Chapter 2
Renewal and Regenerative Processes

Renewal and regenerative processes are models of stochastic phenomena in which an event (or combination of events) occurs repeatedly over time, and the times between occurrences are i.i.d. Models of such phenomena typically focus on determining limiting averages for costs or other system parameters, or establishing whether certain probabilities or expected values for a system converge over time, and evaluating their limits.

The chapter begins with elementary properties of renewal processes, including several strong laws of large numbers for renewal and related stochastic processes. The next part of the chapter covers Blackwell's renewal theorem, and an equivalent key renewal theorem. These results are important tools for characterizing the limiting behavior of probabilities and expectations of stochastic processes. We present strong laws of large numbers and central limit theorems for Markov chains and regenerative processes in terms of a process with regenerative increments (which is essentially a random walk with auxiliary paths). The rest of the chapter is devoted to studying regenerative processes (including ergodic Markov chains), processes with regenerative increments, terminating renewal processes, and stationary renewal processes.

2.1 Renewal Processes

This section introduces renewal processes and presents several examples. The discussion covers Poisson processes and renewal processes that are "embedded" in stochastic processes.

We begin with notation and terminology for point processes that we use in later chapters as well. Suppose $0 \leq T_1 \leq T_2 \leq \ldots$ are finite random times at which a certain event occurs. The number of the times T_n in the interval $(0, t]$ is

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}(T_n \leq t), \qquad t \geq 0.$$

R. Serfozo, *Basics of Applied Stochastic Processes*,
Probability and its Applications.

We assume this counting process is finite valued for each t, which is equivalent to $T_n \to \infty$ a.s. as $n \to \infty$.

More generally, we will consider T_n as points (or locations) in $\mathbb{R}_+$ (e.g., in time, or a physical or virtual space) with a certain property, and $N(t)$ is the number of points in $[0, t]$. The process $\{N(t) : t \geq 0\}$, denoted by $N(t)$, is a *point process* on $\mathbb{R}_+$. The T_n are its *occurrence times* (or point locations). The point process $N(t)$ is *simple* if its occurrence times are distinct: $0 < T_1 < T_2 < \dots$ a.s. (there is at most one occurrence at any instant).

Definition 1. A simple point process $N(t)$ is a *renewal process* if the inter-occurrence times $\xi_n = T_n - T_{n-1}$, for $n \geq 1$, are independent with a common distribution F, where $F(0) = 0$ and $T_0 = 0$. The T_n are called *renewal times*, referring to the independent or renewed stochastic information at these times. The ξ_n are the *inter-renewal times*, and $N(t)$ is the *number of renewals* in $(0, t]$.

Examples of renewal processes include the random times at which: customers enter a queue for service, insurance claims are filed, accidents or emergencies happen, or a stochastic process enters a special state of interest. In addition, T_n might be the location of the nth vehicle on a highway, or the location of the nth flaw along a pipeline or cable, or the cumulative quantity of a product processed in n production cycles. A discrete-time renewal process is one whose renewal times T_n are integer-valued. Such processes are used for modeling systems in discrete time, or for modeling sequential phenomena such as the occurrence of a certain character (or special data packet) in a string of characters (or packets), such as in DNA sequences.

To define a renewal process for any context, one only has to specify a distribution F with $F(0) = 0$ for the inter-renewal times. The F in turn defines the other random variables. More formally, there exists a probability space and independent random variables $\xi_1, \xi_2, \dots$ defined on it that have the distribution F (see Corollary 6 in the Appendix). Then the other quantities are $T_n = \sum_{k=1}^{n} \xi_k$ and $N(t) = \sum_{n=1}^{\infty} \mathbf{1}(T_n \leq t)$, where $T_n \to \infty$ a.s. by the strong law of large numbers (Theorem 72 in Chapter 1).

Here are two illustrations.

Example 2. Scheduled Maintenance. An automobile is lubricated when its owner has driven it L miles or every M days, whichever comes first. Let $N(t)$ denote the number of lubrications up to time t. Suppose the numbers of miles driven in disjoint time periods are independent, and the number of miles in any time interval has the same distribution, regardless of where the interval begins. Then it is reasonable that $N(t)$ is a renewal process. The inter-renewal distribution is $F(t) = P\{\tau \wedge M \leq t\}$, where τ denotes the time to accumulate L miles on the automobile.

This scheduled maintenance model applies to many types of systems where maintenance is performed when the system usage exceeds a certain level L or when a time M has elapsed. For instance, in reliability theory, the *Age*

Replacement model of components or systems, replaces a component with lifetime τ if it fails or reaches a certain age M (see Exercise 19).

Example 3. Service Times. An operator in a call center answers calls one at a time. The calls are independent and homogeneous in that the callers, the call durations, and the nature of the calls are independent and homogeneous. Also, the time needed to process a typical call (which may include post-call processing) has a distribution F. Then one would be justified in modeling the number of calls $N(t)$ that the operator can process in time t as a renewal process. The time scale here refers to the time that the operator is actually working; it is not the real time scale that includes intervals with no calls, operator work-breaks, etc.

Elementary properties of a renewal process $N(t)$ with inter-renewal distribution F are as follows. The times T_n are related to the counts $N(t)$ by

$$\{N(t) \geq n\} = \{T_n \leq t\},$$
$$T_{N(t)} \leq t < T_{N(t)+1}.$$

In addition, $N(T_n) = n$ and

$$N(t) = \max\{n : T_n \leq t\} = \min\{n : T_{n+1} > t\}.$$

These relations (which also hold for simple point processes) are used to derive properties of $N(t)$ in terms of T_n, and vice versa.

We have a good understanding of $T_n = \sum_{k=1}^{n} \xi_k$, since it is a sum of independent variables with distribution F. In particular, by properties of convolutions of distributions (see the Appendix), we know that

$$P\{T_n \leq t\} = F^{n\star}(t),$$

which is the n-fold convolution of F. Then $\{N(t) \geq n\} = \{T_n \leq t\}$ yields

$$P\{N(t) \leq n\} = 1 - F^{(n+1)\star}(t). \tag{2.1}$$

Also, using $E[N(t)] = \sum_{n=1}^{\infty} P\{N(t) \geq n\}$ (see Exercise 1), we have

$$E[N(t)] = \sum_{n=1}^{\infty} F^{n\star}(t). \tag{2.2}$$

The following result justifies that this mean and all moments of $N(t)$ are finite. Properties of moment generating functions are in the Appendix.

Proposition 4. *For each $t \geq 0$, the moment generating function $E[e^{\alpha N(t)}]$ exists for some α in a neighborhood of 0, and hence $E[N(t)^m] < \infty$, $m \geq 1$.*

Proof. It is clear that if $0 \leq X \leq Y$ and Y has a moment generating function on an interval $[0, \varepsilon]$, then so does X. Therefore, to prove the assertion it suffices to find a random variable larger than $N(t)$ whose moment generating function exists.

To this end, choose $x > 0$ such that $p = P\{\xi_1 > x\} > 0$. Consider the sum $S_n = \sum_{k=1}^{n} 1(\xi_k > x)$, which is the number of successes in n independent Bernoulli trials with probability of success p. The number of trials until the mth success is $Z_m = \min\{n : S_n = m\}$.

Clearly $xS_n < T_n$, and so

$$N(t) = \max\{n : T_n \le t\} \le \max\{n : S_n = \lfloor t/x \rfloor\} \le Z_{\lfloor t/x \rfloor + 1}.$$

Now Z_m has a negative binomial distribution with parameters m and p, and its moment generating is given in Exercise 2. Thus, $Z_{\lfloor t/x \rfloor + 1}$ has a generating function, and hence $N(t)$ has one as well. Furthermore, this existence ensures that all moments of $N(t)$ exist (a basic property of moment generating functions for nonnegative random variables).

Keep in mind that the preceding properties of the renewal process $N(t)$ are true for any distribution F with $F(0) = 0$. When this distribution has a finite mean μ and finite variance σ^2, the distribution of $N(t)$, for large t, is approximately a normal distribution with mean t/μ and variance $t\sigma^2/\mu^3$ (this follows by the central limit theorem in Example 67 below). Refined asymptotic approximations for the mean of $N(t)$ are given in Proposition 84.

The rest of this section is devoted to examples of renewal processes. The most prominent renewal process is as follows.

Example 5. Poisson Process. Suppose the i.i.d. inter-renewal times of the renewal process $N(t)$ have the exponential distribution $F(t) = 1 - e^{-\lambda t}$ with rate λ (its mean is λ^{-1}). Then as we will see in the next chapter, $N(t)$ is *Poisson process* with rate λ.

In this case, by properties of convolutions

$$P\{T_n \le t\} = F^{n\star}(t) = \int_0^t \lambda^n x^{n-1} \frac{e^{-\lambda x}}{(n-1)!} dx.$$

This is a gamma distribution with parameters n and λ. Alternatively,

$$P\{T_n \le t\} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

This is justified by noting that the derivative of the summation equals the integrand (the gamma density) in the preceding integral. Then using the relation $\{N(t) \ge n\} = \{T_n \le t\}$, we arrive at

$$P\{N(t) \le n\} = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

This is the Poisson distribution with mean $E[N(t)] = \lambda t$.

Poisson processes are very important in the theory and applications of stochastic processes. We will discuss them further in Chapter 3. Note that the discrete-time analogue of a Poisson process is the Bernoulli process described in Exercise 2.

Example 6. Delayed Renewal Process. Many applications involve a renewal process $N(t)$ with the slight difference that the first renewal time ξ_1 does not have the same distribution as the other ξ_n, for $n \geq 2$. We call $N(t)$ a *delayed renewal process*. Elementary properties of delayed renewal processes are similar to those for renewal processes with the obvious changes (e.g., if ξ_1 has distribution G, then the time T_n of the nth renewal has the distribution $G \star F^{(n-1)\star}(t)$). More important, we will see that many limit theorems for renewal processes apply to delayed renewal processes.

In addition to being of interest by themselves, renewal processes play an important role in analyzing more complex stochastic processes. Specifically, as a stochastic process evolves over time, it is natural for some event associated with its realization to occur again and again. When the "embedded" occurrence times of the event are renewal times, they may be useful for gleaning properties about the parent process. Stochastic processes with embedded renewal times include discrete- and continuous-time Markov chains, Markov-Renewal processes and more general regenerative processes (which are introduced in later chapters).

The next example describes renewal processes embedded in ergodic Markov chains due the regenerative property of Markov chains.

Example 7. Ergodic Markov Chain. Let X_n denote a discrete-time Markov chain on a countable state space that is ergodic (aperiodic, irreducible and positive recurrent). Consider any state i and let $0 < \nu_1 < \nu_2 < \ldots$ denote the (discrete) times at which X_n enters state i. Theorem 67 in Chapter 1 showed that the times ν_n form a discrete-time renewal process when $X_0 = i$. These times form a delayed renewal process when $X_0 \neq i$. The Bernoulli process in Exercise 2 is a special case.

Example 8. Cyclic Renewal Process. Consider a continuous-time stochastic process $X(t)$ that cycles through states $0, 1, \ldots, K-1$ in that order, again and again. That is, it starts at $X(0) = 0$, and its nth state is j if $n = mK + j$ for some m. For instance, in modeling the status of a machine or system, $X(t)$ might be the amount of deterioration of a system, or the number of shocks (or services) it has had, and the system is renewed whenever it ends a sojourn in state $K-1$.

Assume the sojourn times in the states are independent, and let F_j denote the sojourn time distribution for state j, where $F_j(0) = 0$. The time for the process $X(t)$ to complete a cycle from state 0 back to 0 has the distribution $F = F_0 \star F_1 \star \cdots \star F_{K-1}$. Then it is clear that the times at which $X(t)$ enters state 0 form a renewal process with inter-renewal distribution F. We call $X(t)$ a *cyclic* renewal process.

There are many other renewal processes embedded in $X(t)$. For instance, the times at which the process enters any fixed state i form a delayed renewal process with the same distribution F. Another more subtle delayed renewal process is the sequence of times at which the processes $X(t)$ bypasses state 0 by jumping from state $K-1$ to state 1 (assuming $F_0(0) > 0$); see Exercise 7. It is quite natural for a single stochastic process to contain several such embedded renewal processes.

Example 9. Alternating Renewal Process. An *alternating* renewal process is a cyclic renewal process with only two states, say 0 and 1. This might be appropriate for indicating whether a system is working (state 1) or not working (state 0), or whether a library book is available or unavailable for use.

2.2 Strong Laws of Large Numbers

This section begins our study of the long run behavior of renewal and related stochastic processes. In particular, we present a framework for deriving strong laws of large numbers for a variety of processes. We have already seen SLLNs in Chapter 1 for sums of i.i.d. random variables and for functions of Markov chains.[1]

Throughout this section, assume that $N(t)$ is a point process on $\mathbb{R}_+$ with occurrence times T_n. With no loss in generality, assume that $N(t) \uparrow \infty$ a.s. as $t \to \infty$. The first result says that T_n satisfies a SLLN if and only if $N(t)$ does. Here $1/\mu$ is 0 when $\mu = \infty$.

Theorem 10. *For a constant $\mu \leq \infty$ (or random variable), the following statements are equivalent:*

$$\lim_{n\to\infty} n^{-1} T_n = \mu \quad a.s. \tag{2.3}$$

$$\lim_{t\to\infty} t^{-1} N(t) = 1/\mu \quad a.s. \tag{2.4}$$

Proof. Suppose (2.3) holds. We know $T_{N(t)} \leq t < T_{N(t)+1}$. Dividing these terms by $N(t)$ (for large enough t so $N(t) > 0$), we have

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}.$$

Supposition (2.3) along with $N(t) \uparrow \infty$ and $(N(t)+1)/N(t) \to 1$ ensure that the first and last terms in this display converge to μ. Since $t/N(t)$ is sandwiched between these terms, it must also converge to their limit μ. This proves (2.4).

[1] The limit statements here and below are for the a.s. mode of convergence, but we sometimes suppress the term a.s., especially in the proofs.

Conversely, suppose (2.4) holds. When $N(t)$ is simple, $N(T_n) = n$, and so $T_n/n = T_n/N(T_n) \to \mu$, which proves (2.3). When $N(t)$ is not simple, $N(T_n) \geq n$ and (2.3) follows by Exercise 18.

Corollary 11. (SLLN for Renewal Processes) *If $N(t)$ is a renewal process whose inter-renewal times have a mean $\mu \leq \infty$, then*

$$t^{-1}N(t) \to 1/\mu \quad a.s. \text{ as } t \to \infty.$$

Proof. This follows by Theorem 10, since the classical SLLN (Theorem 72 in Chapter 1) ensures that $n^{-1}T_n \to \mu$.

Example 12. Statistical Estimation. Suppose $N(t)$ is a Poisson process with rate λ, but this rate is not known, and one wants to estimate it. One approach is to observe the process for a fixed time interval of length t and record $N(t)$. Then an estimator for λ is

$$\hat{\lambda}_t = t^{-1}N(t).$$

This estimator is unbiased in that $E[\hat{\lambda}_t] = \lambda$. It is also a *consistent estimator* since $\hat{\lambda}_t \to \lambda$ by Corollary 11. Similarly, if $N(t)$ is a renewal process whose inter-renewal distribution has a finite mean μ, then $\hat{\mu}_t = t/N(t)$ is a consistent estimator for μ (but it is not unbiased).

Of course, if it is practical to observe a fixed number n of renewals (rather than observing over a "fixed" time), then $n^{-1}T_n$ is an unbiased and consistent estimator of μ.

We now present a framework for obtaining limiting averages (or SLLNs) for a variety of stochastic processes. Consider a real-valued stochastic process $\{Z(t) : t \geq 0\}$ on the same probability space as the point process $N(t)$. Our interest is in natural conditions under which the limit of its average value $t^{-1}Z(t)$ exists. For instance, $Z(t)$ might denote a cumulative utility (e.g., cost or reward) associated with a system, and one is interested in the utility per unit time $t^{-1}Z(t)$ for large t.

The following theorem relates the limit of the *time average* $t^{-1}Z(t)$ to the limit of the embedded *interval average* $n^{-1}Z(T_n)$. An important quantity is

$$M_n = \sup_{T_{n-1} < t \leq T_n} |Z(t) - Z(T_{n-1})|,$$

which is the maximum fluctuation of $Z(t)$ in the interval $(T_{n-1}, T_n]$. We impose the rather weak assumption that this maximum does not increase faster than n does.

Theorem 13. *Suppose that $n^{-1}T_n \to \mu$ a.s. as $n \to \infty$, where $\mu \leq \infty$ is a constant or random variable. Let a be a constant or random variable that may be infinite when μ is finite, and consider the limit statements*

$$\lim_{t\to\infty} t^{-1}Z(t) = a/\mu \quad a.s. \tag{2.5}$$

$$\lim_{n\to\infty} n^{-1}Z(T_n) = a \quad a.s. \tag{2.6}$$

Statement (2.5) implies (2.6). Conversely, (2.6) implies (2.5) if the process $Z(t)$ is increasing, or if $\lim_{n\to\infty} n^{-1}M_n = 0$ *a.s.*

Proof. Clearly (2.5) implies (2.6) since

$$n^{-1}Z(T_n) = T_n^{-1}Z(T_n)(T_n/n) \to a.$$

Next, suppose (2.6) holds, and consider

$$t^{-1}Z(t) = t^{-1}Z(T_{N(t)}) + r(t).$$

where $r(t) = t^{-1}[Z(t) - Z(T_{N(t)})]$. By Theorem 10, $n^{-1}T_n \to \mu$ implies $N(t)/t \to 1/\mu$. Using the latter and (2.6), we have

$$t^{-1}Z(T_{N(t)}) = [Z(T_{N(t)})/N(t)][N(t)/t] \to a/\mu.$$

Then to prove $t^{-1}Z(t) \to a/\mu$, it remains to show $r(t) \to 0$.

In case $Z(t)$ is increasing, (2.6) and $N(t)/t \to 1/\mu$ ensure that

$$|r(t)| \le \frac{[Z(T_{N(t)+1}) - Z(T_{N(t)})]}{N(t)} \frac{N(t)}{t} \to 0.$$

Also, in the other case in which $n^{-1}M_n \to 0$,

$$|r(t)| \le [M_{N(t)+1}/(N(t)+1)][(N(t)+1)/t] \to 0.$$

Thus $r(t) \to 0$, which completes the proof that (2.6) implies (2.5).

Here is a consequence of Theorem 13 that applies to processes with regenerative increments, which are discussed in Section 2.10.

Corollary 14. *If $N(t)$ is a renewal process, and $(Z(T_n) - Z(T_{n-1}), M_n)$, $n \ge 1$, are i.i.d. with finite means, then*

$$t^{-1}Z(t) \to E[Z(T_1) - Z(0)]/E[T_1] \quad a.s. \text{ as } t \to \infty. \tag{2.7}$$

Proof. By the classical SLLN, $n^{-1}Z(T_n) \to E[Z(T_1) - Z(0)]$. Also, since M_n are i.i.d., it follows by Exercise 33 in the preceding chapter that $n^{-1}M_n \to 0$. Then Theorem 13 yields (2.7).

We will see a number of applications of Theorem 13 throughout this chapter. Here are two elementary examples.

Example 15. Renewal Reward Process. Suppose $N(t)$ is a renewal process associated with a system in which a reward Y_n (or cost or utility value) is received at time T_n, for $n \geq 1$. Then the total reward in $(0, t]$ is[2]

$$Z(t) = \sum_{n=1}^{\infty} Y_n \mathbf{1}(T_n \leq t) = \sum_{n=1}^{N(t)} Y_n, \qquad t \geq 0.$$

For instance, Y_n might be claims received by an insurance company at times T_n, and $Z(t)$ would represent the cumulative claims.

The process $Z(t)$ is a *renewal reward process* if the pairs (ξ_n, Y_n), $n \geq 1$, are i.i.d. (ξ_n and Y_n may be dependent). Under this assumption, it follows by Theorem 13 that the average reward per unit time is

$$\lim_{t\to\infty} t^{-1} Z(t) = E[Y_1]/E[\xi_1] \quad \text{a.s.},$$

provided the expectations are finite. This result is very useful in many diverse contexts. One only has to justify the renewal conditions and evaluate the expectations. In complicated systems with many activities, a little thought may be needed to identify the renewal times as well as the associated rewards.

Example 16. Cyclic Renewal Process. Let $X(t)$ be a cyclic renewal process on $0, \ldots, K-1$ as in Example 8. Recall that the entrance times to state 0 form a renewal process, and the mean inter-renewal time is $\mu = \mu_0 + \cdots + \mu_{K-1}$, where μ_i is the mean sojourn time in state i. Suppose a cost or value $f(i)$ per unit time is incurred whenever $X(t)$ is in state i. Then the average cost per unit time is

$$\lim_{t\to\infty} t^{-1} \int_0^t f(X(s))ds = \frac{1}{\mu} \sum_{i=0}^{K-1} f(i)\mu_i \quad \text{a.s.}. \tag{2.8}$$

This follows by applying Corollary 13 to $Z(t) = \int_0^t f(X(s))ds$ and noting that $E[Z(T_1)] = \sum_{i=0}^{K-1} f(i)\mu_i$.

A particular case of (2.8) says that the portion of time $X(t)$ spends in a subset of states J is

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathbf{1}(X(s) \in J)ds = \frac{1}{\mu} \sum_{j\in J} \mu_j \quad \text{a.s.}.$$

2.3 The Renewal Function

This section describes several fundamental properties of renewal processes in terms of the their mean value functions.

[2] Recall the convention that $\sum_{n=1}^{0}(\cdot) = 0$.

For this discussion, suppose that $N(t)$ is a renewal process with inter-renewal distribution F with a finite mean μ. We begin by showing that the mean value function $E[N(t)]$ contains all the probabilistic information about the process. It is more convenient to use the slight variation of the mean value function defined as follows.

Definition 17. The *renewal function* associated with the distribution F (or the process $N(t)$) is

$$U(t) = \sum_{n=0}^{\infty} F^{n\star}(t), \qquad t \in \mathbb{R}, \tag{2.9}$$

where $F^{0\star}(t) = \mathbf{1}(t \geq 0)$. Clearly $U(t) = E[N(t)] + 1$, for $t \geq 0$, is the expected number of renewals up to time t, including a "fictitious renewal" at time 0.

Note that $U(t)$ is similar to a distribution function in that it is nondecreasing and right-continuous on $\mathbb{R}$, but $U(t) \uparrow \infty$ as $t \to \infty$. Keep in mind that $U(t)$ is 0 for $t < 0$ and it has a unit jump at $t = 0$. Although a renewal function is ostensibly very simple, it has some remarkable uses as we will soon see.

Our first observation is that if the inter-renewal times are continuous random variables, then the renewal function has a density.

Proposition 18. *Suppose the inter-renewal distribution F has a density f. Then $U(t)$ also has a density for $t > 0$, and it is $U'(t) = \sum_{n=1}^{\infty} f^{n\star}(t)$. In addition,*

$$P\{N(t) > N(t-)\} = 0, \qquad t \geq 0. \tag{2.10}$$

Proof. The first assertion follows since $U(t) = \sum_{n=0}^{\infty} F^{n\star}(t)$, and the derivative of $F^{n\star}(t)$ is $f^{n\star}(t)$. The second assertion, which is equivalent to $N(t) - N(t-) = 0$ a.s., will follow if $E[N(t) - N(t-)] = 0$. But the last equality is true since, by the monotone convergence theorem (Theorem 13 in the Appendix) and the continuity of U,

$$E[N(t-)] = E[\lim_{s \uparrow t} N(s)] = \lim_{s \uparrow t} U(s) - 1 = U(t) - 1 = E[N(t)].$$

Expression (2.10) tells us that the probability of a renewal at any time is 0, when the inter-renewal times are continuous. Here is an important case.

Remark 19. If $N(t)$ is a Poisson process with rate λ, then the probability of a jump at any time t is 0.

Some of the results below have slight differences depending on whether the inter-renewal distribution is or is not arithmetic. The distribution F is *arithmetic* (or periodic) if it is piecewise constant and its points of increase are contained in a set $\{0, d, 2d, \ldots\}$; the largest $d > 0$ with this property is the *span*. In this case, it is clear that the distributions $F^{n\star}$ and the renewal function $U(t)$ also have this arithmetic property. If F is not arithmetic, we call it *non-arithmetic*. A distribution with a continuous part is necessarily non-arithmetic.

The rest of this chapter makes extensive use of Riemann-Stieltjes integrals; see the review in the Appendix. In particular, the expectation of a function $g : \mathbb{R} \to \mathbb{R}$ on a finite or infinite interval I with respect to F will be expressed as the *Riemann-Stieltjes integral*[3]

$$\int_I g(t)dF(t).$$

All the functions in this book like g are assumed to be measurable (see the Appendix); we will not repeat this assumption unless emphasis is needed. Riemann-Stieltjes integrals with respect to U are defined similarly, since U is like a distribution function. A typical integral is

$$\int_{[0,b]} g(t)dU(t) = g(0) + \int_{(0,b]} g(t)dU(t).$$

The right-hand side highlights that $g(0)U(0) = g(0)$ is the contribution from the unit jump of U at 0. Since $U(t) = 0$ for $t < 0$, we will only consider integrals with respect to U on intervals in $\mathbb{R}_+$.

An important property of the renewal function $U(t)$ is that it uniquely determines the distribution F. To see this, we will use Laplace transforms. The Laplace-Stieltjes or simply the *Laplace transform* of F is defined by

$$\hat{F}(\alpha) = \int_{\mathbb{R}_+} e^{-\alpha t} dF(t), \qquad \alpha \geq 0.$$

A basic property is that the transform $\hat{F}$ uniquely determines F and vice versa. The Laplace transform $\hat{U}(\alpha)$ of $U(t)$ is defined similarly. Now, taking the Laplace transform of both sides in (2.9), we have

$$\hat{U}(\alpha) = \sum_{n=0}^{\infty} \widehat{F^{n\star}}(\alpha) = \sum_{n=0}^{\infty} \hat{F}(\alpha)^n = 1/(1 - \hat{F}(\alpha)).$$

The last equation follows by Fubini's theorem. This yields the following result.

Proposition 20. The Laplace transforms $\hat{U}(\alpha)$ and $\hat{F}(\alpha)$ determine each other uniquely by the relation $\hat{U}(\alpha) = 1/(1-\hat{F}(\alpha))$. Hence U and F uniquely determine each other.

One can sometimes use this result for identifying that a renewal process is of a certain type. For instance, a Poisson process has a renewal function

[3] This integral is the usual Riemann integral $\int_I g(t)f(t)dt$ when F has a density f. Also, $\int_I h(t)dt$ is written as $\int_a^b h(t)dt$ when I is $(a, b]$ or $[a, b]$ etc.

$U(t) = \lambda t + 1$, and so any renewal process with this type of renewal function is a Poisson process.

Remark 21. A renewal process $N(t)$, whose inter-renewal times have a finite mean, is a Poisson process with rate λ if and only if $E[N(t)] = \lambda t$, for $t \geq 0$.

Other examples of renewal processes with tractable renewal functions are those whose inter-renewal distribution is a convolution or mixture of exponential distributions; see Exercises 6 and 12. Sometimes the Laplace transform $\hat{U}(\alpha) = 1/(1 - \hat{F}(\alpha))$ can be inverted to determine $U(t)$. Unfortunately, nice expressions for renewal processes are the exception rather than the rule.

In addition to characterizing renewal processes as discussed above, renewal functions arise naturally in expressions for probabilities and expectations of functions associated with renewal processes. Such expressions are the focus of much of this chapter.

The next result describes an important family of functions of point processes as well as renewal processes. Expression (2.11) is a special case of Campbell's formula in the theory of point processes (see Theorem 106 in Chapter 4). Here and in other places in the book the phrase "provided the integral exists" means that the Lebesgue (or Riemann-Stieltjes) integral exists. A Lebesgue integral is a generalization of a Riemann-Stieltjes integral; see the Appendix, Section 6.4. The proof below uses the monotone and dominated convergence theorems applied to Riemann-Stieltjes integrals with a Lebesgue integral as the possible limit.

Theorem 22. *Let $N(t)$ be a simple point process with point locations T_n such that $\eta(t) = E[N(t)]$ is finite for each t. Then for any function $f : \mathbb{R}_+ \to \mathbb{R}$,*

$$E\Big[\sum_{n=1}^{N(t)} f(T_n)\Big] = \int_{(0,t]} f(s)d\eta(s), \qquad t \geq 0, \tag{2.11}$$

provided the integral exists. Moreover, if $X_1, X_2, \ldots$ are random variables defined on the same probability space as the process $N(t)$ such that $E[X_n|T_n = s] = f(s)$, independent of n. Then

$$E\Big[\sum_{n=1}^{N(t)} X_n\Big] = \int_{(0,t]} f(s)d\eta(s), \qquad t \geq 0, \tag{2.12}$$

provided the integral exists.

Proof. We will prove (2.11) by a standard approach for proving formulas for integrals. For convenience, denote the equality (2.11) by $\Sigma(f) = I(f)$. First, consider the simple piecewise-constant function

$$f(s) = \sum_{k=1}^{m} a_k \mathbf{1}(s \in (s_k, t_k]),$$

for fixed $0 \le s_1 < t_1 < \cdots \le s_m < t_m \le t$. In this case,

$$\begin{aligned}\Sigma(f) &= E\Big[\sum_{k=1}^{m} a_k[N(t_k) - N(s_k)]\Big] \\ &= \sum_{k=1}^{m} a_k[\eta(t_k) - \eta(s_k)] \;=\; I(f).\end{aligned}$$

Next, for any nonnegative function f one can define simple functions f_m as above such that $f_m(s) \uparrow f(s)$ as $m \to \infty$ for each s. For instance,

$$f_m(s) = m \wedge (\lfloor 2^m f(s) \rfloor / 2^m)\mathbf{1}(s \in [-2^m, 2^m]).$$

Then by the monotone convergence theorem (see the Appendix, Theorem 13) and the first part of this proof,

$$\Sigma(f) = \lim_{m\to\infty} \Sigma(f_m) = \lim_{m\to\infty} I(f_m) = I(f).$$

Thus, (2.11) is true for nonnegative f.

Finally, (2.11) is true for a general function f, since $f(s) = f(s)^+ - f(s)^-$ and the preceding part of the proof for nonnegative functions yield

$$\Sigma(f) = \Sigma(f^+) - \Sigma(f^-) = I(f^+) - I(f^-) = I(f).$$

It suffices to prove (2.12) for nonnegative X_n. Conditioning on T_n, we have

$$\begin{aligned}E\Big[\sum_{n=1}^{N(t)} X_n\Big] &= \sum_{n=1}^{\infty} E\Big[E[X_n 1(T_n \le t)|T_n]\Big] = \sum_{n=1}^{\infty} E\Big[1(T_n \le t)E[X_n|T_n]\Big] \\ &= \sum_{n=1}^{\infty} E\Big[1(T_n \le t)f(X_n)\Big].\end{aligned}$$

Then applying (2.11) to the last term yields (2.12).

Remark 23. Theorem 22 applies to a renewal process $N(t)$ with its renewal function U being equal to η. For instance, (2.12) would be

$$E\Big[\sum_{n=1}^{N(t)} X_n\Big] = \int_{(0,t]} f(s)dU(s).$$

Note that this integral does not include the unit jump of U at 0. An extension that includes a value X_0 with $f(0) = E[X_0]$ would be

$$E\Big[\sum_{n=0}^{N(t)} X_n\Big] = \int_{[0,t]} f(s)dU(s). \qquad (2.13)$$

This remark yields the following special case of a general Wald identity for stopping times in Corollary 25 in Chapter 5.

Corollary 24. (Wald Identity for Renewals) *For the renewal process* $N(t)$*,*

$$E[T_{N(t)+1}] = \mu E[N(t)+1], \qquad t \geq 0.$$

Proof. Using Remark 23 with $f(s) = E[\xi_{n+1}|T_n = s] = \mu$, it follows that

$$\begin{aligned} E[T_{N(t)+1}] &= E\Big[\sum_{n=0}^{N(t)} \xi_{n+1}\Big] \\ &= \mu U(t) = \mu E[N(t)+1]. \end{aligned}$$

In light of this result, one might suspect that $E[T_{N(t)}] = \mu E[N(t)]$. However, this is not the case. In fact, $E[T_{N(t)}] \leq \mu E[N(t)]$; and this is a strict inequality for a Poisson process; see Exercise 22.

Example 25. Discounted Rewards. Suppose a renewal process $N(t)$ has rewards associated with it such that a reward (or cost) Y_n is obtained at the nth renewal time T_n. The rewards are discounted continuously over time and if a reward y occurs a time t, it has a discounted value of $ye^{-\alpha t}$. Then the total discounted reward up to time t is

$$Z(t) = \sum_{n=1}^{N(t)} Y_n e^{-\alpha T_n}.$$

As in Theorem 22, assume there is a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $E[Y_n|T_n = s] = f(s)$, independent of n. Then applying Remark 23 to $X_n = Y_n e^{-\alpha T_n}$ yields

$$E[Z(t)] = \int_{(0,t]} e^{-\alpha s} f(s) dU(s).$$

The next examples describe several systems modeled by renewal processes with the same type of inter-renewal distribution shown in (2.14) below (also see Exercise 16).

Example 26. Single-Server System. Pallets are scheduled to arrive at an automatically guided vehicle (AGV) station according to a renewal process $N(t)$ with inter-arrival distribution F. The station is attended by a single AGV, which can transport only one pallet at a time. Pallets scheduled to arrive when the AGV is already busy transporting a pallet are diverted to another station. Assume the transportation times are independent with common distribution G.

Let us consider the times $\widetilde{T}_n$ at which the AGV begins to transport a pallet (the times at which pallets arrive and the AGV is idle). For simplicity, assume a transport starts at time 0. To describe $\widetilde{T}_1$, let τ denote a transport time for the first pallet. Then $\widetilde{T}_1$ equals τ plus the waiting time $T_{N(\tau)+1} - \tau$ for the next pallet to arrive after transporting the first pallet. That is, $\widetilde{T}_1 = T_{N(\tau)+1}$. When the next pallet arrives at time $T_{N(\tau)+1}$, the system is renewed and these cycles are repeated indefinitely. Thus $\widetilde{T}_n$ are renewal times.

The inter-renewal distribution of $\widetilde{T}_1$ and its mean have reasonable expressions in terms of the arrival process. Indeed, conditioning on τ, which is independent of $N(t)$, yields

$$P\{\widetilde{T}_1 \le t\} = \int_{\mathbb{R}_+} P\{T_{N(x)+1} \le t\} dG(x). \tag{2.14}$$

Also, if F has a finite mean μ, then by Wald's identity,

$$E[\widetilde{T}_1] = \int_{\mathbb{R}_+} E[T_{N(x)+1}] dG(x) = \mu \int_{\mathbb{R}_+} U(x) dG(x).$$

Example 27. $G/G/1/1$ System. Consider a system in which customers arrive at a processing station according to a renewal process with inter-arrival distribution F and are processed by a single server. The processing or service times are independent with the common distribution G, and are independent of the arrival process. Also, customer arrivals during a service time are blocked from being served — they either go elsewhere or go without service. In this context the times $\widetilde{T}_n$ at which customers begin services are renewal times as in the preceding example with inter-renewal distribution (2.14). This system is called a $G/G/1/1$ system: G/G means the inter-arrival and service times are i.i.d. (with general distributions) and $1/1$ means there is one server and at most one customer in the system.

Example 28. Geiger Counters. A classical model of a Geiger counter assumes that electronic particles arrive at the counter according to a Poisson or renewal process. Upon recording an arrival of a particle, the counter is locked for a random time during which arrivals of new particles are not recorded. The times of being locked are i.i.d. and independent of the arrivals. Under these assumptions, it follows that the times $\widetilde{T}_n$ at which particles are recorded are renewal times, and have the same structure as those for the $G/G/1/1$ system described above. This so-called Type I model assumes that particles arriving while the counter is locked do not affect the counter.

A slightly different Type II Geiger counter model assumes that whenever the counter is locked and a particle arrives, that particle is not recorded, but it extends the locked period by another independent locking time. The times at which particles are registered are renewal times, but the inter-renewal distribution is more intricate than that for the Type I counter.

2.4 Future Expectations

We have just seen the usefulness of the renewal function for characterizing a renewal process and for describing some expected values of the process. In the following sections, we will discuss the major role a renewal function plays in describing the limiting behavior of probabilities and expectations associated with renewal and regenerative phenomena. This section outlines what to expect in the next three sections, which cover the heart of renewal theory.

The analysis to follow will use convolutions of functions with respect to the renewal function $U(t)$, such as

$$U \star h(t) = \int_{[0,t]} h(t-s)dU(s) = h(0) + \int_{(0,t]} h(t-s)dU(s),$$

where h is bounded on finite intervals and equals 0 for $t < 0$.

We will see that many probabilities and expectations associated with a renewal process $N(t)$ can be expressed as a function $H(t)$ that satisfies a recursive equation of the form

$$H(t) = h(t) + \int_{[0,t]} H(t-s)dF(s), \qquad t \geq 0.$$

This "renewal equation", under minor technical conditions given in the next section, has a unique solution of the form $H(t) = U \star h(t)$.

The next topic we address is the limiting behavior of such functions as $t \to \infty$. We will present Blackwell's theorem, and an equivalent key renewal theorem, which establishes

$$\lim_{t\to\infty} U \star h(t) = \frac{1}{\mu} \int_{\mathbb{R}_+} h(s)ds.$$

This is for non-arithmetic F; an analogous result holds for arithmetic F. Also, the integral is slightly different from the standard Riemann integral.

We cover the topics outlined above — Renewal Equations, Blackwell's Theorem and the Key Renewal Theorem — in the next three sections. Thereafter, we discuss applications of these theorems that describe the limiting behavior of probabilities and expectations associated with renewal, regenerative and Markov chains.

2.5 Renewal Equations

We begin our discussion of renewal equations with a concrete example.

Example 29. Let $X(t)$ be a cyclic renewal process on $0, 1, \ldots, K-1$, and consider the probability $H(t) = P\{X(t) = i\}$ as a function of time, for a fixed state i. To show $H(t)$ satisfies a renewal equation, the standard approach is to condition on the time T_1 of the first renewal (the first entrance to state 0). The result is

$$H(t) = P\{X(t) = i, T_1 > t\} + P\{X(t) = i, T_1 \leq t\}, \tag{2.15}$$

where the last probability, conditioning on the renewal at T_1, is

$$\int_{[0,t]} P\{X(t) = i | T_1 = s\} dF(s) = \int_{[0,t]} H(t-s) dF(s).$$

Therefore, the recursive equation (2.15) that $H(t)$ satisfies is

$$H(t) = h(t) + F \star H(t),$$

where $h(t) = P\{X(t) = i, T_1 > t\}$. This type of equation is a renewal equation, which is defined as follows.

Definition 30. Let $h(t)$ be a real-valued function on $\mathbb{R}$ that is bounded on finite intervals and equals 0 for $t < 0$. The *renewal equation* for $h(t)$ and the distribution F is

$$H(t) = h(t) + \int_{[0,t]} H(t-s) dF(s), \qquad t \geq 0, \tag{2.16}$$

where $H(t)$ is a real-valued function. That is $H = h + F \star H$. We say $H(t)$ is a *solution of this equation* if it satisfies the equation, and is bounded on finite intervals and equals 0 for $t < 0$.

We first observe that a renewal equation has a unique solution.

Proposition 31. *The function $U \star h(t)$ is the unique solution to the renewal equation (2.16).*

Proof. Clearly $U \star h(t) = 0$ for $t < 0$, and it is bounded on finite intervals since

$$\sup_{s \leq t} |U \star h(s)| \leq \sup_{s \leq t} |h(s)| U(t) < \infty, \qquad t \geq 0.$$

Also, $U \star h$ is a solution to the renewal equation, since by the definition of U and $F^{0\star} \star h = h$,

$$U \star h = \Big(F^{0\star} + F \star \sum_{n=1}^{\infty} F^{(n-1)\star} \Big) \star h = h + F \star (U \star h).$$

To prove $U \star h$ is the unique solution, let $H(t)$ be any solution to the renewal equation, and consider the difference $D(t) = H(t) - U \star h(t)$. From the renewal equation, we have $D = F \star D$, and so iterating this yields $D = F^{n\star} \star D$. Now, the finiteness of $U(t)$ implies $F^{n\star}(t) \to 0$, as $n \to \infty$, and hence $D(t) = 0$ for each t. This proves that $U \star h(t)$ is the unique solution of the renewal equation.

The standard approach for deriving a renewal equation is by conditioning on the first renewal time to obtain the function $h(t)$ (recall Example 29). Upon establishing that a function $H(t)$ satisfies a renewal equation, one automatically knows that $H(t) = U \star h(t)$ by Proposition 31. For instance, Example 29 showed that the probability $P\{X(t) = i\}$ for a cyclic renewal process satisfies a renewal equation for $h(t) = P\{X(t) = i, T_1 > t\}$, and hence

$$P\{X(t) = i\} = U \star h(t). \tag{2.17}$$

Although $H(t) = U \star h(t)$ is a solution of the renewal equation, it is not an explicit expression for the function $H(t)$ in that $h(t)$ generally depends on $H(t)$. For instance, $h(t) = P\{X(t) = i, T_1 > t\}$ in (2.17) is part of the probability $H(t) = P\{X(t) = i\}$.

Only in very special settings is the formula $H(t) = U \star h(t)$ tractable enough for computations. On the other hand, we will see in Section 2.7 that the function $U \star h(t)$ is the framework of the Key Renewal Theorem that yields limit theorems for a variety of stochastic processes.

2.6 Blackwell's Theorem

The next issue is to characterize the limiting behavior of functions of the form $U \star h(t)$ as $t \to \infty$. This is based on the limiting behavior of $U(t)$, which we now consider.

Throughout this section, assume that $N(t)$ is a renewal process with renewal function $U(t)$ and mean inter-renewal time μ, which may be finite or infinite. In Section 2.2, we saw that $N(t)/t \to 1/\mu$ a.s., and so $N(t)$ behaves asymptotically like t/μ as $t \to \infty$ (recall that $1/\mu = 0$ when $\mu = \infty$). This suggests $U(t) = E[N(t)] + 1$ should also behave asymptotically like t/μ. Here is a confirmation.

Theorem 32. (Elementary Renewal Theorem)

$$t^{-1}U(t) \to 1/\mu, \qquad \textit{as } t \to \infty.$$

Proof. For finite μ, using $t < T_{N(t)+1}$ and Wald's identity (Corollary 24),

$$t < E[T_{N(t)+1}] = \mu U(t).$$

This yields the lower bound $1/\mu < t^{-1}U(t)$. Also, this inequality holds trivially when $\mu = \infty$. With this bound in hand, to finish proving the assertion it suffices to show

$$\limsup_{t\to\infty} t^{-1}U(t) \le 1/\mu. \tag{2.18}$$

To this end, for a constant b, define a renewal process $\overline{N}(t)$ with inter-renewal times $\overline{\xi}_n = \xi_n \wedge b$. Define $\overline{T}_n$ and $\overline{U}(t)$ accordingly. Clearly, $U(t) \le \overline{U}(t)$. Also, by Wald's identity and $\overline{T}_{\overline{N}(t)+1} \le t+b$ (since the $\overline{\xi}_n$ are bounded by b),

$$E[\xi_1 \wedge b]\overline{U}(t) = E[\overline{T}_{\overline{N}(t)+1}] \le t+b.$$

Consequently,

$$t^{-1}U(t) \le t^{-1}\overline{U}(t) \le \frac{1+b/t}{E[\xi_1 \wedge b]}.$$

Letting $t\to\infty$ and then letting $b\to\infty$ (whereupon the last fraction tends to $1/\mu$, even when $\mu=\infty$), we obtain (2.18), which finishes the proof.

A more definitive description of the asymptotic behavior of $U(t)$ is given in the following major result.

Theorem 33. (Blackwell) *For non-arithmetic F and $a > 0$,*

$$U(t+a) - U(t) \to a/\mu, \qquad \text{as } t\to\infty.$$

If F is arithmetic with span d, the preceding limit holds with $a = md$ for any integer m.

Proof. A proof for non-arithmetic F using a coupling argument is in Section 2.15 below. A simpler proof for the arithmetic case is as follows.

Suppose F is arithmetic and, for simplicity, assume the span is $d=1$. Then renewals occur only at integer times, and $p_i = F(i) - F(i-1)$ is the probability that an inter-renewal time is of length i, where $p_0 = 0$.

We will represent the renewal times by the backward recurrence time process $\{A(t) : t = 0,1,2,\ldots\}$, which we know is a Markov chain with transition probabilities

$$p_{i0} = \frac{p_i}{\sum_{j=i}^{\infty} p_j} = 1 - p_{i,i+1}, \qquad i \ge 0.$$

(recall Example 20 and Exercises 37 and 38 in Chapter 1). This chain is irreducible, and hits state 0 at and only at the renewal times. Then the chain is ergodic since the time between renewals has a finite mean μ. Theorem 54 in Chapter 1 yields $P\{A(t)=0\} \to 1/\mu$ (the representation for π_0).

Then because $U(t+m) - U(t)$ is the expected number of renewals exactly at the times $t+1,\ldots,t+m$, it follows that

$$U(t+m) - U(t) = \sum_{k=1}^{m} P\{A(t+k)=0\} \to m/\mu, \qquad \text{as } t\to\infty.$$

Blackwell's theorem says that the renewal function $U(t)$ is asymptotically linear. This raises the question: "Does the asymptotic linearity of $U(t)$ lead to a nice limit for functions of the form $U \star h(t)$?" The answer is yes, as we will see shortly.

As a preliminary, let us investigate the limit of $U \star h(t)$ for a simple piecewise-constant function

$$h(s) = \sum_{k=1}^{m} a_k \mathbf{1}(s \in [s_k, t_k)),$$

where $0 \le s_1 < t_1 \le s_2 < t_2 < \cdots \le s_m < t_m < \infty$. In this case,

$$\begin{aligned} U \star h(t) = \int_{[0,t]} h(t-s)dU(s) \;&=\; \sum_{k=1}^{m} a_k \int_0^t \mathbf{1}(t-s \in [s_k, t_k))dU(s) \\ &= \sum_{k=1}^{m} a_k [U(t-s_k) - U(t-t_k)]. \end{aligned} \tag{2.19}$$

The last equality follows since the integral is over $s \in [t-t_k, t-s_k)$, and $U(t) = 0$ when $t < 0$. By Theorem 33, we know

$$U(t-s_k) - U(t-t_k) \to (t_k - s_k)/\mu.$$

Applying this to (2.19) yields

$$\lim_{t\to\infty} U \star h(t) = \frac{1}{\mu} \sum_{k=1}^{m} a_k (t_k - s_k) = \frac{1}{\mu} \int_{\mathbb{R}_+} h(s)ds. \tag{2.20}$$

This result suggests that a limit of this form would also be true for general functions $h(t)$. That is what we will establish next.

2.7 Key Renewal Theorem

This section will complete our development of renewal functions and solutions of renewal equations. The issue here is to determine limits of functions of the form $U \star h(t)$ as $t \to \infty$.

We begin with preliminaries on integrals of functions on the infinite axis $\mathbb{R}_+$. Recall that the Riemann integral $\int_0^t h(s)ds$ is constructed by Riemann sums on grids that become finer and finer (see Definition 86 below). The integral exists when h is continuous on $[0,t]$, or is bounded and has a countable number of discontinuities. Furthermore, the Riemann integral of h on $\mathbb{R}_+$ is defined by

$$\int_{\mathbb{R}_+} h(s)ds = \lim_{t\to\infty} \int_0^t h(s)ds, \tag{2.21}$$

provided the limit exits. In that case, h is *Riemann integrable* on $\mathbb{R}_+$.

The Key Renewal Theorem requires a slightly different notion of a function being *directly Riemann integrable* on $\mathbb{R}_+$. A DRI function is defined in Section 2.17, where an integral is constructed "directly" on the entire axis $\mathbb{R}_+$ by Riemann sums, analogously to the construction of a Riemann integral on a finite interval. A DRI function is Riemann integrable in the usual sense, but the converse is not true; see Exercise 32.

For our purposes, we only need the following properties from Proposition 88 below (also see Exercise 33).

Remark 34. A function $h : \mathbb{R}_+ \to \mathbb{R}$ is DRI in the following cases.
(a) $h(t) \geq 0$ is decreasing and Riemann integrable.
(b) h is continuous except possibly on a set of Lebesgue measure 0, and $|h(t)| \leq b(t)$, where b is DRI.

Here is the main result. Its proof is in Section 2.17.

Theorem 35. (Key Renewal Theorem) *If F is non-arithmetic and $h(t)$ is DRI, then*

$$\lim_{t\to\infty} U \star h(t) = \frac{1}{\mu} \int_{\mathbb{R}_+} h(s)ds. \tag{2.22}$$

Remark 36. This theorem is equivalent to Blackwell's Theorem 33, which asserts that $U(t+a) - U(t) \to a/\mu$. Indeed, Section 2.17 shows that Blackwell's theorem implies the Key Renewal Theorem. Conversely, (2.22) applied to $h(s) = 1(a < s < t + a)$ (so $h(t - s) = 1(t - a < s \leq t)$) yields Blackwell's theorem.

An analogous key renewal theorem for arithmetic F is as follows. It can also be proved by Blackwell's renewal theorem — with fewer technicalities — as suggested in Exercise 31.

Theorem 37. (Arithmetic Key Renewal Theorem) *If F is arithmetic with span d, then for any $u < d$,*

$$\lim_{n\to\infty} U \star h(u + nd) = \frac{d}{\mu} \sum_{k=0}^{\infty} h(u + kd),$$

provided the sum is absolutely convergent.

The next order of business is to show how the limit statement in the key renewal theorem applies to limits of time-dependent probabilities and expected values of stochastic processes. We know that any function $H(t)$ that satisfies a renewal equation has the form $H(t) = U \star h(t)$. It turns out that this functional form is "universal" in the following sense.

Proposition 38. *Any function $H(t)$ that is bounded on finite intervals and is 0 for $t < 0$ can be expressed as*

$$H(t) = U \star h(t), \qquad \textit{where} \quad h(t) = H(t) - F \star H(t).$$

Proof. This follows since $U = F^{0\star} + U \star F$, and so

$$H = F^{0\star} \star H = (U - U \star F) \star H = U \star (H - F \star H).$$

Knowing that $U \star h(t)$ is a universal form for any function that is bounded on finite intervals, the remaining issue is, "How to relate $U \star h(t)$ to probabilities and expectations of stochastic processes?" A natural vehicle is the following type of stochastic process.

Definition 39. A real-valued stochastic process $X(t)$ is *crudely regenerative* at a positive random time T if

$$E[X(T+t)|T] = E[X(t)], \qquad t \geq 0, \tag{2.23}$$

and these expectations are finite.

An important connection between crudely regenerative processes and functions $U \star h(t)$ is as follows.

Proposition 40. *Suppose that $X(t)$ is a crudely regenerative process at T, which has the distribution F. If $E[X(t)]$ is bounded on finite intervals, then*

$$E[X(t)] = U \star h(t), \qquad \textit{where} \qquad h(t) = E[X(t)\mathbf{1}(T > t)].$$

Proof. Applying Proposition 38 to $H(t) = E[X(t)]$, it follows that $E[X(t)] = U \star h(t)$, where

$$h(t) = H(t) - F \star H(t) = E[X(t)] - \int_{[0,t]} E[X(t-s)]dF(s).$$

By the crude regeneration property, $E[X(t)|T = s] = E[X(t-s)]$, $s \leq t$, and

$$h(t) = E[X(t)] - \int_{[0,t]} E[X(t)|T = s]dF(s) = E[X(t)\mathbf{1}(T > t)].$$

This completes the proof.

The family of crudely regenerative processes is very large; it includes ergodic Markov chains in discrete and continuous time, regenerative processes, and many functions of these processes as well. More details on these processes are in the next sections. Typically, $X(t)$ is a real-valued function of one or more stochastic processes. An important example is a probability $P\{Y(t) \in A\} = E[X(t)]$, when $X(t) = \mathbf{1}(Y(t) \in A)$.

The following major result is a version of the key renewal theorem that characterizes limiting distributions and expectations. Many applications in the next sections are based on this formulation.

Theorem 41. (Crude Regenerations) *Suppose that $X(t)$ is a real-valued process that is crudely regenerative at T, and define $M = \sup\{|X(t)| : t \leq T\}$. If T is non-arithmetic and M and MT have finite means, then*

$$\lim_{t\to\infty} E[X(t)] = \frac{1}{\mu}\int_{\mathbb{R}_+} h(s)ds, \tag{2.24}$$

where $h(t) = E[X(t)\mathbf{1}(T > t)]$.

Proof. Since $E[X(t)] = U \star h(t)$ by Proposition 40, where T has the non-arithmetic distribution F, the assertion (2.24) will follow by the key renewal theorem provided $h(t)$ is DRI.

To prove this, note that $|h(t)| \leq b(t) = E[M\mathbf{1}(T > t)]$. Now, by the dominated convergence theorem in the Appendix and $E[M] < \infty$, we have $b(t) \downarrow 0$. Also,

$$\int_{\mathbb{R}_+} b(s)ds = E\Big[\int_0^T M\,ds\Big] = E[MT] < \infty. \tag{2.25}$$

Then $b(t)$ is DRI by Remark 34 (a), and so $h(t)$ is DRI by Remark 34 (b).

2.8 Regenerative Processes

The primary use of the key renewal theorem is in characterizing the limiting behavior of regenerative processes and their relatives via Theorem 41. This section covers limit theorems for regenerative processes, and the next three sections cover similar results for Markov chains, and processes with regenerative increments.

We begin by defining regenerative processes. Loosely speaking, a discrete- or continuous-time stochastic process is regenerative if there is a renewal process such that the segments of the process between successive renewal times are i.i.d. More precisely, let $\{X(t) : t \geq 0\}$ denote a continuous-time stochastic process with a state space S that is a metric space (e.g., the Euclidean space $\mathbb{R}^d$ or a Polish space; see the Appendix). This process need not be a jump process like the continuous-time Markov chains we discuss later. However, we assume that the sample paths of $X(t)$ are right-continuous with left-hand limits a.s. This ensures that the sample paths are continuous except possibly on a set of Lebesgue measure 0.

Let $N(t)$ denote a renewal process on $\mathbb{R}_+$, defined on the same probability space as $X(t)$, with renewal times T_n and inter-renewal times $\xi_n = T_n - T_{n-1}$, which have a distribution F with a finite mean μ.

Definition 42. For the process $\{(N(t), X(t)) : t \geq 0\}$, its sample path in the time interval $[T_{n-1}, T_n)$ is described by

$$\zeta_n = (\xi_n, \{X(T_{n-1} + t) : 0 \leq t < \xi_n\}). \tag{2.26}$$

This ζ_n is the nth *segment* of the process. The process $X(t)$ is *regenerative over the times* T_n if its segments ζ_n are i.i.d.

Classic examples of regenerative processes are ergodic Markov chains in discrete and continuous time. An important fact that follows directly from the definition is that functions of regenerative processes inherit the regenerative property.

Remark 43. Inheritance of Regenerations. If $\widetilde{X}(t)$ with state space $\widetilde{S}$ is regenerative over T_n, then $X(t) = f(\widetilde{X}(t))$ is also regenerative over T_n, for any $f : \widetilde{S} \to S$.

For instance, we can express the distribution of a regenerative process $\widetilde{X}(t)$ as the expectation $P\{\widetilde{X}(t) \in B\} = E[X(t)]$, where $X(t) = \mathbf{1}(\widetilde{X}(t) \in B)$ (a function of $\widetilde{X}$) is a real-valued regenerative process.

To include the possibility that the first segment of the process $X(t)$ in the preceding definition may differ from the others, we say $X(t)$ is a *delayed* regenerative process if ζ_n are independent, and $\zeta_2, \zeta_3, \ldots$ have the same distribution, which may be different from the distribution of ζ_1. We discuss more general regenerative-like processes with stationary segments in Section 2.19.

Remark 44. Regenerative processes are crudely regenerative, but not vice versa.

Indeed, if $X(t)$ is regenerative over the times T_n, then $X(t)$ is crudely regenerative at T_1. Next, consider the process $X(t) = X_n(t)$, if $t \in [n-1, n]$ for some n, where $\{X_n(t) : t \in [0,1]\}$ for $n \geq 1$, are independent stochastic processes with identical mean functions ($E[X_n(t)] = E[X_1(t)]$ for each n), but non-identical variance functions. Clearly X is crudely regenerative at $T = 1$, but it is not regenerative.

To proceed, a few comments are in order concerning convergence in distribution. For a process $X(t)$ on a countable state space S, a probability measure P on S is the limiting distribution of $X(t)$ if

$$\lim_{t\to\infty} P\{X(t) \in B\} = \mathrm{P}(B), \qquad B \subset S. \tag{2.27}$$

This definition, however, is too restrictive for uncountable S, where (2.27) is not needed for all subsets B. In particular, when the state space S is the Euclidean space $\mathbb{R}^d$, then P on $S = \mathbb{R}^d$ is defined to be the limiting distribution of $X(t)$ if (2.27) holds for $B \in \mathcal{S}$ (the Borel sets of S) such that $\mathrm{P}(\partial B) = 0$, where ∂B is the boundary of B.

Equivalently, P on S is the *limiting distribution* of $X(t)$ if

$$\lim_{t\to\infty} E[f(X(t))] = \int_S f(x)\mathrm{P}(dx), \tag{2.28}$$

for any continuous function $f : S \to [0,1]$. This means that the distribution of $X(t)$ converges weakly to P (see Section 6.9 in the Appendix for more details on weak convergence).

We are now ready to apply Theorem 41 to characterize the limiting distribution of regenerative processes. For simplicity, assume throughout this section that the inter-renewal distribution F (for the times between regenerations) is non-arithmetic.

Theorem 45. (Regenerative Processes) *Suppose the process $X(t)$ on a metric state space S (e.g. $\mathbb{R}^d$) with Borel σ-field $\mathcal{S}$ is regenerative over T_n. For $f : S \to \mathbb{R}$ define $M = \sup\{|f(X(t))| : t \le T_1\}$. If M and MT_1 have finite means, then*

$$\lim_{t\to\infty} E[f(X(t))] = \frac{1}{\mu} E\Big[\int_0^{T_1} f(X(s))ds\Big]. \tag{2.29}$$

In particular, the limiting distribution of $X(t)$ is

$$\mathrm{P}(B) = \lim_{t\to\infty} P\{X(t) \in B\} = \frac{1}{\mu} E\Big[\int_0^{T_1} \mathbf{1}(X(s) \in B)ds\Big], \quad B \in \mathcal{S}. \tag{2.30}$$

Proof. Assertion (2.29) follows by Theorem 41, since $f(X(t))$ is regenerative over T_n and therefore it satisfies the crude-regeneration property. Clearly, (2.30) is a special case of (2.29).

Theorems 41 and 45 provide a framework for characterizing limits of expectations and probabilities of regenerative processes. For expectations, one must check that the maximum M of the process during an inter-renewal interval has a finite mean. The main step in applying these theorems, however, is to evaluate the integrals $\int_{\mathbb{R}_+} h(s)ds$ or $\int_S f(x)\mathrm{P}(dx)$. Keep in mind that one need not set up a renewal equation or check the DRI property for each application — these properties have already been verified in the proof of Theorem 41.

Theorem 45 and most of those to follow are true, with slight modifications, for delayed regenerative processes. This is due to the property in Exercise 42 that the limiting behavior of a delayed regenerative process is the same as the limiting behavior of the process after its first regeneration time T_1. Here is an immediate consequence of Theorem 45 and Exercise 42.

Corollary 46. (Delayed Regenerations) *Suppose the process $X(t)$ with a metric state space S is a delayed regenerative process over T_n. If $f : S \to \mathbb{R}$ is such that the expectations of $M = \sup\{|f(X(t))| : T_1 \le t \le T_2\}$ and $M\xi_2$ are finite, then*

$$\lim_{t\to\infty} E[f(X(t))] = \frac{1}{\mu} E\Big[\int_{T_1}^{T_2} f(X(s))\,ds\Big].$$

In particular, the limiting distribution of $X(t)$ is

$$\mathrm{P}(B) = \frac{1}{\mu} E\Big[\int_{T_1}^{T_2} \mathbf{1}(X(s) \in B)ds\Big], \quad B \in \mathcal{S}.$$

We end this section with applications of Theorem 45 to three regenerative processes associated with a renewal process.

Definition 47. *Renewal Process Trinity.* For a renewal process $N(t)$, the following three processes provide more information about renewal times:

$A(t) = t - T_{N(t)}$, the *backward recurrence time* at t (or the *age*), which is the time since the last renewal prior to t.

$B(t) = T_{N(t)+1} - t$, the *forward recurrence time* at t (or the *residual renewal time*), which is the time to the next renewal after t.

$L(t) = \xi_{N(t)+1} = A(t) + B(t)$, *length of the renewal interval* covering t.

For instance, a person arriving at a bus stop at time t would have to wait $B(t)$ minutes for the next bus to arrive, or a call-center operator returning to answer calls at time t would have to wait for a time $B(t)$ before the next call. Also, if a person begins analyzing an information string at a location t looking for a certain character (or pattern), then $A(t)$ and $B(t)$ would be the distances to the left and right of t where the next character occurs.

Note that the three-dimensional process $(A(t), B(t), L(t))$ is regenerative over T_n, and so is each process by itself. Each of the processes $A(t)$ and $B(t)$ is a continuous-time Markov process with piece-wise deterministic paths on the state space $\mathbb{R}_+$; see Exercises 34 and 35. A convenient expression for their joint distribution is, for $0 \le x < t$, $y \ge 0$,

$$P\{A(t) > x, B(t) > y\} = P\{N(t+y) - N((t-x)-) = 0\}. \tag{2.31}$$

This is simply the probability of no renewals in $[t-x, t+y]$. Although this probability is generally intractable, one can show that it is the solution of a renewal equation, and so it has the form $U \star h(t)$; see Exercises 36 and 37.

Example 48. Trinity in Equilibrium. One can obtain the limiting distributions of $A(t)$ and $B(t)$ separately from Theorem 45. Instead, we will derive their joint limiting distribution. Since $(A(t), B(t))$ is regenerative over T_n, Theorem 41 yields

$$\lim_{t\to\infty} P\{A(t) > x, B(t) > y\} = 1 - \frac{1}{\mu}\int_0^{x+y} [1 - F(s)]ds, \tag{2.32}$$

since, by the definitions of the variables,

$$h(t) = P\{A(t) > x, B(t) > y, T_1 > t\} = P\{T_1 > t + y\}\mathbf{1}(t > x).$$

From (2.32), it immediately follows that

$$\lim_{t\to\infty} P\{A(t) \le x\} = \lim_{t\to\infty} P\{B(t) \le x\} = \frac{1}{\mu}\int_0^{x} [1 - F(s)]ds. \tag{2.33}$$

This limiting distribution, which is called the *equilibrium distribution* associated with F, is important in other contexts. We will see its significance in Section 2.15 for stationary renewal processes.

One can also obtain the limiting distribution of $L(t) = A(t) + B(t)$ by Theorem 41. Namely,

$$\lim_{t\to\infty} P\{L(t) \le x\} = \frac{1}{\mu}\int_{[0,x]} s dF(s), \tag{2.34}$$

since

$$h(t) = P\{L(t) \le x, T_1 > t\} = P\{T_1 \le x, T_1 > t\} = (F(x) - F(t))\mathbf{1}(x > t).$$

Alternatively, one can derive (2.34) directly from (2.32).

Additional properties of the three regenerative processes $A(t)$, $B(t)$ and $L(t)$ are in Exercises 34–41. These processes are especially nice for a Poisson process.

Example 49. Poisson Recurrence Times. If $N(t)$ is a Poisson process with rate λ, then from (2.31)

$$P\{A(t) > x, B(t) > y\} = e^{-\lambda(x+y)}, \qquad 0 \le x < t, y \ge 0, \tag{2.35}$$

which is the Poisson probability of no renewals in an interval of length $x+y$. In particular, setting $x = 0$, and then $y = 0$, yields

$$P\{B(t) > y\} = e^{-\lambda y}, \qquad P\{A(t) > x\} = e^{-\lambda x}\mathbf{1}(x < t).$$

Thus $B(t)$ is exponentially distributed with rate λ; this also follows by the memoryless property of the exponential distribution (Exercise 1 in Chapter 3). Note that $A(t)$ has the same exponential distribution, but it is truncated at $x = t$. The limiting distribution of each of these processes, however, is exponential with rate λ. Since $L(t) = A(t) + B(t)$, its distribution can be obtained from (2.35); its mean is shown in Exercise 39.

Even though recurrence time processes $A(t)$ and $B(t)$ are typically not tractable for a fixed t, their equilibrium distribution F_e in (2.33) may be.

Example 50. Uniformly Distributed Renewals. Suppose $N(t)$ is a renewal process with uniform inter-renewal distribution $F(x) = x$, for $x \in [0, 1]$. Its associated equilibrium distribution (2.33) is simply $F_e(x) = 2x - x^2$.

Interestingly, $F_e(x) \ge F(x)$ for each x. That is, the distribution F_e for the forward recurrence time $B(t)$ in equilibrium is greater than the distribution F of the forward recurrence time $B(0) = \xi_1$ at time 0. This means that $B(t)$ in equilibrium is *stochastically smaller* than $B(0)$. This is due to the fact that the failure rate $F'(x)/(1 - F(x)) = 1/(1 - x)$ of F is increasing. Compare this property with the inspection paradox in Exercise 39.

2.9 Limiting Distributions for Markov Chains

This section covers the classical renewal argument for determining the limiting distributions of ergodic Markov chains. The argument uses limit theorems in the preceding section, which are manifestations of the key renewal theorem for regenerative processes. We present a similar characterization of limiting distributions for continuous-time Markov chains in Chapter 4.

Assume that X_n is an ergodic Markov chain on a countable state space S, with limiting distribution

$$\pi_j = \lim_{n\to\infty} P\{X_n = j\}, \qquad j \in S,$$

which does not depend on X_0. Recall that Theorems 59 and 54 in Chapter 1 established that the limiting distribution is also the stationary distribution and it is the unique distribution π that satisfies the balance equation $\pi = \pi \mathrm{P}$. They also showed (via a coupling proof) that the stationary distribution has the following form, which we will now prove by a classical renewal argument.

Theorem 51. (Markov Chains) *The ergodic Markov chain X_n has a unique limiting distribution given as follows: for a fixed $i \in S$,*

$$\pi_j = \frac{1}{\mu_i} E\Big[\sum_{n=0}^{\tau_1(i)-1} \mathbf{1}(X_n = j)\Big| X_0 = i\Big], \qquad j \in S, \tag{2.36}$$

where $\mu_i = E[\tau_1(i)|X_0 = i]$. Another expression for this probability is

$$\pi_j = \frac{1}{\mu_j}, \qquad j \in S. \tag{2.37}$$

Proof. We will prove this by applying the key renewal theorem. The main idea is that the strong Markov property ensures that X_n is a (discrete-time) delayed regenerative process over the times $0 < \tau_1(i) < \tau_2(i) < \ldots$ at which X_n enters a fixed state i (Theorem 67 in Chapter 1). In light of this fact, the assertion (2.36) follows by Corollary 46. Also, setting $i = j$ in (2.36) yields (2.37), since the sum in (2.36) is the sojourn time in state j, which is 1.

2.10 Processes with Regenerative Increments

Many cumulative cost or utility processes associated with ergodic Markov chains and regenerative processes can be formulated as processes with regenerative increments. These processes are basically random walks with auxiliary paths or information. We will show that the classical SLLN and central limit theorem for random walks extend to processes with regenerative increments.

This section presents a SLLN based on material in Section 2.2, and Section 2.13 presents a central limit theorem. Functional central limit theorems for random walks and processes with regenerative increments are the topic of Section 5.9 in Chapter 5.

For this discussion, $N(t)$ will denote a renewal process whose inter-renewal times $\xi_n = T_n - T_{n-1}$ have a distribution F and finite mean μ.

Our focus will be on the following processes that are typically associated with cumulative information of regenerative processes.

Definition 52. Let $Z(t)$ be a real-valued process with $Z(0) = 0$ defined on the same probability space as a renewal process $N(t)$. For the two-dimensional process $\{(N(t), Z(t)) : t \geq 0\}$, its increments in the time interval $[T_{n-1}, T_n)$ are described by

$$\zeta_n = (\xi_n, \{Z(t + T_{n-1}) - Z(T_{n-1}) : 0 \leq t < \xi_n\}).$$

The process $Z(t)$ has *regenerative increments over the times* T_n if ζ_n are i.i.d. A process with "delayed" regenerative increments is defined in the obvious way, where the distribution ζ_1 is different from the others.

Under this definition, $(T_n - T_{n-1}, Z(T_n) - Z(T_{n-1}))$ are i.i.d. These i.i.d. increments of N and Z leads to many nice limit theorems based on properties of random walks.

A primary example of a process with regenerative increments is a cumulative functional $Z(t) = \int_0^t f(X(s))ds$, where $X(t)$ is regenerative over T_n and $f(i)$ is a cost rate (or utility rate) when the process $X(t)$ is in state i.

Hereafter, assume that $Z(t)$ is a process with regenerative increments over T_n. Keep in mind that $Z(0) = 0$. Although the distribution and mean of $Z(t)$ are generally not tractable for computations, we do have a Wald identity for some expectations.

Proposition 53. (Wald Identity for Regenerations) *For the process* $Z(t)$ *with regenerative increments and finite* $a = E[Z(T_1)]$,

$$E[Z(T_{N(t)+1})] = aE[N(t) + 1], \qquad t \geq 0. \tag{2.38}$$

Proof. By Theorem 22,

$$E[Z(T_{N(t)+1})] = E\left[\sum_{n=0}^{N(t)} [Z(T_{n+1}) - Z(T_n)]\right] = a \cup (t).$$

By the classical SLLN, we know that

$$n^{-1}Z(T_n) = n^{-1}\sum_{k=1}^{n} \left[Z(T_k) - Z(T_{k-1})\right] \to E[Z(T_1)], \qquad \text{a.s. as } n \to \infty.$$

This extends to $Z(t)$ as follows, which is a special case of Corollary 14. Here

$$M_n = \sup_{T_{n-1} < t \leq T_n} |Z(t) - Z(T_{n-1})|, \qquad n \geq 1.$$

Theorem 54. *For the process $Z(t)$ with regenerative increments, suppose the mean of M_n is finite, and $E[T_1]$ and $a = E[Z(T_1)]$ exist, but are not both infinite. Then $t^{-1}Z(t) \to a/\mu$, a.s. as $t \to \infty$.*

The next result is a special case of Theorem 54 for a functional of a regenerative process, where the limiting average is expressible in terms of the limiting distribution of the regenerative process. The convergence of the expected value per unit time is also shown in (2.40); a refinement of this is given in Theorem 85 below.

Theorem 55. *Let $X(t)$ be a regenerative process over T_n with a metric state space S (e.g. $\mathbb{R}^d$), and let P denote the limiting distribution of $X(t)$ given by (2.30), where $\mu = E[T_1]$ is finite. Suppose $f : S \to \mathbb{R}$ is such that $\int_0^{T_1} |f(X(s))|ds$ and $|f(\overline{X})|$ have finite means, where $\overline{X}$ has the distribution P. Then*

$$\lim_{t\to\infty} t^{-1} \int_0^t f(X(s))ds = E[f(\overline{X})], \qquad a.s. \tag{2.39}$$

If, in addition, $E[T_1 \int_0^{T_1} |f(X(s))|ds]$ is finite, and T_1 has a non-arithmetic distribution, then

$$\lim_{t\to\infty} t^{-1} E\Big[\int_0^t f(X(s))ds\Big] = E[f(\overline{X})]. \tag{2.40}$$

Proof. Applying Theorem 54 to $Z(t) = \int_0^t f(X(s))ds$ and noting that

$$E[M_n] \le E[\int_0^{T_1} |f(X(s))|ds] < \infty,$$

we obtain $t^{-1}Z(t) \to E[Z(T_1)]/\mu$. Then (2.39) follows since by expression (2.29) for P,

$$\begin{aligned} E[Z(T_1)]/\mu &= \frac{1}{\mu} E\Big[\int_0^{T_1} f(X(s))ds\Big] \\ &= \int_S f(x)\mathrm{P}(dx) \;=\; E[f(\overline{X})]. \end{aligned}$$

To prove (2.40), note that $E[f(X(t))] \to E[f(\overline{X})]$ by Theorem 45. Then (2.40) follows by the fact that $t^{-1}\int_0^t g(s)ds \to c$ if $g(t) \to c$.

Remark 56. Limiting Averages as Expected Values. The limit (2.39) as an expected value is a common feature of many strong laws of large numbers when $f(X(t)) \xrightarrow{d} f(\overline{X})$. However, there are non-regenerative processes that satisfy the strong law (2.39), but not (2.40).

2.11 Average Sojourn Times in Regenerative Processes

We now show how SLLNs yield fundamental formulas, called Little laws, that relate the average sojourn times in queues to the average input rate and average queue length. We also present similar formulas for average sojourn times of a regenerative process in a region of its state space.

Consider a general service system or input-output system where discrete items (e.g., customers, jobs, data packets) are processed, or simply visit a location for a while. The items arrive to the system at times τ_n that form a point process $N(t)$ on $\mathbb{R}_+$ (it need not be a renewal process). Let W_n denote the total time the nth item spends in the system. Here the waiting or sojourn time W_n includes the item's service time plus any delay waiting in queue for service. The item exits the system at time $\tau_n + W_n$. Then the quantity of items in the system at time t is

$$Q(t) = \sum_{n=1}^{\infty} \mathbf{1}(\tau_n \le t < \tau_n + W_n), \qquad t \ge 0.$$

There are no assumptions concerning the processing or visits of the items or the stochastic nature of the variables W_n and τ_n, other than their existence. For instance, items may arrive and depart in batches, an item may reenter for multiple services, or the items may be part of a larger network that affects their sojourns.

We will consider the following three standard system performance parameters:

$$L = \lim_{t\to\infty} t^{-1} \int_0^t Q(s)ds \qquad \text{average quantity in the system,}$$
$$\lambda = \lim_{t\to\infty} t^{-1} N(t) \qquad \text{arrival rate,}$$
$$W = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} W_k \qquad \text{average waiting time.}$$

There are many diverse systems in which two of the averages L, λ, and W exist, and the issue is whether the third one exists. We will consider this issue under the following assumption, which is very natural for most queueing systems.

Empty-System Assumption. Let T_n denote the nth time at which an item arrives to an empty system, i.e., $Q(T_n-) = 0$ and $Q(T_n) > 0$. Assume the times T_n form a point process on $\mathbb{R}_+$ such that the limit $\mu = \lim_{n\to\infty} n^{-1} T_n$ exists and is positive.[4] This simply says that the system empties out infinitely often, and it does so at times that have a limiting average.

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[4] Keep in mind that the arrival process $N(t)$ is "not" the counting process associated with these empty times T_n.

Theorem 57. (Little Law) *Suppose the system described above satisfies the empty-system assumption. If any two of the averages L, λ or W exists, then the other one also exists, and $L = \lambda W$.*

Proof. With no loss in generality, we may assume the system is empty at time 0 and an item arrives. We begin with the key observation that in the time interval $[0, T_n)$, all of the $\nu_n = N(T_n-)$ items that arrive in the interval also depart by the empty-system time T_n, and their total waiting time is

$$\sum_{k=1}^{\nu_n} W_k = \sum_{k=1}^{\infty} \int_0^{T_n} \mathbf{1}(\tau_k \le s < \tau_k + W_k)ds = \int_0^{T_n} Q(s)ds. \tag{2.41}$$

The first equality follows since the system is empty just prior to T_n, and the second equality follows from the definition of $Q(t)$.

Also, observe that under the assumptions $t^{-1}N(t) \to \lambda$ and $T_n/n \to \mu$,

$$n^{-1}\nu_n = T_n^{-1}N(T_n-)(n^{-1}T_n) \to \lambda\mu. \tag{2.42}$$

First assume that λ and W exist. Then by (2.41), we have

$$n^{-1}\int_0^{T_n} Q(s)\,ds = (\nu_n/n)\nu_n^{-1}\sum_{k=1}^{\nu_n} W_k \to \lambda\mu W.$$

Therefore, an application of Theorem 13 to the nondecreasing process $Z(t) = \int_0^t Q(s)\,ds$ and the times T_n yields

$$L = \lim_{t\to\infty} t^{-1}Z(t) = \lambda W.$$

Next, assume that λ and L exist. Then by (2.41),

$$n^{-1}\sum_{k=1}^{\nu_n} W_k = (n^{-1}T_n)\Big(T_n^{-1}\int_0^{T_n} Q(s)ds\Big) \to \mu L, \qquad \text{a.s. as } t \to \infty.$$

Now, by a discrete-time version of Theorem 13 for the nondecreasing process $Z'_n = \sum_{k=1}^n W_k$ and integer-valued indices ν_n, which satisfy (2.42), it follows that

$$W = \lim_{n\to\infty} n^{-1}Z'_n = L/\lambda.$$

Thus, W exists and $L = \lambda W$.

Exercise 23 shows that if L and W exist then λ exists and $L = \lambda W$.

The preceding Little law applies to a wide variety of queueing systems as long as two of the averages λ, L or W exist. Here are a few examples.

Example 58. Regenerative Processing System. Suppose the system described above satisfies the empty-system assumption, the arrival process is a renewal process with a finite mean $1/\lambda$, and $Q(t)$ is a regenerative process over the empty-system times T_n. Assume that T_1 and $\int_0^{T_1} Q(s)ds$ have finite means.

By the SLLN for the renewal input process, the arrival rate is λ. Also, applying Theorem 54 to $Z(t) = \int_0^t Q(s)ds$, we have $L = E[\int_0^{T_1} Q(s)ds]/E[T_1]$. Therefore, by Theorem 57, the average waiting time W exists and $L = \lambda W$; that is, $W = E[\int_0^{T_1} Q(s)ds]/(\lambda E[T_1])$.

In some queueing systems, the Little law $L = \lambda W$ we have been discussing for averages has an analogue in which the averages are means.

Example 59. Little Laws for Means. Consider the system in the preceding example with the additional assumption that the sequence of sojourn times W_n is regenerative over the discrete times $\nu_n = N(T_n-)$. Since $Q(t)$ is regenerative over T_n, and W_n is regenerative over ν_n,

$$Q(t) \xrightarrow{d} \overline{Q} \text{ as } t \to \infty, \quad \text{and} \quad W_n \xrightarrow{d} \overline{W} \text{ as } n \to \infty,$$

where the distributions of $\overline{Q}$ and $\overline{W}$ are described in Theorem 45. Furthermore, by Theorem 55,

$$L = \lim_{t\to\infty} t^{-1} \int_0^t Q(s)ds = E[\overline{Q}] \quad \text{a.s.},$$
$$W = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} W_k = E[\overline{W}] \quad \text{a.s.}$$

Also, the renewal arrival rate λ can be represented as $\lambda = E[\tilde{N}(1)]$, where $\tilde{N}(t)$ is a stationary version of $N(t)$ as described in Theorem 76 below. Then the Little law $L = \lambda W$ that holds for averages has the following analogue for expected values:

$$E[\overline{Q}] = E[\tilde{N}(1)]E[\overline{W}].$$

Example 60. $G/G/1$ Queueing System. A general example of a regenerative queueing system is a $G/G/1$ system, where arrivals form a renewal process with mean inter-arrival time $1/\lambda$, the services times are i.i.d., independent of the arrivals, and customers are served by a single server under a first-in-first-out (FIFO) discipline. Assume that the mean service time is less than the mean inter-arrival time, and that T_1 and $\int_0^{T_1} Q(s)ds$ have finite means. In this case, the sojourn times W_n are regenerative over $\nu_n = N(T_n-)$, and W exists by Theorem 118 in Chapter 4. Then it follows by Theorem 57 that the average queue length L exists and $L = \lambda W$.

Special cases of the $G/G/1$ system are an $M/G/1$ system when the arrival process is a Poisson process, a $G/M/1$ system when the service times are exponentially distributed, and an $M/M/1$ system when the arrivals are Poisson and the service times are exponential.

Theorem 57 also yields expected waiting times in Jackson networks, which we discuss in Chapter 5.

There are several Little laws for input-output systems and general utility processes not related to queueing [101]. The next result is an elementary but very useful example.

Let $X(t)$ be a regenerative process over T_n with state space S. Assume $X(t)$ is a pure jump process (piecewise constant paths, etc.) with a limiting distribution $p(B) = \lim_{n\to\infty} P\{X(t) \in B\}$, which is known. Let B denote a fixed subset of the state space whose complement B^c is not empty. The expected number of times that $X(t)$ enters B between regenerations is

$$\gamma(B) = E\Big[\sum_n \mathbf{1}\Big(X(\tau_{n-1}) \in B^c, X(\tau_n) \in B, \tau_n \in (T_1, T_2]\Big)\Big],$$

where τ_n is the time of the nth jump of $X(t)$. The expected number of transitions of $X(t)$ between regenerations is $\gamma(S)$, which we assume is finite.

Consider the average sojourn time of $X(t)$ in B defined by

$$W(B) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} W_k(B),$$

where $W_n(B)$ is its sojourn time in B at its nth visit to the set.

Proposition 61. (Sojourns in Regenerative Processes) *For the regenerative process $X(t)$ defined above, its average sojourn time in B exists and is $W(B) = p(B)\gamma(S)/\gamma(B)$.*

Proof. Consider $Q(t) = \mathbf{1}(X(t) \in B)$ as an artificial queueing process that only takes values 0 or 1. Clearly $Q(t)$ is regenerative over T_n, since $X(t)$ is regenerative over T_n; and $Q(t)$ satisfies the empty-system assumption. Now, the limiting average of $Q(t)$ is

$$L = \lim_{t\to\infty} t^{-1} \int_0^t \mathbf{1}(X(s) \in B)\, ds = p(B).$$

The arrival rate λ is the rate $\gamma(B)/\gamma(S)$ at which $X(t)$ enters B. Thus, Theorem 57 yields $p(B) = \lambda W(B) = (\gamma(B)/\gamma(S))W(B)$, which proves the assertion.

2.12 Batch-Service Queueing System

For service systems that process items in batches, a basic problem is to determine when to serve batches and how many items should be in the batches. This is a dynamic control problem or a Markov decision problem. We will address this problem for a particular setting and show how to obtain certain control parameters by using a SLLN for regenerative processes.

Consider a single-server station that serves items or customers in batches as follows. Items arrive to the station according to a Poisson process with rate λ and they enter a queue where they wait to be served. The server can serve items in batches, and the number of items in a batch can be any number less than or equal to a fixed number $K \le \infty$ (the service capacity). The service times of the batches are independent, identically distributed and do not depend on the arrival process or the batch size (think of a computer, bus, or truck). Only one batch can be served at a time and, during a service, additional arrivals join the queue.

The server observes the queue length at the times at which an arrival occurs and the server is idle, or whenever a service is completed. At each of these observation times, the server takes one of the following actions:

• No items are served.

• A batch consisting of all or a portion of the items waiting is served (the batch size cannot exceed $i \wedge K$, where i is the queue length).

These actions control the batch sizes and the timing of the services. If the server takes the first action, the next control action is taken when the next item arrives, and if the server takes the second action to serve a batch, the next control action is taken when the service is completed. A *control policy* is a rule for selecting one of these actions at each observation time. The general problem is to find a control policy that minimizes the average cost (or discounted cost) of serving items over an infinite time horizon.

This Markov decision problem was solved in [33] for natural holding and service cost functions for both the average-cost and discounted-cost criteria. In either case, the main result is that there is an optimal M-policy of the following form: At each observation time when the queue length is i, do not serve any items if $i < M$, and serve a batch of $i \wedge K$ items if $i \ge M$. Here M is an "optimal" level that is a function of the costs.

We will now describe an optimal level M for a special case. Suppose the system is to operate under the preceding M-policy, where the capacity K is infinite, and the service times are exponentially distributed with rate γ. Assume there is cost C for serving a batch and a cost hi per unit time for holding i items in the queue.

Theorem 62. *Under the preceding assumptions, the average cost per unit time is minimized by setting the level M to be*

$$M = \min\{m \ge 0 : m(m+1) \ge 2[(\lambda/\gamma)^2 p^m + C\lambda/h]\}, \tag{2.43}$$

where $p = \lambda/(\lambda+\gamma)$.

Proof. Let $X_m(t)$ denote the number of items in the queue at time t, when the system is operated under an m-policy. Let T_n denote the time at which the server initiates the nth service terminates, and let $N(t)$ denote the associated counting process. For simplicity, assume that a service has just been completed at time 0, and let $T_0 = 0$.

We will show that, under the m-policy with exponential service times, the T_n are renewal times; and the service plus holding cost in $[0, t]$ is

$$Z_m(t) = CN(t) + h \int_0^t X_m(s)\, ds.$$

Next, we will establish the existence of the average cost

$$f(m) = \lim_{t\to\infty} t^{-1} Z_m(t),$$

and then show that $f(m)$ is minimized at the M specified in (2.43).

Let Q_n denote the number of items in the system at time T_n of the nth service completion, for $n \geq 0$. Note that Q_n is just the number of arrivals that occur during the nth service period, since all the waiting items are served in the batch. Because of the exponential service times, Q_n, for $n \geq 1$, are i.i.d. with

$$P\{Q_n = i\} = \int_{\mathbb{R}_+} \frac{(\lambda t)^i e^{-\lambda t}}{i!} \gamma e^{-\gamma t}\, dt = p^i(1-p), \qquad i \geq 0. \tag{2.44}$$

For notational convenience, assume the initial queue length Q_0 has this distribution and is independent of everything else.

Next, observe that the quantity Q_n determines the time $\xi_{n+1} = T_{n+1} - T_n$ until the next service initiation. Specifically, if $Q_n \geq m$, then ξ_{n+1} is simply a service time; and if $Q_n = i < m$, then ξ_{n+1} is the time it takes for $m - i$ more items to arrive plus a service time. Since the Q_n are i.i.d., it follows that T_n are renewal times. Furthermore, conditioning on Q_0, the inter-renewal distribution is

$$P\{\xi_1 \leq t\} = P\{Q_0 \geq m\} G_\gamma(t) + \sum_{i=0}^{m-1} P\{Q_0 = i\} G_\lambda^{(m-i)\star} \star G_\gamma(t),$$

where G_λ is an exponential distribution with rate λ.

Then using the distribution (2.44), the inter-renewal distribution and its mean (indexed by m) are:[5]

$$P\{\xi_1 \leq t\} = p^m G_\gamma(t) + (1-p) \sum_{i=0}^{m-1} p^i G_\lambda^{(m-i)\star} \star G_\gamma(t),$$
$$\mu_m = \gamma^{-1} + m\lambda^{-1} - (1-p^m)\gamma^{-1}.$$

Now, the increasing process $Z_m(t)$ is such that $Z_m(T_n) - Z_m(T_{n-1})$, for $n \geq 1$, are i.i.d. with mean

[5] The identity $\sum_{i=1}^k i p^{i-1} = \frac{d}{dp}(\sum_{i=0}^k p^i)$ is used to derive the formula for μ_m.

$$E[Z_m(T_1)] = C + hE\Big[\int_0^{T_1} X_m(s)\,ds\Big].$$

Then by Theorem 13, the average cost, as a function of m, is

$$f(m) = \lim_{t\to\infty} t^{-1} Z_m(t) = \mu_m^{-1} E[Z_m(T_1)].$$

To evaluate this limit, let $\tilde{N}(t)$ denote the Poisson arrival process with exponential inter-arrival times $\tilde{\xi}_n$, and let τ denote an exponential service time with rate γ. Then we can write

$$\int_0^{T_1} X_m(s)\,ds = Q_0\tau + \int_0^{\tau} \tilde{N}(s)\,ds + \sum_{i=0}^{m-1} \mathbf{1}(Q_0 = i) \sum_{k=1}^{m-i} (i+k-1)\tilde{\xi}_k. \quad (2.45)$$

The first two terms on the right-hand side represent the holding time of items during the service period, and the last term represents the holding time of items (which is 0 if $Q_0 \geq m$) prior to the service period. Then from the independence of Q_0 and τ and Exercise 14,

$$E\Big[\int_0^{T_1} X_m(s)\,ds\Big] = \Big[1/(1-p)\gamma + \lambda/\gamma^2 + (1-p)\lambda^{-1} \sum_{i=0}^{m-1} p^i \sum_{k=1}^{m-i} (i+k-1)\Big].$$

Substituting this in the expression above for $f(m)$, it follows from lengthy algebraic manipulations that

$$f(m+1) - f(m) = h(1-p^{m+1})D_m/(\lambda^2 \mu_m \mu_{m+1}),$$

where $D_m = m(m+1) - 2[(\lambda/\gamma)^2 p^m + C\lambda/h]$. Now, D_m is increasing in m and the other terms in the preceding display are positive. Therefore $f(m)$ is monotone decreasing and then increasing and has a unique minimum at $M = \min\{m : D_m \geq 0\}$, which is equivalent to (2.43).

Analysis similar to that above yields a formula for the optimal level M when the service capacity K is finite; see Exercise 52 in Chapter 4.

2.13 Central Limit Theorems

For a real-valued process $Z(t)$ with regenerative increments over T_n, we know that under the conditions in Theorem 54,

$$Z(t)/t \to a = E[Z(T_1)]/E[T_1] \quad \text{a.s.} \quad \text{as } t \to \infty.$$

In other words, $Z(t)$ behaves asymptotically like at. Further information about this behavior can be obtained by characterizing the limiting distribution of the difference $Z(t) - at$ as $t \to \infty$. We will now present a central limit theorem that gives conditions under which this limiting distribution is a normal distribution. Special cases of this result are CLT's for renewal and Markovian processes.

We will obtain the CLT for regenerative processes by applying the following classical CLT for sums of independent random variables (which is proved in standard probability texts). The analysis will involve the notion of convergence in distribution of random variables; see Section 6.9 in the Appendix.

Theorem 63. (Classical CLT) *Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with mean μ and variance $\sigma^2 > 0$, and define $S_n = \sum_{m=1}^{n}(X_m - \mu)$. Then*

$$P\{S_n/n^{1/2} \le x\} \to \int_{-\infty}^{x} \frac{e^{-y^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}\, dy, \qquad x \in \mathbb{R}.$$

This convergence in distribution is denoted by

$$S_n/n^{1/2} \xrightarrow{d} N(0, \sigma^2), \qquad \text{as } n \to \infty,$$

where $N(0, \sigma^2)$ is a normal random variable with mean 0 and variance σ^2.

We will also use the following result for randomized sums; see for instance p.216 in [26]. This result and the ones below are contained in the functional central limit theorems in Chapter 5, which focus on the convergence of entire stochastic processes instead of random variables.

Theorem 64. (Anscombe) *In the context of Theorem 63, let $N(t)$ be an integer-valued process defined on the same probability space as the X_n, where $N(t)$ may depend on the X_n. If $t^{-1}N(t) \xrightarrow{d} c$, where c is a positive constant, then*

$$S_{N(t)}/t^{1/2} \xrightarrow{d} N(0, c\sigma^2), \text{ as } t \to \infty.$$

The following is a regenerative analogue of the classical CLT.

Theorem 65. (Regenerative CLT) *Suppose $Z(t)$ is a real-valued process with regenerative increments over T_n such that $\mu = E[T_1]$ and $a = E[Z(T_1)]/\mu$ are finite. In addition, let*

$$M_n = \sup_{T_{n-1} < t \le T_n} |Z(t) - Z(T_{n-1})|, \quad n \ge 1,$$

and assume $E[M_1]$ and $\sigma^2 = Var[Z(T_1) - aT_1]$ are finite, and $\sigma > 0$. Then

$$(Z(t) - at)/t^{1/2} \xrightarrow{d} N(0, \sigma^2/\mu), \qquad \text{as } t \to \infty. \tag{2.46}$$

Proof. The process $Z(t)$ is "asymptotically close" to $Z(T_{N(t)})$, when dividing them by $t^{1/2}$, because their difference is bounded by $M_{N(t)+1}$, which is a regenerative process that is 0 at regeneration times. Consequently, the normalized process

$$\tilde{Z}(t) = (Z(t) - at)/t^{1/2}$$

should have the same limit as the process

$$Z'(t) = (Z(T_{N(t)}) - aT_{N(t)})/t^{1/2}.$$

Based on this conjecture, we will prove

$$Z'(t) \xrightarrow{d} N(0, \sigma^2/\mu), \qquad \text{as } t \to \infty, \tag{2.47}$$

$$|\tilde{Z}(t) - Z'(t)| \xrightarrow{d} 0, \qquad \text{as } t \to \infty. \tag{2.48}$$

Then it will follow by a standard property of convergence in distribution (see Exercise 53 of Chapter 5), that

$$\tilde{Z}(t) = Z'(t) + (\tilde{Z}(t) - Z'(t)) \xrightarrow{d} N(0, \sigma^2/\mu).$$

To prove (2.47), note that

$$Z'(t) = t^{-1/2} \sum_{n=1}^{N(t)} X_n,$$

where $X_n = Z(T_n) - Z(T_{n-1}) - a(T_n - T_{n-1})$. Since $Z(t)$ has regenerative increments over T_n, the X_n are i.i.d. with mean 0 and variance σ^2. Also, $t^{-1}N(t) \to 1/\mu$ by the SLLN for renewal processes. In light of these observations, Anscombe's theorem above yields (2.47).

To prove (2.48), note that

$$\tilde{Z}(t) - Z'(t) = t^{-1/2}[Z(t) - Z(T_{N(t)}) - a(t - T_{N(t)})].$$

Then letting $Y_n = M_n + a(T_{n+1} - T_n)$, it follows that

$$|\tilde{Z}(t) - Z'(t)| \le t^{-1/2} Y_{N(t)} = \sqrt{N(t)/t}\Big(N(t)^{-1/2} Y_{N(t)}\Big).$$

Since $Z(t)$ has regenerative increments, the Y_n are i.i.d., and so

$$n^{-1/2} Y_n \stackrel{d}{=} n^{-1/2} Y_1 \to 0 \quad \text{a.s.}$$

Using this and $N(t)/t \to 1/\mu$ a.s. in the preceding proves (2.48).

An important use of a CLT is to find confidence intervals for certain parameters. Here is an example.

Example 66. Confidence Interval for the Mean. Under the assumptions of Theorem 65, let us construct a confidence interval for the mean a of the regenerative-increment process $Z(t)$ based on observing the process up to a fixed time t. Assume (which is reasonable) that we do not know the variance parameter $\sigma^2 = \text{Var}[Z(T_1) - aT_1]$.

Note that by applications of SLLNs, it follows that

$$S(t^2) = (N(t)/t)\Big[N(t)^{-1}\sum_{k=1}^{N(t)}(Z(t_k) - Z(T_{k-1})^2 - (Z(t)/t)^2\Big] \to \sigma^2/\mu \text{ a.s.}$$

This combined with Theorem 65 yields

$$(Z(t) - at)/S(t)t^{1/2} \xrightarrow{d} N(0,1).$$

Then an approximate confidence interval for a with confidence coefficient $1-\alpha$ is

$$\Big[Z(t)/t - z_{\alpha/2}S(t)t^{-1/2},\ Z(t)/t + z_{\alpha/2}S(t)t^{-1/2}\Big],$$

where $P\{-z_{\alpha/2} \le N(0,1) \le z_{\alpha/2}\} = 1-\alpha$. This follows since, for large t, we have the approximation

$$\begin{aligned} 1-\alpha &\approx P\{-z_{\alpha/2} \le (Z(t) - at)/S(t)t^{1/2} \le z_{\alpha/2}\} \\ &= P\{Z(t)/t - z_{\alpha/2}S(t)t^{-1/2} \le a \le Z(t)/t + z_{\alpha/2}S(t)t^{-1/2}\}. \end{aligned}$$

Similar asymptotic confidence intervals for sums of i.i.d. variables are in [99].

Insights on simulation procedures for this and related models are in [43].

What would be an analogous confidence interval when $Z(t)$ is observed only at regeneration times? See Exercise 52.

Applying Theorem 65 to a regenerative-increment process involves determining conditions on the process under which the main assumptions are satisfied and then finding expressions for the normalization constants a and σ. Here are some examples.

Example 67. CLT for Renewal Processes. Suppose that $N(t)$ is a renewal process whose inter-renewal distribution has a finite mean μ and variance σ^2. Then $Z(t) = N(t)$ satisfies the assumptions in Theorem 65, and so

$$(N(t) - t/\mu)/t^{1/2} \xrightarrow{d} N(0, \sigma^2/\mu^3), \qquad \text{as } t \to \infty,$$

where

$$a = 1/\mu, \qquad \text{Var}[Z(T_1) - aT_1] = \sigma^2\mu^{-2}.$$

Example 68. CLT for Markov Chains. Let X_n be an ergodic Markov chain on S with limiting distribution π. Consider the sum

$$Z_n = \sum_{m=1}^{n} f(X_m), \qquad n \ge 0,$$

where $f(j)$ is a real-valued cost or utility for the process being in state j. For simplicity, fix an $i \in S$ and assume $X_0 = i$ a.s. Then Z_n has regenerative increments over the discrete times ν_n at which X_n enters state i. We will apply a discrete-time version of Theorem 65 to Z_n.

Accordingly, assume $\mu_i = E[\nu_1]$ and $E\Big[\max_{1\le n\le \nu_1}|Z_n|\Big]$ are finite. The latter is true when $E\Big[\sum_{n=1}^{\nu_1}|f(X_n)|\Big]$ is finite. In addition, assume

$$a = \frac{1}{\mu_i}E_i[Z_{\nu_1}] = \sum_{j\in S}\pi_j f(j), \quad \text{and} \quad \sigma^2 = \frac{1}{\mu_i}\text{Var}[Z_{\nu_1} - a\nu_1]$$

are finite and $\sigma > 0$. Letting $\tilde f(j) = f(j) - a$, Exercise 54 shows that

$$\sigma^2 = E[\tilde f(X_0)^2] + 2\sum_{n=1}^{\infty} E[\tilde f(X_0)\tilde f(X_n)], \tag{2.49}$$

where $P\{X_0 = i\} = \pi_i$. Then Theorem 65 (in discrete time) yields

$$(Z_n - an)/n^{1/2} \xrightarrow{d} N(0,\sigma^2), \qquad \text{as } n \to \infty. \tag{2.50}$$

This result also applies to random functions of Markov chains as follows (see Exercise 33 in Chapter 4 for a related continuous-time version). Suppose

$$Z_n = \sum_{m=1}^{n} f(X_m, Y_m), \qquad n \ge 0,$$

where $f : S \times S' \to \mathbb{R}$, and Y_m are conditionally independent given X_n ($n \ge 0$), and $P\{Y_m \in B | X_n, n \ge 0\}$ only depends on X_m and $B \in \mathcal{S}'$. Here S' need not be discrete. In this setting, the cost or utility $f(X_m, Y_m)$ at time m is partially determined by the auxiliary or environmental variable Y_m. Then the argument above yields the CLT (2.50). In this case, $a = \sum_{j\in S}\pi_j\alpha(j)$, $\alpha(j) = E[f(j, Y_1)]$, and

$$\sigma^2 = E[(f(X_0, Y_1) - \alpha(X_0))^2] + 2\sum_{n=1}^{\infty} E[m(X_0, X_n)],$$

where $m(j,k) = E[(f(j,Y_1) - \alpha(j))(f(k,Y_2) - \alpha(k))]$.

2.14 Terminating Renewal Processes

In this section, we discuss renewal processes that terminate after a random number of renewals. Analysis of these terminating (or transient) renewal processes uses renewal equations and the key renewal theorem applied a little differently than above.

Consider a sequence of renewal times T_n with inter-renewal distribution F. Suppose that at each time T_n (including $T_0 = 0$), the renewals terminate with probability $1-p$, or continue until the next renewal epoch with probability p. These events are independent of the preceding renewal times, but may depend on the future renewal times.

Under these assumptions, the total number of renewals ν over the entire time horizon $\mathbb{R}_+$ has the distribution

$$P\{\nu \geq n\} = p^n, \qquad n \geq 0,$$

and $E[\nu] = p/(1-p)$. The number of renewals in $[0,t]$ is

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}(T_n \leq t, \nu \geq n), \qquad t \geq 0.$$

Of course $N(t) \to \nu$ a.s. Another quantity of interest is the time T_ν at which the renewals terminate.

We will also use the following equivalent formulation of this terminating renewal process. Assume that $N(t)$ counts renewals in which the independent inter-renewal times have an *improper* distribution $G(t)$, with $p = G(\infty) < 1$. Then p is the probability of another renewal and $1-p = 1-G(\infty)$ is the probability that an inter-renewal time is "infinite", which terminates the renewals. This interpretation is consistent with that above since necessarily $G(t) = pF(t)$, where F as described above is the conditional distribution of an inter-renewal time given that it is allowed (or is finite).

Similarly to renewal processes, we will address issues about the process $N(t)$ with the use of its *renewal function*

$$V(t) = \sum_{n=0}^{\infty} G^{n\star}(t) = \sum_{n=0}^{\infty} p^n F^{n\star}(t).$$

We first observe that the counting process $N(t)$ and the termination time T_ν are finite a.s., and their distributions and means are

$$\begin{aligned} P\{N(t) \geq n\} &= G^{n\star}(t), \qquad & E[N(t)] &= V(t) - 1, \\ P\{T_\nu \leq t\} &= (1-p)V(t), \qquad & E[T_\nu] &= p\mu/(1-p). \end{aligned} \tag{2.51}$$

To establish these formulas, recall that the events $\nu = n$ (to terminate at n) and $\nu > n$ (to continue to the $n+1$st renewal) are assumed to be independent of $T_1, \ldots, T_n$. Then

$$\begin{aligned} P\{N(t) \geq n\} = P\{\nu \geq n, T_n \leq t\} &= p^n F^{n\star}(t) = G^{n\star}(t), \\ E[N(t)] = \sum_{n=1}^{\infty} P\{N(t) \geq n\} &= V(t) - 1. \end{aligned}$$

Similarly, using the independence and $T_\nu = \sum_{n=0}^{\infty} \mathbf{1}(\nu = n) T_n$,

$$P\{T_\nu \le t\} = \sum_{n=0}^{\infty} P\{\nu = n, T_n \le t\} \; = \; (1-p)\sum_{n=0}^{\infty} p^n F^{n\star}(t),$$
$$E[T_\nu] = \sum_{n=1}^{\infty} P\{\nu = n\} E[T_n] \; = \; \mu p/(1-p).$$

Although a regular renewal function tends to infinity, the renewal function for a terminating process has a finite limit.

Remark 69. As $t \to \infty$

$$V(t) = \sum_{n=0}^{\infty} p^n F^{n*}(t) \to 1/(1-p).$$

Corollary 71 below describes the convergence rate.

We will now discuss limits of certain functions associated with the terminating renewal process. As in Proposition 31, it follows that $H(t) = V \star h(t)$ is the unique solution to the renewal equation

$$H(t) = h(t) + G \star H(t).$$

We will consider the limiting behavior of $H(t)$ for the case in which the limit

$$h(\infty) = \lim_{t\to\infty} h(t)$$

exists, which is common in applications. Since $V(t) \to 1/(1-p)$ and $h(t)$ is bounded on compact sets and converges to $h(\infty)$, it follows by dominated convergence that

$$\begin{aligned} H(t) = h \star V(t) \; &= \; h(\infty)V(t) + \int_{[0,t]} [h(t-s) - h(\infty)]dV(s) \\ &\to h(\infty)/(1-p) \qquad \text{as } t \to \infty. \end{aligned} \tag{2.52}$$

The next result describes the rate of this convergence under a few more technical conditions. Assume there is a positive β such that

$$\int_{\mathbb{R}_+} e^{\beta t} dG(t) = 1.$$

The existence of a unique β is guaranteed under the weak condition that $\int_{\mathbb{R}_+} e^{\beta t} dG(t)$ is finite for some $\beta > 0$. Indeed, this function of β is continuous and increasing and, being finite at one point, its range contains the set $[p, \infty)$; thus, it must equal 1 for some β. We also assume the distribution

$$F^{\#}(t) = \int_{[0,t]} e^{\beta s} dG(s)$$

is non-arithmetic and has a mean $\mu^{\#}$.

Theorem 70. *In addition to the preceding assumptions, assume the function* $e^{\beta t}[h(t) - h(\infty)]$ *is DRI. Then*

$$H(t) = h(\infty)/(1-p) + ce^{-\beta t}/\mu^{\#} + o(e^{-\beta t}), \quad \text{as } t \to \infty, \tag{2.53}$$

where $c = \int_{\mathbb{R}_+} e^{\beta s}[h(s) - h(\infty)]\, ds - h(\infty)/\beta$.

Proof. Multiplying the renewal equation $H = h + G \star H$ by $e^{\beta t}$ yields the renewal equation $H^{\#} = h^{\#} + F^{\#} \star H^{\#}$ where $H^{\#}(t) = e^{\beta t}H(t)$ and $h^{\#}(t) = e^{\beta t}h(t)$.

We can now describe the limit of $H(t) - h(\infty)/(1-p)$ by the limit of $H^{\#}(t) - v(t)$, where $v(t) = e^{\beta t}h(\infty)/(1-p)$. From Lemma 83 below,

$$H^{\#}(t) = v(t) + \frac{1}{\mu^{\#}} \int_{\mathbb{R}_+} \overline{h}(s)\, ds + o(1), \quad \text{as } t \to \infty, \tag{2.54}$$

provided $\overline{h}(t) = h^{\#}(t) - v(t) + F^{\#} \star v(t)$ is DRI. In this case,

$$\overline{h}(t) = e^{\beta t}[h(t) - h(\infty)] - \Big[\frac{h(\infty)e^{\beta t}}{1-p}(p - G(t))\Big]. \tag{2.55}$$

Now, the first term on the right-hand side is DRI by assumption. Also,

$$e^{\beta t}(p - G(t)) \le \int_{(t,\infty)} e^{\beta s} dG(s) = 1 - F^{\#}(t).$$

This bound is decreasing to 0 and its integral is $\mu^{\#}$, and so the last term in brackets in (2.55) is DRI. Thus $\overline{h}(t)$ is DRI. Finally, an easy check shows that $\int_{\mathbb{R}_+} \overline{h}(s)\, ds = c$, the constant in (2.53). Substituting this in (2.54) and dividing by $e^{\beta t}$ yields (2.53).

Corollary 71. *Under the assumptions preceding Theorem 70,*

$$V(t) = 1/(1-p) - e^{-\beta t}/(\beta\mu^{\#}) + o(e^{-\beta t}),$$
$$P\{T_\nu > t\} = (1-p)e^{-\beta t}/(\beta\mu^{\#}) + o(e^{-\beta t}), \quad \text{as } t \to \infty.$$

Proof. The first line follows by Theorem 70 with $h(t) = 1$, since by its definition, $V(t) = 1 + G \star V(t)$. The second follows from the first line and (2.51).

Example 72. Waiting Time for a Gap in a Poisson Process. Consider a Poisson process with rate λ that terminates at the first time a gap of size $\ge c$ occurs. That is, the termination time is T_ν, where $\nu = \min\{n : \xi_{n+1} \ge c\}$, where $\xi_n = T_n - T_{n-1}$ and T_n are the occurrence times of the Poisson process. Now, at each time T_n, the process either terminates if $\xi_{n+1} \ge c$, or it continues until the next renewal epoch if $\xi_{n+1} < c$. These events are clearly independent of $T_1, \ldots, T_n$.

Under these assumptions, the probability of terminating is

$$1 - p = P\{\xi_{n+1} \geq c\} = e^{-\lambda c}.$$

The conditional distribution of the next renewal period beginning at T_n is

$$F(t) = P\{\xi_{n+1} \leq t | \xi_{n+1} < c\} = p^{-1}(1 - e^{-\lambda t}), \qquad 0 \leq t \leq c.$$

Then from (2.51), the distribution and mean of the waiting time for a gap of size c are

$$P\{T_\nu \leq t\} = e^{-\lambda c} V(t), \qquad E[T_\nu] = (e^{\lambda c} - 1)/\lambda.$$

Now, assume $\lambda c > 1$. Then the condition $\int_{\mathbb{R}_+} e^{\beta t} p dF(t) = 1$ above for defining β reduces to $\lambda e^{(\beta - \lambda)c} = \beta$, for $\beta < \lambda$. Such a β exists as in Figure 1.3 in Chapter 1 for the branching model. Using this formula and integration by parts, we have

$$\mu^{\#} = \int_{[0,c]} t e^{\beta t} p dF(t) = (c\beta - 1)/(\beta - \lambda).$$

Then by Corollary 71,

$$P\{T_\nu > t\} = \Big(\frac{1 - \beta/\lambda}{1 - \beta c}\Big) e^{-\beta(t+c)} + o(e^{-\beta t}), \quad \text{as } t \to \infty.$$

Example 73. Cramér-Lundberg Risk Model. Consider an insurance company that receives capital at a constant rate c from insurance premiums, investments, interest etc. The company uses the capital to pay claims that arrive according to a Poisson process $N(t)$ with rate λ. The claim amounts $X_1, X_2, \ldots$ are i.i.d. positive random variables with mean μ, and are independent of the arrival times. Then the company's capital at time t is

$$Z_x(t) = x + ct - \sum_{n=1}^{N(t)} X_n, \qquad t \geq 0,$$

where x is the capital at time 0.

An important performance parameter of the company is the probability

$$R(x) = P\{Z_x(t) \geq 0, t \geq 0\},$$

that the capital does not go negative (the company is not ruined). We are interested in approximating this survival probability when the initial capital x is large. Exercise 25 shows that $R(x) = 0$, regardless of the initial capital x, when $c < \lambda\mu$ (the capital input rate is less than the payout rate).

We will now consider the opposite case $c > \lambda\mu$. Conditioning on the time and size of the first claim, one can show (e.g., see [37, 92, 94]) that $R(x)$ satisfies a certain differential equation whose corresponding integral equation is the renewal equation

$$R(x) = R(0) + R \star G(x), \tag{2.56}$$

where $R(0) = 1 - \lambda\mu/c$ and

$$G(y) = \lambda c^{-1} \int_0^y P\{X_1 > u\}\, du.$$

The G is a defective distribution with $G(\infty) = \lambda\mu/c < 1$. Then applying (2.52) to $R(x) = h \star V(x) = R(0)V(x)$, we have

$$R(x) \to R(0)/(1 - \lambda\mu/c) = 1, \qquad \text{as } x \to \infty.$$

We now consider the rate at which the "ruin" probability $1 - R(x)$ converges to 0 as $x \to \infty$. Assume there is a positive β such that

$$\lambda c^{-1} \int_{\mathbb{R}_+} e^{\beta x} P\{X_1 > x\}\, dx = 1,$$

and that

$$\mu^{\#} = \lambda c^{-1} \int_{\mathbb{R}_+} x e^{\beta x} P\{X_1 > x\}\, dx < \infty.$$

Then by Theorem 70 (with $R(x)$, $R(0)$ in place of $H(t)$, $h(t)$), the probability of ruin has the asymptotic form

$$1 - R(x) = \frac{1}{\beta\mu^{\#}}(1 - \lambda\mu/c)e^{-\beta x} + o(e^{-\beta x}), \quad \text{as } x \to \infty.$$

2.15 Stationary Renewal Processes

Recall that a basic property of an ergodic Markov chain is that it is stationary if the distribution of its state at time 0 is its stationary distribution (which is also its limiting distribution). This section addresses the analogous issue of determining an appropriate starting condition for a delayed renewal process so that its increments are stationary in time.

We begin by defining the notion of stationarity for stochastic processes and point processes. A continuous-time stochastic process $\{X(t) : t \geq 0\}$ on a general space is *stationary* if its finite-dimensional distributions are invariant under any shift in time: for each $0 \leq s_1 < \ldots < s_k$ and $t \geq 0$,

$$(X(s_1 + t), \ldots, X(s_k + t)) \stackrel{d}{=} (X(s_1), \ldots, X(s_k)). \tag{2.57}$$

Remark 74. A Markov process $X(t)$ is stationary if $X(t) \stackrel{d}{=} X(0)$, $t \geq 0$. This simpler criterion follows as in the proofs of Proposition 52 in Chapter 1 and Exercise 55.

Now, consider a point process $N(t) = \sum_n \mathbf{1}(\tau_n \leq t)$ on $\mathbb{R}_+$, with points at $0 < \tau_1 < \tau_2 < \ldots$ Another way of representing this process is by the family $N = \{N(B) : B \in \mathbb{B}_+\}$, where $N(B) = \sum_n \mathbf{1}(\tau_n \in B)$ is the number of points τ_n in the Borel set B. We also define $B + t = \{s + t : s \in B\}$. The process N is *stationary* (i.e., it has *stationary increments*) if, for any $B_1, \ldots, B_k \in \mathbb{B}_+$,

$$(N(B_1 + t), \ldots, N(B_k + t)) \stackrel{d}{=} (N(B_1), \ldots, N(B_k)), \qquad t \geq 0. \tag{2.58}$$

A basic property of a stationary point process is that its mean value function is linear.

Proposition 75. If N is a stationary point process and $E[N(1)]$ is finite, then $E[N(t)] = tE[N(1)]$, $t \geq 0$.

Proof. To see this, consider

$$E[N(s+t)] = E[N(s)] + E[N(s+t) - N(s)] = E[N(s)] + E[N(t)].$$

This is a linear equation $f(s+t) = f(s) + f(t)$, $s, t \geq 0$. The only nondecreasing function that satisfies this linear equation is $f(t) = ct$ for some c. In our case, $c = f(1) = E[N(1)]$, and hence $E[N(t)] = tE[N(1)]$.

We are now ready to characterize stationary renewal processes. Assume that $N(t)$ is a delayed renewal process, where the distribution of ξ_1 is G, and the distribution of ξ_n, $n \geq 2$, is F, which has a finite mean μ. The issue is how to select the initial distribution G such that N is stationary. The answer, according to (iv) below, is to select G to be F_e, which is the limiting distribution of the forward and backward recurrence times for a renewal process with inter-renewal distribution F. The following result also shows that the stationarity of N is equivalent to the stationarity of its forward recurrence time process.

Theorem 76. *The following statements are equivalent.*
(i) *The delayed renewal process N is stationary.*
(ii) *The forward recurrence time process $B(t) = T_{N(t)+1} - t$ is stationary.*
(iii) *$E[N(t)] = t/\mu$, for $t \geq 0$.*
(iv) *$G(t) = F_e(t) = \frac{1}{\mu}\int_0^t [1 - F(s)]ds$.*
When these statements are true, $P\{B(t) \leq x\} = F_e(x)$, for $t, x \geq 0$.

Proof. (i) $\Leftrightarrow$ (ii): Using $T_n = \inf\{u : N(u) = n\}$, we have

$$\begin{aligned} B(t) = T_{N(t)+1} - t &= \inf\{u - t : N(u) = N(t) + 1\} \\ &\stackrel{d}{=} \inf\{t' : N((0, t'] + t) = 1\}. \end{aligned} \tag{2.59}$$

Consequently, the stationarity property (2.58) of N implies $B(t) \stackrel{d}{=} B(0)$, $t \geq 0$. Then B is stationary by Remark 74, because it is a Markov process (Exercise 55).

Conversely, since N counts the number of times $B(t)$ jumps upward,

$$N(A+t) = \sum_{u \in A} \mathbf{1}(B(u+t) > B((u+t)-)). \tag{2.60}$$

Therefore, the stationarity of B implies N is stationary.

(i) $\Rightarrow$ (iii): If N is stationary, Proposition 75 ensures $E[N(t)] = tE[N(1)]$. Also, $E[N(1)] = 1/\mu$ since $t^{-1}E[N(t)] \to 1/\mu$ by Proposition 32. Therefore, $E[N(t)] = t/\mu$.

(iii) $\Rightarrow$ (iv): Assume $E[N(t)] = t/\mu$. Exercise 53 shows $U \star F_e(t) = t/\mu$, and so $E[N(t)] = U \star F_e(t)$. Another expression for this expectation is

$$E[N(t)] = \sum_{n=1}^{\infty} G \star F^{(n-1)\star}(t) = G \star U(t).$$

Equating these expressions, we have $U \star F_e(t) = G \star U(t)$. Taking the Laplace transform of this equality yields

$$\hat{U}(\alpha)\hat{F}_e(\alpha) = \hat{G}(\alpha)\hat{U}(\alpha), \tag{2.61}$$

where the hat symbol denotes Laplace transform; e.g., $\hat{G}(\alpha) = \int_{\mathbb{R}_+} e^{-\alpha t} dG(t)$. By Proposition 20, we know $\hat{U}(\alpha) = 1/(1 - \hat{F}(\alpha))$ is positive. Using this in (2.61) yields $\hat{F}_e(\alpha) = \hat{G}(\alpha)$. Since these Laplace transforms uniquely determine the distributions, we obtain $G = F_e$.

(iv) $\Rightarrow$ (ii): By direct computation as in Exercise 37, it follows that

$$P\{B(t) > x\} = 1 - G(t+x) + \int_{[0,t]} [1 - F(t+x-s)] dV(s), \tag{2.62}$$

where $V(t) = E[N(t)] = G \star U(t)$. Now, the assumption $G = F_e$, along with $U \star F_e(t) = t/\mu$ from Exercise 53, yield

$$V(t) = G \star U(t) = U \star G(t) = U \star F_e(t) = t/\mu.$$

Using this in (2.62), along with a change of variable in the integral, we have

$$P\{B(t) > x\} = 1 - G(t+x) + F_e(x+t) - F_e(x). \tag{2.63}$$

Since $G = F_e$, this expression is simply $P\{B(t) > x\} = 1 - F_e(x)$, $t \geq 0$. Thus, the distribution of $B(t)$ is independent of t. This condition is sufficient for $B(t)$ to be stationary since it is a Markov process (see Exercise 55).

Example 77. Suppose the inter-renewal distribution for the delayed renewal process N is the beta distribution

$$F(t) = 30 \int_0^t s^2(1-s)^2\, ds, \qquad t \in [0,1].$$

The equilibrium distribution associated with F is clearly

$$F_e(t) = 2t - 5t^4 + 6t^5 - 2t^6, \qquad t \in [0,1].$$

Then by Theorem 76, N is stationary if and only if $G = F_e$.

One consequence of Theorem 76 is that Poisson processes are the only non-delayed renewal processes (whose inter-renewal times have a finite mean) that are stationary.

Corollary 78. *The renewal process $N(t)$ with no delay, and whose inter-renewal times have a finite mean, is stationary if and only if it is a Poisson process.*

Proof. By Theorem 76 (vi), $N(t)$ is stationary if and only if $E[N(t)] = t/\mu$, $t \geq 0$, which is equivalent to $N(t)$ being a Poisson process by Remark 21.

An alternate proof is to apply Theorem 76 (iii) and use the fact (Exercise 4 in Chapter 3) that $F = F_e$ if and only if F is an exponential distribution.

Here is another useful stationarity property.

Remark 79. If $N(t)$ is a stationary renewal process, then

$$E\Big[\sum_{n=1}^{N(t)} f(T_n)\Big] = \frac{1}{\mu}\int_0^t f(s)ds.$$

This follows by Theorem 22 and $E[N(t)] = t/\mu$.

Many stationary processes arise naturally as functions of stationary processes (two examples are in the proof of Theorem 76). A general statement to this effect is as follows; it is a consequence of the definition of stationarity.

Remark 80. Hereditary Property of Stationarity. Suppose $X(t)$ is a stationary process. Then the process $Y(t) = f(X(t))$ is also stationary, where f is a function on the state space of X to another space. More generally, $Y(t) = g(\{X(s+t) : s \geq 0\})$ is stationary, where g is a function on the space of sample paths of X to some space. Analogously, N is a stationary point process if, for any bounded set B and $t > 0$,

$$N(B+t) = g(\{X(s+t) : s \geq 0\}, B) \tag{2.64}$$

(see for instance (2.59) and (2.60)).

Example 81. Let $X(t)$ be a delayed regenerative process (e.g., a continuous-time Markov chain as in Chapter 4) over the times $0 < T_1 < T_2 < \ldots$ at which $X(t)$ enters a special state x^*. Let N denote the point process of these times. If $X(t)$ is stationary, then N is a stationary renewal process. This follows since, like (2.64),

$$N(B+t) = \sum_{s \in B} \mathbf{1}(X((s+t)-) \neq x^*, X(s+t) = x^*).$$

Although the bounded set B may be uncountable, only a finite number of its values will contribute to the sum.

Because a stationary renewal process $N(t)$ has a stationary forward recurrence time process, it seems reasonable that the backward recurrence time process $A(t) = t - T_{N(t)}$ would also be stationary. This is not true, since the distribution of $A(t)$ is not independent of t; in particular, $A(t) = t$, for $t < T_1$. However, there is stationarity in the following sense.

Remark 82. Stationary Backward Recurrence Time Process. Suppose the stationary renewal process is extended to the negative time axis with (artificial or virtual) renewals at times $\ldots < T_{-1} < T_0 < 0$. One can think of the renewals occurring since the beginning of time at $-\infty$. Consistent with the definition above, the backward recurrence process is

$$A(t) = t - T_n, \qquad \text{if } t \in [T_n, T_{n+1}), \text{ for some } n \in \mathbb{R}.$$

Assuming N is stationary on $\mathbb{R}_+$, the time $A(0) = T_1$ to the first renewal has the distribution F_e. Then one can show, as we proved (i) $\Leftrightarrow$ (ii) in Theorem 76, that the process $\{A(t) : t \in \mathbb{R}\}$ is stationary with distribution F_e.

2.16 Refined Limit Laws

We will now describe applications of the key renewal theorem for functions that are not asymptotically constant.

The applications of the renewal theorem we have been discussing are for limits of functions $H(t) = U \star h(t)$ that converge to a constant (i.e., $H(t) = c + o(1)$). However, there are many situations in which $H(t)$ tends to infinity, but the key renewal theorem can still be used to describe limits of the form $H(t) = v(t) + o(1)$ as $t \to \infty$, where the function $v(t)$ is the asymptotic value of $H(t)$.

For instance, a SLLN $Z(t)/t \to b$ suggests $E[Z(t)] = bt + c + o(1)$ might be true, where the constant c gives added information on the convergence. In this section, we discuss such limit theorems.

We first note that an approach for considering limits $H(t) = v(t) + o(1)$ is simply to consider a renewal equation for the function $H(t) - v(t)$ as follows.

Lemma 83. *Suppose $H(t) = U \star h(t)$ is a solution of a renewal equation for a non-arithmetic distribution F, and $v(t)$ is a real-valued function on $\mathbb{R}$ that* *is bounded on finite intervals and is* 0 *for negative* t*. Then*

$$H(t) = v(t) + \frac{1}{\mu}\int_{\mathbb{R}_+} \overline{h}(s)\,ds + o(1), \quad \text{as } t \to \infty, \tag{2.65}$$

provided $\overline{h}(t) = h(t) - v(t) + F \star v(t)$ *is DRI. In particular, for a linear function* $v(t) = bt$*,*

$$H(t) = bt + \frac{b(\sigma^2 + \mu^2)}{2\mu} + \frac{1}{\mu}\int_{\mathbb{R}_+} (h(s) - b\mu)\,ds + o(1), \quad \text{as } t \to \infty, \tag{2.66}$$

where σ^2 *is the variance of* F*, provided* $h(t) - b\mu$ *is DRI.*

Proof. Clearly $H - v$ satisfies the renewal equation

$$H - v = (h - v + F \star v) + F \star (H - v).$$

Then $H - v = U \star \overline{h}$ by Proposition 31, and its limit (2.65) is given by the key renewal theorem.

Next, suppose $v(t) = bt$ and $h(t) - b\mu$ is DRI. Then using $\mu = \int_{\mathbb{R}_+}[1 - F(x)]\,dx$ and the change of variable $x = t - s$ in the integral below, we have

$$\begin{aligned}\overline{h}(t) &= h(t) - bt + b\int_0^t F(t-s)\,ds \\ &= h(t) - b\mu + bg(t),\end{aligned}$$

where $g(t) = \int_t^\infty [1 - F(x)]\,dx$. Now $g(t)$ is continuous and decreasing and

$$\int_0^\infty g(t)\,dt = \frac{1}{2}\int_{\mathbb{R}_+} t^2 dF(t) = \frac{\sigma^2 + \mu^2}{2}. \tag{2.67}$$

Then $g(t)$ is DRI by Proposition 88 (a), and hence $\overline{h}(t) = h(t) - b\mu + bg(t)$ is DRI. Thus, by what we already proved, (2.65) is true but it reduces to (2.66) in light of (2.67).

Our first use of the preceding result is a refinement of $t^{-1}U(t) \to 1/\mu$ from Proposition 32.

Proposition 84. *If* $N(t)$ *is a renewal process whose inter-renewal times have a non-arithmetic distribution with mean* μ *and variance* σ^2*, then*

$$U(t) = t/\mu + (\sigma^2 + \mu^2)/2\mu^2 + o(1), \qquad \text{as } t \to \infty.$$

Proof. This follows by Lemma 83 with $H(t) = U(t)$, $h(t) = 1$, and $v(t) = t/\mu$ (that $h(t) - b\mu$ is DRI need not be verified since it equals 0).

We will now apply Lemma 83 to a real-valued stochastic process $Z(t)$ whose sample paths are right-continuous with left-hand limits. Assume that $Z(t)$, has *crude regenerative increments at* T in the sense that

$$E[Z(T+t) - Z(T)|T] = E[Z(t)], \qquad t \geq 0. \tag{2.68}$$

If $Z(t)$ has regenerative increments over T_n, then $Z(t)$ has crude regenerative increments at T_1.

Theorem 85. *For the process* $Z(t)$ *defined above, let*

$$M = \sup\{|Z(T) - Z(t)| : t \leq T\}.$$

If the expectations of M, MT, T^2, $|Z(T)|$, *and* $\int_0^T |Z(s)|ds$ *are finite, then*

$$E[Z(t)] = at/\mu + a(\sigma^2 + \mu^2)/2\mu^2 + c + o(1), \qquad \text{as } t \to \infty, \tag{2.69}$$

where $a = E[Z(T)]$ *and* $c = \frac{1}{\mu}E\Big[\int_0^T Z(s)ds - TZ(T)\Big]$.

Proof. Because $Z(t)$ has crude regenerative increments, it would be natural that that $t^{-1}E[Z(t)] \to a/\mu$. So to prove (2.69), we will apply Lemma 83 with $v(t) = at/\mu$.

We first derive a renewal equation for $E[Z(t)]$. Conditioning on T,

$$E[Z(t)] = E[Z(t)\mathbf{1}(T > t)] + \int_{[0,t]} E[Z(t)|T = s]dF(s).$$

Using $E[Z(t)|T = s] = E[Z(t-s)] + E[Z(s)|T = s]$ from assumption (2.68) and some algebra, it follows that the preceding is a renewal equation $H = h + F \star H$, where $H(t) = E[Z(t)]$ and

$$h(t) = a + E\Big[(Z(t) - Z(T))\mathbf{1}(T > t)\Big].$$

Now, by Lemma 83 for $v(t) = at/\mu$, we have

$$E[Z(t)] = at/\mu + \frac{\sigma^2 + \mu^2}{2\mu^2} + \frac{1}{\mu}\int_{\mathbb{R}_+} g(s)\,ds + o(1), \quad \text{as } t \to \infty, \tag{2.70}$$

provided $g(t) = h(t) - a = E\Big[(Z(t) - Z(T))\mathbf{1}(T > t)\Big]$ is DRI. Clearly

$$|g(t)| \leq b(t) = E[M\mathbf{1}(T > t)].$$

Now, $b(t) \downarrow 0$; and as in (2.25), $\int_{\mathbb{R}_+} b(s)ds = E[MT]$ is finite. Then $b(t)$ is DRI by Proposition 88 (a). Hence $g(t)$ is also DRI by Proposition 88 (c). Finally, observe that

$$\int_{\mathbb{R}_+} g(t)\,dt = E\Big[\int_0^T Z(s)ds - TZ(T)\Big].$$

Substituting this formula in (2.70) proves (2.69).

2.17 Proof of the Key Renewal Theorem*

This section proves the key renewal theorem by applying Blackwell's theorem, which is proved in the next section.

The key renewal theorem involves real-valued functions that are integrable on the entire axis $\mathbb{R}_+$ as follows.

Definition 86. Similarly to the definition of a Riemann integral on a finite interval, it is natural to approximate the integral of a real-valued function $h(t)$ on the entire domain $\mathbb{R}_+$ over a grid $0, \delta, 2\delta, \ldots$ by the upper and lower Riemann sums

$$I^\delta(h) = \delta \sum_{k=0}^{\infty} \sup\{h(s) : k\delta \le s < (k+1)\delta\},$$
$$I_\delta(h) = \delta \sum_{k=0}^{\infty} \inf\{h(s) : k\delta \le s < (k+1)\delta\}.$$

The function $h(t)$ is *directly Riemann integrable* (DRI) if $I^\delta(h)$ and $I_\delta(h)$ are finite for each δ, and they both converge to the same limit as $\delta \to 0$. The limit is necessarily the usual Riemann integral

$$\int_{\mathbb{R}_+} h(s)ds = \lim_{t\to\infty} \int_0^t h(s)ds,$$

where the last integral is the limit of the Riemann sums on $[0, t]$.

A DRI function is clearly Riemann integrable in the usual sense, but the converse is not true; see Exercise 28. From the definition, it is clear that $h(t)$ is DRI if it is Riemann integrable and it is 0 outside a finite interval. Also, $h(t)$ is DRI if and only if its positive and negative parts $h^+(t)$ and $h^-(t)$ are both DRI. Further criteria for DRI are given in Proposition 88 and Exercise 33.

We are now ready for the main result.

Theorem 87. (Key Renewal Theorem) *If $h(t)$ is DRI and F is non-arithmetic, then*

$$\lim_{t\to\infty} U \star h(t) = \frac{1}{\mu} \int_{\mathbb{R}_+} h(s)ds.$$

Proof. Fix $\delta > 0$ and define $\overline{h}_k = \sup\{h(s) : k\delta \le s < (k+1)\delta\}$ and

$$\overline{h}(t) = \sum_{k=0}^{\infty} \overline{h}_k \mathbf{1}(k\delta \le t < (k+1)\delta).$$

* The star at the end of a section title means the section contains advanced material that need not be covered in a first reading.

Define $\underline{h}(t)$ and $\underline{h}_k$ similarly, with sup replaced by inf. Obviously,

$$U \star \underline{h}(t) \le U \star h(t) \le U \star \overline{h}(t). \tag{2.71}$$

Letting $d_k(t) = U(t - k\delta) - U(t - (k+1)\delta)$, we can write (like (2.19))

$$U \star \overline{h}(t) = \sum_{k=0}^{\infty} \overline{h}_k d_k(t).$$

Now $\lim_{t\to\infty} d_k(t) = \delta/\mu$ by Theorem 33, and $d_k(t) \le U(\delta)$ by Exercise 28. Then by the dominated convergence theorem (see the Appendix, Theorem 14) and the DRI property of h,

$$\begin{aligned}\lim_{\delta\to 0}\lim_{t\to\infty} U \star \overline{h}(t) &= \lim_{\delta\to 0} \frac{\delta}{\mu} \sum_{k=0}^{\infty} \overline{h}_k \\ &= \lim_{\delta\to 0} \frac{1}{\mu} I^{\delta}(h) = \frac{1}{\mu} \int_{\mathbb{R}_+} h(s)ds.\end{aligned}$$

This (double) limit is the same with $\overline{h}(t)$ and $I^{\delta}(h)$ replaced by $\underline{h}(t)$ and $I_{\delta}(h)$. Therefore, the upper and lower bounds in (2.71) for $U \star h(t)$ have the same limit $\frac{1}{\mu}\int_{\mathbb{R}_+} h(s)ds$, and so $U \star h(t)$ must also have this limit. This proves the assertion.

We end this section with criteria for a function to be DRI.

Proposition 88. *Any one of the following conditions is sufficient for $h(t)$ to be DRI.*
(a) $h(t) \ge 0$ *is decreasing and is Riemann integrable on* $\mathbb{R}_+$.
(b) $h(t)$ *is Riemann integrable on* $[0, a]$ *for each* a, *and* $I^{\delta}(h) < \infty$ *for some* $\delta > 0$.
(c) $h(t)$ *is continuous except possibly on a set of Lebesgue measure* 0, *and* $|h(t)| \le b(t)$, *where* $b(t)$ *is DRI.*

Proof. Suppose condition (a) holds. Since the usual Riemann integral of h on $\mathbb{R}_+$ exists, we have

$$I_{\delta}(h) \le \int_{\mathbb{R}_+} h(s)ds \le I^{\delta}(h).$$

Also, the decreasing property of $h(t)$ implies $I^{\delta}(h) - I_{\delta}(h) = \delta h(0) \to 0$ as $\delta \to 0$. These observations prove $h(t)$ is DRI.

Next, suppose (b) holds. We will write

$$I^{\delta}(h) = \mathcal{I}^{\delta}[0, a/\delta) + \mathcal{I}^{\delta}[a/\delta, \infty),$$

where $\mathcal{I}^{\delta}[x, \infty) = \delta \sum_{k=\lceil x-1 \rceil}^{\infty} \sup\{h(s) : k\delta \le s < (k+1)\delta\}$. We will use a similar expression for $I_{\delta}(h)$. Since $h(t)$ is Riemann integrable on $[0, a]$, it follows that $\mathcal{I}^\delta[0, a/\delta)$ and $\mathcal{I}_\delta[0, a/\delta)$ both converge to $\int_0^a h(s)ds$ as $\delta \to 0$. Therefore,

$$I^\delta(h) - I_\delta(h) = o(1) + \mathcal{I}^\delta[a/\delta, \infty) - \mathcal{I}_\delta[a/\delta, \infty), \qquad \text{as } \delta \to 0. \tag{2.72}$$

Let γ be such that $I^\gamma(h) < \infty$. Then for any $\varepsilon > 0$, there is a large enough a such that $\mathcal{I}^\gamma[a/\gamma, \infty) < \varepsilon$. Then clearly, for sufficiently small δ,

$$\mathcal{I}_\delta[a/\delta, \infty) \le \mathcal{I}^\delta[a/\delta, \infty) \le \mathcal{I}^\gamma[a/\gamma, \infty) < \varepsilon.$$

Using this in (2.72), we have

$$I^\delta(h) - I_\delta(h) \le o(1) + 2\varepsilon, \qquad \text{as } \delta \to 0.$$

Since this holds for any ε, it follows that $h(t)$ is DRI.

Finally, (c) implies (b) since $I^\delta(h) \le I^\delta(b)$. Thus $h(t)$ is DRI.

2.18 Proof of Blackwell's Theorem*

This section describes a coupling proof of Blackwell's theorem. The proof is more complicated than the one we presented above for arithmetic inter-renewal times.

The classical proof of Blackwell's theorem based on analytical properties of the renewal function and integral equations is in Feller (1971). Lindvall (1977) and Athreya, McDonald and Ney (1978) gave another probabilistic proof involving "coupling" techniques. A nice review of various applications of coupling is in Lindvall (1992). A recent refinement of the coupling proof is given in Durrett (2005). The following is a sketch of his presentation when the inter-renewal time has a finite mean (he gives a different proof for the case of an infinite mean).

Let $N(t)$ be a renewal process with renewal times T_n whose inter-renewal times ξ_n have a non-arithmetic distribution and a finite mean μ. For convenience, we will write Blackwell's theorem (Theorem 33) as

$$\lim_{t\to\infty} E[N(t, t+a]] = a/\mu, \tag{2.73}$$

where $N(t, t+a] = N(t+a) - N(t)$. Now, this statement would trivially hold if $N(t)$ were a stationary renewal process, since in this case $E[N(t, t+a]]$ would equal a/μ by Proposition 75. So if one could construct a version of $N(t)$ that approximates a stationary process as close as possible, then (2.73) would be true. That is the approach in the proof that we now describe.

On the same probability space as $N(t)$, let $N'(t)$ be a stationary renewal process with renewal times T'_n, whose inter-renewal ξ'_n times for $n \ge 2$ have the same distribution as the ξ_n. The first and most subtle part of the proof is to construct a third renewal process $N''(t)$ on the same probability space that is equal in distribution to the original process $N(t)$ and approximates the stationary process $N'(t)$. We will not describe the construction of these processes, but only specify their main properties.

In particular, for a fixed $\varepsilon > 0$, the proof begins by defining random indices ν and ν' such that $|T_\nu - T_{\nu'}| < \varepsilon$. Then a third renewal process $N''(t)$ is defined (on the same probability space) with inter-renewal times $\xi_1, \ldots, \xi_\nu, \xi'_{\nu'}, \xi'_{\nu'+1} \ldots$ This process has the following properties:

(a) $\{N''(t) : t \geq 0\} \stackrel{d}{=} \{N(t) : t \geq 0\}$ (i.e., their finite-dimensional distributions are equal).

(b) On the event $\{T_\nu \leq t\}$,

$$N'(t+\varepsilon, t+a-\varepsilon] \leq N''(t, t+a] \leq N'(t-\varepsilon, t+a+\varepsilon]. \tag{2.74}$$

This construction is an ε-*coupling* in that $N''(t)$ is a coupling of $N(t)$ that is within ε of the targeted stationary version $N'(t)$ in the sense of condition (b).

With this third renewal process in hand, the rest of the proof is as follows. Consider the expectation

$$E[N(t, t+a]] = E[N''(t, t+a]] = V_{\mathbf{1}}(t) + V_2(t), \tag{2.75}$$

where

$$V_{\mathbf{1}}(t) = E\Big[N''(t, t+a]\mathbf{1}(T_\nu \leq t)\Big], \qquad V_2(t) = E\Big[N''(t, t+a]\mathbf{1}(T_\nu > t)\Big].$$

Condition (b) and $E[N'(c, d]] = (d-c)/\mu$ (due to the stationarity) ensure

$$V_{\mathbf{1}}(t) \leq E\Big[N'(t-\varepsilon, t+a+\varepsilon]\mathbf{1}(T_\nu \leq t)\Big] \leq (a+2\varepsilon)\mu.$$

Next, observe that $E[N''(t, t+a]|T_\nu > t] \leq E[N''(a)]$, since the worse-case scenario is that there is a renewal at t. This and condition (b) yield

$$V_2(t) \leq P\{T_\nu > t\}E[N''(a)].$$

Similarly,

$$\begin{aligned} V_{\mathbf{1}}(t) &\geq E\Big[N'(t+\varepsilon, t+a-\varepsilon] - N''(t, t+a]\mathbf{1}(T_\nu > t)\Big] \\ &\geq (a-2\varepsilon)/\mu - P\{T_\nu > t\}E[N''(a)]. \end{aligned}$$

Here we take $\varepsilon < a/2$, so that $t + \varepsilon < t + a - \varepsilon$. Combining the preceding inequalities with (2.75), and using $P\{T_\nu > t\} \to 0$ as $t \to \infty$, it follows that

$$(a-2\varepsilon)/\mu + o(1) \leq E[N(t, t+a]] \leq (a+2\varepsilon)/\mu + o(1).$$

Since this is true for arbitrarily small ε, we obtain $E[N(t, t+a]] \to a/\mu$, which is Blackwell's result.

2.19 Stationary-Cycle Processes*

Most of the results above for regenerative processes also apply to a wider class of regenerative-like processes that we will now describe.

For this discussion, suppose $\{X(t) : t \geq 0\}$ is a continuous-time stochastic process with a general state space S, and $N(t)$ is a renewal process defined on the same probability space. As in Section 2.8, we let

$$\zeta_n = (\xi_n, \{X(T_{n-1} + t) : 0 \leq t < \xi_n\})$$

denote the segment of these processes on the interval $[T_{n-1}, T_n)$. Then $\{\zeta_{n+k} : k \geq 1\}$ is the *future of* $(N(t), X(t))$ *beginning at time* T_n. This is what an observer of the processes would see beginning at time T_n.

Definition 89. The process $X(t)$ is a *stationary-cycle process* over the times T_n if the future $\{\zeta_{n+k} : k \geq 1\}$ of $(N(t), X(t))$ beginning at any time T_n is independent of $T_1, \ldots, T_n$, and the distribution of this future is independent of n. Discrete-time and delayed stationary-cycle processes are defined similarly.

The defining property ensures that the segments ζ_n form a stationary sequence, whereas for a regenerative process, the segments are i.i.d. Also, for a regenerative process $X(t)$, its future $\{\zeta_{n+k} : k \geq 1\}$ beginning at any time T_n is independent of the entire past $\{\zeta_k : k \leq n\}$ (rather than only $T_1, \ldots, T_n$ as in the preceding definition).

All the strong laws of large numbers for regenerative processes in this chapter also hold for stationary-cycle processes. A law's limiting value would be a constant as usual when ζ_n is ergodic (as in Section 4.18 in Chapter 4), but the value would be random when ζ_n is not ergodic. We will not get into these details.

As in Section 2.10, one can define processes with stationary-cycle increments. Most of the results above such as the CLT have obvious extensions to these more complicated processes.

We end this section by commenting on limiting theorems for probabilities and expectations of stationary-cycle processes.

Remark 90. Theorem 45 and Corollary 46 are also true for stationary-cycle processes. This follows since such a process satisfies the crude-regeneration property in Theorem 41 leading to Theorem 45 and Corollary 46.

There are many intricate stationary-cycle processes that arise naturally from systems that involve stationary and regenerative phenomena. Here is an elementary illustration.

Example 91. Regenerations in a Stationary Environment. Consider a process $X(t) = g(Y(t), Z(t))$ where $Y(t)$ and $Z(t)$ are independent processes and g is a function on their product space. Assume $Y(t)$ is a regenerative process over the times T_n (e.g., an ergodic continuous-time Markov chain as in Chapter 4) with a metric state space S. Assume $Z(t)$ is a stationary process. One can regard $X(t) = g(Y(t), Z(t))$ as a regenerative-stationary reward process, where $g(y, z)$ is the reward rate from operating a system in state y in environment z. Now, the segments ζ_n defined above form a stationary process, and hence $X(t)$ is a stationary-cycle process.

In light of Remark 90, we can describe the limiting behavior of $X(t)$ as we did for regenerative processes. In particular, assuming for simplicity that g is real-valued and bounded, Theorem 45 for stationary-cycle processes tells us that

$$\lim_{t\to\infty} E[X(t)] = \frac{1}{\mu} E\Big[\int_0^{T_1} g(Y(s), Z(s))ds\Big].$$

2.20 Exercises

Exercise 1. Show that if X is nonnegative with distribution F, then

$$E[X] = \int_{\mathbb{R}_+} (1 - F(x))dx.$$

One approach is to use $E[X] = \int_{\mathbb{R}_+} \big(\int_0^x dy\big)dF(x)$. (For an integer-valued X, the preceding formula is $E[X] = \sum_{n=0}^{\infty} P\{X > n\}$.)

For a general X with finite mean, use $X = X^+ - X^-$ to prove

$$E[X] = \int_{\mathbb{R}_+} (1 - F(x))dx - \int_{-\infty}^{0} F(x)dx.$$

Exercise 2. *Bernoulli Process*. Consider a sequence of independent Bernoulli trials in which each trial results in a success or failure with respective probabilities p and $q = 1 - p$. Let $N(t)$ denote the number of successes in t trials, where t is an integer. Show that $N(t)$ is a discrete-time renewal process, called a Bernoulli Process. (The parameter t may denote discrete-time or any integer referring to sequential information.) Justify that the inter-renewal times have the geometric distribution $P\{\xi_1 = n\} = pq^{n-1}$, $n \geq 1$. Find the distribution and mean of $N(t)$, and do the same for the renewal time T_n. Show that the moment generating function of T_n is

$$E[e^{\alpha T_n}] = \Big(\frac{pe^{\alpha}}{1 - qe^{\alpha}}\Big)^n, \qquad 0 < \alpha < -\log q.$$

Exercise 3. Exercise 1 in Chapter 3 shows that an exponential random variable X satisfies the *memoryless property*

$$P\{X > s + t | X > s\} = P\{X > t\}, \qquad s, t > 0.$$

Prove the analogue $P\{X > \tau + t | X > \tau\} = P\{X > t\}$, for $t > 0$, where τ is a positive random variable independent of X. Show that, for a Poisson process $N(t)$ with rate λ, the forward recurrence time $B(t) = T_{N(t)+1} - t$ at time t has an exponential distribution with rate λ. Hint: condition on $T_{N(t)}$.

Consider the forward recurrence time $B(\tau)$ at a random time τ independent of the Poisson process. Show that $B(\tau)$ also has an exponential distribution with rate λ.

Exercise 4. A system consists of two components with independent lifetimes X_1 and X_2, where X_1 is exponentially distributed with rate λ, and X_2 has a uniform distribution on $[0, 1]$. The components operate in parallel, and the system lifetime is $\max\{X_1, X_2\}$ (the system is operational if and only if at least one component is working). When the system fails, it is replaced by another system with an identical and independent lifetime, and this is repeated indefinitely. The number of system renewals over time forms a renewal process $N(t)$. Find the distribution and mean of the system lifetime. Find the distribution and mean of $N(t)$ (reduce your formulas as much as possible). Determine the portion of time that (a) two components are working, (b) only type 1 component is working, and (c) only type 2 component is working.

Exercise 5. *Continuation.* In the context of the preceding exercise, a typical system initially operates for a time $Y = \min\{X_1, X_2\}$ with two components and then operates for a time $Z = \max\{X_1, X_2\} - Y$ with one component. Thereupon it fails. Find the distributions and means of Y and Z. Find the distribution of Z conditioned that $X_1 > X_2$. You might want to use the memoryless property of the exponential distribution in Exercise 3. Find the distribution of Z conditioned that $X_2 > X_1$.

Exercise 6. Let $N(t)$ denote a renewal process with inter-renewal distribution F and consider the number of renewals $N(T)$ in an interval $(0, T]$ for some random time T independent of $N(t)$. For instance, $N(T)$ might represent the number of customers that arrive at a service station during a service time T. Find general expressions for the mean and distribution of $N(T)$. Evaluate these expressions for the case in which T has an exponential distribution with rate μ and $F = G^{2\star}$, where G is an exponential distribution with rate λ.

Exercise 7. Let $X(t)$ denote the cyclic renewal process in Example 8, where $F_0, \dots, F_{K-1}$ are the sojourn distributions in states $0, 1, \dots, K-1$. Assume $p = F_0(0) > 0$, but $F_i(0) = 0$, for $i = 1, \dots, K-1$. Let T_n denote the times at which the process $X(t)$ jumps from state $K-1$ directly to state 1 (i.e., it spends no time in state 0). Justify that the T_n form a delayed renewal process with inter-renewal distribution

$$F(t) = p \sum_{j=0}^{\infty} F_1 \star \cdots \star F_{K-1} \star \tilde{F}^{j\star}(t),$$

where $\tilde{F}(t) = \tilde{F}_0 \star F_1 \star \cdots \star F_{K-1}(t)$, and $\tilde{F}_0(t)$ is the conditional distribution of the sojourn time in state 0 given it is positive. Specify a formula for $\tilde{F}_0(t)$, and describe what $\tilde{F}(t)$ represents.

Exercise 8. *Large Inter-renewal Times.* Let $N(t)$ denote a renewal process with inter-renewal distribution F. Of interest are occurrences of inter-renewal times that are greater than a value c, assuming $F(c) < 1$. Let $\tilde{T}_n$ denote the subset of times T_n for which $\xi_n > c$. So $\tilde{T}_n$ equals some T_k if $\xi_k > c$. (Example 72 addresses a related problem of determining the waiting time for a gap of size c in a Poisson process.) Show that $\tilde{T}_n$ are delayed renewal times and the inter-renewal distribution has the form

$$\tilde{F}(t) = \sum_{k=0}^{\infty} F_c^{k\star} \star G(t),$$

where $F_c(t) = F(t)/F(c)$, $0 \le t \le c$ (the conditional distribution of an inter-renewal time given that it is $\le c$), and specify the distribution $G(t)$ as a function of F.

Exercise 9. *Partitioning and Thinning of a Renewal Process.* Let $N(t)$ be a renewal process with inter-renewal distribution F. Suppose each renewal time is independently assigned to be a type i renewal with probability p_i, for $i = 1, \ldots, m$, where $p_1 + \cdots + p_m = 1$. Let $N_i(t)$ denote the number of type i renewals up to time t. These processes form a partition of $N(t)$ in that $N(t) = \sum_{i=1}^{m} N_i(t)$. Each $N_i(t)$ is a thinning of $N(t)$, where p_i is the probability that a point of $N(t)$ is assigned to $N_i(t)$.

Show that $N_i(t)$ is a renewal process with inter-renewal distribution

$$F_i(t) = \sum_{k=1}^{\infty} (1 - p_i)^{k-1} p_i F^{k\star}(t).$$

Show that, for $n = n_1 + \cdots n_m$,

$$P\{N_1(t) = n_1, \ldots, N_m(t) = n_m\} \\ = \frac{n!}{n_1! \cdots n_m!} p_1^{n_1} \cdots p_m^{n_m} \Big[F^{(n)\star}(t) - F^{(n+1)\star}(t) \Big].$$

For $m = 2$, specify an F for which $N_1(t)$ and $N_2(t)$ are not independent.

Exercise 10. *Multi-type Renewals.* An infinite number of jobs are to be processed one-at-a-time by a single server. There are m types of jobs, and the probability that any job is of type i is p_i, where $p_1 + \cdots + p_m = 1$. The service time of a type i job has a distribution F_i with mean μ_i. The service times and types of the jobs are independent. Let $N(t)$ denote the number of jobs completed by time t. Show that $N(t)$ is a renewal process and specify its inter-renewal distribution and mean. Let $N_i(t)$ denote the number of type i jobs processed up to time t. Show that $N_i(t)$ is a delayed renewal process and specify $\lim_{t\to\infty} t^{-1}N_i(t)$.

Exercise 11. *Continuation.* In the context of Exercise 10, let $X(t)$ denote the type of job being processed at time t. Find the limiting distribution of $X(t)$. Find the portion of time devoted to type i jobs.

Exercise 12. *Continuation.* Consider the multi-type renewal process with two types of renewals that have exponential distributions with rates λ_i, and type i occurs with probability p_i, , $i = 1, 2$. Show that the renewal function has the density

$$U'(t) = \frac{\lambda_1\lambda_2 + p_1p_2(\lambda_1 - \lambda_2)^2 e^{-(p_1\lambda_2+p_2\lambda_1)t}}{p_1\lambda_2 + p_2\lambda_1}, \qquad t > 0.$$

Exercise 13. *System Availability.* The status of a system is represented by an alternating renewal process $X(t)$, where the mean sojourn time in a working state 1 is μ_1 and the mean sojourn time in a non-working state 0 is μ_0. The system *availability* is measured by the portion of time it is working, which is $\lim_{t\to\infty} t^{-1}\int_0^t X(s)ds$. Determine this quantity and show that it is equal to the *cycle-availability* measured by $\lim_{n\to\infty} T_n^{-1}\int_0^{T_n} X(s)ds$.

Exercise 14. *Integrals of Renewal Processes.* Suppose $N(t)$ is a renewal process with renewal times T_n and $\mu = E[T_1]$. Prove

$$E\Big[\int_0^{T_n} N(s)\, ds\Big] = \mu n(n-1)/2.$$

For any non-random $t > 0$, it follows by Fubini's theorem that

$$E\Big[\int_0^{t} N(s)\, ds\Big] = \int_0^t E[N(s)]\, ds.$$

Assuming τ is an exponential random variable independent of N with rate γ, prove

$$E\Big[\int_0^{\tau} N(s)\, ds\Big] = \int_{\mathbb{R}_+} e^{-\gamma t} E[N(t)]\, dt.$$

Show that if N is a Poisson process with rate λ, then the preceding expectation equals λ/γ^2. (Integrals like these are used to model holding costs; see Section 2.12 and the next exercise.)

Exercise 15. *Continuation.* Items arrive to a service station according to a Poisson process $N(t)$ with rate λ. The items are stored until m have accumulated. Then the m items are served in a batch. The service time is exponentially distributed with rate γ. During the service, items continue to arrive. There is a cost hi per unit time of holding i customers in the system. Assume the station is empty at time 0. Find the expected cost C_1 of holding the customers until m have arrived. Find the expected cost C_2 for holding the added arrivals in the system during the service.

Exercise 16. Customers arrive to a service system according to a Poisson process with rate λ. The system can only serve one customer at a time and, while it is busy serving a customer, arriving customers are blocked from getting service (they may seek service elsewhere or simply go unserved). Assume the service times are independent with common distribution G and are independent of the arrival process. For instance, a contractor may only be able to handle one project at a time (or a vehicle can only transport one item at a time). Determine the following quantities:
(a) The portions of time the system is busy, and not busy.
(b) The number of customers per unit time that are served.
(c) The portion of customers that are blocked from service.

Exercise 17. *Delayed Renewals.* A point process $N(t)$ is an *m-step delayed renewal process* if the inter-occurrence times ξ_{m+k}, for $k \geq 1$, are independent with a common distribution F, and no other restrictions are placed on $\xi_1, \ldots, \xi_m$. That is, $N_m(t) = N(t) - N(T_m)$, for $t \geq T_m$ is a renewal process. Show that Corollary 11 and Theorem 13 hold for such processes. Use the fact that $N(t)$ is asymptotically equivalent to $N_m(t)$ in that

$$N_m(t)/N(t) = 1 - N(T_m)/N(t) \to 1, \quad \text{a.s. as } t \to \infty.$$

Exercise 18. For a point process $N(t)$ that is not simple, show that if $t^{-1}N(t) \to 1/\mu$ as $t \to \infty$, then $n^{-1}T_n \to \mu$, as $n \to \infty$. Hint: For a fixed positive constant c, note that $N((T_n - c)^+) \leq n \leq N(T_n)$. Divide these terms by T_n and take limits as $n \to \infty$.

Exercise 19. *Age Replacement Model.* An item (e.g., battery, vehicle, tool, or electronic component) whose use is needed continuously is replaced whenever it fails or reaches age a, whichever comes first. The successive items are independent and have the same lifetime distribution G. The cost of a failure is c_f dollars and the cost of a replacement at age a is c_r. Show that the average cost per unit time is

$$C(a) = [c_f G(a) + c_r(1 - G(a))] / \int_0^a (1 - G(s))ds.$$

Find the optimal age a that minimizes this average cost.

Exercise 20. *Point Processes as Jump Processes.* Consider a point process $N(t) = \sum_{k=1}^{\infty} \mathbf{1}(T_k \le t)$, where $T_1 \le T_2 \le \cdots$. It can also be formulated as an integer-valued jump process of the form

$$N(t) = \sum_{n=1}^{\infty} \nu_n \mathbf{1}(\hat{T}_n \le t),$$

where $\hat{T}_n$ are the "distinct" times at which $N(t)$ takes a jump, and ν_n is the size of the jump. That is, $\hat{T}_n = \min\{T_k : T_k > \hat{T}_{n-1}\}$, where $\hat{T}_0 = 0$, and $\nu_n = \sum_{k=1}^{\infty} \mathbf{1}(T_k = \hat{T}_n)$, $n \ge 1$.

For instance, suppose T_n are times at which data packets arrive to a computer file. Then $\hat{T}_n$ are the times at which batches of packets arrive, and at time $\hat{T}_n$, a batch of ν_n packets arrive. Suppose $\hat{T}_n$ are renewal times, and ν_n are i.i.d. and independent of the $\hat{T}_n$. Show that the number of packets that arrive per unit time is $E[\nu_1]/E[\hat{T}_1]$ a.s., provided these expectations are finite. Next, assume $\hat{T}_n$ form a Poisson process with rate λ, and ν_n has a Poisson distribution. Find $E[N(t)]$ by elementary reasoning, and then show that $N(t)$ has a Poisson distribution.

Exercise 21. *Batch Renewals.* Consider times $T_n = \sum_{k=1}^{n} \xi_k$, where the ξ_k are i.i.d. with distribution F and $F(0) = P\{\xi_k = 0\} > 0$. The associated point process $N(t)$ is a renewal process with *instantaneous renewals* (or batch renewals). In the notation of Exercise 20, $N(t) = \sum_{n=1}^{\infty} \nu_n \mathbf{1}(\hat{T}_n \le t)$, where ν_n is the number of renewals exactly at time $\hat{T}_n$. Specify the distribution of ν_n. Are the ν_n i.i.d.? Are they independent of $\hat{T}_n$? Specify the distribution of $\hat{T}_1$ in terms of F.

Exercise 22. Prove $E[T_{N(t)}] = \mu E[N(t)+1] - E[\xi_{N(t)+1}]$. If $N(t)$ is a Poisson process, show that $E[T_{N(t)}] < \mu E[N(t)]$.

Exercise 23. *Little Law.* In the context of the Little law in Theorem 57, show that if L and W exist, then λ exists and $L = \lambda W$.

Exercise 24. *Superpositions of Renewal Processes.* Let $N_1(t)$ and $N_2(t)$ be independent renewal processes with the same inter-renewal distribution, and consider the sum $N(t) = N_1(t) + N_2(t)$ (sometimes called a *superposition*). Assuming that $N(t)$ is a renewal process. prove that it is a Poisson process if and only if $N_1(t)$ and $N_2(t)$ are Poisson processes.

Exercise 25. *Production-Inventory Model.* Consider a production-inventory system that produces a product at a constant rate of c units per unit time and the items are put in inventory to satisfy demands. The products may be discrete or continuous (e.g., oil, chemicals). Demands occur according to a Poisson process $N(t)$ with rate λ, and the demand quantities $X_1, X_2, \ldots$ are independent, identically distributed positive random variables with mean μ, and are independent of the arrival times. Then the inventory level at time t would be

$$Z_x(t) = x + ct - \sum_{n=1}^{N(t)} X_n, \qquad t \geq 0,$$

where x is the initial inventory level. Consider the probability $R(x) = P\{Z_x(t) \geq 0, t \geq 0\}$ of never running out of inventory. Show that if $c < \lambda\mu$, then $R(x) = 0$ no matter how high the initial inventory level x is. Hint: apply a SLLN to show that $Z_x(t) \to -\infty$ as $t \to \infty$ if $c < \lambda\mu$, where x is fixed. Find the limit of $Z_x(t)$ as $t \to \infty$ if $c > \lambda\mu$. (The process $Z_x(t)$ is a classical model of the capital of an insurance company; see Example 73.)

Exercise 26. Let $H(t) = E[N(t) - N(t-a)\mathbf{1}(a \leq t)]$. Find a renewal equation that H satisfies.

Exercise 27. *Non-homogeneous Renewals.* Suppose $N(t)$ is a point process on $\mathbb{R}_+$ whose inter-point times $\xi_n = T_n - T_{n-1}$ are independent with distributions F_n. Assuming it is finite, prove that $E[N(t)] = \sum_{n=1}^{\infty} F_1 \star \cdots \star F_n(t)$.

Exercise 28. *Subadditivity of Renewal Function.* Prove that

$$U(t + a) \leq U(a) + U(t), \qquad a, t \geq 0.$$

Hint: Use $a \leq T_{N(a)+1}$ in the expression

$$N(t + a) - N(a) = \sum_{k=1}^{\infty} \mathbf{1}(T_{N(a)+k} \leq t + a).$$

Exercise 29. *Arithmetic Blackwell Theorem.* The proof of Theorem 33 for arithmetic inter-arrival distributions was proved under the standard condition that $p_0 = F(0) = 0$. Prove the same theorem when $0 < p_0 < 1$. Use a similar argument including the fact that renewals occur in batches and a batch size has a geometric distribution with parameter $1 - p_0$.

Exercise 30. *Elementary Renewal Theorem via Blackwell.* Prove the elementary renewal theorem (Theorem 32) by an application of Blackwell's theorem. One approach, for non-arithmetic F, is to use

$$E[N(t)] = \sum_{k=1}^{\lceil t \rceil} [U(k) - U(k-1)] + E[N(\lceil t \rceil)] - E[N(t)].$$

Then use the fact $n^{-1} \sum_{k=1}^{n} c_k \to c$ when $c_k \to c$.

Exercise 31. *Arithmetic Key Renewal Theorem.* Represent $U \star h(u + nd)$ as a sum like (2.19), and then prove Theorem 37 by applying Blackwell's theorem.

Exercise 32. Let $h(t) = \sum_{n=1}^{\infty} a_n \mathbf{1}(n - \varepsilon_n \le t < n + \varepsilon_n)$, where $a_n \to \infty$ and $1/2 > \varepsilon_n \downarrow 0$ such that $\sum_{n=1}^{\infty} a_n \varepsilon_n < \infty$. Show that h is Riemann integrable, but not DRI.

Exercise 33. Prove that a continuous function $h(t) \ge 0$ is DRI if and only if $I^{\delta}(h) < \infty$ for some $\delta > 0$.

The next eight exercises concern the renewal process trinity: the backward and forward recurrence times $A(t) = t - T_{N(t)}$, $B(t) = T_{N(t)+1} - t$, and the length $L(t) = \xi_{N(t)+1} = A(t) + B(t)$ of the renewal interval containing t. Assume the inter-renewal distribution is non-arithmetic.

Exercise 34. Draw a typical sample path for each of the processes $A(t)$, $B(t)$, and $L(t)$.

Exercise 35. Prove that $B(t)$ is a Markov process by showing it satisfies the following Markov property, for $x, y, t, u \ge 0$:

$$P\{B(t+u) \le y | B(s) : s < t, B(t) = x\} = P\{B(u) \le y | B(0) = x\}.$$

Exercise 36. Formulate a renewal equation that $P\{B(t) > x\}$ satisfies.

Exercise 37. *Bypassing a renewal equation.* Use Proposition 40 (without using a renewal equation) to prove $P\{B(t) > x\} = \int_{[0,t]} [1 - F(t+x-s)] dU(s)$.

Exercise 38. Prove $E[B(t)] = \mu E[N(t) + 1] - t$. Assuming F has a finite variance σ^2, prove

$$\lim_{t\to\infty} E[A(t)] = \lim_{t\to\infty} E[B(t)] = \frac{\sigma^2 + \mu^2}{2\mu}.$$

Is this limit also the mean of the limiting distribution $F_e(t) = \frac{1}{\mu} \int_0^t [1 - F(s)] ds$ of $A(t)$ and $B(t)$?

Exercise 39. *Inspection Paradox.* Consider the length $L(t) = \xi_{N(t)+1}$ of the renewal interval at any time t (this is what an inspector of the process would see at time t). Prove the paradoxical result that $L(t)$ is *stochastically larger* than the length ξ_1 of a typical renewal interval; that is

$$P\{L(t) > x\} \ge P\{\xi_1 > x\}, \qquad t, x \ge 0.$$

This inequality is understandable upon observing that the first probability is for the event that a renewal interval bigger than x "covers" t, and this is more likely to happen than a fixed renewal interval being bigger than x. A consequence of this result is $E[L(t] \ge E[\xi_1]$, which is often a strict inequality.

Suppose $\mu = E[T_1]$ and $\sigma^2 = \mathrm{Var}[T_1]$ are finite. Recall from (2.34) that the limiting distribution of $L(t)$ is $\frac{1}{\mu} \int_0^x s dF(s)$. Derive the mean of this distribution (as a function of μ and σ^2), and show it is $\ge \mu$.

Show that if $N(t)$ is a Poisson process with rate λ, then

$$E[L(t)] = \lambda^{-1}[2 - (1 + \lambda t)e^{-\lambda t}].$$

In this case, $E[L(t] > E[\xi_1]$.

Exercise 40. Prove $\lim_{t\to\infty} P\{A(t)/L(t) \le x\} = x$, $0 \le x \le 1$. Prove this result with $B(t)$ in place of $A(t)$.

Exercise 41. Show that

$$\lim_{t\to\infty} E[A(t)^k B(t)^\ell (A(t) + B(t))^m] = \frac{E[T_1^{k+\ell+m}]}{\mu(k+\ell+1)\binom{k+\ell}{k}}.$$

Find the limiting covariance, $\lim_{t\to\infty} \mathrm{Cov}(A(t), B(t))$.

Exercise 42. *Delayed Versus Non-delayed Regenerations.* Let $\tilde{X}(t)$ be a real-valued delayed regenerative process over T_n. Then $X(t) = \tilde{X}(T_1+t)$, $t \ge 0$ is a regenerative process. Assuming $\tilde{X}(t)$ is bounded, show that if $\lim_{t\to\infty} E[X(t)]$ exists (such as by Theorem 45), then $E[\tilde{X}(t)]$ has the same limit. Hint: Take the limit as $t \to \infty$ of

$$E[\tilde{X}(t)] = \int_{[0,t]} E[X(t-s)]dF(s) + E[\tilde{X}(t)\mathbf{1}(T_1 > t)].$$

Exercise 43. *Dispatching System.* Items arrive at a depot (warehouse or computer) at times that form a renewal process with finite mean μ between arrivals. Whenever M items accumulate, they are instantaneously removed (dispatched) from the depot. Let $X(t)$ denote the number of items in the depot at time t. Find the limiting probability that there are p_j items in the system ($j = 0, \ldots, M-1$). Find the average number of items in the system over an infinite time horizon.

Suppose the batch size M is to be selected to minimize the average cost of running the system. The relevant costs are a cost C for dispatching the items, and a cost h per unit time for holding an item in the depot. Let $C(M)$ denote the average dispatching plus holding cost for running the system with batch size M. Find an expression for $C(M)$. Show that the value of M that minimizes $C(M)$ is an integer adjacent to the value $M^* = \sqrt{2C/h\mu}$.

Exercise 44. *Continuation.* In the context of the preceding exercise, find the average time W that a typical item waits in the system before being dispatched. Find the average waiting time $W(i)$ in the system for the ith arrival in the batch.

Exercise 45. Consider an ergodic Markov chain X_n with limiting distribution π_i. Prove

$$\lim_{n\to\infty} P\{X_n = j, X_{n+1} = \ell\} = \pi_j p_{j\ell}.$$

One can show that (X_n, X_{n+1}) is a two-dimensional Markov chain that is ergodic with the preceding limiting distribution. However, establish the limit above only with the knowledge that X_n has a limiting distribution.

Exercise 46. Items with volumes $V_1, V_2, \ldots$ are loaded on a truck one at a time until the addition of an arriving item would exceed the capacity v of the truck. Then the truck leaves to deliver the items. The number of items that can be loaded in the truck before its volume v is exceeded is

$$N(v) = \min\{n : \sum_{k=1}^{n} V_k > v\} - 1.$$

Assume the V_n are independent with identical distribution F that has a mean μ and variance σ^2. Suppose the capacity v is large compared to μ. Specify a single value that would be a good approximation for $N(v)$. What would be a good approximation for $E[N(v)]$? Specify how to approximate the distribution of $N(v)$ by a normal distribution. Assign specific numerical values for μ, σ^2, and v, and use the normal distribution to approximate the probability $P\{a \le N(v) \le b\}$ for a few values of a and b.

Exercise 47. *Limiting Distribution of a Cyclic Renewal Process.* Consider a cyclic renewal process $X(t)$ on the states $0, 1, \ldots, K-1$ as described in Example 8. Its inter-renewal distribution is $F = F_0 \star \cdots \star F_{K-1}$, where F_i is distribution of a sojourn time in state i having a finite mean μ_i. Assume one of the F_i is non-arithmetic. Show that F is non-arithmetic. Prove

$$\lim_{t\to\infty} P\{X(t) = i\} = \frac{\mu_i}{\mu_0 + \ldots + \mu_{K-1}}.$$

Is this limiting distribution the same as $\lim_{t\to\infty} t^{-1}E[\int_0^t \mathbf{1}(X(s) = i)ds]$, the average expected time spent in state i? State any additional assumptions needed for the existence of this limit.

Exercise 48. Consider a $G/G/1$ system as in Example 60. Let W'_n denote the length of time the nth customer waits in the queue prior to obtaining service. Determine a Little law for the average wait $W' = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} W'_k$.

Exercise 49. *System Down Time.* Consider an alternating renewal process that represents the up and down states of a system. Suppose the up times have a distribution G with mean μ and variance σ^2, and the down times have a distribution G_0 with mean μ_0 and variance σ_0^2. Let $D(t)$ denote the length of time the system is down in the time interval $[0, t]$. Find the average expected down time $\beta = \lim_{t\to\infty} t^{-1}E[D(t)]$. Then show $(D(t) - \beta t)/t^{1/2} \xrightarrow{d} N(0, \gamma^2)$ and specify γ.

Exercise 50. *Congestion in a Running Race.* The following model was developed by Georgia Tech undergraduate students to assess the congestion in the 10-kilometer Atlanta Road Race, which is held every July 4th. After the pack of elite runners begins the race, the rest of the runners start the race a little later as follows. The runners are partitioned into m groups, with r_k runners assigned to group k, $1 \le k \le m$, depending on their anticipated completion times (the runners in each group being about equal in ability). The groups are released every τ minutes, with group k starting the race at time $k\tau$ (the groups are ordered so that the faster runners go earlier). Although the group sizes r_k are random, assume for simplicity that they are not. Typical numbers are 10 groups of 5000 runners in each group. The aim was to design the race so that the congestion did not exceed a critical level that would force runners to walk. To do this, the students developed a model for computing the probability that the congestion would be above the critical level. (They used this model to determine reasonable group sizes and their start times under which the runners would start as soon as possible, with a low probability of runners being forced to walk.)

The students assumed the velocity of each runner is the same throughout the race, the velocities of all the runners are independent, and the velocity of each runner in group k has the same distribution F_k. The distributions F_k were based on empirical distributions from samples obtained in prior races. Using pictures of past races, it was determined that if the number of runners in an interval of length ℓ in the road was greater than b, then the runners in that interval would be forced to walk. This was based on observing pictures of congestion in past races where the runners had to slow down to a walk.

Under these assumptions, the number of runners in group k that are in an interval $[a, a+\ell]$ on the road at time t is

$$Z_a^k(t) = \sum_{n=1}^{r_k} \mathbf{1}(V_{kn}(t - k\tau) \in [a, a+\ell]),$$

where $V_{k1}, \ldots, V_{kr_k}$ are the independent velocities of the runners in group k that have the distribution F_k. Then the total number of runners that are in $[a, a+\ell]$ at time t is

$$Z_a(t) = \sum_{k=1}^{m} Z_a^k(t).$$

Specify how one would use the central limit theorem to compute the probability $P\{Z_a(t) > b\}$ that the runners in $[a, a+\ell]$ at time t would be forced to walk.

Exercise 51. *Confidence Interval.* In the context of Example 66, suppose the regenerative-increment process $Z(t)$ is not observed continuously over time, but only observed at its regeneration times T_n. In this case, the information observed up to the nth regeneration time T_n is $\{Z(t) : t \le T_n\}$. First, find the a.s. limit of $Z(T_n)/n$, and the limiting distribution of $(Z(T_n) - aT_n)/n^{1/2}$.

Then find an approximate confidence interval for the mean a analogous to that in Example 66.

Exercise 52. *Continuation.* Use the CLT in Examples 67 and 68 to obtain approximate confidence intervals for a renewal process and a Markov chain comparable to the confidence interval in Example 66.

Exercise 53. Consider a delayed renewal process $N(t)$ with initial distribution $F_e(x) = \frac{1}{\mu}\int_0^x [1 - F(s)]ds$. Prove $E[N(t)] = U \star F_e(t) = t/\mu$ by a direct evaluation of the integral representing the convolution, where $U = \sum_{n=0}^{\infty} F^{n\star}$.

Exercise 54. Justify expression (2.49), which in expanded form is

$$\sigma^2 = \mu_i^{-1}\mathrm{Var}[Z_{\nu_1} - a\nu_1] = \sum_{j\in S} \pi_j \tilde{f}(j)^2 + 2\sum_{j\in S} \pi_j \tilde{f}(j) \sum_{k\in S} \sum_{n=1}^{\infty} p_{jk}^n \tilde{f}(k).$$

First show that $E[Z_{\nu_1} - a\nu_1)] = 0$, and then and use the expansion

$$\begin{aligned} \mathrm{Var}[Z_{\nu_1} - a\nu_1] &= E_i\Big[[\sum_{n=1}^{\nu_1} \tilde{f}(X_n)]^2\Big] \\ &= E_i\Big[\sum_{n=1}^{\nu_1} \tilde{f}(X_n)^2\Big] + 2E_i\Big[\sum_{n=1}^{\nu_1} V_n\Big], \end{aligned} \tag{2.76}$$

where $V_n = \tilde{f}(X_n)\sum_{\ell=n+1}^{\nu_1} \tilde{f}(X_\ell)$. Apply Proposition 69 from Chapter 1 to the last two expressions in (2.76)(noting that $\sum_{n=1}^{\nu_1} V_n = \sum_{n=0}^{\nu_1-1} V_n$). Use the fact that

$$E_i[V_n|X_n = j, \nu_1 \geq n] = \tilde{f}(j)h(j),$$

where $h(j) = E_j[\sum_{n=1}^{\nu_1} \tilde{f}(X_n)]$ satisfies the equation

$$h(j) = \sum_{k\in S} p_{jk}\tilde{f}(k) + \sum_{k\in S} p_{jk}h(k),$$

and hence $h(j) = \sum_{k\in S}\sum_{n=1}^{\infty} p_{jk}^n \tilde{f}(k)$.

Exercise 55. In the context of Theorem 76, show that if the distribution of the residual time $B(t)$ is independent of t, then it is a stationary process. Hint: For any s_i, x_i and t, let

$$\Gamma_t = \{B(s_1 + t) \leq x_1, \ldots, B(s_k + t) \leq x_k\}.$$

Show that $P\{\Gamma_t|B(t) = x\} = P\{\Gamma_0|B(0) = x\}$, and use this equality to prove $P(\Gamma_t)$ is independent of t.

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