J_S^S \eta - J_S^S \zeta) \hat{\alpha} \hat{\gamma} \sin \gamma \\
& = J_S^S \hat{\alpha} \cos \gamma - J_S^S \hat{\alpha} \hat{\gamma} \sin \gamma \\
\end{align*}
\]
(2.108)

\[
\begin{align*}
M_{\text{kin}}^S \eta & \equiv J_S^S \hat{\eta} \hat{\omega}_\eta - (J_S^S \zeta - J_S^S \xi) \omega_\xi \omega_\zeta \\
& = J_S^S (\hat{\alpha} \sin \gamma - \hat{\alpha} \hat{\gamma} \cos \gamma) - (J_S^S \zeta - J_S^S \xi) \hat{\alpha} \hat{\gamma} \cos \gamma \\
& = -J_S^S \hat{\alpha} \sin \gamma - J_S^S \hat{\alpha} \hat{\gamma} \cos \gamma \\
\end{align*}
\]
(2.109)

\[
\begin{align*}
M_{\text{kin}}^S \zeta & \equiv J_S^S \hat{\zeta} \hat{\omega}_\zeta - (J_S^S \xi - J_S^S \eta) \omega_\xi \omega_\eta \\
& = J_S^S \hat{\gamma}.
\end{align*}
\]
(2.110)

---

**Fig. 2.10** Forces and moments at a pivoting frame with a rotor

The forces and moments are shown in the opposite direction to the body-fixed coordinate directions in Fig. 2.10c. They are transformed into the \(x^*-y^*-z^*\) coordinate system and balanced with the applied moments and reaction moments:
\[ M_{\text{kin}x}^S \equiv -M_{\text{kin}y}^S \sin \gamma + M_{\text{kin}z}^S \cos \gamma = J^S_{\gamma x} - M_{\gamma x}^S \]  
\[ M_{\text{kin}y}^S \equiv M_{\text{kin}y}^S \cos \gamma + M_{\text{kin}z}^S \sin \gamma = -J^S_{\gamma y} \dot{\gamma} = M_{\gamma y}^S \]  
\[ M_{\text{kin}z}^S \equiv M_{\text{kin}z}^S = J^S_{\gamma z} = M_{\gamma z}^S \]  

See the depiction in Fig. 2.10a. The moment \( M_{\gamma \text{in}}^\gamma \) causes the angular acceleration \( \dot{\gamma} \) and supports itself against the frame. The components of the kinetic moments of the moving rotor with respect to the \( x^*-y^*-z^* \) coordinate system are entered in the opposite direction to the positive coordinate directions. The same sign convention as explained before in Sect. 2.3.2 was also applied to the inertia forces, which can be calculated using (2.107). The following applies at the rotor (and due to the equilibrium of forces at the frame as well) for the reaction forces:

\[ F_y = m \ddot{y}_S = m \left[ -\dot{\alpha}(l \sin \alpha + h \cos \alpha) - \dot{\alpha}^2(l \cos \alpha - h \sin \alpha) \right] \]  
\[ F_z = m \ddot{z}_S = m \left[ \dot{\alpha}(l \cos \alpha - h \sin \alpha) - \dot{\alpha}^2(l \sin \alpha + h \cos \alpha) \right]. \]

From the equilibrium conditions for the frame, one finds the reaction moments relative to the origin \( O \), see Fig. 2.10a:

\[ M^O_x = M_{\text{an}x}^O = -F_y(l \sin \alpha + h \cos \alpha) + F_z(l \cos \alpha - h \sin \alpha) - M_{\gamma x}^S \]  
\[ = m(l^2 + h^2) + J^S_{\alpha x} \dot{\alpha} \]  
\[ M^O_y = M_{\text{an}y}^O = M_{\gamma y}^O \cos \alpha - M_{\gamma x}^S \sin \alpha = -J^S_{\gamma y} (\dot{\alpha} \dot{\gamma} \cos \alpha + \dot{\gamma} \sin \alpha) \]  
\[ M^O_z = M_{\text{an}z}^O = M_{\gamma z}^O \sin \alpha + M_{\gamma x}^S \cos \alpha = J^S_{\gamma z} (\dot{\alpha} \dot{\gamma} \sin \alpha + \dot{\gamma} \cos \alpha). \]

It can be seen from (2.116) that the expression in the square brackets represents the moment of inertia about \( O \) (parallel-axis theorem). The terms that depend on \( \dot{\alpha}^2 \) do not exert any influence on the moment about the origin since the resultant centrifugal force acts in radial direction and has no leverage with respect to \( O \). A gyroscopic moment emerges at constant angular velocities that has an effect on the \( y \) and \( z \) axes, see Section 2.3.

### 2.4 Equations of Motion

The equations of motion for a body that rotates about a fixed \( \zeta \) axis are given in (2.90) with respect to the center of gravity. Using the component of the angular velocity \( \omega_\zeta = \dot{\varphi} \), the result is

\[ J_{\zeta \zeta}^S \ddot{\varphi} - J_{\varphi \zeta}^S \dot{\varphi}^2 = M_\zeta^S \]  
\[ J_{\varphi \varphi}^S \ddot{\varphi} + J_{\varphi \zeta}^S \dot{\varphi}^2 = M_\varphi^S \]  
\[ J_{\zeta \varphi}^S \dot{\varphi} = M_\zeta^S. \]

The kinetic moments are on the left side of these equations, and the external moments that stem from the bearing forces and the input torque that puts the rotor into this state of motion are on the right side.

The components of the resultant external moment from the bearing forces and input torque are (see Figs. 2.9a and 2.11):

\[ M_\zeta^S = -F_{A\zeta} a + F_{B\zeta} b \]  
\[ M_\varphi^S = -F_{A\varphi} a + F_{B\varphi} b \]  
\[ M_\zeta^S = M_{\text{an}} + (F_{A\zeta} + F_{B\zeta}) \eta_S - (F_{A\eta} + F_{B\eta}) \xi_S. \]
so that one obtains three equations for the unknown components of the bearing forces and the input torque from (2.119) to (2.124):

\[-F_A a + F_B b = J^S_{\xi \phi} \ddot{\phi} - J^S_{\eta \phi} \phi^2 \quad (2.125)\]

\[F_A a - F_B b = J^S_{\eta \phi} \ddot{\phi} + J^S_{\xi \phi} \phi^2 \quad (2.126)\]

\[M_{an} + (F_A + F_B) \eta_S - (F_A + F_B) \xi_S = J^S_{\xi \phi} \phi \quad (2.127)\]

Fig. 2.11 Free-body diagram

Three other equations for the unknown quantities follow from the center-of-gravity theorem. (2.79) applies in the space-fixed reference system. However, it is useful to express it in body-fixed components in accordance with (2.82). The following applies because of \( \ddot{\vec{r}}_{SG} \equiv \vec{0} \), see Fig. 2.11:

\[m(\ddot{\vec{\omega}} + \ddot{\vec{\omega}}) \vec{l}_S = \vec{F}. \quad (2.128)\]

Since the external forces are the unknown bearing forces in \( A \) and \( B \), one can also write:

\[\vec{F}_A + \vec{F}_B = m(\ddot{\vec{\omega}} + \ddot{\vec{\omega}}) \vec{l}_S. \quad (2.129)\]

In detail, this equation with the vector \( \vec{l}_S = (\xi_S, \eta_S, \zeta_S)^T \) and the tensor matrices is as follows:

\[
\tilde{\omega} = \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\omega} = \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\quad (2.130)
\]

and after multiplication of the three matrices, respectively.
\[ F_{\Lambda \xi} + F_{B \xi} = m(\dot{\varphi} \eta_S - \dot{\varphi}^2 \xi_S) \]
\[ F_{\Lambda \eta} + F_{B \eta} = m(\dot{\varphi} \xi_S - \dot{\varphi}^2 \eta_S). \]

The components in the \( \zeta \) direction are zero. There are two linear equations each for two unknown quantities that can easily be solved: (2.125), (2.126), (2.131), and (2.132). The components of the bearing forces in the body-fixed reference system are derived from

\[ F_{\Lambda \xi} = \frac{J^S_{\nu \xi} \ddot{\varphi} + J^S_{\xi \xi} \dot{\varphi}^2 - mb(\dot{\varphi} \eta_S + \dot{\varphi}^2 \xi_S)}{a + b} \]
\[ F_{B \xi} = \frac{-J^S_{\nu \xi} \ddot{\varphi} - J^S_{\xi \xi} \dot{\varphi}^2 - ma(\dot{\varphi} \eta_S + \dot{\varphi}^2 \xi_S)}{a + b} \]
\[ F_{\Lambda \eta} = \frac{-J^S_{\nu \eta} \ddot{\varphi} - J^S_{\xi \eta} \dot{\varphi}^2 + mb(\dot{\varphi} \xi_S - \dot{\varphi}^2 \eta_S)}{a + b} \]
\[ F_{B \eta} = \frac{J^S_{\nu \eta} \ddot{\varphi} - J^S_{\xi \eta} \dot{\varphi}^2 + ma(\dot{\varphi} \xi_S - \dot{\varphi}^2 \eta_S)}{a + b} \]

(2.133)

The respective formulae for the force components with respect to fixed directions can be found in Sect. 2.6.2 (Balancing of rigid rotors) for the special case \( \dot{\varphi} = \Omega = \text{const.} \), see (2.327).

The input torque can be found from (2.127) if one inserts the forces from (2.131) and (2.132):

\[ M_{an} = \left[ J^S_{\xi \zeta} + m(\xi_S^2 + \eta_S^2) \right] \ddot{\varphi}. \]

(2.134)

The matrix (2.75) with (2.96) corresponds to the moment of inertia tensor of the circular cylinder that is positioned symmetrically in the \( \xi_1 - \eta_1 - \zeta_1 \) system and whose central axis is the \( \xi_1 \) axis:

\[ J^{S*} = \begin{bmatrix} J_\alpha & 0 & 0 \\ 0 & J_\alpha & 0 \\ 0 & 0 & J_p \end{bmatrix}. \]

(2.135)

Transformation into the \( \xi - \eta - \zeta \) coordinate system is performed using the matrix \( A^* \) from (2.69) that results from the angles

\[ \alpha_{\xi k} = \gamma; \quad \beta_{\xi k} = \frac{\pi}{2}; \quad \gamma_{\xi k} = \frac{\pi}{2} - \gamma \]
\[ \alpha_{\eta k} = \frac{\pi}{2}; \quad \beta_{\eta k} = 0; \quad \gamma_{\eta k} = \frac{\pi}{2} \]
\[ \alpha_{\zeta k} = \frac{\pi}{2} + \gamma; \quad \beta_{\zeta k} = \frac{\pi}{2}; \quad \gamma_{\zeta k} = \gamma \]

(2.136)

and is structured similar to \( A_2 \) in (2.11):

\[ A^* = \begin{bmatrix} \cos \alpha_{\xi k} & \cos \beta_{\xi k} \cos \gamma_{\xi k} & \cos \alpha_{\eta k} \cos \beta_{\eta k} \cos \gamma_{\eta k} \\ \cos \alpha_{\eta k} \cos \beta_{\eta k} \cos \gamma_{\eta k} & \cos \alpha_{\zeta k} \cos \beta_{\zeta k} \cos \gamma_{\zeta k} \end{bmatrix} = \begin{bmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \end{bmatrix}. \]

(2.137)

Matrix multiplications according to (2.70) result in the following moment of inertia tensor with respect to the directions of the \( \xi - \eta - \zeta \) system:

\[ J^S = A^* J^{S*} A^*^T = \begin{bmatrix} J_\alpha \cos^2 \gamma + J_p \sin^2 \gamma & 0 & (J_p - J_\alpha) \sin \gamma \cos \gamma \\ 0 & J_\alpha & 0 \\ (J_p - J_\alpha) \sin \gamma \cos \gamma & 0 & J_\alpha \sin^2 \gamma + J_p \cos^2 \gamma \end{bmatrix}. \]

(2.138)
Thus the elements of the moment of inertia tensor required to calculate the bearing forces in (2.133) and (2.134) are (by comparing coefficients)

\[
\begin{align*}
J_{\xi\xi}^S &= (J_p - J_a) \sin \gamma \cos \gamma; \\
J_{\eta\zeta}^S &= 0; \\
J_{\zeta\zeta}^S &= J_a \sin^2 \gamma + J_p \cos^2 \gamma.
\end{align*}
\]

(2.139)

The kinetic moment that acts on the bearings would change its sign if \(\gamma\) became negative, i.e. the tilting were the other way. The product of inertia \(J_{\xi\zeta}^S\) is “trying to tilt” the rotor from its axis of rotation. The effect of the products of inertia can vividly be explained by the centrifugal forces that cause a moment about the negative \(\eta\) axis in such a tilted rotor position.

It is also relevant whether the rotor is a flat thin disk \((J_p > J_a)\) or a long roller \((J_p < J_a)\), see Fig. 5.5 and Table 5.2. For a circular cylinder, the difference of the principal moments of inertia is \(J_p - J_a = m(3R^2 - L^2)/12\) in accordance with (2.96) and (2.120), that is, the sign of \(J_{\xi\zeta}^S\) and the direction of the “tilting moment” depend on whether the rotor is thick \((L < \sqrt{3}R)\) or thin \((L > \sqrt{3}R)\).

### 2.4 Kinetics of Multibody Systems

#### 2.4.1 Mechanisms with Multiple Drives

##### 2.4.1.1 Spatial Rigid-Body Mechanisms

Rigid-body mechanisms are systems of rigid bodies that perform planar or spatial motions, depending on the motions of their input links. So-called **generalized coordinates** \(q_k\) are used to describe the motion of such a system, each drive being assigned a coordinate and a **generalized force** \(Q_k\) \((k = 1, 2, \ldots, n)\). The position of each link of a mechanism with multiple drives depends on these \(n\) drive coordinates

\[
q = (q_1, q_2, \ldots, q_n)^T.
\]

(2.140)

Each \(q_k\) is a translation (then \(Q_k\) is a force) or an angle (then a moment is assigned to it). The number \(n\) of independent drives is termed **mobility** in mechanism theory to distinguish it from the degree of freedom that may, for example, be related to elastic deformations.

A mechanism with rigid links is called a **constrained mechanism**, if – in all positions of the mechanism – the position of any link is unambiguously related to the positions of the other links. The relationship between the number of moving links, the number of kinematic pairs and the degree of freedom of a constrained mechanism is usually called Gruebler’s criterion. **Mobility** is the number of independent coordinates needed to define the configuration of a constrained mechanism. In practice, the conditions of rigid operation are met – and the calculation model of rigid-body mechanisms (**rigid machine**) can be applied to such machines and their sub-assemblies – if the influences of clearances, elastic deformations and oscillations of the links of the mechanism are negligibly small.
The kinematic analysis of planar and spatial mechanisms with multiple drives is typically performed using appropriate software from the field of multibody dynamics. Such programs can be used to analyze the dynamic loads of very complex mechanisms, taking into account arbitrary time functions of the applied forces and moments (such as input forces, spring forces, damper forces, processing forces). In design practice, the engineer has to study the software description thoroughly to be able to utilize these powerful tools efficiently.

The following discussion will touch upon just a few general connections that exist between the drive motions (that is, time functions of the translations or angles) and the generalized forces of the drives. This is relevant for understanding the range of applicability of such programs, for their proper use, and for evaluating the results of calculations. For most practical problems, the driving forces are given by the motor characteristic, so that it becomes necessary to integrate the equations of motion, see Sect. 2.4.3.

A mechanism consists of $I$ links, of which the frame is given the index 1 and the movable bodies are given the indices $i = 2, 3, \ldots, I$, and index $I$ is usually assigned to an output link. Figure 2.12 shows some examples of rigid-body mechanisms with multiple drives. The gyroscope in Fig. 2.14 can also be interpreted in such a way that the position of the rigid body is determined by the three “input coordinates” $q_1, q_2, q_3$.

The center-of-gravity coordinates $r_{Si}$ of the $i$th link of a mechanism show an (often nonlinear) dependence on the so-called kinematic dimensions and the positions of the $n$ input links:

$$r_{Si}(q) = [x_{Si}(q), y_{Si}(q), z_{Si}(q)]^T.$$  \hfill (2.141)

Their velocities can also be calculated according to the chain rule:

$$\frac{d(r_{Si})}{dt} = v_{Si} = \sum_{k=1}^{n} \frac{\partial r_{Si}}{\partial q_k} \dot{q}_k = \sum_{k=1}^{n} r_{Si,k} \dot{q}_k; \quad i = 2, 3, \ldots, I. \hfill (2.142)$$

(2.142) spells out in detail as

$$\dot{x}_{Si} = \sum_{k=1}^{n} x_{Si,k} \dot{q}_k; \quad \dot{y}_{Si} = \sum_{k=1}^{n} y_{Si,k} \dot{q}_k; \quad \dot{z}_{Si} = \sum_{k=1}^{n} z_{Si,k} \dot{q}_k. \hfill (2.143)$$

The partial derivatives with respect to the coordinates $q_k$ are abbreviated by the letter $k$ after the comma.

The components of the angular velocities of each link in a rigid mechanism show a linear dependence on the velocities of the input coordinates. The following applies to the body-fixed components of the vector $\omega_i = (\omega_{\xi i}, \omega_{\eta i}, \omega_{\zeta i})^T$ of the angular velocity of the $i$th link:

$$\omega_{\xi i} = \sum_{k=1}^{n} u_{\xi ik} \dot{q}_k; \quad \omega_{\eta i} = \sum_{k=1}^{n} u_{\eta ik} \dot{q}_k; \quad \omega_{\zeta i} = \sum_{k=1}^{n} u_{\zeta ik} \dot{q}_k. \hfill (2.144)$$
Fig. 2.12 Examples of rigid-body mechanisms with multiple drives; a) planar Stewart platform \((n = 3)\), b) spatial Stewart platform \((n = 6)\), c) hydraulic drives of a dredging shovel \((n = 3)\), d) lift truck \((n = 2)\), e) cable-guided handling system \((n = 6)\), f) welding robot \((n = 3)\).

Compare, for example, (2.163) and (2.164).

The linear relations can be expressed according to (2.142) and (2.144) using a \textbf{Jacobian matrix} for each. The following relations apply to the translation of the centers of gravity and to rotation:

\[
\dot{r}_{Si} = Y_i(q)\dot{q}; \quad \omega_i = Z_i(q)\dot{q}; \quad i = 2, 3, \ldots, I. \tag{2.145}
\]

\(Y_i(q)\) and \(Z_i(q)\) are the Jacobian matrices for translation and rotation of the \(i\)th rigid body (link). They are rectangular matrices with three rows and \(n\) columns. Which elements are contained in these matrices follows from (2.142) and (2.144):
\[ Y_i(q) = \left[ \begin{array}{ccc} x_{Si,1} & x_{Si,2} & \cdots & x_{Si,n} \\ y_{Si,1} & y_{Si,2} & \cdots & y_{Si,n} \\ z_{Si,1} & z_{Si,2} & \cdots & z_{Si,n} \end{array} \right] ; \quad Z_i(q) = \left[ \begin{array}{ccc} u_{\xi i1} & u_{\xi i2} & \cdots & u_{\xi in} \\ u_{\eta i1} & u_{\eta i2} & \cdots & u_{\eta in} \\ u_{\zeta i1} & u_{\zeta i2} & \cdots & u_{\zeta in} \end{array} \right] \]

(2.146)

\( Z_i(q) \) is found by differentiating the angular velocities with respect to the input velocities \( \dot{q}_k \) or simply by a comparison of coefficients, see for example (2.30) from Sect. 2.2.3.

Position functions and Jacobian matrices can be explicitly stated in analytical form for open linkages, such as in Examples c, d and f in Fig. 2.12. Mechanisms with a loop structure, such as in cases a, b and e in Fig. 2.12, in which the constraint equations cannot be solved in closed form, the Jacobian matrices can be numerically calculated as a function of position (using a PC and existing software). The elements of the Jacobian matrices typically depend on the position of the input coordinates. They are also called first-order position functions.

A mechanism consists of \( I - 1 \) movable rigid bodies whose dynamic properties are captured by 10 mass parameters, respectively, that are contained in the parameter vector

\[ p_i = [m_i, \xi_{Si}, \eta_{Si}, \zeta_{Si}, J_{\xi\xi i}^S, J_{\eta\eta i}^S, J_{\zeta\zeta i}^S, J_{\xi\eta i}^S, J_{\xi\zeta i}^S, J_{\eta\zeta i}^S]^T; \quad i = 2, 3, \ldots, I \]

(2.147)

For the \( i \)th body, these are the mass \( m_i \), the three static moments \( (m_i \xi_{Si}, m_i \eta_{Si}, m_i \zeta_{Si}) \) and the six elements of the moment of inertia tensor \( (J_{\xi\xi i}^S, J_{\eta\eta i}^S, J_{\zeta\zeta i}^S, J_{\xi\eta i}^S, J_{\xi\zeta i}^S, J_{\eta\zeta i}^S) \), if one refers to the axes of gravity. If one knows the position of the principal axes (through the center of gravity), the moment of inertia tensor includes only the three principal moments of inertia \( J_{\xi\xi i}^S, J_{\eta\eta i}^S, J_{\zeta\zeta i}^S \).

The kinetic energy of the rigid-body system is the sum of the kinetic energies of all its individual bodies consisting of the translational energy and the rotational energy, see (2.57). Using the Jacobian matrices according to (2.146), the kinetic energy with the moment of inertia tensors in analogy with (2.57) becomes

\[ W_{\text{kin}} = \frac{1}{2} \dot{q}^T \sum_{i=2}^I (m_i Y_i^T Y_i + Z_i^T Z_i^S) \dot{q} = \frac{1}{2} \dot{q}^T M \dot{q} \]

(2.148)

\[ = \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n m_{kl}(q) \dot{q}_k \dot{q}_l \]

The symmetrical mass matrix \( M \) only depends on \( q \) if the Jacobian matrices contain terms that depend on \( q \). The matrix \( M \) has \( n^2 \) elements \( m_{kl} \) that are called generalized masses:

\[ m_{kl}(q) = m_{kl}(q) = \sum_{i=2}^I \left\{ m_i(x_{Si,k}x_{Si,l} + y_{Si,k}y_{Si,l} + z_{Si,k}z_{Si,l}) + J_{\xi\xi i}^S u_{\xi ik} u_{\xi il} + J_{\eta\eta i}^S u_{\eta ik} u_{\eta il} + J_{\zeta\zeta i}^S u_{\zeta ik} u_{\zeta il} + 2(J_{\eta\zeta i}^S u_{\xi ik} u_{\eta il} + J_{\zeta\xi i}^S u_{\eta ik} u_{\zeta il} + J_{\xi\zeta i}^S u_{\xi ik} u_{\zeta il}) \right\} \]

(2.149)
Due to $z_{Si} = \text{const.}$ and $\omega_{\xi i} = \omega_{\eta i} = 0$, $u_{\xi ik} = u_{\eta ik} = 0$ always applies to planar mechanisms, all links of which move parallel to the $x$-$y$ plane. The angles $\varphi_i$ are defined in Fig. 2.15. Because of $\omega_{\zeta i} = \dot{\varphi}_i$, $u_{\zeta ik} = \varphi_{i,k}$ and therefore $J_{\zeta i}^S = J_{Si}^S$:

$$m_{kl}(q) = m_{lk}(q) = \sum_{i=2}^l \left[ m_i(x_{Si,k}x_{Si,l} + y_{Si,k}y_{Si,l}) + J_{Si}^\varphi_{i,k}\varphi_{i,l} \right]$$  \hspace{1cm} (2.150)

The generalized masses are independent of the state of motion, but do depend on the position of the input coordinates in mechanisms with variable transmission ratio.

The partial derivatives are designated by the short notation already introduced in (2.142) (comma and index of the coordinate):

$$\frac{\partial(m_{kl})}{\partial q_p} = m_{kl,p}$$  \hspace{1cm} (2.151)

The so-called Christoffel symbols of the first kind, that occur when deriving the equations of motion using Lagrange’s equations of the second kind are calculated from the partial derivatives of the generalized masses as follows:

$$\Gamma_{klp} = \Gamma_{lkp} = \frac{1}{2} \left( m_{lp,k} + m_{pk,l} - m_{kl,p} \right)$$  \hspace{1cm} (2.152)

Several applied forces and moments that act on arbitrary points of the $i$th body are summarized by resultants that act on the $i$th center of gravity. These resultants of the applied forces $F_i^{(e)}$ and moments $M_i^{S(e)}$ that act on all links of the mechanism are then referred to the input coordinates $q$ using the principle of virtual work:

$$\delta W^{(e)} = \delta q^T \sum_i (Y_i^T F_i^{(e)} + Z_i^T M_i^{S(e)}) = \delta q^T Q$$  \hspace{1cm} (2.153)

The generalized forces thus result from:

$$Q = (Q_1, Q_2, \ldots, Q_n)^T = \sum_i (Y_i^T F_i^{(e)} + Z_i^T M_i^{S(e)})$$  \hspace{1cm} (2.154)

Each component $Q_p$ of the generalized forces thus follows from the applied forces and the Jacobian matrices that are characteristic for a mechanism. The generalized force quantities are not distinguished based on whether they follow from the derivative of a potential (such as potential energy, deformation energy, magnetic energy) (such as weight, spring forces, electromagnetic forces) or not (such as input forces, input moments, braking torques, friction and damping forces), a distinction that is occasionally found in the technical literature.

The equations of motion for rigid-body mechanisms with $n$ drives result from the kinetic energy (2.148) and the generalized forces obtained from (2.154) when using Lagrange’s equations of the second kind:
2.4 Kinetics of Multibody Systems

\[ Q_{p,\text{kin}} \equiv \sum_{l=1}^{n} m_{pl}(q)\ddot{q}_l + \sum_{k=1}^{n} \sum_{l=1}^{n} \Gamma_{k lp}(q)\dot{q}_k \dot{q}_l = Q_p; \quad p = 1, 2, \ldots, n \quad (2.155) \]

They each express the equilibrium between the applied forces and the kinetic forces (caused by inertia) with respect to the respective coordinates \( q_p \). The forces that result from inertia of the rigid bodies of the rigid system are called kinetic (or traditionally “kinetostatic”) forces, in contrast to the general “vibration forces”, that arise due to the oscillations of rigid bodies, see the other chapters of this book. The kinetostatic forces can be calculated for given drive motions \( q(t) \) – from the left side of (2.155). They depend on the mass parameters, the geometric conditions, and the state of motion, see for example (2.90), (2.92), (2.95), and (2.97) to (2.99).

(2.155) can be looked at from various points of view:

- If the kinematic parameters are given, the generalized forces \( Q \) “in the direction of the input coordinates” have to act in such a way that the state of motion described by the kinematic parameters \( \dot{q}(t) \) and \( \ddot{q}(t) \) is achieved. Or:

- If the generalized forces \( Q \) are known, e.g. as functions of the coordinates and velocities, (2.155) represents a system of ordinary nonlinear differential equations that one has to integrate if one wishes to calculate \( q(t) \) and its time derivatives.

- If some of the generalized forces are driving forces, the others can be considered reactions and calculated after determining \( q(t) \) and its time derivatives, see for example (2.159) and (2.160).

It is evident from (2.155) that the kinetostatic forces do not only depend on the accelerations of the drive motions. This means that inertia forces also act when the drives move at constant velocities. This is not surprising since the masses in rigid-body systems are indeed accelerated and/or decelerated. This phenomenon is known from simple cases: the centrifugal force increases with the square of the angular velocity, and the Coriolis force is proportional to the product of velocity and angular velocity. (2.155) shows that the products of all input velocities can in general occur in combination.

For the special case of a mechanism with two drives \( (n = 2) \), it follows from (2.155):

\[ m_{11} \ddot{q}_1 + m_{12} \ddot{q}_2 + \Gamma_{111} \dot{q}_1^2 + 2\Gamma_{121} \dot{q}_1 \dot{q}_2 + \Gamma_{221} \dot{q}_2^2 = Q_1 \quad (2.156) \]
\[ m_{21} \ddot{q}_1 + m_{22} \ddot{q}_2 + \Gamma_{112} \dot{q}_1^2 + 2\Gamma_{122} \dot{q}_1 \dot{q}_2 + \Gamma_{222} \dot{q}_2^2 = Q_2 \quad (2.157) \]

The following applies

\[ \Gamma_{111} = \frac{1}{2} m_{11,1}; \quad \Gamma_{121} = \frac{1}{2} m_{11,2}; \quad \Gamma_{221} = m_{12,2} - \frac{1}{2} m_{22,1} \]
\[ \Gamma_{112} = m_{12,1} - \frac{1}{2} m_{11,2}; \quad \Gamma_{122} = \frac{1}{2} m_{22,1}; \quad \Gamma_{222} = \frac{1}{2} m_{22,2} \quad (2.158) \]

If, for example, the input coordinate \( q_2 = \text{const.} \), the relationship between the force referred to the drive \( Q_1 \) and the kinetostatic forces follows from (2.156)
\[ m_{11}\ddot{q}_1 + \Gamma_{111}\dot{q}_1^2 = Q_1 \]  
\[ (2.159) \]

The kinetostatic forces also act in the direction of coordinate \( q_2 \):

\[ m_{21}\ddot{q}_1 + \Gamma_{112}\dot{q}_1^2 = m_{21}\dot{q}_1 + \left( m_{12,1} - \frac{1}{2} m_{11,2} \right) \dot{q}_1^2 = Q_2 \]  
\[ (2.160) \]

This lets one draw important conclusions regarding the parameter dependence of inertia forces. Since \( q_2 \) can be any coordinate in the direction of which the velocity and acceleration are zero, \( Q_2 \) accordingly there is a force or a moment at this “stationary site” of the mechanism. Such a force in the direction of an arbitrary coordinate can be interpreted as a generalized reaction force inside the mechanism. So it can be a joint force or an axial force if a coordinate \( q_2 \) matches its direction. Equations of the type of (2.160) are rarely used for calculating such internal forces, but they can provide explanations for the following important conclusions:

All reaction forces and moments at all places in any links of mechanisms with a single drive \((n = 1)\) result from two terms, one of which is proportional to the acceleration, the other to the square of the velocity of the drive. They all show a linear dependence on the mass parameters of each rigid body, see (2.147) and (2.149).

### 2.4.1.2 Equations of Motion of a Planetary Gear Mechanism

Planetary or epicyclic gear mechanisms are used because of their applicability as speed-transforming, superposition, and switching gear mechanisms in many fields of drive engineering. This type of gear mechanism has proven its worth, in particular, in automotive engineering and shipbuilding, where large outputs have to be transmitted at high speeds. These mechanisms allow extremely high and low gear ratios while requiring little installation space. The distribution of the static and dynamic forces over several gears and the low bearing loads at coaxial positioning of the input and output shafts are advantages. This enables the superposition of the speeds and torques of multiple drives, and they are also used as differential gears, see VDI Guideline 2157 [36].

The equations of motion that describe the relationships between the input moments and angular accelerations shall now be established for a simple planetary gear mechanism as outlined in Fig. 2.13. This superposition gear mechanism (also called gear set, transfer, or differential gear) has two degrees of freedom (mobility \( n = 2 \)). It consists of the sun gear \( 2 \), the ring gear \( 3 \), three planetary gears and the arm that carries the planetary gears \( 5 \), all of which pivot about the \( z \) axis. The radii \( r_2 \) and \( r_4 \), the moments of inertia \( J_2, J_3 \) and \( J_5 \) with respect to the fixed axis of rotation, the mass \( m_4 \) of each gear \( 4 \) and the moment of inertia \( J_4 \) of one of the gears \( 4 \) about its bearing axis are given. The center of gravity \( S \) of each planet gear is in its bearing axis. The torques that act on the shafts of the members \( 2, 3, \) and \( 5 \) are to be taken into account.
The constraints form the starting point for the kinematic and dynamic analysis. They are established in general form based on Fig. 2.13 so that they also apply to special cases of spur differentials with one degree of freedom (e.g. \( \dot{\varphi}_3 = 0 \) or \( \dot{\varphi}_5 = 0 \)).

The constraints result from the fact that the relative velocities of the engaged gears are zero at their contact points (the pitch points). Therefore, the following applies:

\[
\begin{align*}
& r_2 \dot{\varphi}_2 - [r_2 \dot{\varphi}_5 - r_4 (\dot{\varphi}_4 - \dot{\varphi}_5)] = 0 \quad (2.161) \\
& (r_2 + 2r_4) \dot{\varphi}_3 - [(r_2 + 2r_4) \dot{\varphi}_5 + r_4 (\dot{\varphi}_4 - \dot{\varphi}_5)] = 0 \quad (2.162)
\end{align*}
\]

Four position coordinates were introduced here, but only two independent coordinates exist due to the two constraints. The angles of rotation of the sun gear (\( q_1 = \varphi_2 \)) and ring gear (\( q_2 = \varphi_3 \)) are used as the two independent input coordinates. The dependent angular velocities can be found from (2.161) and (2.162):

\[
\begin{align*}
\dot{\varphi}_4 &= -\frac{r_2}{2r_4} \dot{q}_1 + \frac{r_2 + 2r_4}{2r_4} \dot{q}_2 = u_{41} \dot{q}_1 + u_{42} \dot{q}_2 \quad (2.163) \\
\dot{\varphi}_5 &= \frac{r_2}{2(r_2 + r_4)} \dot{q}_1 + \frac{r_2 + 2r_4}{2(r_2 + r_4)} \dot{q}_2 = u_{51} \dot{q}_1 + u_{52} \dot{q}_2 \quad (2.164)
\end{align*}
\]

These equations have the form of (2.144) with \( \omega_{\xi_i} \equiv 0 \), \( \omega_{\eta_i} \equiv 0 \) and \( \omega_{\zeta_i} = \dot{\varphi}_i = \sum u_{ik} \dot{q}_k \). The **gear ratios** \( u_{ik} \) express the ratio of angular velocity \( \dot{\varphi}_i \) to angular velocity \( \dot{q}_k \). The kinetic energy is the sum of the rotational energies of the sun gear (2), arm (5), planet gears (4), and ring gear (3), as well as the translational energy of the three planet gears (4):

\[
2W_{\text{kin}} = J_2 \dot{\varphi}_2^2 + J_3 \dot{\varphi}_3^2 + 3J_4 \dot{\varphi}_4^2 + J_5 \dot{\varphi}_5^2 + 3m_4 (r_2 + r_4)^2 \dot{\varphi}_5^2 \quad (2.165)
\]

To specify the kinetic energy as a function of the velocities \( \dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2)^T \), (2.163) and (2.164) are used to eliminate the angular velocities \( \dot{\varphi}_4 \) and \( \dot{\varphi}_5 \). Formally, one
could also proceed in accordance with the description in Sect. 2.4.1.1 using the
Jacobian matrices according to (2.146), which are independent of the gear position
for mechanisms with constant transmission ratio and which upon limiting to \( \omega_\zeta \) are
reduced to:

\[
Z_2 = [1 \ 0]; \quad Z_3 = [0 \ 1]; \quad Z_4 = [u_{41} \ u_{42}]; \quad Z_5 = [u_{51} \ u_{52}] (2.166)
\]

Then, the kinetic energy is, according to (2.148):

\[
W_{\text{kin}} = \frac{1}{2} \dot{q}^T M \dot{q} \quad (2.167)
\]

with the mass matrix \( M \) according to (2.148) that includes the following general-
ized masses as elements:

\[
m_{11} = J_2 + 3J_4 u_{41}^2 + [J_5 + 3m_4(r_2 + r_4)^2]u_{51}^2
\]

\[
m_{12} = m_{21} = 3J_4 u_{41} u_{42} + [J_5 + 3m_4(r_2 + r_4)^2]u_{51}u_{52}
\]

\[
m_{22} = J_3 + 3J_4 u_{42}^2 + [J_5 + 3m_4(r_2 + r_4)^2]u_{52}^2 (2.168)
\]

The generalized masses are constant here, so their partial derivatives and thus all
Christoffel symbols equal zero. The generalized forces \( Q_1 \) and \( Q_2 \) depend on the
moments \( M_2, M_3, \) and \( M_5 \). The virtual work of the input torques must be equal to
that of the generalized forces. Therefore,

\[
\delta W^{(e)} = M_2 \delta q_2 + M_3 \delta q_3 + M_5 \delta q_5
\]

\[
= M_2 \delta q_1 + M_3 \delta q_2 + M_5 (u_{51} \delta q_1 + u_{52} \delta q_2)
\]

\[
= (M_2 + M_5 u_{51}) \delta q_1 + (M_3 + M_5 u_{52}) \delta q_2 = Q_1 \delta q_1 + Q_2 \delta q_2
\]

(2.169)

A comparison of coefficients for \( \delta q_1 \) and \( \delta q_2 \) yields

\[
Q_1 = M_2 + M_5 u_{51}; \quad Q_2 = M_3 + M_5 u_{52} (2.170)
\]

Thus the equations of motion are as follows in accordance with (2.156) and (2.157)
and taking into consideration (2.168) and (2.170):

\[
m_{11} \ddot{q}_1 + m_{12} \ddot{q}_2 = M_2 + M_5 u_{51} \quad (2.171)
\]

\[
m_{21} \ddot{q}_1 + m_{22} \ddot{q}_2 = M_3 + M_5 u_{52} \quad (2.172)
\]

This general relationship leaves it still open which of the three moments or which
two angular accelerations are given or sought after. According to (2.163) and
(2.164), conditions for the other angles could be taken into account as well. Since
this mechanism has a mobility \( n = 2 \), three out of the five quantities \( (q_1, q_2, M_2, \]
\( M_3, M_5) \) can be prescribed to calculate the remaining two unknown quantities. By
integrating the differential equations (2.171) and (2.172), various operating states
can be dynamically analyzed, e. g. time functions and dynamic loads during startup,
shifting, and braking processes, if the characteristics of the motors or clutches are
given. The driving powers that result from multiplying the moments and angular velocities can also be calculated.

For example, the following operating states can occur:

**Operating state a):** input at the sun gear 2, ring gear 3 is fixed, output at the arm 5

Given are $M_2$ and $M_5$ as well as $\dot{\varphi}_2 = \dot{\varphi}_3 = 0$. (2.171) provides the angular acceleration at the input link 2

$$\ddot{q}_1 = \frac{M_2 + M_5 u_{51}}{m_{11}}$$  \hspace{1cm} (2.173)

The time function is obtained, taking into account the initial conditions, by integrating this differential equation, and the moment at the ring gear results after insertion into (2.172):

$$M_3 = -M_5 u_{52} + m_{21} \ddot{q}_1 = -M_5 u_{52} + m_{21} \frac{M_2 + M_5 u_{51}}{m_{11}}$$  \hspace{1cm} (2.174)

**Operating state b):** Input at the ring gear 3 and the sun gear 2, output at the arm 5

Given are $M_2$, $M_3$ and $M_5$. The angular accelerations $\ddot{q}_1$ and $\ddot{q}_2$ are obtained from (2.171) and (2.172) by solving the system of linear equations.

The time functions of all angles for given moment functions can then be calculated from $q_1$ and $q_2$ and all other angles from (2.163) and (2.164).

The same equations can be used for the operating state “input at the sun gear 2 and the arm 5, output at the ring gear 3”.

### 2.4.1.3 Gimbal-Mounted Rotor

The expressions for the kinetic energy and the relationships between the moments ($Q_1 = M_x$, $Q_2 = M_y$, $Q_3 = M_z$) and the three cardan angles $q_1$, $q_2$, and $q_3$ are to be established for the gyroscope that is supported as shown in Fig. 2.14. The center of gravity is assumed to be at the origin of the body-fixed coordinate system ($S = O$). Given are all elements of the moment of inertia tensor $\mathbf{J}^S$ with respect to the center of gravity $S$ and the moments of inertia $J_A$ and $J_B$ of the two frames with respect to their bearing axes.

The kinetic energy of the rotation in (2.57) is used as a starting point and must still be complemented by the kinetic energy of the two frames

$$W_{\text{kin, frame}} = \frac{1}{2} (J_A \dot{q}_1^2 + J_B \dot{q}_2^2).$$  \hspace{1cm} (2.175)

If one inserts the components of the angular velocity known from (2.30) into (2.57), it is obtained as a function of the generalized coordinates and their time derivatives:
\[ W_{\text{kin}} = W_{\text{kin frame}} + W_{\text{kin rotor}} = \frac{1}{2} \dot{q}^T M \dot{q} \]

\[ = \frac{1}{2} (m_{11} \ddot{q}_1^2 + m_{22} \ddot{q}_2^2 + m_{33} \ddot{q}_3^2) + m_{12} \ddot{q}_1 \ddot{q}_2 + m_{13} \ddot{q}_1 \ddot{q}_3 + m_{23} \ddot{q}_2 \ddot{q}_3 \]

The elements of the mass matrix \( M \) can be found by comparing coefficients:

\[
m_{11} = J_A + (J_S^{\eta} \cos^2 q_3 + J_S^{\eta} \sin^2 q_3) \cos^2 q_2 + J_S^{\xi} \sin^2 q_2 \]
\[-2 J_S^{\xi} \cos^2 q_2 \cos q_3 \sin q_3 - 2(J_S^{\xi} \sin q_3 - J_S^{\eta} \cos q_3) \cos q_2 \sin q_2 \]

\[
m_{12} = (J_S^{\eta} - J_S^{\eta}) \cos q_2 \sin q_3 \cos q_3 \]
\[+ J_S^{\xi} (\sin^2 q_3 - \sin^2 q_3) \cos q_2 + (J_S^{\eta} \cos q_3 + J_S^{\xi} \sin q_3) \sin q_2 \]

\[
m_{13} = J_S^{\xi} \sin q_2 - (J_S^{\eta} \sin q_3 - J_S^{\xi} \cos q_3) \cos q_2 \]

\[
m_{22} = J_B + J_S^{\eta} \sin^2 q_3 + J_S^{\eta} \cos^2 q_3 + 2 J_S^{\xi} \cos q_3 \sin q_3 \]

\[
m_{23} = J_S^{\eta} \cos q_3 + J_S^{\xi} \sin q_3 \]

\[
m_{33} = J_S^{\xi} \cos q_3 \]

Now the moments \( Q_1, Q_2 \) and \( Q_3 \) that act in the direction of the three angular coordinates \( q_1, q_2 \) and \( q_3 \) are to be calculated from (2.155) for the special case \( J_S^{\xi} = J_S^{\eta} = J_S^{\xi} = 0 \) when the gyroscope rotates at the angular speed \( \omega_\zeta = \dot{q}_3 \) about its \( \zeta \) axis [principal axis of inertia III (\( J_S^{\xi} = J_{III}^{\xi} \))] and the angles of the frame.
are changed in accordance with \( q_1(t) \) and \( q_2(t) \). \( J^S_{\xi \xi} = J^S_1 \) and \( J^S_{\eta \eta} = J^S_1 \) are also principal moments of inertia.

Taking into account the special condition that \( m_{23} = m_{32} = 0 \) and \( m_{33} = J^S_{\zeta \zeta} = \text{const.} \), it follows from (2.152) for the Christoffel symbols

\[
\Gamma_{111} = \Gamma_{331} = \Gamma_{122} = \Gamma_{222} = \Gamma_{332} = \Gamma_{133} = \Gamma_{233} = \Gamma_{333} = 0. \quad (2.178)
\]

It then follows from (2.155)

\[
\begin{align*}
m_{11} \ddot{q}_1 + m_{12} \ddot{q}_2 + m_{13} \ddot{q}_3 + \Gamma_{221} \dot{q}_1^2 \ddot{q}_2 + 2 \Gamma_{121} \dot{q}_1 \dot{q}_2 + 2 \Gamma_{131} \dot{q}_1 \dot{q}_3 + 2 \Gamma_{231} \dot{q}_2 \dot{q}_3 &= Q_1 \\
m_{12} \ddot{q}_1 + m_{22} \ddot{q}_2 + \Gamma_{112} \dot{q}_2^2 + 2 \Gamma_{132} \dot{q}_1 \dot{q}_3 + 2 \Gamma_{232} \dot{q}_2 \dot{q}_3 &= Q_2 \\
m_{13} \ddot{q}_1 + m_{33} \ddot{q}_3 + \Gamma_{113} \dot{q}_1^2 + 2 \Gamma_{123} \dot{q}_1 \dot{q}_2 + \Gamma_{223} \dot{q}_2^2 &= Q_3 
\end{align*}
\]

The following expressions result from the generalized masses in (2.177) for the Christoffel symbols in this special case:

\[
\begin{align*}
\Gamma_{121} &= -\Gamma_{112} = \frac{1}{2} m_{11,2} = -(J^S_{\xi \xi} \cos^2 q_3 + J^S_{\eta \eta} \sin^2 q_3 - J^S_{\zeta \zeta}) \sin q_2 \cos q_2 \\
\Gamma_{221} &= m_{12,2} = -(J^S_{\xi \xi} - J^S_{\eta \eta}) \sin q_2 \sin q_3 \\
\Gamma_{131} &= -\Gamma_{113} = \frac{1}{2} m_{11,3} = -(J^S_{\xi \xi} - J^S_{\eta \eta}) \sin q_3 \cos q_3 \cos^2 q_2 \\
\Gamma_{231} &= \frac{1}{2} m_{13,2} + \frac{1}{2} m_{12,3} = \frac{1}{2} [(J^S_{\xi \xi} - J^S_{\eta \eta})(\cos^2 q_3 - \sin^2 q_3) + J^S_{\zeta \zeta}] \cos q_2 \\
\Gamma_{132} &= -\Gamma_{123} = \frac{1}{2} m_{12,3} - \frac{1}{2} m_{13,2} \\
&= \frac{1}{2} [(J^S_{\xi \xi} - J^S_{\eta \eta})(\cos^2 q_3 - \sin^2 q_3) - J^S_{\zeta \zeta}] \cos q_2 \\
\Gamma_{232} &= -\Gamma_{223} = \frac{1}{2} m_{22,3} = (J^S_{\xi \xi} - J^S_{\eta \eta}) \sin q_3 \cos q_3
\end{align*}
\]

(2.180)

The expressions can be simplified further when taking a more specific look at

\[ J^S_{\xi \xi} = J^S_{\eta \eta} = J^S_\alpha, \quad \text{so} \quad m_{12} = 0 \quad \text{and} \quad \Gamma_{221} = \Gamma_{131} = \Gamma_{113} = \Gamma_{232} = 0 \]

(2.181)
e. g. a rotationally symmetrical rotor. The equations of motion (2.179) then are as follows:

\[
\begin{align*}
(J^S_\alpha + J^S_\xi \cos^2 q_2 + J^S_\zeta \sin^2 q_2) \ddot{q}_1 + J^S_\zeta \sin q_2 \dot{q}_3 \\
-2(J^S_\alpha - J^S_\zeta) \sin q_2 \cos q_2 \dot{q}_1 \dot{q}_2 + J^S_\zeta \cos q_2 \dot{q}_2 \dot{q}_3 &= Q_1 \\
(J^S_B + J^S_\xi) \ddot{q}_2 + (J^S_\alpha - J^S_\zeta) \sin q_2 \cos q_2 \dot{q}_1^2 - J^S_\zeta \cos q_2 \dot{q}_1 \dot{q}_3 &= Q_2 \\
J^S_\zeta \sin q_2 \dot{q}_1 + J^S_\zeta \dot{q}_3 + J^S_\zeta \cos q_2 \dot{q}_1 \dot{q}_2 &= Q_3
\end{align*}
\]

(2.182) - (2.184)

It is remarkable that varying moments have to act about the three axes to maintain this state of motion even if the angular velocities are constant, that is \( \ddot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0 \).
A rotationally symmetrical body frequently rotates about two axes only. The following special cases result from (2.182) to (2.184):

Case 1: $q_1 = \text{const.}; q_2(t)$ and $q_3(t)$ variable.

\[
J_S^S \sin q_2 \ddot{q}_3 + J_S^S \cos q_2 \dot{q}_2 \dot{q}_3 = Q_1
\]

(2.185)

\[
(J_A + J_a^S) \ddot{q}_2 = Q_2
\]

(2.186)

\[
J_S^S \ddot{q}_3 = Q_3
\]

(2.187)

Fall 2: $q_1(t)$ variable; $q_2 = \beta = \text{const.}; q_3(t)$ variable.

\[
(J_A + J_a^S \cos^2 \beta + J_S^S \sin^2 \beta) \ddot{q}_1 + J_S^S \sin \beta \dot{q}_3 = Q_1
\]

(2.188)

\[
(J_a^S - J_S^S \sin \beta \cos \beta \dot{q}_1^2 - J_S^S \cos \beta \dot{q}_3 \dot{q}_1) = Q_2
\]

(2.189)

\[
J_S^S \sin \beta \ddot{q}_1 + J_S^S \ddot{q}_3 = Q_3
\]

(2.190)

All terms of kinetic energy become simpler for the special case defined by (2.181):

\[
W_{\text{kin}} = \frac{1}{2} (J_A + J_a^S \cos^2 q_2 + J_S^S \sin^2 q_2) \dot{q}_1^2 + \frac{1}{2} (J_B + J_a^S) \dot{q}_2^2 + J_S^S \sin q_2 \dot{q}_1 \dot{q}_3 + \frac{1}{2} J_S^S \dot{q}_3^2
\]

(2.191)

The kinetic energy also is not constant at constant input velocities $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = \Omega$ in the case of $J_{\xi} = J_{\eta} = J_{\zeta} = J$ (e.g. rigid body as a homogeneous sphere), but depends on the angle $q_2$: $W_{\text{kin}} = \frac{1}{2} [J_A + J_B + J(3 + 2 \sin q_2)] \Omega^2$. This is due to the fact that a moment $Q_2$ according to (2.189) is acting to cause this state of motion, see also (2.31).

2.4.2 Planar Mechanisms

2.4.2.1 General Perspective

Planar mechanisms with their links moving in parallel planes are used more frequently in mechanical engineering than spatial mechanisms, since they are less complicated to build and more conveniently calculated, especially if they have just a single drive. From a mechanical point of view, they are special cases of the mechanisms discussed in Sect. 2.4.1 to which $n = 1, q_1 = q$, $\dot{z}_{Si} = 0$, $\omega_{\xi i} = \omega_{\eta i} = 0$ and $\omega_{\zeta i} = \dot{\phi}_i$ applies. These planar mechanisms are not dismissed as special cases, but discussed in more detail below to spare the reader who is interested in these objects only the reading of Sect. 2.4.1.

The (kinematically) planar mechanism is assumed to consist of a total of \( I \) rigid bodies that are numbered in such a way that the frame is number 1. The input link is assigned the number \( i = 2 \), and the output link the number \( I \). The geometrical
conditions are determined by the structure of the mechanism and the dimensions of its links.

![Diagram of a rigid body in planar motion](image)

**Fig. 2.15** Nomenclature for a rigid body in planar motion (link \( i \))

Given are the characteristic mass parameters for planar motion of all rigid bodies: the positions of the centers of gravity in the body-fixed \((\xi_{Si}; \eta_{Si})\) or fixed reference system \((x_{Si}; y_{Si})\), the masses \(m_i\) and the moments of inertia with respect to the body-fixed axes through the center of gravity \(J_{Si}\) that were designated as \(J_{\xi\xi i}\) in Sect. 2.3.1. External forces and moments such as input and braking torques, friction forces and moments, cutting and pressing forces, etc. can act on each body (link). The forces applied onto the \( i \)th body are captured with their components in the fixed coordinate directions and labeled as \(F_{xi}\) and \(F_{yi}\). The applied moment on the \( i \)th body is \(M_i\).

Figure 2.15 defines the nomenclature of the applied forces and geometrical dimensions at an arbitrary body. Geometrical relations between the position of the input link identified by the generalized coordinate \(q\) and those coordinates that specify the position of each rigid body can be formulated based on the structure and dimensions of a machine. For mechanisms with a rotating input link as in Figs. 2.15, 2.18, 2.26, 2.29 and 2.20, \(q = \varphi_2\) is often selected, but in principle a translational coordinate can be used as well.

The dependency of the center-of-gravity coordinates and the angles of rotation \(\varphi_i\) on the input coordinate are known in the form of zeroth-order position functions:

\[
\begin{align*}
\xi_{Si} &= \xi_{Si}(q); & \eta_{Si} &= \eta_{Si}(q); & \varphi_i &= \varphi_i(q); & i &= 2, 3, \ldots, I. \quad (2.192)
\end{align*}
\]

Their calculation will be explained with reference to multiple examples in the following Sections.

Starting from the time dependence of the input coordinate \(q = q(t)\), the positions of the links of the mechanism can be determined as time functions, see also (2.142), (2.143) and (2.144):

\[
\begin{align*}
x_{Si}(t) &= x_{Si}[q(t)]; & y_{Si}(t) &= y_{Si}[q(t)]; & \varphi_i(t) &= \varphi_i[q(t)]. \quad (2.193)
\end{align*}
\]
The velocities result from differentiation with respect to time according to the chain rule

\[
\dot{x}_{Si} = \frac{dx_{Si}}{dt} = \frac{dx_{Si}}{dq} \frac{dq}{dt} = x'_{Si} \dot{q}, \quad \dot{y}_{Si} = y'_{Si} \dot{q}, \quad \dot{\varphi}_i = \varphi'_i \dot{q} \quad (2.194)
\]

Derivatives with respect to the input coordinate \( q \) are denoted by a dash, total derivatives with respect to time by a dot. The accelerations are calculated as follows:

\[
\ddot{x}_{Si} = \frac{d^2x_{Si}}{dt^2} = \frac{d\dot{x}_{Si}}{dt} = \frac{d(x'_{Si} \dot{q})}{dt} = \frac{dx_{Si}}{dq} \frac{dq}{dt} + \dot{x}'_{Si} \dot{q} = \ddot{x}_{Si} \dot{q} + \dot{x}'_{Si} \ddot{q} \quad (2.195)
\]

In summary, the following applies:

\[
\ddot{x}_{Si}(q, t) = x''_{Si}(q) \ddot{q}(t) + x'_{Si}(q) \dot{q}(t)
\]

\[
\ddot{y}_{Si}(q, t) = y''_{Si}(q) \ddot{q}(t) + y'_{Si}(q) \dot{q}(t)
\]

\[
\ddot{\varphi}_i(q, t) = \varphi''_i(q) \ddot{q}(t) + \varphi'_i(q) \dot{q}(t) \quad (2.196)
\]

This representation contains a separation of the position functions from zeroth-order position functions \( x_{Si}(q) \), \( y_{Si}(q) \), \( \varphi_i(q) \) to second-order position functions \( x''_{Si}, y''_{Si}, \varphi''_i \), which are independent of the state of motion from the time functions \( q(t) \), \( \dot{q}(t) \), and \( \ddot{q}(t) \) of the input link, which characterize the state of motion.

The position functions can be specified analytically in closed form for simple systems, such as gear mechanisms, slider-crank mechanisms, and others; they can be calculated numerically (software) for more complex systems, such as multi-link mechanisms.

The kinetic energy is derived taking into consideration the translational motions of all centers of gravity and the rotations about the axis through the center of gravity of all moving links

\[
W_{\text{kin}} = \frac{1}{2} \sum_{i=2}^I \left[ m_i \left( \dot{x}_{Si}^2 + \dot{y}_{Si}^2 \right) + J_{Si} \dot{\varphi}_i^2 \right] \quad (2.197)
\]

If the relationships (2.194) are used, the following results from (2.197) as a special case of (2.148)

\[
W_{\text{kin}} = \frac{1}{2} \dot{q}^2 \sum_{i=2}^I \left[ m_i \left( x''_{Si} + y''_{Si} \right) + J_{Si} \dot{\varphi}_i^2 \right] = \frac{1}{2} J(q) \dot{q}^2, \quad (2.198)
\]

if the generalized mass, that is also called a reduced moment of inertia, is introduced in the form of

\[
J_{\text{red}} = J(q) = \sum_{i=2}^I \left[ m_i \left( x''_{Si} + y''_{Si} \right) + J_{Si} \dot{\varphi}_i^2 \right] \quad (2.199)
\]
(special case of \( m_{11} \)). If one compares (2.198) with (2.197), it becomes evident that the kinetic energy of the generalized mass is equal to the kinetic energy of all moving links. The generalized mass \( J(q) \) has the dimension of a moment of inertia if the generalized coordinate \( q \) is an angle, and it has the dimension of a mass if \( q \) is a translation. \( J(q) \) is always positive. It is worth noting that the first-order position function in (2.199) appear in quadratic form and thus \( J(q) \) is independent of the direction of the motion.

Potential energy is often stored in mechanisms in the form of lifting work and/or deformation work of the spring (spring constant \( c \), spring length \( l \), unstretched spring length \( l_0 \))

\[
W_{pot} = \sum \left[ m_i g y_{Si} + \frac{1}{2} c_i (l_i - l_{0i})^2 \right] \tag{2.200}
\]

(the \( y \) axis being directed vertically upwards). The total mass of the moving links and the overall center-of-gravity height \( y_S \) are

\[
m = \sum_{i=2}^{I} m_i \tag{2.201}
\]

\[
y_S = \frac{1}{m} \sum_{i=2}^{I} m_i y_{Si} \tag{2.202}
\]

The applied non-potential forces \( F_{xi}, F_{yi} \) and moments \( M_i \) that act on the links of the mechanism are referred to the generalized coordinate, see (2.153). Their work must be equal to the work of the generalized force \( Q \). Thus

\[
dW = Q dq = \sum_{i=2}^{I} \left( F_{xi} dx_i + F_{yi} dy_i + M_i d\varphi_i \right) \tag{2.203}
\]

It follows for the power of the applied forces:

\[
P = Q \frac{dq}{dt} = \sum_{i=2}^{I} \left( F_{xi} \frac{dx_i}{dt} + F_{yi} \frac{dy_i}{dt} + M_i \frac{d\varphi_i}{dt} \right) \tag{2.204}
\]

Using (2.194) and after dividing by \( \dot{q} \), one finds the sought-after equation for the generalized force

\[
Q = \sum_{i=2}^{I} \left( F_{xi} x_i' + F_{yi} y_i' + M_i \varphi_i' \right) = Q_{an} + Q^{*} \tag{2.205}
\]

\( Q \) mostly is not constant but depends on the position \( (q) \), velocity \( (\dot{q}) \) and/or time \( (t) \). The generalized driving force \( Q_{an} \) (input torque \( M_{an} \) for a rotary drive and the input force \( F_{an} \)) for a linear actuator is not a potential force and contained in \( Q \). It is useful to label and highlight them separately. The other non-potential forces are included in the quantity \( Q^{*} \).

Lagrange’s equation of the second kind for this system with one degree of freedom is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = Q \tag{2.206}
\]
with the Lagrangian:

\[ L = W_{\text{kin}} - W_{\text{pot}}. \]  

(2.207)

The differentiations result in the following with (2.198):

\[
\frac{\partial L}{\partial \dot{q}} = J(q)\dot{q}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{dJ(q)}{dt} \dot{q} + J(q)\ddot{q} + J'(q)\dot{q}^2 + J(q)\ddot{q}
\]

\[
\frac{\partial L}{\partial q} = \frac{1}{2} J'(q)\dot{q}^2 - W'_{\text{pot}}
\]

(2.208)

Equations (2.207), (2.208), and (2.206) result in:

\[ J(q)\ddot{q} + \frac{1}{2} J'(q)\dot{q}^2 + W'_{\text{pot}} = Q_{\text{an}} + Q^* \]  

(2.209)

This equation of motion of the rigid machine with one degree of freedom is a special case of (2.155).

### 2.4.2.2 Hoisting Gear

The hoisting gear of a crane as shown in 2.16 is represents a mechanism with constant transmission ratio, consisting of two gear steps and the translationally moving load. It is assumed that the hoisting cable is massless, has no bending stiffness and is as perfectly rigid (in the axial direction) as all the other components.

The geometrical parameters of the system, i.e. the pitch radii of the gears \((r_2, r_{32}, r_{34}, r_4)\) and the cable length \(l\) (hereinafter not relevant). The mass parameters of the system are the moments of inertia of the gears about their axes through the center of gravity that coincide with their axes of rotation \((J_2; J_3; J_4)\) and the mass of the hoisting load \(m_5\). The moments of inertia \(J_2, J_3, \text{and } J_4\) include those of the motor, the coupling, the cable drum and all other rotating parts. The external force field consists of the static weight of the load (the force \(F_{y5} = -m_5g\) that acts against the coordinate direction \(y\)) and the input torque \(M_{\text{an}}\).

The quantities to be found as functions of the coordinate \(q = \varphi_2\) are the reduced moment of inertia \(J_{\text{red}}\) and the equation of motion.

The solution starts with establishing the constraints. The following geometrical relations can be obtained from Figure 2.16:

\[ r_2\varphi_2 = -r_{32}\varphi_3, \quad r_{34}\varphi_3 = -r_4\varphi_4, \quad y_{S5} = r_4\varphi_4 - l \]

(2.210)

After a brief manipulation, this yields the position functions in the form of the equations (2.192):

\[ \varphi_3 = -\frac{r_2}{r_{32}}\varphi_2, \quad \varphi_4 = \frac{r_2r_{34}}{r_4r_{32}}\varphi_2, \quad y_{S5} = \frac{r_{34}r_2}{r_{32}}\varphi_2 - l \]

(2.211)
The first-order position functions are:

\[
\begin{align*}
\dot{\varphi}_2 &= 1, \\
\dot{\varphi}_3 &= \frac{r_2}{r_{32}} = u_{23}, \\
\dot{\varphi}_4 &= \frac{r_2 r_{34}}{r_4 r_{32}} = u_{24}, \\
y_{S5}' &= \frac{r_2 r_{34}}{r_{32}}.
\end{align*}
\]  

This is where the gear ratios \( u_{2k} \) are introduced, see (2.144), (2.163) and (2.164).

When changing to a slower motion, the value of a gear ratio is greater than one, and it is smaller than one when changing to a faster motion. The sign indicates the direction of rotation relative to the drive direction. The reduced moment of inertia according to (2.199) is

\[
J_{\text{red}} = J_2 \varphi_2'^2 + J_3 \varphi_3'^2 + J_4 \varphi_4'^2 + m_5 y_{S5}'^2,
\]

i.e.,

\[
J_{\text{red}} = J_2 + J_3 u_{23}^2 + J_4 u_{24}^2 + m_5 \left( \frac{r_2 r_{34}}{r_{32}} \right)^2 = \text{const.}
\]  

(2.213)

It can be seen that the gear ratios \( u_{2k} \) enter quadratically into the calculation of the generalized mass so that their signs (direction of rotation) are not relevant. This squared gear ratio entails that the reduced moment of inertia of many gear mechanisms is primarily determined by the moment of inertia of the fast running gear step and that the total moment of inertia of a gear mechanism can often be estimated by taking the moment of inertia of the first step and multiplying it by a factor (e.g. 1.1 to 1.2).

The change in the center of gravity height of the moving masses that determines the hoisting work is \( y_{S5}' = y_{S5}' \), since only the mass \( m_5 \) changes its height, see (2.202). The equation of motion thus results from (2.209) with \( W'_{\text{pot}} = m_5 g y_{S5}' \) and \( Q^* \equiv Q \) to become

\[
\begin{bmatrix}
J_2 + J_3 u_{23}^2 + J_4 u_{24}^2 + m_5 \left( \frac{r_2 r_{34}}{r_{32}} \right)^2
\end{bmatrix} \ddot{\varphi}_2 + m_5 g (r_2 r_{34}/r_{32}) = M_{\text{an}}
\]  

(2.214)
### 2.4.2.3 Four-Bar Linkage

Four-bar linkages with a rotating input link are used in many machines in the form of crank-rocker mechanisms (output link \( q_4 \) rocks back and forth) or double-crank mechanisms (output link \( q_4 \) rotates fully) to generate non-uniform motions.

![Fig. 2.17 Geometrical and mechanical parameters in a four-bar linkage with a translational spring](image)

The given quantities are the lengths \( l_i \) shown in Fig. 2.17, the center-of-gravity coordinates \( \xi_{Si} \) in the body-fixed reference system, the masses \( m_i \) and moments of inertia \( J_{Si} \), which are causing inertia forces and moments. A spring force, the magnitude of which can be calculated from the spring constant \( c \) and the spring deflection, which in turn results from the unstretched spring length \( l_0 \) and the instantaneous spring length \( l \), acts in addition to the input torque \( M_2 \) and the static weight of the links. The spring length depends on the fixed coordinates \( x_{15}, y_{15} \) and the body-fixed coordinates of the pivot point \( (\xi_{35}, \eta_{35}) \).

Sought-after quantities are general formulae for calculating the reduced moment of inertia and the other terms that enter into the equation of motion of the rigid machine for \( q = \varphi_2 \).

The equations for calculating the angles \( \varphi_3 \) and \( \varphi_4 \) are derived from the constraints. These express the fact that the projections of the coordinates of the joint points onto the two coordinate directions form a solid straight line (loop), see Fig. 2.17:

\[
\begin{align*}
    l_3 \cos \varphi_3 &= l_4 \cos \varphi_4 + l_1 - l_2 \cos \varphi_2 \quad (2.215) \\
    l_3 \sin \varphi_3 &= l_4 \sin \varphi_4 - l_2 \sin \varphi_2 \quad (2.216)
\end{align*}
\]

Squaring and adding yields

\[
l_3^2 = l_4^2 + l_1^2 + l_2^2 - 2l_1 l_2 \cos \varphi_2 + 2l_4 (l_1 - l_2 \cos \varphi_2) \cos \varphi_4 - 2l_4 l_2 \sin \varphi_2 \sin \varphi_4. \quad (2.217)
\]
If one solves this equation, the result obtained after some intermediate calculations
with the abbreviations

\[ a_{34} = \frac{2(l_2 \cos \varphi_2 - l_1)l_4}{N} \]
\[ b_{34} = \frac{2l_2l_4 \sin \varphi_2}{N} \]
\[ w_{34} = a_{34}^2 + b_{34}^2 \tag{2.218} \]

and the denominator

\[ N = (l_2 \cos \varphi_2 - l_1)^2 + l_2^2 \sin^2 \varphi_2 + l_4^2 - l_3^2 \tag{2.219} \]
is the sine and cosine of \( \varphi_4 \):

\[ \sin \varphi_4 = \frac{b_{34} - a_{34} \sqrt{w_{34}} - 1}{w_{34}}, \quad \cos \varphi_4 = \frac{a_{34} + b_{34} \sqrt{w_{34}} - 1}{w_{34}}. \tag{2.220} \]

The other unknown trigonometric functions are most easily derived from (2.215) and (2.216) using (2.220):

\[ \cos \varphi_3 = \frac{l_4 \cos \varphi_4 - l_2 \cos \varphi_2 + l_1}{l_3}, \quad \sin \varphi_3 = \frac{l_4 \sin \varphi_4 - l_2 \sin \varphi_2}{l_3}. \tag{2.221} \]

This allows the calculation of all (zeroth-order) position functions of the centers of gravity, see Fig. 2.17:

\[ x_{S2} = \xi_{S2} \cos \varphi_2, \quad y_{S2} = \xi_{S2} \sin \varphi_2 \]
\[ x_{S3} = l_2 \cos \varphi_2 + \xi_{S3} \cos \varphi_3, \quad y_{S3} = l_2 \sin \varphi_2 + \xi_{S3} \sin \varphi_3 \tag{2.222} \]
\[ x_{S4} = l_1 + \xi_{S4} \cos \varphi_4, \quad y_{S4} = \xi_{S4} \sin \varphi_4. \]

The first-order position functions are then derived by differentiation with respect to the input coordinate \( q = \varphi_2 \):

\[ x'_{S2} = -\xi_{S2} \sin \varphi_2, \quad y'_{S2} = \xi_{S2} \cos \varphi_2 \]
\[ x'_{S3} = -l_2 \sin \varphi_2 - \xi_{S3} \varphi_3' \sin \varphi_3, \quad y'_{S3} = l_2 \cos \varphi_2 + \xi_{S3} \varphi_3' \cos \varphi_3 \tag{2.223} \]
\[ x'_{S4} = -\xi_{S4} \varphi_4' \sin \varphi_4, \quad y'_{S4} = \xi_{S4} \varphi_4' \cos \varphi_4. \]

Here, the as yet undetermined first-order position functions of the angles \( \varphi_3 \) and \( \varphi_4 \) appear. They can be calculated from the following system of linear equations that results from differentiating (2.215) and (2.216):

\[ -l_3 \sin \varphi_3 \varphi_3' + l_4 \sin \varphi_4 \varphi_4' = l_2 \sin \varphi_2 \tag{2.224} \]
\[ l_3 \cos \varphi_3 \varphi_3' + l_4 \sin \varphi_4 \varphi_4' = -l_2 \cos \varphi_2. \tag{2.225} \]

This results in

\[ \varphi_3' = \frac{l_2 \sin(\varphi_2 - \varphi_4)}{l_3 \sin(\varphi_4 - \varphi_3)}, \quad \varphi_4' = \frac{l_2 \sin(\varphi_2 - \varphi_3)}{l_4 \sin(\varphi_4 - \varphi_3)}. \tag{2.226} \]
The reduced moment of inertia of the four-bar linkage is derived from (2.199) for $I = 4$

\[
J_{\text{red}} = m_2(x_{S2}^2 + y_{S2}^2) + J_{S2}\phi_2^2 + m_3(x_{S3}^2 + y_{S3}^2) + J_{S3}\phi_3^2
+ m_4(x_{S4}^2 + y_{S4}^2) + J_{S4}\phi_4^2,
\]

(2.227)
yielding the following when using (2.223):

\[
J_{\text{red}} = m_2\xi_{S2}^2 + J_{S2}\phi_2^2 + m_3(l_2^2 + 2l_2\xi_{S3}\cos(\phi_2 - \phi_3)\phi_3' + \xi_{S3}\phi_3^2) + J_{S3}\phi_3^2
+ (m_4\xi_{S4}^2 + J_{S4})\phi_4^2.
\]

(2.228)

The first-order position functions of the angles that appear herein are known from (2.226). According to (2.209), the input torque necessary to overcome the inertia forces depends on the derivative of the reduced moment of inertia and the square of the angular velocity. The static portions of the moment $M_{\text{st}}$ from the weight of the links are derived – without explicitly using the equilibrium conditions – from the position function of the center-of-gravity height:

\[
W'_{\text{pot}} = M_{\text{st}} = mgy_s' = (m_2y_{S2}' + m_3y_{S3}' + m_4y_{S4}')g.
\]

(2.229)

This portion of the moment can be calculated as a function of the crank angle $\phi_2$ using the position functions $y_s'$, known from (2.223).

What remains to be calculated is the portion of the moment that the spring attached to an arbitrary point of the link 3 exerts onto the input link 2. The spring moment is derived from the potential spring energy $W'_{\text{potF}} = c(l - l_0)^2/2$:

\[
W'_{\text{potF}} = M_c = c(l - l_0)l' = cll' \left(1 - \frac{l_0}{\sqrt{l^2}}\right).
\]

(2.230)

The spring length in the loaded condition is calculated from the coordinates of both spring pivot points using the Pythagorean theorem:

\[
l^2 = (x_{35} - x_{15})^2 + (y_{35} - y_{15})^2.
\]

(2.231)

Implicit differentiation results in:

\[
2ll' = 2(x_{35} - x_{15})x_{35}' + 2(y_{35} - y_{15})y_{35}'
\]

(2.232)

and provides the expression required for (2.230). The position functions of the spring pivot point are required for this. One can gather from Fig. 2.17 that the following geometrical relationships apply:

\[
x_{35} = l_2\cos\phi_2 + \xi_{35}\cos\phi_3 - \eta_{35}\sin\phi_3
\]

\[
y_{35} = l_2\sin\phi_2 + \xi_{35}\sin\phi_3 + \eta_{35}\cos\phi_3.
\]

(2.233)

Their partial derivatives are
\[ x'_{35} = -l_2 \sin \varphi_2 - (\xi_{35} \sin \varphi_3 + \eta_{35} \cos \varphi_3)\psi_3' \]
\[ y'_{35} = l_2 \cos \varphi_2 + (\xi_{35} \cos \varphi_3 - \eta_{35} \sin \varphi_3)\psi_3'. \]

(2.234)

The spring moment according to (2.230) thus becomes:

\[ M_c = c \left[ (x_{35} - x_{15}) x'_{35} + (y_{35} - y_{15}) y'_{35} \right] \left[ 1 - \frac{l_0}{\sqrt{(x_{35} - x_{15})^2 + (y_{35} - y_{15})^2}} \right]. \]

(2.235)

Now all portions of the moment that are included in the equation of motion (2.209) are known for the four-bar linkage shown. Based on the expressions that can be calculated from the given parameters according to equations (2.228), (2.229), and (2.235), the equation of motion for \( q = \varphi_2 \) is:

\[ J(q)\ddot{q} + \frac{1}{2} J'(q)q^2 + M_{st}(q) + M_c(q) = M_{an}. \]

(2.236)

### 2.4.2.4 Large Press

The 14-link linkage shown as a schematic in Fig. 2.18a is used in a large press. When designing the input elements, the dynamic forces that occur in the operating states of startup, forming, and braking are of particular importance besides the kinematics. First, a software program was used to determine the function of the reduced moment of inertia and its derivative from the given dimensions and mass parameters, Fig. 2.18b. Then the equation of motion (2.209) was numerically integrated and the joint forces were calculated.

Various force fields have to be taken into account depending on the operating states of interest:

1. When braking or engaging the clutch, moments that depend on time occur due to the pneumatically operated friction clutches or brakes: \( M(t) \)
2. According to the motor characteristic, only a moment that depends on the angular velocity is present under the steady-state operating conditions: \( M(\dot{\varphi}_2) \); see Sect. 1.5.2.
3. During pressing, forces occur that are both displacement- and velocity-dependent and that are to be referred to the input shaft: \( M(\varphi_2, \dot{\varphi}_2) \).
4. The friction forces and moments depend on the joint forces, the relative velocities, and the friction coefficient and are to be captured in a function \( M(\varphi_2, \dot{\varphi}_2) \).

It is often useful for such complex and expensive machines, such as large presses, to develop specific computer programs that capture their design specifics. It is the job of the designer to thoroughly compile the specifications required for such a calculation and to “work” with the software program.
2.4.3 States of Motion of a Rigid Machine

The variation with time of an input motion can generally be obtained by numerical integration of the equation of motion (2.209) if the reduced moment of inertia $J_{\text{red}}$ and the moment $Q_{\text{an}}$ are given. Its solution in closed form is possible for a conservative force field, see 2.4.4. The result of the integration is the function of the input angle $\varphi(t)$ and its time derivatives $\dot{\varphi}(t)$ and $\ddot{\varphi}(t)$, which are required for calculating all other kinematic and dynamic quantities.

The diagram in Fig. 2.19 represents a typical operating cycle of a mechanism. It consists of start-up process, the steady (or stationary) state, and the coast-down process. Machines that work in a nonstationary operating cycle include cranes, excavators, vehicles, conveyors, presses, actuating and transport systems in which starting and braking processes are frequently repeated.

It is mostly the starting and braking times, the starting and braking distances and angles, and the moment variation with time that are most interesting from a practical point of view. The designer uses these quantities to compare various drive systems or to size motors, brakes, couplings, and clutches. The dynamic forces required to size the links and joints (bolts, bearings, gears, etc.) can also be calculated if the actual sequence of motions is known. These dynamic loads during starting and braking processes will often have to be determined to prove the operating strength.
The friction moment of a machine is hard to calculate in advance since it depends on factors that are only determined during assembly or by the operating state, see (1.127). These include reaction forces for a statically indeterminate support and the operating temperature of a bearing (viscosity of the lubricant). The friction moment and friction loss are often approximated using the efficiency or determined experimentally using coast-down tests, see VDI Guideline 2158 [36].

The mechanical efficiency is defined as the ratio of output power $P_m$ to input power $(P_m + P_v)$, where $P_v$ is the friction loss:

$$\eta = \frac{P_m}{P_m + P_v} < 1.$$  \hspace{1cm} (2.237)

The efficiency is specified in the technical literature on machine elements for specific sub-assemblies such as gear mechanisms, block and pulley systems of hoists, etc.

Electric motors are typically selected based on their driving power and heating up, taking into account their duty cycle. The input torque, however, is more meaningful than the driving power when characterizing the mechanical loads on machine elements.

In most machines, multiple drive mechanisms interact in an accurately coordinated sequence of motions. Designers use motion programs that describe the coordinated sequences of motions of all drives of a machine to make major decisions in the blueprint phase that also affect dynamic behavior. Figure 2.20 shows an example of a motion program.

Starting from the minimum engineering requirements, a designer has to determine all sequences of motions with consideration to the dynamic aspects to ensure stable a operation even at high operating speeds. Since each mechanism involves a different set of inertia forces, the one that is most dynamically demanding should be designed, for example, such that the unsteady stages of motion are stretched over a longer periods of time.

In the example shown in Fig. 2.20, the motion stages 1, 3, 4 and 6 exhibit the greatest accelerations. The reduced moment of inertia changes most in these sec-
2.4.4 Solution of the Equations of Motion

The treatment of starting and braking processes involves the mathematical problem of integrating the differential (2.209) under the initial conditions

\[ t = 0 : \quad \varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \omega_0 \]  

(2.238) (angle of rotation \( \varphi = \theta \)).

In physical terms, this means that the sequence of motions \( \varphi(t) \) must be determined if an initial position \( \varphi_0 \) and an initial angular velocity \( \omega_0 \) are given at a specific time. An analytical solution can be specified for any angular dependence \( Q_{\text{an}} = M_{\text{an}}(\varphi) \), \( Q^* = M^*(\varphi) \) that also includes constant values. Since the moments of the inertia forces result from the change in kinetic energy, as can be seen from the following

\[ W'_{\text{kin}} = \frac{dW_{\text{kin}}}{d\varphi} = \frac{d(J\dot{\varphi}^2/2)}{d\varphi} = \frac{1}{2} J'\dot{\varphi}^2 + \frac{1}{2} J \frac{d\dot{\varphi}^2}{d\varphi} = \frac{1}{2} J'\dot{\varphi}^2 + J\ddot{\varphi} \]  

(2.239) (2.209) can be written in this way:
\[ W'_{\text{kin}} = M_{\text{an}} + M^* - W'_{\text{pot}}. \] (2.240)

Integration, starting from the initial state according to (2.238) to an arbitrary position \( \varphi \) yields:

\[
\frac{W_{\text{kin}}}{W_{\text{kin}0}} \int_{W_{\text{kin}0}} dW_{\text{kin}} = \frac{1}{2} J(\varphi) \dot{\varphi}^2 - \frac{1}{2} J(\varphi_0) \omega_0^2 \\
= \int_{\varphi_0}^{\varphi} (M_{\text{an}} + M^*) d\varphi - W_{\text{pot}}(\varphi) + W_{\text{pot}}(\varphi_0).
\] (2.241)

If the work of the applied force field and the potential energy are jointly abbreviated by

\[
W(\varphi, \varphi_0) = \int_{\varphi_0}^{\varphi} (M_{\text{an}} + M^*) d\varphi - W_{\text{pot}}(\varphi) + W_{\text{pot}}(\varphi_0)
\] (2.242)

the first result obtained from (2.241) is

\[
W_{\text{kin}} = \frac{1}{2} J(\varphi) \dot{\varphi}^2 = \frac{1}{2} J(\varphi_0) \omega_0^2 + W(\varphi, \varphi_0)
\] (2.243)

and then the dependence of the angular velocity on the angle of rotation with \( W_{\text{kin}0} = J(\varphi_0) \omega_0^2 / 2 \) is determined:

\[
\dot{\varphi}(\varphi) = \sqrt{\frac{J(\varphi_0) \omega_0^2 + 2W(\varphi, \varphi_0)}{J(\varphi)}} = \omega_0 \sqrt{\frac{J(\varphi_0)}{J(\varphi)} (1 + \frac{W(\varphi, \varphi_0)}{W_{\text{kin}0}})}.
\] (2.244)

If one entirely neglects the work \( W \) of the applied force field, a special case of (2.244) and its derivative follow:

\[
\dot{\varphi}(\varphi) = \omega_0 \sqrt{\frac{J(\varphi_0)}{J(\varphi)}}, \quad \ddot{\varphi}(\varphi) = \frac{-J'(\varphi) J(\varphi_0)}{2J^2(\varphi)} \omega_0^2 = -\frac{J' W_{\text{kin}0}}{J^2}.
\] (2.245)

The state of motion described in (2.245) results when the mechanism is left to its own devices in position \( \varphi_0 \) with the initial energy \( W_{\text{kin}0} \). No input torque is required for this eigenmotion. Check this by insertion into (2.209)!

The periodic motion that can be caused by the variability of \( J(\varphi) \) and/or \( M(\varphi) \) is expressed by the coefficient of speed fluctuation \( \delta \). Such studies of machines were first conducted in the 19th century in connection with the development of steam engines. The coefficient of speed fluctuation expresses the variation of the angular velocity \( \omega = \dot{\varphi} \) of the drive during one operating cycle (usually one full revolution) relative to the mean value:
The coefficient of speed fluctuation is \( \delta = 0 \) at \( \omega_{\text{max}} = \omega_{\text{min}} \) and \( \delta = 2 \) at \( \omega_{\text{min}} = 0 \). The smaller its coefficient of speed fluctuation is the smoother a machine operates.

The extreme angular velocities can be specified according to (2.245) for the approximation \( W \ll W_{\text{kin}0} \). If one assumes the mean moment of inertia \( J(\varphi_0) = J_m \) and the mean angular velocity \( \omega_0 = \omega_m \), one gets:

\[
\begin{align*}
\omega_{\text{min}} &= \omega_0 \sqrt{\frac{J_m}{J_{\text{max}}}}; \\
\omega_{\text{max}} &= \omega_0 \sqrt{\frac{J_m}{J_{\text{min}}}}.
\end{align*}
\]

(2.247)

These relationships are illustrated in Fig. 2.21. (2.246) and (2.247), after brief manipulation result in

\[
\delta = 2 \frac{\sqrt{J_{\text{max}}} - \sqrt{J_{\text{min}}}}{\sqrt{J_{\text{max}}} + \sqrt{J_{\text{min}}}} \approx \frac{\Delta J}{J_m} \left[ 1 + \frac{1}{4} \left( \frac{\Delta J}{J_m} \right)^2 + \cdots \right].
\]

(2.248)

If the external force field (\( W \ll W_{\text{kin}0} \)) is negligibly small, one only needs to determine the function \( J(\varphi) \) for an operating cycle and use the result to determine the difference \( \Delta J \) to be able to specify the coefficient of speed fluctuation. Based on the approximation

\[
\Delta J = \frac{J_{\text{max}} - J_{\text{min}}}{2} \ll J_m; \quad J_m = \frac{1}{2\pi} \int_0^{2\pi} J(\varphi) d\varphi \approx \frac{J_{\text{max}} + J_{\text{min}}}{2}
\]

(2.249)

(2.248) can be simplified by series expansion
\[ \sqrt{J_{\text{max/min}}} = \sqrt{J_m \pm \Delta J} = \sqrt{J_m} \left[ 1 \pm \frac{\Delta J}{2J_m} - \frac{1}{8} \left( \frac{\Delta J}{J_m} \right)^2 + \cdots \right]. \tag{2.250} \]

The approximation for calculating the required mean value for a given coefficient of speed fluctuation results from (2.248):

\[ J_m = \frac{\Delta J}{\delta} \left[ 1 + \frac{\delta^2}{4 + 2\delta^2} + \cdots \right] \approx \frac{\Delta J}{\delta_{\text{zul}}}. \tag{2.251} \]

Consider now the other special case in which the coefficient of speed fluctuation is mainly determined by the work \( W(\varphi, \varphi_0) \) and not by \( J(\varphi) \). The reduced moment of inertia is captured by its mean value \( J_m \) according to (2.249). In the steady state, (2.244) with \( J(\varphi_0) = J(\varphi) = J_m \), the mean angular velocity \( \omega_0 = \omega_m \) and the mean kinetic energy \( J_m \omega_m^2 / 2 \) leads to:

\[ \omega_{\text{min}} = \omega_m \left( 1 + \frac{W_{\text{min}}}{2W_{\text{kin,m}}} \right); \quad \omega_{\text{max}} = \omega_m \left( 1 + \frac{W_{\text{max}}}{2W_{\text{kin,m}}} \right). \tag{2.252} \]

If one inserts the extreme values into (2.246) an alternative for equations (2.248) and (2.251) is obtained:

\[ \delta = \frac{\Delta W}{2T_m} = \frac{\Delta W}{J_m \omega_m^2} \quad \text{or} \quad J_m = \frac{\Delta W}{\omega_m^2 \delta}. \tag{2.253} \]

\( \Delta W = W_{\text{max}} - W_{\text{min}} \) is the surplus work per period.

It follows from (2.251) and (2.253) that for a given surplus work \( \Delta W \), the coefficient of speed fluctuation gets smaller the greater the mean reduced moment of inertia \( J_m \) is. To obtain a more uniform motion, the mean moment of inertia must be increased. This can be achieved using a flywheel.

A flywheel functions as a storage device for kinetic energy. It compensates the coefficient of speed fluctuation by accumulating kinetic energy in the acceleration phase and releasing it under load conditions. It allows the use of a drive motor with a breakdown moment that is smaller than the reduced static input torque. It is often useful to let the flywheel rotate continuously and to engage the clutch of the mechanism during the working cycles or load phases only. Flywheels are primarily used in machines that work in steady-state mode of operation.

It must be distinguished whether the flywheel is placed between the motor and the mechanism or between the mechanism and the work machine, see Fig. 2.27. Installation between the mechanism and the work machine is useful to protect the mechanism from sudden load increases. The flywheel mass decreases, however, if the flywheel is installed on the fast running shaft between motor and mechanism.

A designer has to evaluate the importance of these criteria and decide for the case on hand which arrangement is preferable. A small moment of inertia is preferable for machines with unsteady operation because frequent starting and stopping processes will put larger loads on the motors and brakes (risk of overheating) when the flywheel is bigger.
The kinetic energy of flywheels is not only used to drive toys but also to drive vehicles (gyrobuses in Switzerland since 1945). It has been proven that energy on the order of magnitude of 400,000 N·m/kg can be stored in high-strength fast-rotating flywheels. Values of 1,750,000 N·m/kg are said to be physically possible in theory, which means that superflywheels can store more energy relative to their mass than electrochemical batteries.

In real-world machines, the applied forces and moments depend in a complex manner on the mechanism position, angular velocity, and time. If these functions are known, the equation of motion (2.209) can be solved numerically after solving it for the angular acceleration (\(\ddot{\varphi} = q\)):

\[
\ddot{\varphi} = \frac{1}{J(\varphi)} \left[ M_{\text{an}} + M^* - W'_{\text{pot}} - \frac{1}{2} J'(\varphi) \dot{\varphi}^2 \right]
\] (2.254)

The sequence of motions can be calculated step-by-step for small time increments \(\Delta t\) under the initial conditions of (2.238).

All simulation programs use numerical integration methods that were worked out by mathematicians. There are many methods of interpretation, and a selection of which an engineer does not have to know in detail is implemented in the software. An engineer should know, however, that these methods calculate the function values step by step starting from the initial conditions.

Figure 2.22 illustrates the general approach. Starting from the time \(t_0 = 0\), the function values with respect to time \(t_{k+1} = t_k + \Delta t\) are calculated incrementally from the time values \(t_k\). The accuracy of \(\varphi(t_k)\) depends on the correct selection of the increment \(\Delta t\). It follows from the mathematical analysis that the error changes with \((\Delta t)^5\). This means that the accuracy of the method is improved by a factor of 16 when the increment is cut in half and that it deteriorates by the same factor when the increment is doubled. Since the number of steps is inversely proportional to the increment, this factor is not 32 but 16 for a given interval.

High accuracy is often achieved by selecting small increments but there is a risk that round-off errors add up due to the large number of steps. It can be recommended...
to select at least 20 steps per (estimated) period of a cycle of motion, that is roughly
\[ \Delta t = T_0 / 20 \] for a period of \( T_0 \).

### 2.4.5 Example: Press Drive

A baler consists of a crank-rocker mechanism, the geometrical parameters of which are characterized by \( r_4/l_5 \ll 1 \) and \( r_4/l_4 \ll 1 \), see Fig. 2.23a. The dynamic behavior is to be calculated taking into account the motor characteristic, the processing force at the output and the friction moment, focusing particularly on the analysis of the influence exerted by the flywheel. The mass of link 4 is assumed to be distributed across the adjacent links.

**Given:**
- \( r_2 = 80 \text{ mm} \)
- \( r_3 = 320 \text{ mm} \)
- \( r_4 = 150 \text{ mm} \)
- \( l_5 = 1.0 \text{ m} \)
- \( x_{15} \approx l_4, \xi_{S5} = 1.5 \text{ m} \)
- \( J_2 = 0.03 \text{ kg} \cdot \text{m}^2 \)
- \( J_3 = 10; 25; 100; 200 \text{ kg} \cdot \text{m}^2 \)
- \( m_5 = 40 \text{ kg} \)
- \( J_{S5} = 36 \text{ kg} \cdot \text{m}^2 \)
- Press force \( F_0 = 7.6 \text{ kN} \), see curve in Fig. 2.23b,
- Friction moment \( M_R = (7.5 + 0.022 \dot{\varphi}_2^2) \text{ N} \cdot \text{m} \) referred to angle \( \varphi_2 \), determined experimentally, see also problem P1.6 (\( \dot{\varphi}_2 \) in rad/s)
- Motor torque \( M_{an} = M_0(1 - \dot{\varphi}_2/\Omega) \) with \( M_0 = 10 \text{ 200 N} \cdot \text{m} \) and \( \Omega = 2\pi n \)
- Synchronous motor speed: \( n = 750 \text{ min}^{-1} \)
- Forces from static weights are deemed negligible as compared to inertia forces.

**Find:**
1. Function of reduced moment of inertia \( J(\varphi_2) \)
2. Functions of angular velocity and input torque in steady-state operation
3. Influence of flywheel size on angular velocity and input torque by varying \( J_3 \)
4. Effective power, total power and efficiency of this press drive.

The solution starts by establishing the constraints, see Fig. 2.23:

\[
\begin{align*}
    r_2 \varphi_2 &= -r_3 \varphi_3, & x_{45} &\approx r_4 \cos \varphi_3 + l_4 \\
    x_{45} &= x_{15} + l_5 \sin \left( \frac{\pi}{2} - \varphi_5 \right) &\approx x_{15} + l_5 \cdot \left( \frac{\pi}{2} - \varphi_5 \right) \\
    x_{S5} &= x_{15} + \xi_{S5} \sin \left( \frac{\pi}{2} - \varphi_5 \right) &\approx x_{15} + \xi_{S5} \cdot \left( \frac{\pi}{2} - \varphi_5 \right) \\
    y_{S5} &= \xi_{S5} \cos \left( \frac{\pi}{2} - \varphi_5 \right) &\approx \xi_{S5}
\end{align*}
\]

(2.255)

If the motor angle \( \varphi_2 \) is defined as the generalized coordinate, the following applies:
Fig. 2.23 Press drive; a) schematic of mechanism, b) dependence of press force on angular velocity of rocker

\[
\varphi_3 = -\frac{r_2}{r_3} \varphi_2, \quad \varphi'_3 = -\frac{r_2}{r_3} \frac{r_4}{l_5} \cos \left( \frac{r_2}{r_3} \varphi_2 \right), \\
x_{S5} \approx \xi_{S5} \frac{r_4}{l_5} \cos \left( \frac{r_2}{r_3} \varphi_2 \right) + x_{15}, \quad x'_{S5} \approx -\xi_{S5} \frac{r_4}{l_5} \frac{r_2}{r_3} \sin \left( \frac{r_2}{r_3} \varphi_2 \right), \\
y_{S5} \approx \xi_{S5}, \quad y'_{S5} \approx 0
\]  \hspace{1cm} (2.256)

According to (2.199), the reduced moment of inertia is

\[
J(\varphi_2) = J_2 + J_3 \varphi_3^2 + m_5 (x_{S5}^2 + y_{S5}^2) + J_{S5} \varphi_5^2 
\]

After a brief calculation, the above expressions result in

\[
J(\varphi_2) = J_2 + J_3 \left( \frac{r_2}{r_3} \right)^2 + \left( J_{S5} + m_5 \xi_{S5}^2 \right) \left( \frac{r_4}{l_5} \frac{r_2}{r_3} \right)^2 \sin^2 \left( \frac{r_2}{r_3} \varphi_2 \right). 
\]

When using the given parameter values (\(J_3 = 25 \text{ kg} \cdot \text{m}^2\)), the following applies

\[
\overline{J(\varphi_2)} = \left[ 1, 5925 + 0, 1772 \sin^2 \left( \frac{r_2}{r_3} \varphi_2 \right) \right] \text{ kg} \cdot \text{m}^2 \\
= \left( 1, 6811 - 0, 0886 \cos \frac{\varphi_2}{2} \right) \text{ kg} \cdot \text{m}^2, 
\]

because \(r_3 = 4r_2\) and based on the identity of \(\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)\).
The derivative with respect to $\varphi_2$ is
\[ J'(\varphi_2) = 0.0443 \sin \frac{\varphi_2}{2} \text{ kg} \cdot \text{m}^2. \] (2.260)

The moment of the applied loads referred to the input angle $\varphi_2$ is according to (2.205)
\[ M(\varphi_2) = M_{an}(\varphi_2) + F_5(\varphi_5)x'_{S5} - M_R(\varphi_2). \] (2.261)

Especially the portion that is caused by the processing force $F_5(\varphi_5)$ according to
\[ F_5 = \begin{cases} F_0 & \text{for } \varphi_5 \geq 0 \\ 0 & \text{for } \varphi_5 < 0 \end{cases} \] (2.262)
– designated here as processing moment $M_t = F_5 x'_{S5} - \dot{\varphi}_2$ can be stated as follows due to $\dot{\varphi}_2 > 0$ and $\dot{\varphi}_5 = \varphi'_5 \dot{\varphi}_2$:
\[ M_t(\varphi_2) = F_5 x'_{S5} = \begin{cases} -F_0 \xi_{S5} \frac{r_4}{l_5} \frac{r_2}{r_3} \sin \left( \frac{r_2}{r_3} \varphi_2 \right) & \text{for } \sin \left( \frac{r_2}{r_3} \varphi_2 \right) \geq 0 \\ 0 & \text{for } \sin \left( \frac{r_2}{r_3} \varphi_2 \right) < 0 \end{cases} \] (2.263)
or, using the given parameter values:
\[ M_t(\varphi_2) = \begin{cases} -427.5 \text{ N} \cdot \text{m} \cdot \sin \left( \frac{\varphi_2}{4} \right) \text{ fr} \sin \left( \frac{\varphi_2}{4} \right) \geq 0 \\ 0 \text{ N} \cdot \text{m} \text{ fr} \sin \left( \frac{\varphi_2}{4} \right) < 0 \end{cases} \] (2.264)

The equation of motion of the rigid machine was integrated numerically after it was converted into the form
\[ \ddot{\varphi}_2 = \frac{1}{J(\varphi_2)} \cdot \left( M_0 \cdot \left( 1 - \frac{\varphi_2}{\Omega} \right) + M_t(\varphi_2) - M_R(\varphi_2) - \frac{1}{2} J'(\varphi_2) \dot{\varphi}_2^2 \right) \] (2.265)

The individual portions of the moment including the input torques were calculated from the functions of $\varphi_2(t)$ and $\dot{\varphi}_2(t)$ as determined first. Their curves are shown in Fig. 2.24.

As can be seen, a mean angular velocity of $\omega_{2m} = 76.4 \text{ s}^{-1}$ is obtained (cycle time $T_0 = 8\pi / \omega_{2m} = 0.329 \text{ s}$) which, as was to be expected, is somewhat lower than the synchronous speed of the motor that amounts to $\Omega = 750\pi / 30 = 78.5 \text{ s}^{-1}$. The mean input torque $M_{an} = 272 \text{ N} \cdot \text{m}$, shown in Fig. 2.24c results from an approximate calculation of the sum of the mean friction moment $M_R = 136 \text{ N} \cdot \text{m}$ (Fig. 2.24b) and the mean processing moment $M_t = W_N / 8\pi = 136 \text{ N} \cdot \text{m}$. The function of the speed can be interpreted by comparing it to the curve of the input torque and taking the linear motor characteristic into account. The speed rises/falls when the input torque increases/decreases. The sharp drop in speed at a relatively small $J_3$ is due to the processing moment $M_t$, see Fig. 2.24a.
The input torque $M_{\text{an}}$ is composed of the three components that are represented individually in Fig. 2.24c. The significant influence of the flywheel is clearly visible. While $M_t$ and $M_R$ are influenced to a minor extent only by the speed variation (so that this influence is hidden by the thickness of the line in the figure), the kinetic moment is affected considerably by the flywheel size. $M_{\text{kin}} = J(\varphi_2)\ddot{\varphi}_2 + 0.5J'(\varphi_2)\dot{\varphi}_2^2$ applies, and both the effects of the angular acceleration and of the angular velocity can be seen from the curve in Fig. 2.24c. When the $J_3$ values are small, the portion of the moment that originates from the variability of the moment of inertia is dominant. In the curves in Fig. 2.24a, one can notice the dual variation per period that is due to $J'(\varphi_2)$. When the values of $J_3$ are larger, the influence of the angular acceleration becomes dominant although the angular acceleration itself decreases.

The results show that, as a result of the large flywheel, the peak value of the input torque may be smaller than the one that results from the friction moment and the processing moment. The mean value of the input torque is virtually unaffected by the size of the flywheel. The larger the flywheel is, the smaller are the speed variations in the steady operating state.

A complete operating cycle corresponds to a full crank revolution ($0 < \varphi_3 < 2\pi$), i.e. four revolutions of the motor shaft ($0 < \varphi_2 < 8\pi$). According to (2.242), the effective work during the operating cycle due to the pressing action is

$$W_N = -\int_0^{8\pi} M_t(\varphi_2) d\varphi_2 = F_0 r_4 \frac{\xi_{S5} r_2}{l_5} \int_0^{4\pi} \sin \frac{\varphi_2}{4} d\varphi_2 + \int_0^{8\pi} 0 d\varphi_2$$

$$= 8F_0 r_4 \frac{\xi_{S5} r_2}{l_5} \frac{r_3}{r_3} = 3420 \text{ N} \cdot \text{m}.$$  

(2.266)
At the mean angular velocity of $\omega_{2m} = 76.4 \, \text{s}^{-1}$, the effective mechanical power is $P_m = W_N/T = 3420 \, \text{N} \cdot \text{m}/0.329 \, \text{s} = 10.4 \, \text{kW}$. The mean input torque $M_{an} = 272 \, \text{N} \cdot \text{m}$ that results from all four moment components yields a total power of $P_m + P_v = M_{an}\omega_{2m} = 272 \cdot 76.4 \, \text{W} = 20.8 \, \text{kW}$. According to (2.237), the efficiency of this press drive is only about $\eta = 0.5$.

2.4.6 Problems P2.5 to P2.8

P2.5 Drive of a Belt-Type Stacker for Strip Mining

The grossly simplified calculation model shown in Fig. 2.25 reflects the drive system of a belt-type stacker used for dumping overburden in strip mining. The slewing gear, which consists of a motor, a coupling, and two gear mechanisms, sets the top section into motion.

![Fig. 2.25 Kinematic schematic of a belt-type stacker for strip mining](image)

1. swivel axis; 2 motor; 3 coupling; 4, 5 gear mechanism; 6 top section

Given:
- moments of inertia of the motor: $J_2 = 2.14 \, \text{kg} \cdot \text{m}^2$;
- coupling: $J_3 = 1.12 \, \text{kg} \cdot \text{m}^2$;
- gear mechanism 1: $J_4 = 22.6 \, \text{kg} \cdot \text{m}^2$; referred to gear mechanism 2: $J_5 = 4540 \, \text{kg} \cdot \text{m}^2$; transmission output engine room: $J_6 = 1.185 \cdot 10^8 \, \text{kg} \cdot \text{m}^2$;
- Top section masses: $m_7 = 2.05 \cdot 10^4 \, \text{kg}$; $m_8 = 1.85 \cdot 10^5 \, \text{kg}$;
- Top section lengths: $l_7 = 110 \, \text{m}$; $l_8 = 61 \, \text{m}$;
- Gear ratios: $u_{42} = u_{43} = 627$; $u_{54} = u_{64} = 36.2$

Note that the sequence of indices is relevant for the gear ratio $(\varphi'_k = u_{2k} = 1/u_{k2})$ and that it is defined as the ratio of input to output angular velocities, see (2.212).

Find:
1. Input torque of the motor required to move the top section at an angular acceleration of $\ddot{\varphi}_6 = 0.0007 \, \text{rad/s}^2$.
2. Input torque $M_6$ referred to the swivel axis.
P2.6 Slider-Crank Mechanism

Slider-crank mechanisms are used for transforming rotational into translational motions (and vice versa). The moment of inertia referred to the crank angle is required for dynamic calculations.

**Given:** Dimensions and parameters according to Fig. 2.26a.

**Find:**

1. Reduced moment of inertia $J_{\text{red}}$ using 2 equivalent masses for the connecting rod
2. Mean value $J_m$ for the reduced moment of inertia
3. Input torque at $\dot{\varphi}_2 = \Omega$.

\[ \lambda = \frac{l_2}{l_3} \ll 1 \] applies to the crank ratio. This fact can be utilized by expanding the appearing root expressions into series and neglecting $\lambda^2$ with respect to 1.

P2.7 Flywheel Placement

Two possible placements of the flywheel can be used in a design. Either $J_{S1}$ or $J_{S2}$ are to be used to maintain a specific coefficient of speed fluctuation, see Fig. 2.27.

\[ \delta_{zul} \text{ coefficient of speed fluctuation} \]

\[ u_{21} = \frac{n_1}{n_2} > 1 \] Gear ratio

**Find:**

Formula for calculating the required moments of inertia of the flywheels when the mechanism is idling. Compare the quantities of $J_{S1}$ and $J_{S2}$.
P2.8 Influence of the Flywheel in a Forming Machine

The forming force only acts in a small range of the operating cycle of presses, cutting machines and other forming machines. The drives of forming machines are therefore equipped with flywheels, which release kinetic energy during the forming process and are “re-charged” in the remaining time of each cycle.

The input torque at the motor and the function of the angular velocity of the motor shaft are to be determined for the steady-state operation of a crank press with a basic structure as shown in Fig. 2.28. To simplify the problem, friction can be neglected. It is assumed that the mass of the connecting rod has already been distributed over the adjacent links.

Given:

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link lengths of crank and coupler (connecting rod)</td>
<td>( l_2 = 0.22 \text{ m}, l_3 = 1 \text{ m} )</td>
</tr>
<tr>
<td>Gear ratio ( u = \frac{\dot{\varphi}_M}{\dot{\varphi}_2} )</td>
<td>70</td>
</tr>
<tr>
<td>Mass of the ram ( m_4 )</td>
<td>8000 kg</td>
</tr>
<tr>
<td>Moment of inertia of the flywheel on motor shaft (2 variants) ( J_S )</td>
<td>( \begin{cases} 3.5 \text{ kg} \cdot \text{m}^2, \text{ variant A} \ 39.5 \text{ kg} \cdot \text{m}^2, \text{ variant B} \end{cases} )</td>
</tr>
<tr>
<td>Moment of inertia of the motor armature ( J_M )</td>
<td>0.5 \text{ kg} \cdot \text{m}^2</td>
</tr>
<tr>
<td>Moment of inertia of the gear mechanism (referred to the motor shaft) ( J_G )</td>
<td>0.5 \text{ kg} \cdot \text{m}^2</td>
</tr>
<tr>
<td>Breakdown moment ( M_{Ka} )</td>
<td>19.5 \text{ N} \cdot \text{m}</td>
</tr>
<tr>
<td>Breakdown slippage, see (1.126) ( s_{Ka} )</td>
<td>0.12</td>
</tr>
<tr>
<td>Synchronous speed of the motor ( n_0 )</td>
<td>1500 \text{ rpm} (( \Omega_1 = 157.1 \text{ s}^{-1} ))</td>
</tr>
<tr>
<td>Angular range of the acting forming force ( \Delta \varphi )</td>
<td>( \frac{\pi}{12} \approx 15^\circ )</td>
</tr>
<tr>
<td>Forming force (at ( 2k\pi - \Delta \varphi \leq \varphi_2 \leq 2k\pi )), with ( k = \ldots, -2, -1, 0, 1, 2, \ldots ) (Fig. 2.28)</td>
<td>( F_0 = 3.2 \text{ MN} )</td>
</tr>
</tbody>
</table>
Find:

1. Using the approximations given in Table 2.1, analytical solutions for
   1.1 the moment of inertia \( J(\varphi_2) \) referred to the crank angle
   1.2 the crank moment \( M_{S_k} = M_K (\dot{\varphi}_2 = 0) \) required for \( \dot{\varphi}_2 \approx 0 \) and the forming work \( W \) to be performed per cycle
   1.3 the mean moment \( M_{Stm} \) in the shaft between gear mechanism and crank for \( \dot{\varphi}_2 \approx 0 \)
2. The functions (for steady-state operation) of
   2.1 the input torque \( M_{M} \)
   2.2 the angular velocity \( \dot{\varphi}_M \) of the motor shaft for both flywheel variants using the SimulationX® [34] program

2.4.7 Solutions S2.5 to S2.8

S2.5 This is a mechanism with constant transmission ratio \((J' = 0)\). Thus the input torque follows from (2.209):
\[
M_{an} = J\ddot{\varphi}_2 = (J_A + J_O)\ddot{\varphi}_2 \tag{2.267}
\]
According to (2.205), the relationship \( M_{an} = M_6\dot{\varphi}_6 = M_6u_{26} = M_6u_{24}u_{46} \) applies between the input torque and the moment at the swivel axis. The moment of inertia of the drive referred to the motor shaft is see (2.199) and (2.213):
\[
J_A = J_2 + J_3 + J_4u_{24}^2 + J_5u_{25}^2 = J_2 + J_3 + J_4u_{42}^2 + J_5u_{42}^2u_{54}^2 = 3,26 \text{ kg } \cdot \text{m}^2. \tag{2.268}
\]
The moment of inertia of the top section referred to the motor shaft is
\[
J_O = \frac{J_6 + m_7l_4^2 + m_8l_5^2}{u_{42}^2u_{64}^2} = \frac{1055 \cdot 10^6 \text{ kg } \cdot \text{m}^2}{627^2 \cdot 36.2^2} = 2.05 \text{ kg } \cdot \text{m}^2. \tag{2.269}
\]
As the above numerical values show, the reduced moment of inertia of the drive system is greater (and has a greater influence on the startup behavior) due to the high gear ratio than that of the entire top section with its huge masses. The input torque of the motor therefore is \( \ddot{\varphi}_2 = \dot{\varphi}_6u_{42}u_{64} = 15.89 \text{ rad/s}^2 \) and equals
\[
M_{an} = J\ddot{\varphi}_2 = (3.26 + 2.05) \cdot 15.89 \text{ kg } \cdot \text{m}^2/s^2 = 84.37 \text{ N } \cdot \text{m}. \tag{2.270}
\]
The moment at the swivel axis is
\[
M_6 = u_{42} \cdot u_{64}M_{an} = 22 665 \cdot 84,37 \text{ N } \cdot \text{m} = 1,915 \text{ MN } \cdot \text{m}. \tag{2.271}
\]
S2.6 A brief calculation produces the exact coordinates given in Table 2.1 for the slider-crank mechanism due to simple geometrical relations.
The second and third columns list values that result for \( \lambda = l_2/l_3 < 1 \) from a series expansion.
The reduced moment of inertia is obtained according to (2.199):
\[
J(\varphi_2) = m_2 (x_{S2}^2 + y_{S2}^2) + J_{S2}\varphi_2^2 + m_3 (x_{S3}^2 + y_{S3}^2) + J_{S3}\varphi_3^2 + m_4x_{S4}^2. \tag{2.272}
\]
If one only considers terms up to the second power of \( \lambda \), the values from the third column of Table 2.1 result in
\[ J(\varphi_2) = m_2 J_2 + J_3 + m_4 l_2^2 \left\{ 1 + \left[ -\frac{\xi_3 J_3}{l_3} + \left( \frac{\xi_3 J_3}{l_3} \right)^2 \right] \cos^2 \varphi_2 + \cdots \right\} + J_3 \lambda^2 \cos^2 \varphi_2 + m_4 l_2^2 \sin^2 \varphi_2 (1 + \lambda \cos \varphi_2)^2 + \cdots \]  

(2.273)

One can see from this that the influence of the moment of inertia of the connecting rod \( J_3 \) is small, since the crank ratio is squared. It is therefore logical to neglect this term just like all the other terms with higher powers of \( \lambda \). \( m_3 \) is distributed over two equivalent masses \( m_{32} \) and \( m_{34} \) at the crank and piston bolts so that the mass and the center-of-gravity position remain the same (Fig. 2.29). Then:

\[ m_{32} + m_{34} = m_3, \quad m_{34} \left( l_3 - \xi_3 \right) = m_{32} \xi_3. \]  

(2.274)

The resulting two equivalent masses are:

\[ m_{34} = m_3 \frac{\xi_3}{l_3}, \quad m_{32} = m_3 \left( 1 - \xi_3 \frac{l_3}{l_3} \right). \]  

(2.275)

The reduced moment of inertia can be specified using

\[ J_A = J_2 + m_2 \xi_2^2 \]  

(2.276)

as follows:

\[ J(\varphi_2) = J_A + m_{32} l_2^2 + (m_4 + m_{34}) l_2^2 \sin^2 \varphi_2 (1 + 2 \lambda \cos \varphi_2 + \cdots). \]  

(2.277)

Consequently,

\[ J'(\varphi_2) = (m_4 + m_{34}) l_2^2 \cdot \left[ \sin 2 \varphi_2 - \frac{\lambda}{2} (\sin \varphi_2 - 3 \sin 3 \varphi_2) + 0(\lambda^2) \right]. \]  

(2.278)

A variable input torque must act on the crankshaft for it to rotate at a constant angular velocity \( \varphi_2 = \Omega \). If one inserts \( J(\varphi_2) \) into (2.209), it follows that:
\[ M_{an} = (m_4 + m_{34}) l_t^2 \Omega^2 \left[ -\frac{1}{4} \lambda \sin \Omega t + \frac{1}{2} \sin 2\Omega t + \frac{3}{4} \lambda \sin 3\Omega t \right] + F_{4x_4}. \quad (2.279) \]

A smaller input torque would occur if one did not force a constant speed \( \dot{\varphi}_2 = \Omega \), but allowed a variation of the speed around its mean value.

**Fig. 2.29** Distribution of the connecting rod mass over two equivalent masses

**S2.7** First, the moment of inertia referred to \( \varphi_1 \) is determined according to (2.199):

\[
\begin{align*}
J(\varphi_1) &= J_M + J_{S1} + J_G + u_{12}^2 (J_{S2} + J_0 - J_1 \cos 2\varphi_1) \\
J(\varphi_1) &= J_m - u_{12}^2 J_1 \cos 2\varphi_1; \quad u_{12} = 1/u_{21} 
\end{align*}
\quad (2.280)
\]

Since \( J_1 \ll J_0, u_{12}^2 J_1 \ll J_m \) applies as well; \( \Delta J = u_{12}^2 J_1 \), see (2.249). (2.251) can be used as a first approximation for idling:

\[
J_m = \Delta J = J_M + J_{S1} + J_G + u_{12}^2 (J_{S2} + J_0). \quad (2.281)
\]

The following is found for the flywheel on the fast running shaft \( J_{S1} (J_{S2} = 0) \):

\[
J_{S1} = u_{12}^2 \left( \frac{J_1}{\delta_{zul}} - J_0 \right) - (J_M + J_G) \quad (2.282)
\]

and for the flywheel on the slow running shaft \( J_{S2} (J_{S1} = 0) \):

\[
J_{S2} = \left( \frac{J_1}{\delta_{zul}} - J_0 \right) - \frac{1}{u_{12}^2} (J_M + J_G); \quad J_{S2} = \frac{J_{S1}}{u_{12}^2} = J_{S1} u_{21}^2 > J_{S1}. \quad (2.283)
\]
The moment of inertia \( J(\varphi_2) \) referred to the crank angle is derived according to (2.199):

\[
J(\varphi_2) = (J_M + J_S)\varphi_2^2 + J_G + m_4r_4^2 \approx (J_M + J_S)u^2 + J_G + m_4r_4^2 \sin^2 \varphi_2 (1 + \lambda \cos \varphi_2).
\]

Based on the formula given in Table 2.1,

\[
\frac{dx_4}{d\varphi_2} = x_4' \approx -l_2 \left( \sin \varphi_2 + \frac{\lambda}{2} \sin 2\varphi_2 \right)
\]

was inserted.

When taking a static view (e.g. for extremely slow operation where \( \dot{\varphi}_2 \approx 0 \)), the crank moment would have to be large enough to overcome the forming force. According to (2.205), the following applies with \( k \) as cycle index \((k = 1, 2, \ldots)\)

\[
M_{St} = M_K (\dot{\varphi}_2 \approx 0) \approx -F x_4' \approx -F_0 l_2 \left( \sin \varphi_2 + \frac{\lambda}{2} \sin 2\varphi_2 \right) \quad \text{for} \quad 2k\pi - \Delta \varphi \leq \varphi_2 \leq 2k\pi.
\]

The maximum value in each cycle occurs at the beginning of the load \((\varphi_2 = 2k\pi - \Delta \varphi)\) exerted by the press force:

\[
M_{St \ max} = F_0 l_2 \left( \sin \Delta \varphi + \frac{\lambda}{2} \sin 2\Delta \varphi \right) = 0,313.82 \quad F_0 l_2 = 220,9 \text{ kN} \cdot \text{m}. \quad (2.288)
\]

The work of the processing force has to be performed by the drive motor. It results from the energy balance for a period of the mean values of the input torque from

\[
W = F_0 \left[ x_4 (\varphi_2 = 2\pi - \Delta \varphi) - x_4 (\varphi_2 = 2\pi) \right] = 2\pi M_{St m} \quad (2.289)
\]

\[
= F_0 l_2 \left[ 1 - \cos \Delta \varphi + \frac{\lambda}{4} (1 - \cos 2\Delta \varphi) \right] = 0,04144 \quad F_0 l_2 = 29,2 \text{ kN} \cdot \text{m}.
\]

The mean moment \( M_{St m} \) in the shaft between the gear mechanism and the crank or the input torque \( M_{anm} \) at the motor shaft can be calculated from this energy requirement for the forming process:

\[
M_{St m} = \frac{W}{2\pi} = 4,643 \text{ kN} \cdot \text{m}; \quad M_{anm} = \frac{M_{St m}}{u} = 66,4 \text{ N} \cdot \text{m}. \quad (2.290)
\]

These moments are considerably smaller than the static peak moment, as can be seen from comparing \( M_{St m} \) with \( M_{St \ max} \), see (2.288). The moment \( M_{anm} \) provides an approximate value for the motor selection because it is around this mean value that the variable moments vary due to the impact-like forming force. Apart from the small inertia forces in the case on hand, only this small moment would be required if the forming work could be applied uniformly over the entire angular range.

In the limiting case of very slow angular velocity, the inertia force of the ram is zero, and the motor would only have to produce the static moment \( M_{St m} \).

The steady operating state is described by the equation of motion of the rigid machine that takes the following form when taking into account the motor characteristic:

\[
J(\varphi_2)\ddot{\varphi}_2 + \frac{1}{2} J'(\varphi_2)\dot{\varphi}_2^2 = M_{St} + \frac{2M_K}{s_K} \left( 1 - \frac{\dot{\varphi}_2}{\Omega_1} \right) \quad (2.291)
\]
The expressions from (2.284) and (2.287) must be inserted. There is no analytical solution. The numerical solution is performed using the SimulationX® program, see the model in Fig. 2.30. The results of the simulation calculation are shown in Fig. 2.31.

![Fig. 2.30 Calculation model of the crank press in SimulationX®][34]

![Fig. 2.31 Influence of a flywheel in a forming machine; solid line: \(J_S + J_M = 4.0 \text{ kg} \cdot \text{m}^2\); dashed line: \(J_S + J_M = 40 \text{ kg} \cdot \text{m}^2\)]

- a) Angular velocity of the motor shaft; b) Input torque of the motor

Figure 2.31 shows the simulation results for the two flywheels of variants A and B in comparison. The motor moment at the input shaft results from the forming force according to (2.287) and the inertia force of the ram. The forming process occurs where the moment rises steeply and the angular velocity drops sharply, when passing through the angle \(\Delta \phi\).

The greater the moment of inertia of the flywheel is, the more the peak value of \(M_{St}\) is reduced. The smaller the flywheel is in size, the more the angular velocity varies. The mean value of the motor moment matches the amount calculated in (2.290).

The kinetic energy stored in the flywheel is released for the forming work to be performed in the angular range \(\Delta \phi\). The variation of the motor moment in Fig. 2.31 shows how
the drive motor re-accelerates the system outside the forming process. The drop in moment that can be observed in this process is due to the linear motor characteristic.

This solution shows that the **mechanical energy requirement** is independent of the size of the flywheel and that the motor can be sized based on the mean moment. The drive motor does not have to generate the large force (or moment, respectively) that the forming process requires for a short time. It can feed mechanical energy for a longer period of time into the flywheel so that it does not heat up excessively in permanent operation. The motor heating-up is proportional to the squared mean value of the moment.

The function of the input torque depends on the speed-torque characteristic of the electric motor. A motor with a “soft” characteristic results in a higher drop in speed than a motor with a “hard” characteristic. The motor will heat up less if a bigger flywheel is used on the fast running motor shaft. It would be uneconomical to size the drive motor based on the peak moment if a small mean input torque is sufficient for permanent operation.

### 2.5 Joint Forces and Foundation Loading

#### 2.5.1 General Perspective

The inertia forces that develop inside machines during motions often considerably exceed the static forces from the static weights of the components. The accelerations of mechanism links often amount to a multiple of the gravitational acceleration, see Table 2.2. A designer needs methods to determine the bearing and joint forces in any mechanism and machine from given characteristic values, such as mass parameters, geometric parameters, the external force field, and the sequence of motions \( q(t) \).

This information can be used to configure and size gears, bolts and bearings (surface pressure, deformation, . . .), mechanism links (bending, shear, axial force, . . .) and foundations (vibrations). The kinetostatic forces and moments that act on the frame are relevant for exciting vibrations of the foundation, see Sect. 3.

All mechanisms that fit into the model of the rigid machine exhibit the same connection between velocity and acceleration of the input link and the joint forces that are caused by inertia forces. This important connection, which was shown in (2.160), applies regardless of the structure of a mechanism to each internal force \( Q_p \) for an input motion \( q_1(t) \):

\[
F_{\text{kin}} = Q_p(t) = m_{p1}(q_1)\ddot{q}_1(t) + \Gamma_{11p}(q_1)\dot{q}_1^2(t).
\]

(2.292)

It is important, in this respect, that each joint force is composed of two terms that are associated with the acceleration and the squared velocity of the input link. The factors for these kinematic quantities depend on the mechanism position, the geometrical dimensions, and mass parameters.

For example, it can be concluded from (2.292) that

- for \( \dot{q}_1 = \text{const.} \), all joint forces increase with the square of the input speed
- the joint forces can be influenced using the velocity and acceleration curves, e. g. by controlled drives
repeated calculation for different states of motion can be simplified since the influence of $\ddot{q}_1$ and $\dot{q}_1^2$ results from (2.292).

Mechanisms with a rotating driving crank perform a periodic motion, during which no harmonic but periodic excitation forces with circular excitation frequencies $\Omega; 2\Omega; 3\Omega; \ldots$ are generated:

$$F_{\text{kin}}(t) = \sum_{k=1}^{\infty} (A_k \cos k\Omega t + B_k \sin k\Omega t) = \sum_{k=1}^{\infty} C_k \sin(k\Omega t + \beta_k).$$  \hfill (2.293)

The Fourier coefficients $A_k$ and $B_k$ depend on the mass parameters. They are specified for simple examples in VDI Guideline 2149 Part 1 [35]. They can be calculated based on known mass parameters using known software for more complex mechanisms.

To illustrate the influence of speed, Fig. 2.32 shows the function of a joint force for three different speeds at a ratio of $1:2:3$.

![Fig. 2.32 Function of a periodic joint force component at various speeds; a) 100/min, b) 200/min, c) 300/min](image)

Note that the period ($T_0 = 2\pi/\Omega$) linear $(3:2:1)$ shortens, but the maximum force increases quadratically $(1:4:9)$.

### 2.5.2 Calculating Joint Forces

The dynamic loads in many machines are considerably larger than the static ones, see Table 2.2. Noise caused by vibrations of the mechanism links and the housing, and the risk of interference with the technological flow are the reason why designers have to deal with the occurring dynamic joint forces in greater detail.

Below, a handy method for calculating the joint forces for planar mechanisms that are composed of simple groups of links with revolute joints will be described. The algorithm is based on formulae that apply to a respective dyad, see also VDI Guideline 2729 [36].

The definition of the forces that act onto a mechanism link can be seen from Fig. 2.33. Note that the components of the forces and moments are defined uni-
Table 2.2  Speeds and maximum relative accelerations for some machine types

<table>
<thead>
<tr>
<th>Machine type</th>
<th>Speed n in 1/min</th>
<th>Acceleration ratio $a_{\text{max}}/g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting machines, presses</td>
<td>30 . . . 100</td>
<td>0.3 . . . 3</td>
</tr>
<tr>
<td>Power looms</td>
<td>200 . . . 600</td>
<td>1.0 . . . 10</td>
</tr>
<tr>
<td>Knitting machines</td>
<td>1500 . . . 3500</td>
<td>15 . . . 60</td>
</tr>
<tr>
<td>Marine diesel engines</td>
<td>400 . . . 500</td>
<td>70 . . . 80</td>
</tr>
<tr>
<td>Domestic sewing machines</td>
<td>1000 . . . 2000</td>
<td>50 . . . 100</td>
</tr>
<tr>
<td>Industrial sewing machines</td>
<td>5000 . . . 8000</td>
<td>300 . . . 600</td>
</tr>
</tbody>
</table>

formally in the directions specified for reasons of systematics, which in turn helps meet the requirements of computational processing. The force that is exerted onto link $j$ by link $k$ is designated $F_{jk}$ and its components are defined positive in accordance with the coordinate directions. The equal and opposite counterforce is designated as $F_{kj}$ and is defined in the same way. Therefore,

$$F_{xjk} + F_{xkj} = 0, \quad F_{yjk} + F_{ykj} = 0$$

It is assumed that the coordinates of the joint points $(x_{ji}, y_{ji}, x_{jk} = x_{kj}, y_{kj} = y_{jk}, x_{kl}, y_{kl}, x_{jm}, y_{jm})$ and the centers of gravity $(x_{Sj}, y_{Sj}, x_{Sk}, y_{Sk})$ for the outlined dyad are known from a previous kinematic analysis. The joint force $(F_{xjm}, F_{yjm})$ and the moment $M_{jm}$ act on the joint point $(j, m)$ from the adjacent mechanism link $m$. The force components are defined positive in the direction of the positive coordinate axes.

The following equations result from the equilibrium of moments about the revolute joint $(j, i)$ and about the revolute joint $(k, l)$:
\[-F_{xjk} (y_{jk} - y_{ji}) + F_{yjk} (x_{jk} - x_{ji}) = F_{xjm} (y_{jm} - y_{ji}) - F_{yjm} (x_{jm} - x_{ji})\]
\[+ M_{jm} + m_j \ddot{y}_{sj} (x_{sj} - x_{ji}) - m_j \ddot{x}_{sj} (y_{sj} - y_{ji}) + J_{sj} \ddot{\varphi}_j\]
\[-F_{xkj} (y_{kj} - y_{kl}) + F_{ykj} (x_{kj} - x_{kl}) = m_k \ddot{y}_{sk} (x_{sk} - x_{kl}) - m_k \ddot{x}_{sk} (y_{sk} - y_{kl}) + J_{sk} \ddot{\varphi}_k\]  \hspace{1cm} (2.295)  \hspace{1cm} (2.296)

The four unknown quantities $F_{xjk}$, $F_{xkj}$, $F_{yjk}$, and $F_{ykj}$ can be calculated from (2.294) to (2.296).

If one takes into account the equilibrium of forces in the horizontal and vertical directions, one can determine the remaining joint force components of interest:

\[F_{xji} = m_j \ddot{x}_{sj} - F_{xjk} + F_{xjm}, \quad F_{yji} = m_j \ddot{y}_{sj} - F_{yjk} - F_{yjm}\]
\[F_{xkl} = m_k \ddot{x}_{sk} - F_{xki}, \quad F_{ykl} = m_k \ddot{y}_{sk} - F_{ykj}\]  \hspace{1cm} (2.297)

Efficient algorithms for calculating the joint forces are based on the decomposition of multilink mechanisms into simple (statically determinate) groups of links.

In mechanisms with varying transmission ratio, the kinetic energy of all moving transmission links varies with the position of the mechanism. There is a permanent exchange of kinetic energy among the mechanism links via the joint forces. The work that is performed at the joint $(j, k)$ by the joint force $F_{jk}$ on link $j$ has the same magnitude as the joint force $F_{kj}$ on joint $k$, see Fig. 2.33.

The two forces that are applied at the “joint point” in the free-body diagram have the opposite sign ($F_{kj} = -F_{jk}$). In sum, the work of the joint force that acts onto the adjacent links (action = reaction) equals zero. Reaction forces thus do not perform any work in the overall system.

Now this view is generalized for an arbitrary link $i$ ($i = 2, 3, \ldots, I$). The mechanical work that the joint force $F_{ik}$ on a link $i$ (that is viewed independently and cut free) depends on the path along which this joint travels during the motion. The following applies to the work of all joint forces $F_{ik}$, inertia forces and inertia moments acting on a mechanism link $i$ onto which no applied forces and moments act along differentially small paths and angles according to the conservation of energy principle:

\[dW_i = \sum_{k^*} (F_{xik} \dot{x}_{ik} + F_{yik} \dot{y}_{ik}) - m_i (\ddot{x}_{Si} \dot{x}_{Si} + \ddot{y}_{Si} \dot{y}_{Si}) - J_{Si} \ddot{\varphi}_i \dot{\varphi}_i = 0.\]  \hspace{1cm} (2.298)

The summation (Index $k^*$) is performed over all links connected to link $i$. $\dot{x}_{ik}$ and $\dot{y}_{ik}$ are the velocity components of the joint point $(i, k)$. Furthermore:

\[dW_i = \left[\sum_{k^*} (F_{xik} \dot{x}_{ik} + F_{yik} \dot{y}_{ik}) - m_i (\ddot{x}_{Si} \dot{x}_{Si} + \ddot{y}_{Si} \dot{y}_{Si}) - J_{Si} \ddot{\varphi}_i \dot{\varphi}_i\right] dt = 0.\]  \hspace{1cm} (2.299)

The kinetic energy of link $i$ is, see (2.197) and (2.198),

\[W_{\text{kin, } i} = \frac{1}{2} \left[m_i (\dot{x}_{Si}^2 + \dot{y}_{Si}^2) + J_{Si} \dot{\varphi}_i^2\right] = \frac{1}{2} J_{\text{red, } i} (q) \dot{q}^2.\]  \hspace{1cm} (2.300)
It follows from (2.299) and (2.300) that the time derivative of the kinetic energy, i.e. the kinetic power of the inertia forces and moments is equal to the power that the joint forces of the adjacent links transmit onto the link $i$:

$$P_{\text{kin}i} = \frac{dW_{\text{kin}i}}{dt} = \sum_{k} (F_{xik}\dot{x}_{ik} + F_{yik}\dot{y}_{ik})$$

$$= m_i(\ddot{x}_{Si} \dot{x}_{Si} + \ddot{y}_{Si} \dot{y}_{Si}) + J_{S_i}(\ddot{\varphi}_i \dot{\varphi}_i) = \frac{1}{2} J_{\text{red}i}(q)\dot{\varphi}^3 + J_{\text{red}i}\ddot{\varphi} \dot{\varphi}.$$  

(2.301)

For a constant input speed $\dot{\varphi} = \Omega$, the kinetic power of the $i$th link is

$$P_{\text{kin}i} = \frac{1}{2} J_{\text{red}i}(q)\Omega^3.$$  

(2.302)

The exchange of kinetic energy that takes place between the links is also of interest for assessing the dynamic behavior of a mechanism. The kinetic power on link $i$ is a measure for the variation of the joint forces $F_{ik}$. In addition to $J_{\text{red}i}$, the function of each part of each link is of interest, i.e. the summands $J_{\text{red}i}(q)$ and their derivatives, see Fig. 2.36b.

### 2.5.3 Calculation of the Forces Acting onto the Frame

It is of great practical significance to know the dynamic excitation forces and moments that a machine transmits onto the frame, since these can excite undesirable vibrations in the subsoil or the buildings. The problems of machine foundations and vibration isolation that arise in this respect are discussed in more detail in Sect. 3.

It is not only the maximum value of the periodic forces and moments transmitted by a machine, but the size of each Fourier coefficient that is of interest in conjunction with vibration analyses for foundations, see 3.2.1.3. In multilink mechanisms, even the higher harmonics of the inertia forces are relevant. Often the task is to keep the inertia forces and specific excitation harmonics that are transmitted onto the foundation as small as possible. The respective methods for balancing of mechanisms and balancing of rotors are discussed in 2.6.

Consider a multilink mechanism whose links move in parallel planes that can be offset in the $z$ direction, see, for example, Fig. 2.34. The goal is to determine the resultant forces and moments that are transmitted from the moving machine parts via the frame onto the foundation.

Internal static and kinetostatic forces and moments of the machine, such as spring forces between individual links, processing forces (e.g. cutting and pressing forces in forming machines and polygraphic machines, gas forces in internal combustion engines and compressors), have no influence on the foundation forces since they always occur in pairs and cancel each other out.
In real machines in which the elasticity of the links plays a part, additional inertia forces ("vibration forces") that have an effect on the foundation can occur due to deformation of the links in addition to the kinetostatic forces and moments.

The resultant inertia forces and moments that act from the moving mechanism onto the machine frame are derived from the force and moment balances, see 2.3.2 and the forces and moments in Fig. 2.33. Since the motions are parallel to the $x$-$y$ plane, $F_z = 0$, and the following forces result:

$$
F_x = - \sum_{i=2}^{I} m_i \ddot{x}_{Si} = -m \ddot{x}_S; \quad F_y = - \sum_{i=2}^{I} m_i \ddot{y}_{Si} = -m \ddot{y}_S. \quad (2.303)
$$

The input moment already known from (2.209) can also be stated as follows, see also (2.199):

$$
M_{an} = \sum_{i=2}^{I} m_i (\ddot{x}_{Si} \dot{x}_{Si}' + \ddot{y}_{Si} \dot{y}_{Si}') + J_{Si} \ddot{\varphi}_i \dot{\varphi}_i' + W_{pot}' - Q^*. \quad (2.304)
$$

The position of the overall center of gravity of all moving parts of a planar mechanism results from the individual positions of the centers of gravity from the conditions

$$
x_S \cdot \sum_{i=2}^{I} m_i = \sum_{i=2}^{I} m_i x_{Si}; \quad y_S + \sum_{i=2}^{I} m_i = \sum_{i=2}^{I} m_i y_{Si}. \quad (2.305)
$$
The resultant frame forces can thus be calculated from the acceleration of the overall center of gravity. It follows that these forces only depend on the motion of the overall center of gravity and the overall mass of the links. If the overall center of gravity remains at rest during the motion, the resultant of the frame forces is identical to zero. The individual frame forces have finite values, though, and usually a resulting moment \( M_z \) remains, see also Sect. 2.6.3.

The kinetic moments are, see Figs. 2.35 and (2.90):

\[
M_{\text{kin},x}^O \equiv \sum_{i=2}^{l} \left[ m_i z_{Si} \ddot{y}_{Si} + (J_{\eta \xi}^{S} \ddot{\varphi}_i + J_{\eta \zeta}^{S} \dot{\varphi}_i^2) \sin \varphi_i - (J_{\eta \zeta}^{S} \ddot{\varphi}_i - J_{\eta \xi}^{S} \ddot{\varphi}_i^2) \cos \varphi_i \right]
\]

\[
M_{\text{kin},y}^O \equiv -\sum_{i=2}^{l} \left[ m_i z_{Si} \ddot{x}_{Si} + (J_{\eta \xi}^{S} \ddot{\varphi}_i + J_{\eta \zeta}^{S} \dot{\varphi}_i^2) \cos \varphi_i + (J_{\eta \zeta}^{S} \ddot{\varphi}_i - J_{\eta \xi}^{S} \ddot{\varphi}_i^2) \sin \varphi_i \right]
\]

\[
M_{\text{kin},z}^O \equiv \sum_{i=2}^{l} \left[ m_i (y_{Si} \ddot{x}_{Si} - x_{Si} \ddot{y}_{Si}) - J_{Si} \dddot{\varphi}_i \right]
\]

(2.306)

Note that they depend on the position of the coordinate system relative to the machine. It is recommended to select the center of gravity of the foundation block on which the mechanism is installed as the origin \( O \) of the coordinate system when addressing foundation issues, see Figs. 3.6 and 3.8.

The forces that act on the frame can of course also be calculated from the superposition of all joint forces that act onto the frame. This method is laborious, however, since it requires the calculation of the internal joint forces.

\[\text{Fig. 2.35 } \text{Inertia forces and moments on a link in the fixed and body-fixed reference systems}\]
2.5.4 Joint Forces in the Linkage of a Processing Machine

Thanks to software for dynamic analysis and optimization of mechanisms, designers can obtain an accurate overview of the joint force curves of complicated mechanisms. The compilation of all data from the design documents, e.g. mass parameters of the links, takes the greatest effort in that process.

Figure 2.36 shows the reduced moment of inertia (kinetic energy) and Fig. 2.36c the kinetic power of the overall mechanism as compared to that of two individual links for the eight-bar linkage, shown in Fig. 2.36a. One can see from these curves that the links 7 and 8 cause the variations of the moment of inertia and are essentially responsible for the change in kinetic power. It follows from just this one observation that their joint forces must change dramatically, see Sect. 2.5.2. Figure 2.37 shows the calculated joint forces for three bearings.

Both the representation of the forces with their directions of action (Fig. 2.37a) and the time functions of the resultant force (Fig. 2.37b) are of practical interest. An analysis of these forces provides the basis for drawing conclusions about the dynamic loads in the joint bolts (and thus about friction, lubrication, and wear and
2.5 Joint Forces and Foundation Loading

Fig. 2.37 Periodic joint forces in the linkage according to Fig. 2.36a; a) Polar diagrams of the joint forces $F_{14}$, $F_{16}$ and $F_{18}$; b) Magnitude of the joint forces $F_{14}$, $F_{16}$, and $F_{18}$ as a function of the crank angle $q = \Omega t$; c) Excitation spectrum (Fourier coefficients of the joint forces)

tear of the bearing) and about vibration excitations that will occur. A comparison with values determined experimentally allows decisions on the permissibility of the calculation model used ("rigid machine").

The Fourier coefficients are also of interest for assessing the joint forces (Fig. 2.37), see (2.293). They characterize the periodic excitation and are required for analyzing the forced vibrations, see Sects. 3.2 and 6.6.4. As can be seen from Fig. 2.37c, the higher harmonics are of major significance.

If the measured curves clearly deviate from the calculated ones, which happens frequently in engineering practice, one can deduce the causes for vibrations from the difference between the real and the kinetostatic curves, see VDI Guideline 2149 Part 2 [36].
2.5.5 Problems P2.9 and P2.10

P2.9 Parameter Influences on Joint Forces

It is possible to calculate the joint forces in any rigid-body mechanisms using computer programs. Occasionally the problem arises that the results of such calculations have to be checked. Such checks and plausibility considerations can be performed based on the general relationships.

An arbitrary mechanism is considered assuming that all its links consist of straight rigid rods made of the same material and that the bearing dimensions in the joints have a negligibly small influence on the mass parameters. How do the mass-related joint forces develop when the previous cross-sectional areas $A_i$ of all links are changed by the same factor $\kappa$ and the speed is changed by the factor $\kappa_n$?

Given: All geometric and kinematic dimensions

1. Factor $\kappa$ by which all cross-sectional areas are changed ($A_i^* = \kappa A_i$)
2. Factor $\kappa_n$ by which speed is changed ($n^* = \kappa_n n$)

Find: Influence of the factors $\kappa$ and $\kappa_n$ on all joint forces

P2.10 Favorable Distance of the Point of Force Application

When selecting the kinematic dimensions and mass parameters, there are specific freedoms that can be used to reduce the kinetic loads.

![Diagram](https://via.placeholder.com/150)

**Fig. 2.38** Coupling force at the output link; a) Sketch of the link, b) Free-body diagram for $|\phi| \ll 1$

Figure 2.38a shows the output link of a coupler linkage, which is pivoted at the frame. It is driven by a coupling force so that an angular acceleration $\ddot{\phi}$ occurs. Calculate the horizontal component of the bearing force for small angles $|\phi| \ll 1$ and state the parameter values for which this bearing force component becomes zero.
2.5 Joint Forces and Foundation Loading

Given:
- Mass \( m \)
- Distance to center of gravity \( \xi \)
- Moment of inertia \( J \)
- Coupling force \( F \)

Find:
- Horizontal force \( F_{Ox} \)
- Bearing distance \( \xi \) for which the horizontal force becomes zero.

2.5.6 Solutions S2.9 and S2.10

S2.9 The masses and moments of inertia of the slender rod-shaped links with the lengths \( l_i \) result from
\[
m_i = \varrho A_i l_i; \quad J_{Si} = \frac{m_i l_i^2}{12}; \quad i = 2, 3, \ldots, I. \tag{2.307}
\]
The modified parameters are denoted by the symbol *. If the cross-sections of all links are increased or decreased by the same factor, the following applies to the modified mass parameters because of \( A_i^* = \kappa A_i \):
\[
m_i^* = \varrho A_i^* l_i = \varrho \kappa A_i l_i = \varrho \kappa m_i; \quad J_{Si}^* = \frac{m_i^* l_i^2}{12} = \kappa J_{Si}; \quad i = 2, 3, \ldots, I. \tag{2.308}
\]
The coordinates of the centers of gravity are retained both in the bodies and in the \( x \) and \( y \) directions. An arbitrary joint force (force \( F \) in arbitrary direction \( q \)) results for rigid-body mechanisms in the absence of applied forces according to (2.160) or (2.292):
\[
m_{21} \ddot{q}(t) + \left( m_{12,1} - \frac{1}{2} m_{11,2} \right) \dot{q}^2(t) = F(t). \tag{2.309}
\]
The generalized masses show a linear dependence on the masses and moments of inertia since the following applies to planar mechanisms according to (2.150)
\[
m_{kl}(q) = \sum_{i=2}^{I} \left[ m_i(x_{Si,k} x_{Si,l} + y_{Si,k} y_{Si,l}) + J_{Si} \varphi_{i,k} \varphi_{i,l} \right] \tag{2.310}
\]
and thus because of (2.308)
\[
m_{kl}^*(q) = \sum_{i=2}^{I} \left[ m_i^*(x_{Si,k} x_{Si,l} + y_{Si,k} y_{Si,l}) + J_{Si}^* \varphi_{i,k} \varphi_{i,l} \right] = \kappa m_{kl}(q). \tag{2.311}
\]
This results in the time function of a joint force for modified cross-sectional areas:
\[
F^*(t) = m_{21}^* \ddot{q}(t) + \left( m_{12,1}^* - \frac{1}{2} m_{11,2}^* \right) \dot{q}^2(t) = \kappa \left[ m_{21} \ddot{q}(t) + \left( m_{12,1} - \frac{1}{2} m_{11,2} \right) \dot{q}^2(t) \right] = \kappa F(t). \tag{2.312}
\]

First result:
All joint forces change by the same factor \( \kappa \). If, for example, all widths and heights of the rectangular rod cross sections (or the diameters of circular cross sections) are reduced to \( 2/3 \) of the original values, each cross-sectional area changes by a factor of \( \kappa = (2/3)^2 = 4/9 \). It
follows also that all joint forces then are reduced to only 44.4% of their original magnitudes according to (A2.9/6), but their variation over time remains the same, except for factor $\kappa$.

The speed is proportional to the angular velocity ($\Omega = \pi n/30$). At constant speed, a joint force according to (2.292) or (2.309) amounts to

$$F_p = \Gamma_{11p}(q) \Omega^2$$

(2.313)

For a modified speed of ($n^* = \kappa n$), it is $\Omega^* = \kappa \Omega$ and thus

$$F_p^* = \Gamma_{11p}(q) \Omega^{*2} = \Gamma_{11p}(q) \kappa^2 \Omega^2 = \kappa^2 F_p$$

(2.314)

Second result:
The joint forces change with the square of the speed ratio. When the speed is doubled, for example, all joint forces quadruple if no external forces act and the calculation model of the rigid-body system is still valid.

S2.10 The following results from the equilibrium of moments about point $O$, see Fig. 2.38b:

$$F \xi \approx (m\xi_S^2 + J_S) \ddot{\varphi}$$.  

(2.315)

The balance of forces in the horizontal direction yields for $\varphi \ll 1$

$$F + F_{Ox} = m\ddot{x}_S = m\xi_S \ddot{\varphi}$$

(2.316)

The horizontal component of the joint force can be calculated from these equations:

$$F_{Ox} = \left[ m\xi_S - \frac{m\xi_S^2 + J_S}{\xi} \right] \ddot{\varphi}$$.  

(2.317)

This horizontal component of the joint force is zero if the expression in the square bracket is zero, i.e. if the coupling force acts at the distance of the so-called center of percussion

$$\xi = \xi_S + \frac{J_S}{m\xi_S}$$.  

(2.318)

This distance is greater than the distance to the center of gravity.

One can try to place the mass parameters near the relations described by (2.318) by designing the output link accordingly.

The result that may be baffling at first glance becomes physically understandable if one imagines that an individual force moves a free rigid body both translationally (center-of-gravity) and rotationally (moment equilibrium). In this case, an instantaneous center of rotation exists. If the bearing is placed there, it does not have to transmit a force because the body “wants to rotate” about this point. In the design of links, one should first of all take a look at the extreme positions of the output links, since the angular accelerations take on their maximum values there.
2.6 Methods of Mass Balancing

2.6.1 Objective

It can be achieved by smart distribution of the masses that the resultant inertia forces that a machine transmits onto its foundation become small. All measures aiming at balancing the inertia forces are called mass balancing (or counterbalancing for rotors).

It must be emphasized that mass balancing provides relief for the foundation only. The forces applied to the drive shaft and the dynamic bearing loads on individual joints may even increase due to such balancing and thus limit the capacity of the machine. When using mass balancing methods, the connection of such efforts with other side effects should be considered (such as the influence on the natural frequencies).

In addition to reducing the maximum force

\[ |F_{\text{kin}}|_{\text{max}} = \text{Min.} \]  

(2.319)

the task often is to minimize individual \( k \)th harmonics:

\[ A_k^2 + B_k^2 = \text{Min.} \]  

(2.320)

Problems of mass balancing can be defined using (2.319) and (2.320). Special measures are required when the rotors cannot be assumed to be rigid. Note [29] and the ISO Guideline 11 342 – 1998: Mechanical vibration – Methods and criteria for the mechanical balancing of flexible rotors.

A designer will first attempt to keep the rotating and reciprocating masses small, e.g. by applying the lightweight construction principle or by using light metals or glass fiber-reinforced plastics instead of steel. Additional masses made of heavy metals (sintered tungsten materials) have the smallest dimensions due to their high density (\( \rho = 17 \ldots 19 \text{ g/cm}^3 \)).

2.6.2 Counterbalancing of Rigid Rotors

Almost all machines include rotors, which is why one should thoroughly study balancing techniques if one has to deal with them [29]. Here, only an introduction to the subject can be provided.

Rotors are rotating bodies, the bearing pins of which are supported by bearings. This term includes many machine elements, such as slender shafts, thin disks, oblong drums, regardless of whether they are rigid or elastic. A rotor is rigid if it behaves like an ideal rigid body, i.e. undergoes only negligibly small deformations at its operating speed. For practical purposes, a rotor can often be considered to be rigid as long as its speed is about half of its smallest critical speed, which also de-
Dynamics of Rigid Machines

On the support conditions of the rotor. In an elastic rotor, the balancing state changes with its speed due to deformation.

Unbalances often arise as a result of manufacturing inaccuracies and material inhomogeneities. An axisymmetrically designed component does not really have an ideal axisymmetric mass distribution. An unbalance is defined as the product of a point mass \( m_i \) and its distance \( r_i \) from the axis of rotation, see Fig. 2.39:

\[
U_i = m_i r_i
\]  

For a single rotating point mass, a centrifugal force

\[
F_i = m_i r_i \Omega^2 = U_i \Omega^2
\]

occurs at the angular velocity \( \Omega \). The unbalances in a rotor are typically distributed unevenly and randomly in space. Since there are always unbalances, dynamic forces develop when rotors rotate. These forces can have an adverse effect on

1. the bearing forces (surface pressure, wear, service life . . .)
2. the loads on frame and foundation (excitation of vibrations)
3. loads inside of the rotor.

It is therefore advisable, especially for fast running rotors, to counterbalance any unbalances.
Balancing is the process in which the mass distribution of a body is checked and adjusted by mass balancing (removal or addition of material) to ensure that the dynamic bearing forces are within their predefined limits at operating speed. Balancing is performed using balancing machines or special measuring instruments in the original bearings.

A rotor is completely balanced when its mass is distributed so that it does not transmit any dynamic forces onto the bearings. This ideal state cannot be achieved in practice. Residual unbalances within specific limits are often specified in regulations, see, for example, the standards DIN ISO 1940-1, DIN ISO 11 342, ISO 19 499 or OENORM S9032.

A quantity that is independent of the mass of the balancing body is required to characterize the balancing state. For this purpose, the eccentricity is defined as follows
\[ e = \frac{m u r u}{m} = \frac{U}{m} \]  

(2.323)

\( U \) is the overall unbalance and \( m \) the overall mass of the rotor. \( e \) is directly provided by the balancing machine for deflection-measuring balancing machines that are operating below resonance. The evaluation criterion is provided by the product of \( e \Omega \). To give an idea of its order of magnitude, some values are listed in Table 2.3.

<table>
<thead>
<tr>
<th>( e \Omega ) in ( \text{mm/s} )</th>
<th>Rotor or machine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1600</td>
<td>Crank mechanism of rigidly mounted slow-running marine diesel engines</td>
</tr>
<tr>
<td>100</td>
<td>Crank mechanisms of rigidly and elastically mounted engines</td>
</tr>
<tr>
<td>16</td>
<td>Crank mechanisms of car and truck engines, car tires, rims, wheelsets, cardan shafts</td>
</tr>
<tr>
<td>2.5</td>
<td>Centrifuge drums, fans, flywheels, Electric motor armatures, machine-tool parts</td>
</tr>
<tr>
<td>0.4</td>
<td>Gas and steam turbines, machine-tool drives, Capstan drives</td>
</tr>
<tr>
<td></td>
<td>Precision grinding machine armatures, shafts, and disks, gyroscopes</td>
</tr>
</tbody>
</table>

Table 2.3 Permissible values for the product of eccentricity and angular velocity, see \( e \) in (2.323)

The bearing forces of a rotor for body-fixed coordinates were calculated in Sect. 2.3.5, see (2.133). These can be converted into fixed components using the rotational matrix \( A = A_3 \) from (2.11). Using
\[ A = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]  

(2.324)

the components in the fixed reference system result from (2.15) due to the known relation \( F = AF \) :

\[ F_{Ax} = F_{A\xi} \cos \varphi - F_{A\eta} \sin \varphi; \quad F_{Bx} = F_{B\xi} \cos \varphi - F_{B\eta} \sin \varphi \]

\[ F_{Ay} = F_{A\xi} \sin \varphi + F_{A\eta} \cos \varphi; \quad F_{By} = F_{B\xi} \sin \varphi + F_{B\eta} \cos \varphi. \]  

(2.325)
These change harmonically over time with $\varphi(t)$, and
\[ F_{Ax}(\varphi - \pi/2) = F_{Ay}; \quad F_{Bx}(\varphi - \pi/2) = F_{By}. \quad (2.326) \]

For the special case of a constant angular velocity $\dot{\varphi} = \Omega$, the kinetic forces that are exerted by the rotor onto the bearings, see (2.133) and Fig. 2.40, are:

\[
F_{Ax} = \left[ (J_{\xi \xi}^S - mb\xi_S) \cos \Omega t - (J_{\eta \xi}^S - mb\eta_S) \sin \Omega t \right] \frac{\Omega^2}{a + b},
\]
\[
F_{Ay} = \left[ (J_{\xi \xi}^S - mb\xi_S) \sin \Omega t + (J_{\eta \xi}^S - mb\eta_S) \cos \Omega t \right] \frac{\Omega^2}{a + b},
\]
\[
F_{Bx} = \left[ -(J_{\xi \xi}^S + ma\xi_S) \cos \Omega t + (J_{\eta \xi}^S + ma\eta_S) \sin \Omega t \right] \frac{\Omega^2}{a + b},
\]
\[
F_{By} = \left[ -(J_{\xi \xi}^S + ma\xi_S) \sin \Omega t - (J_{\eta \xi}^S + ma\eta_S) \cos \Omega t \right] \frac{\Omega^2}{a + b}. \quad (2.327)
\]

The factors that determine the dynamic bearing forces of a rigid body can be seen from these equations. Any unbalance distribution in a rigid rotor corresponds to a shift in the center of gravity and an inclined orientation of the principal axes of inertia relative to the axis of rotation. It can be recognized from this that the dynamic forces rotate at the angular frequency in the bearings $A$ and $B$, but are not in phase when the products of inertia do not equal zero.

Figure 2.40 shows an example of bearing force curves of an unbalanced rigid rotor where different amplitudes and phases occur at the same angular frequency.

A rigid rotor is thus completely balanced if its center of gravity is on the axis of rotation ($\xi_S = \eta_S = 0$) and if its central principal axis of inertia coincides with the axis of rotation ($J_{\xi \xi}^S = J_{\eta \eta}^S = 0$). Then $F_A = F_B = 0$.

A static unbalance occurs if the center of gravity is not located on the axis of rotation. A dynamic unbalance of the rotor occurs when the central principal axis of inertia (that passes through the center of gravity) does not coincide with the axis of rotation. Both phenomena are always superposed in practice. During balancing, the mass distribution of the rigid rotor is adjusted by balancing in two planes so that the static and dynamic unbalances are compensated. $m_1$, $m_2$, $\xi_1$, $\xi_2$, $\eta_1$, and $\eta_2$ are determined from the measured bearing forces or displacements, see Fig. 2.39.

The following analysis is to show that two balancing masses in two different balancing planes are generally sufficient to completely balance an arbitrary rigid rotor.

According to (2.325), the bearing forces that vary harmonically with the angular frequency $\Omega$ depend on four components $F_{A\xi}$, $F_{A\eta}$, $F_{B\xi}$, and $F_{B\eta}$. Balancing is based on the idea that these four components (that have to be determined by experiment for a given real rotor) must be produced in the opposite direction by additional balancing masses and thus to compensate their sum on each bearing. If one defines the position of the balancing planes by the distances $\zeta_1$ and $\zeta_2$, four independent components of the bearing forces can be produced using the four static moments of two balancing masses ($m_1\xi_1$, $m_1\eta_1$, $m_2\xi_2$ and $m_2\eta_2$). The relationship results from
2.6 Methods of Mass Balancing

Fig. 2.40 Typical curve of the bearing forces of a rigid rotor for $\dot{\varphi} = \Omega = \text{const.};$ a) Components of the bearing forces due to the static and dynamic unbalances; b) Curve of the bearing forces

The equilibrium at the rotor:

\[
F_{A\xi} = -(m_1\xi_1 + m_2\xi_2) \frac{\omega^2}{a + b} \\
F_{B\xi} = [m_1\xi_1(a + b - \zeta_1) + m_2\xi_2(a + b - \zeta_2)] \frac{\omega^2}{a + b} \\
F_{A\eta} = -(m_1\eta_1 + m_2\eta_2) \frac{\omega^2}{a + b} \\
F_{B\eta} = [-m_1\eta_1(a + b - \zeta_1) + m_2\eta_2(a + b - \zeta_2)] \frac{\omega^2}{a + b}
\]  (2.328)

These are two equations each for two unknown quantities, so one can “generate” the four force components with four static moments (bold print). Balancing machines determine the positions and magnitudes of the balancing masses “automatically”, that is, using internal software. Balancing takes a few seconds in the mass production of motors and other small rotors produced in large numbers, while it takes several hours for large turbogenerators.

In view of the arising bending moments within the rotor, unbalances should be compensated, if possible, near the plane in which they occur. Figure 2.41 illustrates the influence of the selected balancing plane on the moment distribution in the rotor for a uniformly distributed unbalance. Balancing in two arbitrary planes is no longer sufficient for so-called “elastic-shaft rotors” that run close to one of their critical speeds. For these cases, balancing meth-
ods in three or more planes were developed that require a considerable amount of extra computational and experimental effort, as compared to the balancing of rigid rotors.

The following aspects should be taken into account when selecting the balancing planes:

1. The balancing planes should be as far away from each other as possible.
2. When balancing assembled rotors with balancing planes on different components, it should be ensured by appropriate means that the unique relative positioning is ensured (positive-locking fastening of the parts against each other, e.g. using pins).
3. Balancing should not impair the strength of the component.

Figure 2.42 gives an overview of the adjustment options when balancing: cutting off, grinding or milling off material (Fig. 2.42a, e). Breaking off segments from disks provided for balancing inside (Fig. 2.42d) or outside (Fig. 2.42f) are examples of subtractive adjustment. Dosing the adjustment unbalance by screws of various lengths or diameters (Fig. 2.42c) or by inserting lead wire into rotating grooves (Fig. 2.42b) are some of the additive methods. The technological conditions of mass production should be taken into account when selecting a method. It may also be useful to weld or solder on strips of sheet metal. The designer has to specify the balancing planes from the outset and cannot leave the selection of the balancing method to chance. In practice, balancing is performed using balancing machines with which the position and magnitude of the unbalance is determined from the bearing responses of the rotor. Without going into too much detail, let us just mention that, depending on the size and speed of the rotor, “deflection-measuring” or “force-measuring” balancing machines are common, see Fig. 2.43. The rotor is rigidly supported in “force-measuring” balancing machines, and the bearing forces...
are measured in the subcritical speed range. Their practical operating range is at speeds from 200 to 3000 \text{ 1/min}.

Fig. 2.42 Examples for methods of mass balancing on rotors; a) Cutting from fan blade, b) Inserting lead wire into groove, c) Screwing in bolts of various lengths, d) Breaking off segments inside, e) Milling off molded-on studs from front end, f) Breaking off segments from specially designed outside disks

Fig. 2.43 Operating principle of balancing machines; a) Force-measuring balancing methods, b) Deflection-measuring balancing methods
2.6.3 Mass Balancing of Planar Mechanisms

2.6.3.1 Complete and Harmonic Balancing

The frame forces and moments resulting from inertia forces can be influenced by the mass parameters \((m_i, \xi S_i, \eta S_i, J_{\xi S_i}, J_{\eta S_i})\), that is, by the mass distribution on the moving links. The goal of mass balancing is to reduce the dynamic forces and moments in such a way that they load the frame within permissible limits only.

These resulting inertia forces and inertia moments can be calculated for planar mechanisms without determining the joint forces inside the mechanism. In the considerations below, reference is made to Fig. 2.15 and to equations (2.192) to (2.196) that will be used again below.

The resultant frame force components \(F_x, F_y\) and the component \(M_0^z\) of the resulting frame moment known from (2.303) and (2.306) are derived from the principle of conservation of linear momentum and from the principle of conservation of angular momentum

\[
F_x = -\frac{dI_x}{dt} = -\frac{d}{dt} \left( \sum m_i \ddot{x}_S i \right)
= -\sum m_i \ddot{x}_S i = -m \ddot{x} = -m_x(q)\ddot{q} - m'_x(q)\dot{q}^2 \quad (2.329)
\]

\[
F_y = -\frac{dI_y}{dt} = -\frac{d}{dt} \left( \sum m_i \ddot{y}_S i \right)
= -\sum m_i \ddot{y}_S i = -m \ddot{y} = -m_y(q)\ddot{q} - m'_y(q)\dot{q}^2 \quad (2.330)
\]

\[
M_0^z = -\frac{dL_0^z}{dt} = -\frac{d}{dt} \left( \sum m_i \left[ (\dot{y}_S i x_S i - \dot{x}_S i y_S i) + J_{\xi S_i} \dot{\phi}_i \right] \right)
= -m_{\phi}(q)\ddot{q} - m'_\phi(q)\dot{q}^2 \quad (2.331)
\]

The generalized masses that depend on the generalized coordinate \(q\) (and thus on the position of the mechanism) can be determined as follows:

\[
m_x(q) = \sum m_i x'_S i, \quad m_y(q) = \sum m_i y'_S i, \quad m_{\phi}(q) = \sum m_i \left[ (y'_S i x_S i - x'_S i y_S i) + J_{\xi S_i} \dot{\phi}_i \right] \quad (2.332)
\]

Note that in equations (2.329) to (2.331) the coefficients in front of \(\dot{q}^2\) represent the derivatives of the ones in front of \(\ddot{q}\), with respect to the generalized coordinate \(q\).

The generalized mass \(m_{\phi}(q)\) referred to the \(z\) axis has the dimension of a moment of inertia in the case of an input angle \((q = \phi)\) and a similar form as the reduced moment of inertia known from (2.199), but must not be confused with that!
While the reduced moment of inertia $J_{\text{red}}(\phi)$ is linked to the kinetic energy and the input torque, the $m_\phi(q)$ that results from the angular momentum is required for calculating the frame moment.

![Diagram](image)

**Fig. 2.44** Influence of mass parameters on the frame forces of a crank-rocker mechanism

- **a)** Kinematic schematic and center-of-mass trajectories, including the reduced masses $m_i^* = m_i/m$; **b)** Polar diagrams of joint forces $F_{12}$ and $F_{14}$

<table>
<thead>
<tr>
<th>Variant 1:</th>
<th>Variant 2:</th>
<th>Variant 3:</th>
</tr>
</thead>
<tbody>
<tr>
<td>———-</td>
<td>– – – –</td>
<td>– – – –</td>
</tr>
</tbody>
</table>

The overall center of gravity of a mechanism normally moves along a trajectory as shown in Fig. 2.44; for an example, see (2.305). In addition to the center-of-mass trajectories, it shows the polar diagrams of the two frame forces of a crank-rocker mechanism for three variants of mass distribution. The masses $m_2$ and $m_4$ and their distances to the center of gravity $\xi_{S2}$ and $\xi_{S4}$ were varied. These curves are obtained at a constant angular velocity of the drive.

The relationship of the center-of-mass trajectories to the polar diagrams is interesting. At a smaller extension of the center-of-mass trajectory, the joint forces for variant 2 also become smaller than for variant 1. In variant 3 where the center-of-mass trajectory contracts into a point, the sum of the joint forces is zero according to (2.303), but the individual joint forces exist.

Any planar mechanism can, in principle, be configured by an appropriate mass distribution of its links such that its center of gravity remains at rest despite arbitrary motion. Complete mass balancing is achieved when the resultant inertia forces and the kinetic inertia moment are zero:
It follows from (2.329) to (2.331) that these conditions are met, if regardless of the state of motion
\[ m_x(q) \equiv 0; \quad m_y(q) \equiv 0; \quad m_\phi(q) \equiv 0 \] (2.334)

Complete inertia force balancing is achieved when the center of gravity remains at rest despite arbitrary motion of the mechanism. The condition for this results from (2.329) and (2.330) if the conditional equations for the position of the center of gravity of a multibody system are considered:
\[ \ddot{x}_S \equiv 0; \quad \ddot{y}_S \equiv 0 \] (2.335)

They can theoretically be satisfied for all planar rigid multilink mechanisms with one drive, see VDI Guideline 2149 Part 1 [35].

A complete balancing of forces and moments is rarely performed in practice because it has the following disadvantages:

- It mostly leads to bulky mechanisms with large dimensions that are hardly realistic from a designer’s point of view.
- The mass of the links has often to be changed significantly, which entails a considerable increase of their moments of inertia.
- The individual bearing and joint forces and joint moments may increase.

In cyclically operating mechanisms, the frame forces and moments are periodic, even if the angular velocity of the drive is variable. They often excite forced vibrations in the frame. The prerequisite is that the frame motions only have an insignificant influence on the motions of the mechanism, otherwise parameter-excited oscillations would be generated.

The goal of harmonic mass balancing is to reduce the amplitudes of critical excitation harmonics in the spectrum of the dynamic forces. Harmonic mass balancing is effective in steady-state operation and can ensure that no dangerous resonance amplitudes occur in the range of operating speeds. Harmonic balancing requires the use of special software programs because analytical solutions are only possible for simple mechanisms, such as slider-crank mechanisms, see Problem P2.12.

Balancing individual harmonics of the periodic excitation that results from mechanisms with variable transmission ratios in steady-state operation is of the greatest significance for mechanical engineering practice since it minimizes the excitation of vibrations. In cam mechanisms, cams with so-called HS profiles have proven their worth [4]. Mechanisms can be found using computer-aided synthesis, in which specific harmonics of the excitation forces and moments have a minimal magnitude. The solution is not always in balancing the first or second harmonics, often balancing of higher harmonics is of practical significance.

In addition to complete and harmonic mass balancing, the following practical measures can help to achieve an improvement:
• Generation of an equivalent countermotion, i.e. compensation by equal and opposite inertia forces of an additional mechanism (Fig. 2.45) or by additional dyad.
• Balancing of specific harmonics using compensatory mechanisms (Fig. 2.45).
• In multicylinder machines by placing counterweights at various crank angles, by offsetting the mechanism planes relative to the axis, and possibly by varying the crank radii and piston masses, see Sect. 2.6.3.3.
• Optimal balancing, taking into account secondary design conditions (requires the use of software).

Fig. 2.45 Examples of mass balancing by using a compensatory mechanism moving in the opposite direction (solid line: original mechanism; dashed line: compensatory mechanism)

2.6.3.2 Mass Balancing for a Slider-Crank Mechanism

Slider-crank mechanisms are used in many machines for converting rotating into reciprocating motions (and vice versa) so that mass balancing has met with special interest for quite some time.

Let us first derive the conditions for the complete inertia force balancing in a slider-crank mechanism without offset. Complete balancing of the resultant frame
moment can be achieved using additional rotational inertia, see VDI Guideline 2149 Part 1 [35].

The position of the center of gravity of the slider-crank mechanism follows from the link positions according to Fig. 2.26:

\[(m_2 + m_3 + m_4) \mathbf{r}_S = m_2 \mathbf{r}_{S2} + m_3 \mathbf{r}_{S3} + m_4 \mathbf{r}_{S4} \quad (2.336)\]

If the motion plane is identified with the plane of complex numbers for the purpose of compact mathematical treatment, then \( \mathbf{r} = x + jy = r \cdot e^{j\varphi} = r \cdot (\cos \varphi + j \sin \varphi) \) (\( j = \sqrt{-1} \)) applies and the position of the centers of gravity of the links can be stated as follows in accordance with Fig. 2.26:

\[
\begin{align*}
\mathbf{r}_{S2} &= \xi_{S2} e^{j\varphi_2}, \\
\mathbf{r}_{S3} &= l_2 e^{j\varphi_2} + \xi_{S3} e^{j\varphi_3}, \\
\mathbf{r}_{S4} &= l_2 e^{j\varphi_2} + l_3 e^{j\varphi_3} \\
\end{align*}
\quad (2.337)\]

Insertion into (2.336) provides the center-of-mass trajectory:

\[
\begin{align*}
(m_2 + m_3 + m_4) \mathbf{r}_S &= m_2 \xi_{S2} e^{j\varphi_2} + m_3 (l_2 e^{j\varphi_2} + \xi_{S3} e^{j\varphi_3}) \\
&\quad + m_4 (l_2 e^{j\varphi_2} + l_3 e^{j\varphi_3}) \\
&= e^{j\varphi_2} (m_2 \xi_{S2} + m_3 l_2 + m_4 l_2) + e^{j\varphi_3} (m_3 \xi_{S3} + m_4 l_3) \\
\end{align*}
\quad (2.338)\]

The center of gravity remains at rest (\( \ddot{\mathbf{r}}_S = \mathbf{0} \)), and there are no resultant inertia forces that act onto the frame if the following balancing conditions are satisfied (setting the expressions in parentheses to zero):

\[
\begin{align*}
m_2 \xi_{S2} + (m_3 + m_4) l_2 &= 0 \\
m_3 \xi_{S3} + m_4 l_3 &= 0 \\
\end{align*}
\quad (2.339, 2.340)\]

This yields the distances to the center of gravity for complete balancing:

\[
\begin{align*}
\xi_{S2} &= -\frac{m_3 + m_4}{m_2} l_2; \\
\xi_{S3} &= -\frac{m_4}{m_3} l_3 \\
\end{align*}
\quad (2.341)\]

As a result, the common center of gravity of the masses \( m_3 \) and \( m_4 \) is placed in the joint (2, 3).

If only (2.340) is satisfied, the center of gravity moves along a circular path and causes harmonic excitation forces. If the center of gravity of the masses \( m_3 \) and \( m_4 \) is at joint (2, 3), it can be shifted by a counterweight \( m_2^* \) to point (1, 2) so that it does not change its position. The time functions of the forces \( F_x, F_y \) and the moment \( M_z \) are composed of several harmonics at a constant input angular velocity \( \dot{\varphi}_2 = \Omega \). The first harmonic component is called first-order inertia force. The second term in the Fourier expansion varies at twice the frequency and is therefore called second-order inertia force. The first-order inertia force, that is, the first harmonic of the force \( F_x \), is balanced when condition (2.339) is satisfied, i.e. if a balancing mass is attached to the crank only. According to (2.341), the balancing mass is then located on the opposite side of the crank. It is, in practice, often designed as a segment of a circle,
2.6 Methods of Mass Balancing

see Fig. 2.29. The first-order inertia force of $F_y$ is balanced when the balancing condition

$$m_2 \xi S_2 + m_3 l_2 \left(1 - \frac{\xi S_3}{l_3}\right) = 0$$

is satisfied.

Fig. 2.46 Design options for balancing individual harmonics (compensatory mechanism); a) and c) forces and moment (1st harmonic), b) forces (1st harmonic), d) forces (1st and 2nd harmonics)

2.6.3.3 Harmonic Balancing in Multicylinder Machines

Multicylinder machines in which multiple slider-crank mechanisms are connected by a common shaft are frequently used in engines and compressors. Balancing of some harmonics is possible if the relative position of the individual mechanism planes and the relative orientation of the crank angles are favorably selected.

It is assumed for the following derivations that the cylinder axes and the crank-shaft axis are in one plane, the $y$-$z$ plane. This covers the case of an in-line engine with $k$ cylinders. The interesting case of a V-engine or radial engine in which the piston directions are arranged at a specific angle is excluded from these considera-
tions, see [1]. It is also assumed that all rotating masses, that is, the crankshaft with the rotating portions of the connecting rod (see \( m_{32} \), Fig. 2.29), are completely balanced. All cylinders should also be identical (equal masses and geometry) and only have different crank angles, see Fig. 2.47. The angle between the first crank \((j = 1)\) and the \(j\)th crank is denoted as \(\gamma_j\).

\[ F_j(t) = \sum_{k=1}^{\infty} \left[ A_k \cos k(\Omega t + \gamma_j) + B_k \sin k(\Omega t + \gamma_j) \right]. \quad (2.343) \]

\(k\) identifies the order of the harmonic. If it is assumed that the crankshaft revolves at constant angular velocity, the crank angles are \(\varphi_j = \Omega t + \gamma_j\). \(\gamma_1 = 0\) applies to the first cylinder. The Fourier coefficients \((A_k; B_k)\) of a mechanism are assumed to be known.

The resultant dynamic forces and moments that are transmitted onto the foundation are derived both for the \(x\) and for the \(y\) components (which is why the index is omitted) for \(J\) cylinders:

\[ F = \sum_{j=1}^{J} F_j; \quad M^0 = \sum_{j=1}^{J} F_j z_j \quad (2.344) \]

\(z_j\) is the distance of the respective mechanism plane from the \(x-y\) plane of the coordinate system, as shown in Figs. 2.34, 2.35, and 2.47.
Insertion of \( F_j \) from (2.343) into (2.344), using certain trigonometric identities and some conversions, provides

\[
F = \sum_j \sum_k \left[ A_k (\cos k\Omega t \cos k\gamma_j - \sin k\Omega t \sin k\gamma_j) + B_k (\sin k\Omega t \cos k\gamma_j + \cos k\Omega t \sin k\gamma_j) \right]
\]

\[
F = \sum_k \left[ (A_k \cos k\Omega t + B_k \sin k\Omega t) \sum_j \cos(k\gamma_j) + \sum_k \left[ -A_k \sin k\Omega t + B_k \cos k\Omega t \right] \sum_j \sin(k\gamma_j) \right]
\]

(2.345)

It follows from a comparison of coefficients that the \( k \)th-order harmonic of the resultant force of a multicylinder machine is completely balanced when the following two conditions are satisfied:

\[
\sum_{j=1}^{J} \cos k\gamma_j = 0; \quad \sum_{j=1}^{J} \sin k\gamma_j = 0
\]

(2.346)

Similarly, the \( k \)th-order harmonics of the moment \( M^0 \) are completely balanced if:

\[
\sum_{j=1}^{J} z_j \cos k\gamma_j = 0; \quad \sum_{j=1}^{J} z_j \sin k\gamma_j = 0
\]

(2.347)

These are the important balancing conditions of \( k \)th-order inertia forces of multicylinder machines. Interestingly, the masses of the links, the speed and geometrical dimensions are not included in these equations. Thus there are four transcendental equations for each order \( k \) for calculating the crank angles \( \gamma_j \) and the distances \( z_j \) (these are \( 2J - 1 \) unknown quantities for \( J \) mechanisms) that are required for complete mass balancing.

**2.6.4 Problems P2.11 to P2.14**

**P2.11 Harmonic Balancing of a Compressor**

The inertia forces that occur in mechanisms with a varying transmission ratio can be the cause of frame vibrations. Particularly dangerous are those components of the excitation spectrum, the frequency of which matches the natural frequency of the frame.

The additional balancing mass \( m_a \) in the form of an annulus segment of constant thickness, to be attached to the driving crank of the compressor (slider-crank mechanism without offset, see Fig. 2.48), is to be sized in such a way that the first harmonic of the resultant frame force component \( F_x \) is completely balanced at constant input angular speed.

**Given:**

\[
\begin{align*}
l_2 & = 40 \text{ mm} \quad \text{crank length} \\
l_3 & = 750 \text{ mm} \quad \text{coupler length} \\
\xi s_2 & = 12 \text{ mm} \quad \text{distance of the center of gravity of the crank from } O
\end{align*}
\]
\[ J_{S2} = 6,1 \cdot 10^{-3} \text{ kg} \cdot \text{m}^2 \]

moment of inertia referred to the axis through the center of gravity of the crank

\[ m_2 = 4.8 \text{ kg} \]

 crank mass

\[ m_4 = 14 \text{ kg} \]

 piston mass

\[ r = 20 \text{ mm} \]

 inner radius of the balancing mass

\[ R_{\text{max}} = 140 \text{ mm} \]

 maximum outer radius of the balancing mass (installation space!)

\[ b = 40 \text{ mm} \]

 thickness of the balancing mass

\[ \varrho_G = 7250 \text{ kg/m}^3 \]

density of cast iron

\[ \varrho_Z = 8900 \text{ kg/m}^3 \]

density of tin bronze

\[ \varrho_W = 9800 \text{ kg/m}^3 \]

density of white metal

Note: The mass parameters of the coupler 3 have been approximately included in the calculation of those of links 2 and 4, see Fig. 2.29.

Find:

1. Resultant frame force components \( F_x \) and \( F_y \) in general form for an arbitrary input motion \( \varphi_2(t) \), taking into account the balancing mass.

2. Required dimensions \((R, \beta)\), mass \( m_{an} \), and moment of inertia \( J^O_{oa} \) of the balancing mass so that both the first harmonic of the frame force \( F_x \) is balanced and \( J^O_{oa} \) becomes as small as possible. The materials that can be selected are cast iron, tin bronze, and white metal.

**P2.12 Compensatory Mechanism for Slider-Crank Mechanism**

Calculate the \( x \) component of the bearing force \((F_{x12})\) and the input torque \( M_{an} \) for a slider-crank mechanism for which the inertia forces are to be balanced using an balancing mass and a compensatory mechanism arranged as shown in Fig. 2.45. State the balancing conditions for the first and second harmonics of \( F_{x12} \) and \( M_{an} \) in general form. What values should be selected for the angles \( \alpha \) and \( \gamma \) and the balancing masses \( m_4 \) and \( m_5 \) so that the first two harmonics of these forces are compensated? Assume \( \lambda = l_2/l_3 = l_2/l_3 \ll 1 \) and \( \varphi = \Omega t \).
### P2.13 Crankshaft of a Four-Cylinder Machine

The following quantities are up for discussion for balancing individual harmonics in a four-cylinder machine, see Fig. 2.51:

**Variante a)**: \( \gamma_1 = 0^\circ; \quad \gamma_2 = 90^\circ; \quad \gamma_3 = 270^\circ; \quad \gamma_4 = 180^\circ \)

**Variante b)**: \( \gamma_1 = 0^\circ; \quad \gamma_2 = 180^\circ; \quad \gamma_3 = 180^\circ; \quad \gamma_4 = 0^\circ \)

Find out which orders of the forces and moments are balanced by these variants if the cylinder distances and cylinders are the same.

### P2.14 Mass Balancing of Crank Shears

Increasing the speed of rolling stock also requires higher speeds of the shears that cut a rolled bar “on the fly” to a specified length. The inertia forces of the crank shears excite undesirable vibrations in the machine frame and cause inadmissibly large loads to the anchoring of the machine, so that measures for mass balancing are required.

Perform mass balancing on the crank shears as shown in Fig. 2.49 in such a way that complete balancing of the forces is achieved by adding balancing masses to the crank and to the rocker.

![Fig. 2.49 Crank shears; a) Kinematic schematic, b) Calculation model with mass parameters](image)

The crank shears consist of two rigidly coupled crank-rocker mechanisms that are symmetrically arranged relative to the rolled bar and move in opposite directions. Since the two crank-rocker mechanisms are almost the same with regard to their dimensions and mass parameters and are arranged symmetrically, it is sufficient to consider just one mechanism, see the calculation model in Fig. 2.49.

**Given:**

Parameter values:

<table>
<thead>
<tr>
<th>( i )</th>
<th>Name</th>
<th>( l_i ) in m</th>
<th>( m_i ) in kg</th>
<th>( \xi_{Si} ) in m</th>
<th>( \eta_{Si} ) in m</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Crank</td>
<td>0.100</td>
<td>41.5</td>
<td>0.021</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Coupler (blade holder)</td>
<td>0.205</td>
<td>53.2</td>
<td>0.074</td>
<td>0.018</td>
</tr>
<tr>
<td>4</td>
<td>Rocker</td>
<td>0.147</td>
<td>17.7</td>
<td>0.065</td>
<td>0</td>
</tr>
</tbody>
</table>

**Find:**

Magnitudes \( U_2 = m_{a2} \cdot r_{a2}; \quad U_4 = m_{a4} \cdot r_{a4} \) and angular positions \( (\beta_2, \beta_4) \) of the balancing masses to be attached to the crank and to the rocker to achieve complete balancing of the resultant frame forces.
2.6.5 Solutions S2.11 to S2.14

S2.11 According to (2.303), and taking into account the functions \( x_{S_1} = x_{S_1}(\varphi_2(t)) \) and \( y_{S_1} = y_{S_1}(\varphi_2(t)) \), one can write:

\[
F_x = -\ddot{\varphi}_2 \sum_{i=2}^{I} m_i x''_{S_1} - \dddot{\varphi}_2 \sum_{i=2}^{I} m_i x_{S_1} + F_y = -\ddot{\varphi}_2 \sum_{i=2}^{I} m_i y''_{S_1} - \dddot{\varphi}_2 \sum_{i=2}^{I} m_i y_{S_1} \tag{2.348}
\]

Since the nomenclature used here mostly coincides with that in Fig. 2.26a, the first-order position functions can be taken directly from Table 2.1; the prerequisite stated there regarding the crank ratio \( \lambda \) is satisfied.

\( \lambda = l_2/l_3 = 0,04 \text{ m/0,75 m} = 0,0533 \ll 1 \) is obtained for the given parameter values. Using the functions from Table 2.1, and after another differentiation with respect to \( \varphi_2 \), one obtains:

\[
\begin{align*}
x''_{S_2} &= -\xi_{S_2} \cdot \cos \varphi_2; & y''_{S_2} &= -\xi_{S_2} \cdot \sin \varphi_2 \\
x''_{S_4} &= -l_2 \cdot (\cos \varphi_2 + \lambda \cos 2\varphi_2)
\end{align*}
\tag{2.349}
\]

The balancing mass has the following center-of-gravity coordinates in the fixed system, see Fig. 2.48:

\[
\begin{align*}
x_{S_a} &= r_a \cdot \cos (\varphi_2 + \pi) = -r_a \cdot \cos \varphi_2 \\
y_{S_a} &= r_a \cdot \sin (\varphi_2 + \pi) = -r_a \cdot \sin \varphi_2
\end{align*}
\tag{2.350}
\]

The derivatives with respect to the crank angle are

\[
\begin{align*}
x'_{S_a} &= r_a \cdot \sin \varphi_2; & x''_{S_a} &= r_a \cdot \cos \varphi_2 \\
y'_{S_a} &= -r_a \cdot \cos \varphi_2; & y''_{S_a} &= r_a \cdot \sin \varphi_2
\end{align*}
\tag{2.351}
\]

If one inserts the expressions from (2.349) and (2.351) into the relationship (2.348) for the forces, the following is obtained:

\[
F_x = \left( m_2 \xi_{S_2} - m_a r_a + m_4 l_2 \right) \cdot \sin \varphi_2 + m_4 l_2 \cdot \sin 2\varphi_2 \ldots \cdot \ddot{\varphi}_2 \\
+ \left( m_2 \xi_{S_2} - m_a r_a + m_4 l_2 \right) \cdot \cos \varphi_2 + m_4 l_2 \cdot \cos 2\varphi_2 \ldots \cdot \ddot{\varphi}_2 \tag{2.352}
\]

\[
F_y = - \left( m_2 \xi_{S_2} - m_a r_a \right) \cdot \left( \ddot{\varphi}_2 \cos \varphi_2 - \dot{\varphi}_2^2 \sin \varphi_2 \right)
\]

The equation for the force \( F_x \) is an approximation because due to \( \lambda \ll 1 \), the terms that contain higher powers of \( \lambda \) have already been neglected in the equations for \( x'_{S_4}, x''_{S_4} \).

The first harmonic of \( F_x \) becomes identical to zero if the expression in parentheses vanishes. Thus the balancing condition is:

\[
m_2 \xi_{S_2} - m_a r_a + m_4 l_2 = 0 \tag{2.353}
\]

The unbalance of the balancing mass is derived as:

\[
U_a = m_a r_a = m_2 \xi_{S_2} + m_4 l_2 = 0,6176 \text{ kg} \cdot \text{m} \tag{2.354}
\]

The general equations for mass, position of the center of gravity, and moment of inertia for the given form of an annulus segment are required for sizing the balancing mass. One can find the following in handbooks:
\[ m_a = \varrho_b \cdot (R^2 - r^2) \cdot \beta; \quad r_a = \frac{2}{3} \cdot \frac{R^3 - r^3}{R^2 - r^2} \cdot \sin \beta \] (2.355)

\[ J_a^O = \frac{\varrho_b}{2} \cdot (R^4 - r^4) \cdot \beta \] (2.356)

It follows from (2.355) for the unbalance, the required magnitude of which is known from (2.354):

\[ U_a = m_a r_a = \frac{2}{3} \varrho_b \cdot (R^3 - r^3) \cdot \sin \beta \] (2.357)

\( n \) It still contains two variables, namely \( R \) and \( \beta \). If one solves (2.357) for \( \sin \beta \), the following relation is obtained:

\[ \sin \beta = \frac{3 \cdot U_a}{2 \varrho_b \cdot (R^3 - r^3)} \leq 1 \] (2.358)

This inequation results in the following for the outer radius \( R \) in conjunction with the limitation set in the problem statement:

\[ R_{\text{min}} = \sqrt{r^3 + \frac{3 \cdot U_a}{2 \varrho_b}} \leq R \leq R_{\text{max}} \] (2.359)

Taking the given parameter values and the unbalance according to (2.354), the minimum radii are:

- cast iron \( R_{\text{min}} = 147.4 \) mm
- tin bronze \( R_{\text{min}} = 137.7 \) mm
- white metal \( R_{\text{min}} = 133.4 \) mm.

A comparison of these minimum radii with the permissible maximum value \( R_{\text{max}} = 140 \) mm shows that cast iron is not an option here. White metal (\( \varrho_b = 392 \) kg/m\(^2\)) is selected as the material and \( R = 135 \) mm as the radius. These selections let one determine the angle \( \beta \) from (2.358). First, \( \sin \beta = 0.96366 \) is obtained. That angle of the two possible solutions

\[ \beta_1 = 1,30044 \text{ rad} \quad (\cong 74^\circ) \quad \text{and} \quad \beta_2 = 1,8412 \text{ rad} \quad (\cong 105^\circ) \] (2.360)

is used for which the mass and the moment of inertia are the smallest. As both have a linear dependency on the angle \( \beta \), only the smaller of the two angles qualifies.

Now one can calculate the mass and moment of inertia of the balancing mass from (2.355) and (2.356):

\[ m_a = 9.09 \text{ kg}; \quad J_a^O = 0.0846 \text{ kg} \cdot \text{m}^2 \] (2.361)

Only the **first harmonic** of a force component can be balanced in a mechanism by a balancing mass that rotates with the drive. More complex balancing mechanisms are required to balance multiple force components and harmonics.

**S2.12** The bearing force of a slider-crank mechanism with an unbalance mass \( m_5 \) has (see Table 2.1) the \( x \) component

\[ F_{x_{12}} = -m_5 \ddot{x}_5 - m_4 \ddot{x}_4 \]

\[ = \Omega^2 \left[ m_5 l_5 \cos(\varphi + \gamma) + m_4 l_2 (\cos \varphi + \lambda \cos 2\varphi + \cdots) \right] \] (2.362)

Only the first two harmonics of the Fourier series were given since the higher ones are smaller than the \( \lambda^2 \) order.

Therefore, the following results for the compensatory mechanism with a crank offset by an angle \( \alpha \) (since the angle \( \varphi + \alpha \) is written here instead of \( \varphi \)):

\[ F'_{x_{12}} = m_4 l_2 \Omega^2 \left[ \cos(\varphi + \alpha) + \lambda \cos 2(\varphi + \alpha) + \cdots \right] \] (2.363)
The sum \( F_x = F_{x12} + F_{x12} \) is then, after using some trigonometric identities, sorted by the order of the harmonics:

\[
F_x = \Omega^2 \cos \varphi \left( m_4 l_2 + \overline{m}_4 \overline{l}_2 \cos \alpha + m_5 l_5 \cos \gamma \right) \\
- \sin \varphi \left( \overline{m}_4 \overline{l}_2 \sin \alpha + m_5 l_5 \sin \gamma \right) \\
+ \lambda \cos 2\varphi \left( m_4 l_2 + \overline{m}_4 \overline{l}_2 \cos 2\alpha \right) + \lambda \sin 2\varphi \left( \overline{m}_4 \overline{l}_2 \sin 2\alpha \right) + \cdots .
\] (2.364)

Those harmonics of \( F_x \) and \( M_{an} \) for which the expressions in parentheses are set to zero vanish. One can select those balancing conditions that are relevant for the respective application and take design measures accordingly.

The first harmonic of \( F_x \) can be balanced without a compensatory mechanism using an unbalance mass \( m_5 \). The balancing conditions are derived from the first two parentheses of \( F_x \) with \( m_4 = 0 \):

\[
m_4 l_2 + m_5 l_5 \cos \gamma = 0, \quad m_5 l_5 \sin \gamma = 0.
\] (2.366)

Both are satisfied for the values \( \gamma = \pi \) and \( m_5 l_5 = m_4 l_2 \), see Fig. 2.29.

Combined mass and power balancing is achieved when setting the first harmonic of \( F_x \) and the dominant second harmonic of \( M_{an} \) to zero. The solution follows from the corresponding four balancing conditions:

\[
\begin{align*}
m_4 l_2 + \overline{m}_4 \overline{l}_2 \cos \alpha + m_5 l_5 \cos \gamma &= 0, \\
\overline{m}_4 \overline{l}_2 \sin \alpha + m_5 l_5 \sin \gamma &= 0, \\
\overline{m}_4 \overline{l}_2 \sin 2\alpha &= 0, \\
m_4 l_2^2 + \overline{m}_4 \overline{l}_2^2 \cos 2\alpha &= 0
\end{align*}
\] (2.367)
as $m_4 = m_4$, $l_2 = l_2$, $m_5 l_5 = \sqrt{2}m_4 l_2$ and either $\alpha = 3\pi/2$, $\gamma = 3\pi/4$ (Fig. 2.50a) or $\alpha = \pi/2$, $\gamma = 5\pi/4$ (Fig. 2.50b). This balances the second harmonic of $F_x$ as well. Check by insertion whether the balancing conditions are satisfied.

**S2.13** If the coordinate system is placed in the first cylinder (Fig. 2.51), $z_1 = 0$ applies. The general balancing conditions according to (2.346) and (2.347) for a four-cylinder machine are then:

$$k\text{-th order forces}$$
$$1 + \cos k\gamma_2 + \cos k\gamma_3 + \cos k\gamma_4 = 0$$
$$0 + \sin k\gamma_2 + \sin k\gamma_3 + \sin k\gamma_4 = 0$$

$$k\text{-th order moments}$$
$$0 + z_2 \cos k\gamma_2 + z_3 \cos k\gamma_3 + z_4 \cos k\gamma_4 = 0$$
$$0 + z_2 \sin k\gamma_2 + z_3 \sin k\gamma_3 + z_4 \sin k\gamma_4 = 0$$

The following applies to variant a) with the specified angles for

\[
\begin{align*}
  k = 1 : & \\
  1 + 0 + 0 + 1 &= 0 \\
  0 + 1 + 1 + 0 &= 0 \\
  0 + 0 + 0 - z_4 &\neq 0 \\
  0 + z_2 - z_3 + 0 &\neq 0
\end{align*}
\]

\[
\begin{align*}
  k = 2 : & \\
  1 - 1 - 1 + 1 &= 0 \\
  0 + 0 + 0 + 0 &= 0 \\
  0 - z_2 - z_3 + z_4 &= 0 \\
  0 + 0 + 0 + 0 &= 0
\end{align*}
\]

Thus the first-order and second-order forces and second-order moments can be balanced. The first-order moments are not balanced because the conditions $z_4 = 0$ and $z_2 = z_3$ cannot be satisfied.

The following results for variant b) with $\gamma_2 = \gamma_3 = \pi, \gamma_4 = 0$:
The first-order forces and moments are balanced in this variant, while the second-order forces and moments cannot be balanced.

S2.14 Analogous to (2.338) for the slider-crank mechanism, a complex equation can be formulated for the center-of-mass trajectory of the four-bar linkage shown in Fig. 2.51b. With the unbalances $U_2$ and $U_4$ defined in the problem statement, see VDI Guideline 2149 Part 1 [35], it takes the following form:

\[
(m_2 + m_{a2} + m_3 + m_4 + m_{a4}) r_S = m_2 \left( jy_{12} + (\xi_S + j\eta_S) e^{j\varphi_2} \right) + U_2 \left( \frac{y_{12}}{r_{a2}} + e^{j(\varphi_2 + \beta_2)} \right) + m_3 \left( jy_{12} + l_2 e^{j\varphi_2} + (\xi_S + j\eta_S) e^{j\varphi_3} \right) + m_4 \left( x_{14} + (\xi_S + j\eta_S) e^{j\varphi_4} \right) + U_4 \left( \frac{x_{14}}{r_{a4}} + e^{j(\varphi_4 + \beta_4)} \right)
\]

The constraint in complex form resulting from Fig. 2.51b must be considered:

\[
jy_{12} + l_2 e^{j\varphi_2} + l_3 e^{j\varphi_3} = x_{14} + l_4 e^{j\varphi_4}
\]

If $e^{j\varphi_4}$ is obtained from this and inserted into (2.372), the center-of-mass trajectory results as a function of the two angles $\varphi_2$ and $\varphi_3$. The center of gravity remains stationary during any motion if the factors in front of the variable terms of $e^{j\varphi_2}$ and $e^{j\varphi_3}$ are set to zero. Four real balancing conditions result from separating the real and imaginary parts:

\[
m_2 \xi_S + U_2 \cdot \cos \beta_2 + m_3 l_2 + (m_4 \xi_S + U_4 \cdot \cos \beta_4) \cdot \frac{l_2}{l_4} = 0 \tag{2.374}
\]
\[
m_2 \eta_S + U_2 \cdot \sin \beta_2 + (m_4 \eta_S + U_4 \cdot \sin \beta_4) \cdot \frac{l_2}{l_4} = 0 \tag{2.375}
\]
\[
m_3 \xi_S + (m_4 \xi_S + U_4 \cdot \cos \beta_4) \cdot \frac{l_3}{l_4} = 0 \tag{2.376}
\]
\[
m_3 \eta_S + (m_4 \eta_S + U_4 \cdot \sin \beta_4) \cdot \frac{l_3}{l_4} = 0. \tag{2.377}
\]

Their solution for the unbalance components and insertion of terms from (2.376) and (2.377) into (2.374) and (2.375) provides:

\[
U_{2\xi} = U_2 \cdot \cos \beta_2 = -m_2 \xi_S - m_3 l_2 \cdot \left( 1 - \frac{\xi_S}{l_3} \right) \tag{2.378}
\]
\[
U_{2\eta} = U_2 \cdot \sin \beta_2 = -m_2 \eta_S + m_3 \eta_3 \cdot \frac{l_3}{l_4} \tag{2.379}
\]
\[
U_{4\xi} = U_4 \cdot \cos \beta_4 = -m_4 \xi_S - m_3 \xi_3 \cdot \frac{l_3}{l_4} \tag{2.380}
\]
\[
U_{4\eta} = U_4 \cdot \sin \beta_4 = -m_4 \eta_S - m_3 \eta_3 \cdot \frac{l_3}{l_4} \tag{2.381}
\]

Now the unbalance values and their angular positions can be calculated from the unbalance components. From (2.378) to (2.381), it follows with $k = 2$ and 4: that

\[
U_k = \sqrt{U_{k\xi}^2 + U_{k\eta}^2}, \quad \cos \beta_k = \frac{U_{k\xi}}{U_k}, \quad \sin \beta_k = \frac{U_{k\eta}}{U_k}. \tag{2.382}
\]
Using the parameter values given in the problem statement, the result is:

\[ U_{2\xi} = -4,2711 \text{ kg} \cdot \text{m}; \quad U_{2\eta} = 0,46712 \text{ kg} \cdot \text{m}. \] (2.383)

The magnitude of the unbalance at the crank, therefore, is

\[ U_2 = m_{a2} \cdot r_{a2} = 4,2966 \text{ kg} \cdot \text{m}. \] (2.384)

and its angular position is defined as:

\[ \cos \beta_2 = -0,99407; \quad \sin \beta_2 = 0,10872 \quad \Rightarrow \quad \beta_2 = 173,76^\circ. \] (2.385)

The following results for the corresponding quantities at the rocker:

\[ U_{4\xi} = -3,9735 \text{ kg} \cdot \text{m}; \quad U_{4\eta} = -0,68667 \text{ kg} \cdot \text{m}. \] (2.386)

This provides the basis for calculating the magnitude of the unbalance at the rocker:

\[ U_4 = m_{a4} \cdot r_{a4} = 4,0324 \text{ kg} \cdot \text{m}. \] (2.387)

at an angular position:

\[ \cos \beta_4 = -0,98539; \quad \sin \beta_4 = 0,17029 \quad \Rightarrow \quad \beta_4 = 189,8^\circ. \] (2.388)

The mechanism is shown true to scale with additional unbalances in Fig. 2.52.

The design of the balancing masses depends on the specific conditions, such as the available installation space. Balancing masses are often designed so that their moments of inertia are minimized.

In the four-bar linkage, complete mass balancing can be achieved by adding one additional unbalance each to the crank and to the rocker. Their influence on the input torque as well as the individual bearing and joint forces should be checked by proper calculations.
Dynamics of Machinery
Theory and Applications
Dresig, H.; Holzweißig, F.
2010, XII, 544 p. With online files/update., Softcover
ISBN: 978-3-540-89939-6