Chapter 2
CAP Localization

In this chapter, we will express the \( \pi \)-isotypic Lefschetz numbers of Hecke operators acting on the cohomology of symmetric spaces \( S_K(G) \) attached to reductive groups \( G \) in terms of so-called elliptic traces \( T_{\ell_G} \), provided the underlying representation \( \pi \) is not a cuspidal representation associated with a parabolic subgroup (CAP representation) of \( G(\mathbb{A}) \). In the following two chapters we derive from these formulas all the essential information required.

For a connected reductive group \( G \) over \( \mathbb{Q} \), let \( K_\infty \) be a maximal compact subgroup of \( G_\infty = G(\mathbb{R}) \) and let \( A_G(\mathbb{R})^0 \) be the topologically connected component of the maximal \( \mathbb{Q} \)-split component \( A_G \) of the center \( Z_G \) of \( G \). Then \( X_G = G_\infty / \tilde{K}_\infty \) for \( \tilde{K}_\infty = K_\infty \cdot A_G(\mathbb{R})^0 \) will be called the connected symmetric space attached to \( G \). For a compact open subgroup \( K \subseteq G(\mathbb{A}_{\text{fin}}) \)

\[
S_K(G) = G(\mathbb{Q}) \setminus (X_G \times G(\mathbb{A}_{\text{fin}})) / K = G(\mathbb{Q}) \setminus (X_G \times G(\mathbb{A}_{\text{fin}})) / K
\]

is a disjoint union of arithmetic quotients of \( X_G \).

Example 2.1. For \( G = GSp(4, \mathbb{Q}) \) we have \( X_G = H \cup -H \) for the Siegel upper half-space \( H \) of genus 2. Hence, \( S_K(G) \) does not coincide with the Shimura variety \( S_K(\mathbb{C}) \), which is an unramified covering of \( S_K(G) \).

Assumption Regarding \( G_{\text{der}} \). In this chapter assume that the derived group \( G_{\text{der}} \) of \( G \) is simply connected. This property is inherited by the Levi subgroups \( L \) of \( G \).

Proof: \( G = G_{\text{der}}Z(G) \) and \( Z(G) \subseteq L \) implies \( L_{\text{der}} = (G_{\text{der}} \cap L)_{\text{der}} \). \( L \cap G_{\text{der}} \) is a Levi group of \( G_{\text{der}} \), since this holds for the Lie algebras by characterizing Levi subgroups as centralizers. So it is enough to consider the semisimple case to see that \( L_{\text{der}} \) is simply connected. For this case see [99], Lemma 5.3 or Theorem 5.8, p. 208, which proves the claim. Since all groups \( L_{\text{der}} \) are simply connected implies that the centralizers \( L_{\gamma} \) of semisimple elements in the Levi groups \( L \) are connected reductive groups.

Lefschetz Numbers. An irreducible complex representation of the group \( G_\infty = G(\mathbb{R}) \) with highest weights \( \lambda \) restricts us to a representation of \( G(\mathbb{Q}) \), which defines
a coefficient system \( V_{\lambda} \) on \( S_K(G) \). The cohomology groups \( H^\nu(S_K(G), V_{\lambda}) \) are modules under the Hecke algebra of \( K \)-bi-invariant functions on \( G(A_{\text{fin}}) \) with compact support. Assume \( K = \prod_v K_v \). Fix a finite set \( S \) of non-Archimedean places such that for all non-Archimedean places \( v \notin S \) the group \( K_v \) is a special, good maximal compact subgroup of \( G_v \). Let

\[
\pi^S = \otimes_{v \notin S} \pi_v
\]

be an irreducible spherical automorphic representation of \( G(A^S_f) \). The \( \pi^S \)-isotypic generalized eigenspace of the \( \nu \)th cohomology group

\[
H^\nu(S_K(G), V_{\lambda})(\pi^S)
\]

is a module under the Hecke algebra \( \mathcal{H}_{S,K} \subseteq \mathcal{H}_S \), defined by the locally constant \( K_S \)-bi-invariant functions on \( G(A_S^f) \) with compact support. A simple formula for the trace of Hecke operators \( f_S \in \mathcal{H}_S = \otimes_{v \in S} \mathcal{H}_v \) in the subspace \( \mathcal{H}_{S,K} \) of the Hecke algebra (see Appendices 1 and 2) defined by

\[
\text{tr}_S(f_S) = \sum_{\nu} (-1)^{\nu} \text{tr} \left( f_S, H^\nu(S_K(G), V_{\lambda})(\pi^S) \right)
\]

is provided by the topological trace formula of Goresky and MacPherson. Assume that the unramified automorphic spherical representation \( \pi^S \) of \( G(A^S_f) \) is not isomorphic to a subquotient of an induced representation \( \text{Ind}^G_{S,P}(\sigma^S) \) for all proper parabolic subgroups \( P \neq G \) with Levi component \( L \), and all irreducible automorphic representations \( \sigma^S \) of \( L(A^S_f) \). In this case \( \pi^S \) is cuspidal, and \( \pi^S \) is not a CAP representation in the sense of [69, 97]. With these assumptions, the formula for the trace of \( f_S \) is further simplified (Sects. 2.6, 2.8).

Of special interest is the case where \( G_{\infty} \) has discrete series representations (Sect. 2.9). In this case the formula for the trace becomes the following (see Corollary 2.6): If the group \( K = \prod_v K_v \) is small and \( \pi^S \) is not CAP, the trace \( \text{Tr}_S(f_S) \) of \( f_S \) is equal to

\[
d(G) \cdot \sum_{\gamma \in G(\mathcal{Q})/\sim} \tau(G_\gamma) O_{G(K)}(f_S f_\infty) f_\infty.
\]

The sum is over all strongly elliptic semisimple conjugacy classes in \( G(\mathcal{Q}) \) (see page 46); \( G_\gamma \) denotes the centralizer of \( \gamma \) in \( G \), which by our assumptions is a connected reductive group. The coefficients \( O_{G(K)} \) are adelic orbital integrals. Measures are such that \( \text{vol}_{d_{g\gamma}}(K) = 1 \) and \( \text{vol}_{d_{g\infty}d_{g\gamma}}(G(\mathcal{Q}) \setminus G(A)) = \tau(G) \) is the Tamagawa number. The function \( f_\pi^S \) is a suitable chosen good \( \pi^S \)-projector depending on the fixed \( f_S \) (see Sect. 2.8), and \( f_\infty \) is a suitable linear combination of pseudocoefficients of discrete series representations with respect to the measure \( d_{g\infty} \) (see Sect. 2.9). The corresponding \( L \)-packet is determined by the representation

\[1\) In this chapter we consider the dual \( V_\lambda \) of the coefficient system \( E_\lambda \) of Chap. 1.
\(\lambda\) defining the coefficient system \(V_\lambda\). \(d(G)\) denotes the number of discrete series representations in this \(L\)-packet.

**Remark 2.1.** If \(\tilde{K}_\infty\) is replaced by a subgroup \(U\) of finite index such that \(G_\infty/U \rightarrow G_\infty/\tilde{K} = X_G\) is a finite unramified covering of degree \(d\), then the trace formula also holds for \(G(\mathcal{O}) \setminus (G_\infty/U \times G(A_{1,\infty}))/\tilde{K}\) except that the formula above has to be multiplied by the degree \(d\) of the covering. This applies for Shimura varieties \((G, h)\) attached to a reductive \(\mathcal{O}\)-group, for which \(Z(G)/A_G\) is \(\mathbb{R}\)-anisotropic, since in this case the centralizer \(Z(h)\) of the structure homomorphism \(h\) of the Shimura variety is a subgroup of finite index in \(\tilde{K} = K_\infty \cdot \bar{A}_G(\mathbb{R})^0\). See page 53.

### 2.1 Standard Parabolic Subgroups

Fix a minimal \(\mathcal{O}\)-parabolic subgroup \(P_0\). For a \(\mathcal{O}\)-rational parabolic subgroup \(P = LN\) containing \(P_0\), and \(\gamma \in P(\mathcal{O})\) let \(\gamma_L\) denote the image of \(\gamma\) under the projection \(P(\mathcal{O}) \rightarrow L(\mathcal{O})\) to the Levi component.

**Contractive Elements.** A semisimple element \(\gamma \in L(\mathcal{O})\), which is contained in a real torus \(T\) of \(L\), which is \(\mathbb{R}\)-anisotropic modulo \(A_L\) is called \(P\)-contractive, if \(|\gamma_L^\sigma|_\infty > 1\) holds for all simple roots \(\sigma\) (over the algebraic closure), which occur in the Lie algebra of the nilpotent radical of \(P\), restricted to the maximal \(\mathcal{O}\)-split torus \(A_L\) (in the center of \(L\)). In fact, it does not matter if we consider the absolute root system or the \(\mathcal{O}\)-root system. Since \(\gamma_L = a_\infty \cdot x_\infty k_\infty x_\infty^{-1}\) for \(a_\infty \in A_L(\mathbb{R})^0\), \(k_\infty \in K_L\), this implies \(|\gamma_L^\sigma|_\infty = |a_\infty^\sigma|_\infty\) for all roots \(\sigma\). Hence, \(\gamma_L\) is \(P\)-contractive if and only if the central component \(a_\infty\) is \(P\)-contractive and this notion depends only on the \((L(\mathcal{O}))\)-conjugacy class of the element \(\gamma\). Suppose \(P = P_0 = LN_\theta\) is a \(\mathcal{O}\)-rational standard parabolic subgroup defined by a subset \(\theta\) of the simple positive \(\mathcal{O}\)-roots. Then by definition \(|\alpha(\gamma_L)|_\infty = |a_\infty^\alpha|_\infty = 1\) holds for all simple roots \(\alpha \in \theta\). Since the roots in \(Lie(N_\theta)\) are the positive roots which are not linear combinations of the roots in \(\theta\), the condition defining the notion \(P\)-contractive may be replaced by the stronger condition: \(|\gamma_L^\alpha|_\infty \geq 1\) holds for all positive roots in \(\Phi^+\), and \(|\gamma_L^\alpha|_\infty = 1\) holds if and only if \(\alpha\) is a root which occurs in \(Lie(L_\theta)\), or alternatively this could also be replaced by the condition \(|a_\infty^\alpha|_\infty > 1\) for all simple \(\mathcal{O}\)-roots \(\alpha \notin \theta\).

**The Set \(W\).** Let \(\Phi_G = \Phi = \Phi^+ \cup \Phi^-\) be the decomposition into the positive and negative roots of the absolute root system. Define \(W'\) as a subset of the absolute Weyl group \(W\) (considered over the algebraic closure) to consist of the elements \(w \in W\) for which \(\Phi^+ \cap w\Phi^- \subseteq \Phi(Lie(N_P))\) [33], p. 474, or equivalently \(w\Phi^- \cap \Phi_L^+ = \emptyset \Leftrightarrow w^{-1}(\Phi_L^+) \subseteq \Phi_G^+\). Then \(W' = W^P\) is the set of all \(w \in W\) such that \(w^{-1}(\alpha) > 0\) holds for all \(\alpha \in \theta\). By a result obtained by Kostant, \(W\) is the disjoint union of the cosets \(W_L \cdot w\) for representatives \(w \in W^P\); hence \(|W^P| = |W|/|W_L|\). Here \(W_L\) denotes the absolute Weyl group of \(L\), considered as a subgroup of \(W = W_G\). The representatives \(w \in W^P\) are uniquely characterized as the representatives of minimal length in the \(W_L\) left cosets of \(W_G\).
Inductivity. Notice the following inductive property of the sets $W^P \subseteq W$. Let $P = P_\theta \subseteq Q = P_{\theta_2} \subseteq G$ be standard parabolic subgroups, corresponding to $\theta_1 \subseteq \theta_2$. Let $L = L_{\theta_2}$ be the standard Levi component of $P_{\theta_2}$. Then $P_\theta \cap L = P'$ is a standard parabolic subgroup of $L$ with Levi component $L' = L_{\theta_1}$. In particular, $W^{P'} \subseteq W_L$ is defined. Then

$$W^{P'} \cdot W^Q = W^P.$$

In fact $w_1 \cdot w_2 = w'_1 \cdot w'_2$ with $w_1, w'_1 \in W^{P'}$ and $w_2, w'_2 \in W^Q$ implies $W_{L_{\theta_2}} w_2 = W_{L_{\theta_2}} w'_2$; hence, $w_2 = w'_2$ and therefore also $w_1 = w'_1$. By the above-mentioned formula for the cardinalities it is enough to show that the product set on the left side is contained in the right side. But this is clear. Every $w_1^{-1}$ for $w_1 \in W^{P'}$ maps $\Phi(L_{\theta_1})^+$ to $\Phi(L_{\theta_2})^+$ and every $w_2^{-1}$ for $w_2 \in W^{P_{\theta_2}}$ maps $\Phi(L_{\theta_2})^+$ to $\Phi^+ = \Phi(G)^+$; thus, $w_1 w_2 \in W^{P_\theta}$.

Characters. For a dominant weight $\lambda$ of $L$ let $\chi_\lambda$ denote the character of the finite-dimensional irreducible complex representation of $L$ with highest weight $\lambda$. Let $\rho_G$ denote half of the sum of the roots in $\Phi^+$. Similarly define $\rho_L$ for the reductive group $L$. Put $\rho_P = \rho_G - \rho_L$ as characters of $L$. If $\lambda$ is a dominant weight for $G$, then $w(\rho_G + \lambda) - \rho_G$ is dominant for $L$ (see [15], Sect. III.1.4 and Sect. III.3.1, and [45]). Using the Coxeter lengths $l(w)$, define

$$\Psi(\gamma, \lambda) = \sum_{w \in W^P} (-1)^{l(w)} \psi_{w(\lambda + \rho_G) - \rho_G}(\gamma L^1).$$

Since $-\rho_G + \rho_P = -\rho_L$, we have for $\gamma \in L(\mathbb{Q})$

$$|\gamma|_\infty^{-\rho_P} \cdot \Psi(\gamma, \lambda) = \sum_{w \in W^P} (-1)^{l(w)} \psi_{w(\lambda + \rho_G) - \rho_L}(\gamma L^1).$$

The Function $r(\gamma)$. Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_{\text{fin}}$ the ring of finite adeles. Let $K = \prod_{i} K_{i}$, $K_i$ be a compact open subgroup of $G(\mathbb{A}_{\text{fin}})$. For a $\mathbb{Q}$-rational parabolic $P = LN$ and for semisimple $\gamma \in P(\mathbb{Q})$ define $\Gamma = G(\mathbb{Q}) \cap K$, $\Gamma_N = \Gamma \cap N$, $\Gamma' = \Gamma \cap \gamma^{-1} \Gamma \gamma$, $\Gamma_N = \Gamma' \cap N$. Then

$$r = r(\gamma) = [\Gamma_N : \Gamma_N^r] = [\Gamma_N : \Gamma_N \cap \gamma^{-1} \Gamma_N \gamma],$$

$$s = s(\gamma) = [\gamma^{-1} \Gamma_N \gamma : \Gamma_N^s] = [\Gamma_N : \gamma \Gamma_N \gamma^{-1}].$$

satisfy $s(\gamma) = [\Gamma_N : \gamma(\Gamma_N \cap \gamma^{-1} \Gamma_N \gamma) \gamma^{-1}] = [\Gamma_N : \gamma \Gamma_N \gamma^{-1} \cap \Gamma_N] = r(\gamma^{-1})$; hence,

Lemma 2.1. $s(\gamma) = r(\gamma^{-1})$, which only depends on $\gamma_L$.

Lemma 2.2. $s(\gamma) / r(\gamma) = \gamma^{2\rho_P}$ or $|\gamma^{2\rho_P}|_\infty r(\gamma) = |\gamma^{-\rho_P}|_\infty r(\gamma^{-1})$.

Proof. The quotient $[\Gamma_N : \Gamma_N \cap \gamma \Gamma_N \gamma^{-1}]/[\Gamma_N : \Gamma_N \cap \gamma^{-1} \Gamma_N \gamma]$ is the virtual index $[\Gamma_N \cap \gamma^{-1} \Gamma_N \gamma : \Gamma_N \cap \gamma \Gamma_N \gamma^{-1}] / [\Gamma_N \cap \gamma^{-1} \Gamma_N \gamma \gamma^{-1} \cap \Gamma_N \gamma^{-1}] = |\gamma^{2\rho_P}|_\infty$. □
2.2 The Adelic Reductive Borel–Serre Compactification

As a set, the adelic reductive Borel–Serre compactification is

$$(S^G_K)^+ = G(\mathbb{Q}) \setminus \bigcup_X X_L \times (G(A_{fin})/K) = G(\mathbb{Q}) \setminus \bigcup_X X_L \times G(A_{fin})/K,$$

a disjoint union over all $\mathbb{Q}$-rational parabolic subgroups $P$ of $G$. $X_L = L/\hat{K}_{L,\infty}$ is the connected symmetric domain attached to $L$, i.e., $\hat{K}_{L,\infty} = K_L \otimes A_L(\mathbb{R})^0$, where $K_L$ denotes a maximal compact subgroup of $L_\infty$ and $A_L(\mathbb{R})^0$ the topologically connected component of the maximal $\mathbb{Q}$-split subtorus $A_L$ in the center $Z(L)$ of $L$. Elements $g \in G(A_{fin})$ act on the projective limit $(S^G)^+ = lim_{\leftarrow} (S^G_K)^+ = G(\mathbb{Q}) \setminus \bigcup_X X_L \times G(A_{fin})$ by $x \mapsto xg^{-1}$. This defines a left action of $G(A_{fin})$ on $(S^G)^+$, which induces a right action on cohomology groups. Now consider

$$T(g^{-1}) : G(\mathbb{Q})x_\infty x_{fin} \mapsto G(\mathbb{Q})x_\infty(x_{fin}g^{-1}).$$

Here $x_\infty \in \bigcup_X X_L$ and $x_{fin} \in G(A_{fin})$. On the quotients $(S^G_K)^+$ this defines Hecke correspondences. Put

$$K' = K \cap g^{-1}Kg.$$

Then the induced Hecke correspondence is given by two maps $c_1 = T(1)$ and $c_2 = T(g^{-1})$ (see Appendix 1)

$$(S^G_K)^+ \xrightarrow{c_1} (S^G_K)^+ \xrightarrow{c_2} (S^G_K)^+.$$

The action of $G(\mathbb{Q})$ on the $\mathbb{Q}$-parabolic subgroups by conjugation is transitive on the minimal $\mathbb{Q}$-parabolic subgroups. Fixing a minimal parabolic $P_0$, every $\mathbb{Q}$-parabolic is conjugate over $\mathbb{Q}$ to one and only one standard $\mathbb{Q}$-subgroup $P$ with respect to $P_0$. Since the stabilizer of $P$ under conjugation with $G(\mathbb{Q})$ is $P(\mathbb{Q})$, $(S^G_K)^+$ is a union over the finitely many standard $\mathbb{Q}$-parabolic subgroups $P = P_0$ containing $P_0$:

$$(S^G_K)^+ = \bigcup_{P_0 \leq P} S^P_K, \quad \text{where} \quad S^P_K = P(\mathbb{Q}) \setminus [X_L \times G(A_{fin})]/K.$$

Goresky and MacPherson [33] deduced a formula for the alternating trace

$$tr_\lambda(T(g^{-1}); H^\cdot(S_K(G), V_\lambda))$$

from the Grothendieck–Verdier–Lefschetz fixed-point formula which they applied for the reductive Borel–Serre compactification $(S^G_K)^+$ of $S_K(G)$. They used the property that the cohomology groups $H^\cdot(S_K(G), V_\lambda)$ coincide with the cohomology groups $H^\cdot((S^G_K)^+, i, V_\lambda)$ of the reductive Borel–Serre compactification $(S^G_K)^+$, where $i : S_K(G) \hookrightarrow (S^G_K)^+$ is the inclusion. As in [33], Theorem
(version 0), the Lefschetz fixed-point theorem of Grothendieck, Verdier, and Illusie therefore expresses the Lefschetz number as a sum of “local” contributions $LC(F)$

$$\sum_P \sum_F LC(F)$$

for the connected components $F$ of the intersection of the fixed-point set of the correspondence with the boundary strata $S^P_K$ attached to the rational parabolic group $P$.

**Rational Hecke Correspondences.** We say a double coset $KgK$ or the corresponding Hecke correspondence is rational if a representative $g$ can be chosen to be $g = \gamma_{fin}$ for some $\gamma = \gamma_\infty \gamma_{fin} \in G(\mathbb{Q})$. In this case the correspondence $T(g^{-1})$ defined on $(S^G)^+ = G(\mathbb{Q}) \setminus \bigcup_P X_L \times G(\mathbb{A}_{fin})$ satisfies $G(\mathbb{Q})x_\infty \mapsto G(\mathbb{Q})_{\gamma_\infty x_\infty} = G(\mathbb{Q})_{\gamma_\infty x_\infty}$; hence, it induces the Hecke correspondence considered in [33], p. 467, defined by $c_1(\Gamma' y) = \Gamma y$ and $c_2(\Gamma' y) = \Gamma \gamma_\infty y$.

### 2.2.1 Components

First consider the connected components of $S^P_K$. Since $X_L$ is topologically connected, the topologically connected components $h$ of the stratum $S^P_K$ are the fibers of the map

$$S^P_K = P(\mathbb{Q}) \setminus [X_L \times (G(\mathbb{A}_{fin})/K)] \longrightarrow \pi_0(S^P_K) = P(\mathbb{Q}) \setminus G(\mathbb{A}_{fin})/K.$$  

For each component $h = P(\mathbb{Q})x_{fin}K$ in $\pi_0(S^P_K)$ put

$$\Gamma_p = P(\mathbb{Q}) \cap x_{fin}Kx_{fin}^{-1} \quad \text{and} \quad \Gamma_{N_h} = N(\mathbb{Q}) \cap x_{fin}Kx_{fin}^{-1}.$$  

For $K_N(h) := N(\mathbb{A}_{fin}) \cap x_{fin}Kx_{fin}^{-1}$ and $K'_N(h) := N(\mathbb{A}_{fin}) \cap x_{fin}K'_x_{fin}x_{fin}^{-1}$ then obviously $[K_N(h) : K'_N(h)] = [\Gamma_{N_h} : \Gamma'_{N_h}]$, where $\Gamma'_{N_h} := K'_N(h) \cap N(\mathbb{Q})$.

**Fixed Components.** Now consider the connected components $F$ of the fixed-point locus of a Hecke correspondence within $S^P_K$, for fixed $P$. Then

$$F \subseteq h$$

for some unique component $h$ of $S^P_K$. If $F$ is fixed, then $h$ is also fixed. So we first determine the fixed components $h$ of the Hecke correspondence, and then the fixed components $F$ in $h$.

### 2.2.2 Fixed Components $h$

The component $h = P(\mathbb{Q})(X_L \times \{x_{fin}\})K$ is fixed

$$T(g^{-1})h = h,$$
if and only if $x_{fin}g^{-1}K = \gamma x_{fin}K$ holds for some $\gamma \in P(\mathfrak{q})$ and some $k \in K$.

Recall $gK'g^{-1} \subseteq K$. Hence, $\gamma^{-1}x_{fin}g^{-1} = x_{fin}k$ implies $\gamma^{-1}x_{fin}K'x_{fin}^{-1} \subseteq x_{fin}Kx_{fin}^{-1}$. Thus,

$$\gamma^{-1}\Gamma_h \gamma := P(\mathfrak{q}) \cap \gamma^{-1}x_{fin}K'(\gamma^{-1}x_{fin})^{-1} \subseteq P(\mathfrak{q}) \cap x_{fin}Kx_{fin}^{-1} =: \Gamma_{ph}.'$$

$x_{fin}K'x_{fin}^{-1} = x_{fin}Kx_{fin}^{-1} \cap x_{fin}g^{-1}K(x_{fin}g^{-1})^{-1} = x_{fin}Kx_{fin}^{-1} \cap (x_{fin}Kx_{fin}^{-1})^{-1}$ again using $x_{fin}g^{-1} = \gamma x_{fin}k$. Hence, the intersection with $P(\mathfrak{q})$ is $\Gamma'_{ph} = \Gamma_{ph} \cap \gamma \Gamma_{ph} \gamma^{-1}$. In particular, $\Gamma'_{Nh} = (\Gamma_{Nh} \cap \gamma \Gamma_{Nh} \gamma^{-1})$. Hence, the fixed equation $T(g^{-1})h = h$ given by $\gamma^{-1}x_{fin}g^{-1} = x_{fin}k$ implies.

**Lemma 2.3.** $[K_N(h) : K_N'(h)] = [\Gamma_{Nh} : \Gamma_{Nh}'] = [\Gamma_{Nh} : (\Gamma_{Nh} \cap \gamma \Gamma_{Nh} \gamma^{-1})] = r(\gamma^{-1})$. For fixed $g, K$ this number only depends on $P$ and the coset $P(\mathfrak{a}_{fin})x_{fin}$.

**Rationality.** To simplify the notation we now replace $K$ by $x_{fin}Kx_{fin}^{-1}$, and $g^{-1}$ by $x_{fin}g^{-1}x_{fin}^{-1}$, which allows us to assume $x_{fin} = 1$ without restriction of generality. Then the fixed-component equation becomes $\gamma \in g^{-1}K$. Hence, the coset $g^{-1}K \subseteq Kg^{-1}K$ has a rational point, and the Hecke correspondence defined by $Kg^{-1}K = K\gamma K$ is rational. For a fixed component $h$ one can thus reduce the local computations of the local term $LC(F)$ for $F$ to the classical setting considered in [33].

### 2.2.3 Another Formulation

The action of $G(\mathfrak{q})$ on the $\mathfrak{q}$-parabolic subgroups by conjugation is transitive on the minimal $\mathfrak{q}$-parabolic subgroups. Hence, choosing a minimal $\mathfrak{q}$-parabolic $P_0$, every $\mathfrak{q}$-parabolic is conjugate over $\mathfrak{q}$ to one and only one standard parabolic $\mathfrak{q}$-subgroup $P$ with respect to $P_0$. Since the stabilizer of $P$ under conjugation with $G(\mathfrak{q})$ is $P(\mathfrak{q})$, $(S_G^b)^+$ is a union over the finitely many standard $\mathfrak{q}$-parabolic subgroups $P = P_0$ containing $P_0$,

$$\left(S_G^b\right)^+ = \bigcup_{P_0 \subset P} P(\mathfrak{q}) \setminus [X_L \times G(\mathfrak{a}_{fin})]/K.$$

Since $gKg^{-1} \cap N_P(\mathfrak{a}_{fin})$ is open in $N_P(\mathfrak{a}_{fin})$ for $P = L_P N_P$, for the strata $S_K^P = P(\mathfrak{q}) \setminus [X_L \times G(\mathfrak{a}_{fin})]/K$ of

$$\left(S_G^b\right)^+ = \bigcup_{P_0 \subset P} S_K^P$$

an easy density argument gives the formula $S_K^P = L_P(\mathfrak{q}) \setminus [L_\infty \times G(\mathfrak{a}_{fin})]/K_{\infty}K$ or

$$S_K^P = L_P(\mathfrak{q}) N_P(\mathfrak{a}_{fin}) \setminus [L_\infty \times G(\mathfrak{a}_{fin})]/K_{\infty}K.$$
Hence, \( \pi \in (S_K^G)^+ \) is a double coset represented by some \( x = x_\infty x_{\text{fin}} \in L_\infty \times G(A_{\text{fin}}) \). Iwasawa decomposition \( G(A_{\text{fin}}) = P(A_{\text{fin}}) \cdot \Omega \) for some maximal compact group \( \Omega \) containing the group \( K \) gives a finite decomposition \( G(A_{\text{fin}}) = \bigcup_g P(A_{\text{fin}})gK \). Therefore, the set \( \pi_0(S_K^G) \) of the topologically connected components is finite, since by a result obtained by Borel and Harish-Chandra [14], \( M(Q) \setminus M(A_{\text{fin}})/K \) is finite for any reductive \( Q \)-group \( M \) and any compact open group \( K_M \subseteq M(A_{\text{fin}}) \). Of course we may choose the representatives elements \( x_{\text{fin}} = k \in \Omega \).

### 2.2.4 Small Groups

Consider compact open subgroups \( K \subseteq G(A_f) \) and \( \tilde{K}_\infty = K_\infty Z_G^0, \) where \( K_\infty \) is maximal compact in \( G_\infty, K \subseteq G(A_f) \) will be called small if

\[
x^{-1} n\gamma x \in KK_\infty Z_{L,\infty}
\]

for \( x \in G(A), n \in N(A), \gamma \in P(Q) \) and any \( Q \)-parabolic \( P = L \cdot N \) with unipotent radical \( N \) implies \( \gamma L \in Z_L(Q) \) (image in the Levi component is contained in the center) and in addition implies \( \gamma L = 1 \) if \( \gamma L \) is a torsion element.

**Remark 2.2.** Of course it is enough to demand this for all standard parabolic groups containing a fixed \( P_0 \).

**Remark 2.3.** “Small” implies “neat” in the sense that \( L(Q)_{\text{tor}} \cap (xKx^{-1} \cap P)_L = 1 \).

Small-level groups \( K \) exist: \( G(A) \) is a finite union of cosets \( P(A)kKK_\infty \) for \( k \in G(A_{\text{fin}}) \). This allows us to replace \( K \) by some conjugate \( K_k \), and \( x \) by some \( p \in P(A) \), and gives equations \( p^{-1} n\gamma p \in K_k \) for \( p \in P(A) \) instead of \( x \in G(A) \). Equivalently, \( m^{-1} \gamma L m \in (K_k \cap P)_L \) for \( m \in L(A) \), where the index \( L \) indicates projection from \( P \) to the Levi component \( L \). \( \gamma L \) is semisimple since modulo the center it is contained in a maximal compact subgroup of \( L_\infty \). The groups \( L \) and \( L_{\text{ad}} = L/Z_L \) are connected reductive groups; hence, by embedding \( L_{\text{ad}} \) into some linear group and using for \( L_{\text{ad}} \) the argument at the beginning of the proof [44], Proposition 8.2, one can show that only finitely many \( L(Q) \) conjugacy classes of semisimple elements \( \gamma_L \) in \( L_{\text{ad}}(Q) \) meet \( (K_k \cap P)_L \). Shrinking \( K \) leaves us, considering eigenvalues, with the unique \( Q \)-conjugacy class \{1\}. Thus, \( \gamma_L \in Z_L(Q) \). Finally, \( Z_L(Q)_{\text{tor}} \) is finite (consider a splitting field of \( Z_L \)). Since it is enough to consider the finitely many standard parabolic groups \( P \) and for each finitely many cosets \( k \), shrinking \( K \) therefore allows us to assume \( Z_L(Q)_{\text{tor}} \cap (K_k \cap P)_L = \{1\} \) for the finitely many relevant cases.
2.3 Fixed Points

Now we want to determine the fixed points \( \gamma \) of the Hecke correspondence \( T(g^{-1}) \) in the reductive Borel–Serre compactification \( (S^P_K)^+ \). They are described by the equations \( c_1(\gamma) = T(1)\gamma \) and \( c_2(\gamma) = T(g^{-1})\gamma \) in \( (S^P_K)^+ \). The unique component \( h \) containing \( \gamma \) is necessarily a fixed component. \( h \) is contained in a stratum \( S^P_K \). Now fix the standard parabolic \( P = L_P N_P \), or \( P = LN \) for short.

Suppose \( \gamma \in S^P_K \) is represented by \( x = x_\infty x_f \in L_\infty \times G(\mathfrak{A}_{fin}) \). Then \( \gamma \) is a fixed point of \( T(g^{-1}) \) if and only if \( xg^{-1} = \gamma \cdot x \cdot k \) holds for some \( \gamma \in P(\mathbb{Q}) \) and \( k \in \bar{K}_{L,\infty} K \), or equivalently if and only if

\[
x^{-1} \gamma x \in g^{-1} K \bar{K}_{L,\infty}.
\]

We may replace \( x \) by another representative \( \delta^{-1} x, \delta \in P(\mathbb{Q}) \). Then instead of \( \gamma \) its conjugate \( \delta \gamma \delta^{-1} \) appears in the fixed-point equation. Moreover

**Lemma 2.4.** The element \( \gamma \) is semisimple and \( \mathbb{R} \)-elliptic. For small \( K \) the \( (\mathbb{Q},\mathbb{Q}) \)-conjugacy class of the image \( \gamma_L \) of \( \gamma \) in \( L(\mathbb{Q}) \) is uniquely determined by the fixed point \( \gamma \in S^P_K \).

**Proof.** The equation \( x^{-1}_\infty \gamma_L x_\infty \in \bar{K}_{L,\infty} \) implies that \( \gamma_L \in L(\mathbb{Q}) \) is \( \mathbb{R} \)-elliptic, hence semisimple. Now choose an equivalent representative \( \delta nxk' \) for \( \gamma \) for some \( n \in N(\mathfrak{A}_{fin}), \delta \in P(\mathbb{Q}), k' \in \bar{K}_{L,\infty} K' \). Suppose \( xg^{-1} = \gamma_1 xk_1 \) and \( (\delta nxk')g^{-1} = \gamma_2(\delta nxk')k_2 \) holds for \( \gamma_i \in P(\mathbb{Q}), k_i \in \bar{K}_{L,\infty} K \). Replacing \( k_2 \) by \( k'k_2(gk'g^{-1})^{-1} \) allows us to assume \( k' = 1 \). Replacing \( \gamma_2 \) by \( \delta^{-1} \gamma_2 \delta \) allows us to assume \( \delta = 1 \). Hence, \( \gamma_L xk_1 = xg^{-1} = n^{-1} \gamma_2 nxk_2 \). Since \( K \) is small, this implies \( \gamma_L \in Z_L(\mathbb{Q}) \) for \( \gamma = \gamma_1^{-1} \gamma_2 \) and hence \( \gamma \) commutes with \( x_\infty \), which then implies \( \gamma_L \in K_{L,\infty} K_L \), where \( K_L \) is the image of \( K \cap P(\mathfrak{A}_{fin}) \) in \( L(\mathfrak{A}_{fin}) \). Thus, \( \gamma_L \in Z_L(\mathbb{Q}) \cap \bar{K}_{L,\infty} K_L \). Looking at the Archimedean place and the non-Archimedean places separately, this forces \( \gamma_L \) to be a torsion element. Therefore, \( \gamma_L = 1 \), since \( K \) is small.

This lemma gives a decomposition of the fixed-point set in the stratum \( S^P_K \), according to the conjugacy classes \( \gamma_L \in L(\mathbb{Q})/\sim \).

**Fixing \( \gamma_L/\sim \).** We want to determine the set \( Fix(\gamma_L) \) of all fixed points \( \gamma \in S^P_K \) of \( T(g^{-1}) \), where in the fixed-point equation for some representative an element \( \gamma \) appears whose projection to \( L(\mathbb{Q}) \) belongs to the fixed conjugacy class \( \gamma_L/\sim \). To unburden the notation we also write \( Fix(\gamma) \) instead of \( Fix(\gamma_L) \),

\[
Fix(\gamma) \subseteq S^P_K = L_P(\mathbb{Q}) \setminus [(L_\infty/\bar{K}_{L,\infty}) \times (N(\mathfrak{A}_{fin}) \setminus G(\mathfrak{A}_{fin}))/K'].
\]

For \( x = x_\infty x_f \in L_\infty \times G(\mathfrak{A}_{fin}) \) the double coset \( \gamma = L(\mathbb{Q})N(\mathfrak{A}_{fin})x \bar{K}_{L,\infty} K' \) is in \( Fix(\gamma) \) if and only if there exist \( n \in N(\mathfrak{A}_f), \delta \in P(\mathbb{Q}), k \in \bar{K}_{L,\infty} K \) such that \( n(\delta \gamma \delta^{-1})xk = xg^{-1} \) holds, or equivalently if and only if there exist \( n \in N(\mathfrak{A}_f), \delta \in P(\mathbb{Q}), k \in \bar{K}_{L,\infty} K \), such that

\[
(*) \quad x^{-1} n \delta \gamma \delta^{-1} x \in g^{-1} \bar{K}_{L,\infty} K,
\]

since we are free to replace \( n \) by \( n \delta \gamma \delta^{-1} \).
By abuse of notation we do not distinguish between global elements $\gamma, \delta$ in $G(\mathbb{Q})$ and their images in $G_v$ or $G(\mathbb{A}_{fin})$. Since $x$ is considered in $S^{\mathcal{P}} = P(\mathbb{Q}) \setminus [X_L \times G(\mathbb{A}_{fin})]/K'$, we may replace $x$ by $\delta^{-1}x$ and $n$ by $n\delta$, which simplifies the equations for $Fix(\gamma)$. Hence, we get

**Lemma 2.5.** We have $Fix(\gamma) \cong L(\mathbb{Q}) \setminus \left( L(\mathbb{Q}) \cdot \prod_v Sol_v(\gamma) \right)$, where

$$Sol_v(\gamma) = \{ x_v \in N(\mathbb{Q}_v) \setminus G(\mathbb{Q}_v)/K'_v \mid x_v^{-1} n_\gamma x_v \in g_v^{-1} K_v \text{ holds for some } n \in N_v \}$$

at the non-Archimedean places $v$ and $K'_v = K_v \cap g_v^{-1} K_v g_v$, and where

$$Sol_{\infty}(\gamma) = \{ x_\infty \in L_\infty(\mathbb{R})/K_{L,\infty} \mid x_\infty^{-1} \gamma x_\infty \in \tilde{K}_{L,\infty} \}$$

at the Archimedean place $\infty$.

By abuse of notation we write $L_\gamma$ for the centralizer $L_\gamma$ of the element $\gamma_L$ in $L$, which is a connected reductive group by the assumption that $G_{der}$ is simply connected.

**Corollary 2.1.** For small $K$ we obtain $Fix(\gamma) = L_\gamma(\mathbb{Q}) \setminus Sol(\gamma)$ for $Sol(\gamma) = \prod_v Sol_v(\gamma)$.

**Proof.** $L(\mathbb{Q})$-equivalent solutions in $Sol(\gamma)$, say, $x^{-1} n_1 \gamma x$ and $x^{-1} n_2 \delta^{-1} \gamma \delta x$ in $g^{-1} K K_{\infty}$ for suitable $n_1, n_2 \in N(\mathbb{A}_{fin})$ and $\delta \in P(\mathbb{Q})$, satisfy $x^{-1} n_2 \delta^{-1} \gamma \delta^{-1} \gamma^{-1} n_1 x \in K K_{\infty}$ for some $n \in N(\mathbb{A})$. We may then assume $n_1 = 1$, and since $K$ is small, this implies $\delta^{-1} \gamma \delta \gamma^{-1} \in Z_L(\mathbb{Q})$. Since the commutator $\delta^{-1} \gamma L \delta \gamma L^{-1}$ is in $L_{der}(\mathbb{Q})$, and since $Z_L(\mathbb{Q}) \cap L_{der}(\mathbb{Q})$ is finite, the commutator is a torsion element, and hence is 1 since $K$ is small. This implies $\delta \in L_\gamma(\mathbb{Q})$ and completes the proof. \(\square\)

### 2.3.1 Archimedean Place

$Sol_{\infty}(\gamma) \cong L_{\gamma,\infty}/(L_{\gamma,\infty} \cap \tilde{K}_{L,\infty})$ by the corollary in Appendix 2, unless it is empty. If it is nonempty, the Archimedean fixed-point condition shows that $\gamma_L$ is $L_{\infty}$-conjugate to a point in $\tilde{K}_{L,\infty}$. To determine $L_{\gamma,\infty}$ we may therefore assume $\gamma_L \in \tilde{K}_{L,\infty}$ without restriction of generality. Hence, the centralizer $L_{\gamma,\infty}$ becomes $\theta$-stable for the Cartan involution $\theta$ (see Appendix 2). Therefore, $K_{\infty} \cap L_{\gamma,\infty}$ is a maximal compact subgroup $K_{L,\gamma,\infty}$ of $L_{\gamma,\infty}$. Since $A_L(\mathbb{R})^0 \subseteq L_{\gamma,\infty}$, $Sol_{\infty}(\gamma) = L_{\gamma,\infty}/(K_{L,\gamma,\infty} A_L(\mathbb{R})^0)$ admits a smooth surjective map to the symmetric space $X_{L,\gamma} = L_{\gamma,\infty}/(K_{L,\gamma,\infty} A_L(\mathbb{R})^0)$ of the centralizer $L_\gamma$, which defines a trivial fibration by the Euclidean space $A_L(\mathbb{R})^0/A_L(\mathbb{R})^0$, and hence a homotopy equivalence. See the Remark 2.15 in Appendix 2.
2.3 Fixed Points

2.3.2 Non-Archimedean Places

Recall that \( T(g^{-1}) \) and \( \gamma/\sim \) are now fixed. Let \( \Omega_v \) be the stabilizer of a special point in the Bruhat–Tits building. Special points always exist, and \( \Omega_v \) is a maximal compact subgroup of \( G_v \). We now assume \( K_v \subseteq \Omega_v \); hence, \( K'_v \subseteq \Omega_v \). Then by the Iwasawa decomposition \( G_v = P_v \cdot \Omega_v \) (see [103], Sect. 3.3.2). For \( k \in \Omega_v \) put \( g_v^{-1} = k g^{-1} k^{-1} \), \( K_k = k K_v k^{-1} \), and \( K'_k = k K'_v k^{-1} \). Elements \( x_v \in \text{Sol}_v(\gamma) \) may be written \( x_v = p \cdot k \) for \( p \in P_v \) and \( k \in \Omega_v \). The coset \((P_v \cap \Omega_v)kK'_v\) is uniquely determined by \( x_v \), and \( G/K' = \bigcup_{k \in P_v \cap \Omega_v} \Omega_v/(P_v \cap kK'_v k^{-1}) \).

Therefore, \( \text{Sol}_v(\gamma) = \bigcup_{k \in P_v \cap \Omega_v} \text{Sol}_v(\gamma, k) \). Here \( \text{Sol}_v(\gamma, k) = \{ p \in N_v \setminus P_v/(P_v \cap kK'_v k^{-1}) \mid p L^{-1} \gamma L p \in (g_v^{-1} K_k \cap P)_v \} \) or

\[
S_v(\gamma, k) = \left\{ m \in L_v \mid m^{-1} L^{-1} m \in (g_v^{-1} K_k \cap P)_v \right\} = \bigcup_{\xi_v} L(\xi_v) \cdot L_v \cdot K'(k)_v.
\]

This is a finite (possibly empty) union over representatives \( \xi_v \in L_v \). From [53], Propositions 7.1 and 8.2, there is only one representative \( \xi_v = 1 \) for almost all \( v \).

2.3.3 Globally

With this notation

\[
\text{Sol}(\gamma, k) = S(\gamma, k)/K'(k)_A \quad \text{for} \quad K'_A(k) = \bar{K}_L \cdot \prod_{v \text{fin}} K'(k)_v,
\]

where \( S(\gamma, k) = \left\{ m \in L(A) \mid m^{-1} L^{-1} m \in (g_v^{-1} K_k \cap P(A_{\text{fin}}))_L \right\} \cdot L_\gamma(A) \) acts on \( S(\gamma, k) \) from the left. Choose a decomposition

\[
S(\gamma, k) = \bigcup_{\xi} L(\xi) \cdot \xi \cdot K'(k)_A
\]

with representatives \( \xi \in L(A) \), where representatives \( \xi = \prod_v \xi_v \) are chosen to be products of corresponding local non-Archimedean representatives \( \xi_v \) for \( L(\Omega_v) \setminus S_v(\gamma, k)/K'(k)_v \), and \( \xi_\infty = 1 \). Then
Lemma 2.6. For small $K$ the contribution of a fixed conjugacy class $\gamma / \sim$ in $L(\mathbb{Q})$ to the fixed-point locus of $T(g^{-1})$ is

$$\text{Fix}(\gamma) \cong \bigsqcup_{k \in P(\mathbb{A}_{\text{fin}}) \cap \Omega / K'} \text{Fix}(\gamma, k),$$

$$\text{Fix}(\gamma, k) \cong \bigsqcup_{\xi} L_{\gamma}(\mathbb{Q}) \setminus L_{\gamma}(\mathbb{A})/(\xi K'(k)A_k^{-1} \cap L_{\gamma}(\mathbb{A})).$$

Of course $L_{\gamma}(\mathbb{Q}) \setminus L_{\gamma}(\mathbb{A})/(\xi K'(k)A_k^{-1} \cap L_{\gamma}(\mathbb{A})) = \bigsqcup_{F} F_{\nu}$ is a finite union of arithmetic quotients $F_{\nu}$.

2.4 Lefschetz Numbers

The Lefschetz number becomes

$$\sum_{P} \sum_{\gamma / \sim} \sum_{k} \sum_{\xi} \sum_{\nu} LC(F_{\nu}),$$

where $k \in P \cap \Omega \setminus \Omega / K'$. Put $F = F_{\nu}$. For the local terms $LC(F)$ Goresky and MacPherson gave an explicit description as a product $\chi(F) r(\gamma F) \Psi(\gamma F, \lambda)$ if $\gamma F$ is $P$-contractive, and it vanishes otherwise. See [33], pp. 470–471 and Theorem (version 3a), p. 474. Here $\gamma F = \gamma^{-1}$ is the characteristic element defined in [33], p. 469, which is the inverse of the element $\gamma$ defined in Lemma 2.4. Hence, if it is nonvanishing, the local number $LC(F)$ is the product of:

- The Euler characteristic $\chi(F)$
- $|\gamma F|^{\rho_P} \cdot r(\gamma F, k)$
- $|\gamma F|^{\rho_L} \cdot \Psi(\gamma F, \lambda) = \sum_{w \in W_{P}} (-1)^{(w)} \psi_{w(\lambda + \rho_G) - \rho_L (\gamma F^{-1})}$

2.4.1 Euler Characteristics

We may sum the terms $\sum_{\nu} \chi(F_{\nu})$ for fixed $P, \gamma / \sim, k, \xi$, which gives the Euler characteristic

$$\chi(L_{\gamma}(\mathbb{Q}) \setminus L_{\gamma}(\mathbb{A})/(\xi K'(k)A_k^{-1} \cap L_{\gamma}(\mathbb{A}))).$$

To compute it we may replace $L_{\gamma, \infty}/K_{L, \infty}$ by $X_{L_\gamma} = L_{\gamma, \infty}/K_{L, \infty}$. See page 44. Notice $L_{\gamma}(\mathbb{Q}) \cap \xi K_{L_{\gamma, \infty}} K'(k)A_k^{-1}$ is contained in the center of $L_{\gamma}$, since $K$ is small. Hence, the intersection is discrete and compact, and hence we have a finite
group. By our assumption $K$ is small; hence, the intersection is trivial. Thus, we obtain

$$
\sum_{\nu} \chi(F_{\nu}) = \frac{\chi(L_{\gamma}, dg_{fin})}{vol_{dg_{fin}}(\xi K'(k)_{A_{fin}})}
$$

for a constant $\chi(L_{\gamma}) = \chi(L_{\gamma}, dg_{fin})$ depending only on $L_{\gamma}$ and on the choice of the Haar measure $dg_{fin}$ on $L_{\gamma}(A_{fin})$.

**Remark 2.4.** Observe that $\gamma$ is $R$-elliptic, and hence is $Q$-elliptic. For the ambiguity of this notion, see [44], p. 392. We show that in our situation this ambiguity does not cause problems, since the Euler characteristic of the corresponding summands vanishes unless both notions agree. Consider the group $L$ or better $L/A_{L}$. The center $Z(L_{\gamma})$ is $Q$-anisotropic modulo $A_{L}$. If the quotient were not anisotropic over $R$, the corresponding global quotient space $X$ would be a nontrivial torus fibration, whose Euler characteristic would therefore vanish. Similarly the Euler characteristic vanishes for locally symmetric arithmetic quotients of semisimple groups unless the $R$-rank of the maximal compact subgroup equals the $R$-rank of the group. Considering the map $L_{der} \rightarrow L$, we can assume that $(L_{\gamma}/A_{L})(R)$ contains an $R$-anisotropic torus of maximal rank, or otherwise the Euler characteristic vanishes and the corresponding summand does not contribute to the trace formula.

**Definition 2.1.** Call $\gamma \in L(Q)$ strongly elliptic if $\gamma$ is $L(R)$-conjugate to an element in $K_{L,\infty} \cdot A_{L}(R)^{0}$ such that the Euler characteristic $\chi(L_{\gamma})$ does not vanish.

**Remark 2.5.** For connected reductive groups $L$ over $Q$, for which the connected component of the center modulo $A_{L}$ is anisotropic over $R$, one also wants to compare $\chi(L, dg_{fin})$ with the Tamagawa number. At the moment we do not need to carry through this comparison. When we need it later, it can be obtained directly from a comparison between the topological $L^{2}$-trace formula and Arthur’s $L^{2}$-trace formula. On the other hand, it should not be difficult to obtain it by reduction to the case of semisimple groups (Harder’s theorem [37]) adapting the argument of [68], pp. 129–131, with a $z$-extension $T' \rightarrow L' \rightarrow L$ replacing the sequence $(V)$, and $L = (L')_{der} \rightarrow L' \rightarrow T$ replacing the sequence $(H)$ in [68].

Only (semisimple) strongly elliptic elements $\gamma$ contribute to the Lefschetz number. Let $\chi^{G}_{P}$ be the characteristic function of the $P$-contractive elements. We obtain for the Lefschetz number the expression

$$
\sum_{P} \sum_{\gamma/\sim} \chi(L_{\gamma}, dg_{fin}) \chi^{G}_{P}(\gamma_{\infty}^{-1}) \sum_{w \in W_{P}} (-1)^{l(w)} \psi_{w(\lambda + \rho_{G})} - \rho_{L}(\gamma) \cdot O_{\gamma}
$$

$$
= \sum_{P} \sum_{\gamma/\sim} \chi(L_{\gamma}, dg_{fin}) \chi^{G}(\gamma_{\infty}) \sum_{w \in W_{P}} (-1)^{l(w)} \psi_{w(\lambda + \rho_{G})} - \rho_{L}(\gamma^{-1}) \cdot O_{\gamma^{-1}},
$$

where

$$
O_{\gamma} = \sum_{k} \sum_{\xi} \frac{|\gamma^{-1}|_{R}^{\rho_{P} \cdot r(\gamma^{-1}, k)}}{vol_{dg_{fin}}(\xi K'(k)_{A_{fin}})}.
$$
Here we used \( r(\gamma, k) = r(\delta \gamma \delta^{-1}, k) \) for \( \delta \in L(\mathbb{Q}) \). To show this, recall \( k = x_{\text{fin}} \) describes the component \( h \) of the \( P \)-stratum, which contains \( F \). Recall \( r(\gamma, x_{\text{fin}}) = r(\delta \gamma \delta^{-1}, \delta x_{\text{fin}}) \) for \( \delta \in P(\mathbb{Q}) \). Since \( r(\gamma, x_{\text{fin}}) \) depends only on \( P \) and the coset \( P(A_{\text{fin}})x_{\text{fin}} \) (Lemmas 2.1 and 2.3), replacing \( \gamma \) by a conjugate does not change \( r(\gamma, k) \).

### 2.4.2 Computation of \( O_\gamma \)

Notice \(|\gamma^{-1}|^p \cdot r(\gamma^{-1}) = \prod_{v \neq \infty} |\gamma|^{p_v} [N_v \cap K_k : N_v \cap K'_k] \) by Lemma 2.3. Since

\[
\sum_{k \in (P_v \cap \Omega_v) \Omega_v/K'_v} \sum_{\xi_v} f(k) = \sum_{k \in \Omega_v/K_v} \frac{f(k)}{[P_v \cap \Omega_v : (P_v \cap K'_v)]},
\]

this allows us to write \( O_\gamma \) as a product \( \prod_{v \neq \infty} O_{\gamma, v} \) of non-Archimedean local terms

\[
O_{\gamma, v} = \sum_{k \in \Omega_v/K'_v} \sum_{\xi_v} \frac{|\gamma|^{p_v} \cdot [N_v \cap K_k : N_v \cap K'_k]}{[P_v \cap \Omega_v : (P_v \cap K'_v)] \cdot vol_{dg_v}(\xi_v K'(k) \xi_v^{-1} \cap L_{\gamma, v})}.
\]

Since

\[
0 \to N_v \cap K'_k \to P_v \cap K'_v \to K'(k)_v \to 0
\]

is exact, this gives

\[
O_{\gamma, v} = \sum_{k \in \Omega_v/K'_v} \sum_{\xi_v} \frac{|\gamma|^{p_v} \cdot vol_{N_v}(N_v \cap K_k) \cdot vol_{L_v}(K'(k)_v)}{vol_{dg_v}(\xi_v K'(k)_v \xi_v^{-1} \cap L_{\gamma, v})},
\]

where measures are normalized such that \( vol(\Omega_v \cap P_v) = 1 \) and \( vol(\Omega_v \cap N_v) = 1 \). In Sect. 2.5 we show that this expresses \( O_\gamma \) as an orbital integral

\[
O_\gamma = O^L_\gamma(f^{(P)}),
\]

of the characteristic function \( f \) of the set \( K g^{-1} K \) up to a normalization factor.

### 2.4.3 Conclusion

The computations in Sects. 2.4.1 and 2.4.2 describe the right action of \( 1_{K g K}/vol_{dg}(K) \) on the cohomology. Any \( K \)-bi-invariant function \( f \) is a linear combination of functions \( f \) as above. However, we should keep in mind that so far we have used a left action of \( G(A_{\text{fin}}) \) on \( S^G \), where \( g \in G(A_{\text{fin}}) \) acts by the formula on page 23; hence, the cohomology becomes a right module under the Hecke algebra.
Theorem 2.1. Assume the derived group of $G$ is simply connected and $K$ is small. Then the Lefschetz number of the right action of a $K$-bi-invariant Hecke operator $f \in C_c^\infty(G(A_{f,\mathrm{fin}}))$ on the cohomology $H^*(S_K(G), V_\lambda)$ is given by

$$L(f, V_\lambda) = \sum_{P} \sum_{\gamma \in L(\mathbb{Q})/\sim} \chi(L_\gamma, dg_{f,\mathrm{fin}}) O^L_{\gamma}(f(P)) \cdot |\gamma|_\infty^{-\rho} \Psi(\gamma, \lambda).$$

The sum extends over all standard $\mathbb{Q}$-parabolic subgroups $P = LN$ containing the fixed minimal $\mathbb{Q}$-parabolic $P_0$ and all $L(\mathbb{Q})$-conjugacy classes $\gamma \in L(\mathbb{Q})/\sim$ of semisimple, strongly elliptic elements in $L(\mathbb{Q})$ with $P$-contractive representatives.

Example 2.2. $G = G_m$ for the representation $x \mapsto x^r$ of weight $\lambda = r$ on $V = \mathbb{C}$, and $K$ maximal compact. Then $S_K$ is a single point. The element $g = (g_v) \in A_{f,\mathrm{fin}}^\times$, where $g_p = p$ and $g_v = 1$ for $v \neq p$, acts on $V_\lambda$ by $1 \times 1 \mapsto 1 \times g^{-1} \simeq p^{-r} \times 1$ in $V \times A_{f,\mathrm{fin}}^\times$. It acts on the cohomology via multiplication by $p^{-r}$ (right action on cohomology) or $p^r$ (left action on cohomology).

Remark 2.6. Notice we used

$$O^L_{\gamma^{-1}}(f(P)) = O^L_{\gamma}(f^{-1}(P)).$$

For the comparison of trace formulas with those in [64] in Chap. 3, we may turn the right action of the Hecke algebra on the cohomology groups into a left action by the substitution $f(x) \mapsto f^-(x) = f(x^{-1})$. This makes the formula compatible with that in [64], p. 197.

Remark 2.7. The factor $\chi(L_\gamma, dg_{f,\mathrm{fin}}) O^L_{\gamma}(\cdot)$ does not depend on the choice of the fixed Haar measure $dg_{f,\mathrm{fin}}$ on $L_\gamma(A_{f,\mathrm{fin}})$; therefore, we do not mention the choice of $dg_{f,\mathrm{fin}}$ in the following.

Remark 2.8. The condition imposed in Theorem 2.1 that $\gamma \in L(\mathbb{Q})/\sim$ contains a $P$-contractive representative $\gamma \in P(\mathbb{Q})$ can be replaced by the stronger condition that $|\alpha(\gamma)|_\infty \geq 1$ holds for all positive roots $\alpha$ of $G$ and $|\alpha(\gamma)|_\infty = 1$ holds if and only if $\alpha$ is a root from $L$ as explained after the definition of contractiveness. Of course it is enough to consider $\mathbb{Q}$-roots, since $G$ and $P$ are defined over $\mathbb{Q}$ and $\gamma$ is a $\mathbb{Q}$-rational element. Therefore, the condition in Theorem 2.1 can be replaced by the condition $|\alpha(\gamma)|_{f,\mathrm{fin}} \leq 1$ holds for all positive $\mathbb{Q}$-roots $\alpha$ and $|\alpha(\gamma)|_{f,\mathrm{fin}} = 1$ holds if and only if $\alpha$ is a root from $L$.

Remark 2.9. For a standard $\mathbb{Q}$-parabolic group $P \supset P_0$ with Levi decomposition $P = LN$ let $X^*(P)_{\mathbb{Q}} = X^*(L)_{\mathbb{Q}} = \text{Hom}_{\mathbb{Q}-\text{alg}}(L, G_m)$ be the group of characters defined over $\mathbb{Q}$. Then $X^*_L = \text{Hom}(X^*(L)_{\mathbb{Q}}, \mathbb{R})$ can be canonically identified with the Lie algebra of $A_L$, and hence with $A_L(\mathbb{R})^0$ by the exponential map. One defines the Harish-Chandra homomorphism

$$H_P : L(\mathbb{A}) \rightarrow X^*_L$$
by $\exp((H_P(l), \chi)) = ||\chi(l)||$, where $||.|| : \mathbb{A}^+ \to \mathbb{R}^+$ is the idele norm, and $\chi \in X^*(L)_{\mathbb{Q}}$. For the minimal $\mathbb{Q}$-parabolic $P = P_0$ we write $A_P = A$. Let $\Delta$ be a basis of the simple $\mathbb{Q}$-roots. Then the standard $\mathbb{Q}$-parabolic subgroups $P = P_0$ correspond uniquely to the subsets $\theta \subseteq \Delta$. The roots in $F$ are the roots of the Lie algebra of the unipotent radical $N_\theta$ are the roots in $\Delta \setminus F$. In fact, since $\gamma$ is strongly elliptic, the condition for $\gamma \in L(\mathbb{Q}) = L_\theta(\mathbb{Q})$, given in Remark 2.8, could also be replaced by the condition

$$|\alpha(H_P(\gamma))|_{\text{fin}} < 1$$

for all simple $\mathbb{Q}$-roots $\alpha$ not in $\theta$.

The Decomposition $\mathcal{X} = \mathcal{X}_L \oplus \mathcal{X}_L^+$ [1]. Let $\beta_j$ denote the dual roots such that $\langle \beta_j, \alpha_i \rangle = \delta_{ij}$, both considered as elements of $\mathcal{X}$. Then there exists a natural orthogonal decomposition $\mathcal{X} = \mathcal{X}_L \oplus \mathcal{X}_L^+$ such that $\mathcal{X}_L$ is the span $\sum_{j \in F} \mathbb{R} \beta_j$ and $\mathcal{X}_L^+ = \sum_{i \in F} \mathbb{R} \alpha_i - \sum_{j \in F} \mathbb{R} \beta_j$. The projection $pr_L : \mathcal{X} \to \mathcal{X}_L$ is $pr_L(\sum_{j \in F} x_j \beta_j + \sum_{i \in F} y_i \alpha_i) = \sum_{j \in F} x_j \beta_j$. The image under $pr_L$ of the open positive Weyl chamber $\mathcal{X}^+ = \sum_{i \in \Delta} \mathbb{R}_{>0} \beta_j \subseteq \mathcal{X}$ defines the open Weyl chamber $\mathcal{X}_L^+$ in $\mathcal{X}_L$; the image of the obtuse Weyl chamber $\mathcal{X} = \sum_{i \in \Delta} \mathbb{R}_{>0} \alpha_i \subseteq \mathcal{X}$ defines the obtuse open Weyl chamber in $\mathcal{X}_L$. Obviously $\mathcal{X}_L^+ = \sum_{i \in F} \mathbb{R}_{>0} \alpha_i$. Then $\mathcal{X}_L^+ = pr(\mathcal{X}^+) = \sum_{i \in F} \mathbb{R}_{>0} \beta_j$, since $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$ and $\langle \beta_i, \beta_j \rangle \geq 0$. In fact $\sum_{j \in F} x_j \beta_j + \sum_{i \in F} y_i \alpha_i \in \sum_{i \in F} \mathbb{R}_{>0} \beta_j$ therefore implies $y_i \geq 0, i \in F$; hence, $x_j > 0, j \notin F$. Also $\mathcal{X}^+ \subseteq \mathcal{X}_L$; therefore, $\mathcal{X}_L^+ \subseteq \mathcal{X}_L$. Finally notice $\mathcal{X}_L^+ \cap -\mathcal{X}_L = \{0\}$.

2.5 Computation of an Orbital Integral

We write the terms $O_\gamma$ in the formula for the Lefschetz numbers as an orbital integral $O_\gamma^L(f^{(P)})$. This is done in steps 1–3. The final result is formulated in step 4.

Step 1. Assume measures are normalized by $\text{vol}_\mathbb{Q}(\Omega) = 1$. Recall $K' = K \cap g^{-1}Kg$ and $g \in G(A_{\text{fin}})$ is fixed. The characteristic function $1_{g^{-1}K}(y)$ of the set $g^{-1}K$ is then $K'$-bi-invariant. Furthermore, $k^{-1}xk \in g^{-1}K \Longleftrightarrow x \in kg^{-1}kKk^{-1} =: gK_k$. Hence,

$$\int_\Omega 1_{g^{-1}K}(k^{-1}xk)dk = \left[ \Omega : K' \right]^{-1} \sum_{k \in \Omega/K'} 1_{g^{-1}K}(k^{-1}xk)$$

$$= \text{vol}_\mathbb{Q}(K') \cdot \sum_{k \in \Omega/K'} 1_{g^{-1}K_k}(x).$$

$$\int_K 1_{g^{-1}K}(k^{-1}xk)dk = \int_K 1_{g^{-1}K}(k^{-1}x)dk = \text{vol}_\mathbb{Q}(g^{-1}Kg \cap K) 1_{g^{-1}K}(x) = \text{vol}_\mathbb{Q}(K') 1_{Kg^{-1}K}(x)$$

holds for $x \in G(A_{\text{fin}})$. Hence, $\int_\Omega = \text{vol}_\mathbb{Q}(K')^{-1} \int_\Omega \int_K$ implies
Computation of an Orbital Integral

\[
\int_{\Omega} 1_{g^{-1}K}(k^{-1}xk)dk = vol_\Omega(K)^{-1} \int_{\Omega} vol_\Omega(K')1_{Kg^{-1}K}(k^{-1}xk)dk.
\]

Comparison of the right sides thus gives for the $\Omega$-average

**Definition 2.2.** $\overline{f}(x) = \int_{\Omega} f(k^{-1}xk)dk, \quad x \in G(\mathbb{A}_{fin}),$ of the normalized characteristic function.

**Definition 2.3.** $f(x) = vol_\Omega(K)^{-1}1_{Kg^{-1}K}(x)$.

**Lemma 2.7.** $\overline{f}(x) = \sum_{k \in \Omega/K'} 1_{g_k^{-1}K_k}(x)$.

**Step 2.** For $x \in P(\mathbb{A}_{fin})$ in a standard parabolic subgroup $P = LN$ by Lemma 2.7

\[
\int_{N(\mathbb{A}_{fin})} \overline{f}(xn)dn = \sum_{k \in \Omega/K'} \int_{N(\mathbb{A}_{fin})} 1_{g_k^{-1}K_k}(xn)dn.
\]

If $x_L$ is not in $(g_k^{-1}K_k \cap P(\mathbb{A}_{fin}))_L$, the corresponding integral on the right side is zero. Otherwise $xn_0 = g_k^{-1}k_0$ holds for some $n_0 \in N(\mathbb{A}_{fin})$ and $k_0 \in K_k$, and in this case the integral becomes $\int_{N(\mathbb{A}_{fin})} 1_{g_k^{-1}K_k}(g_k^{-1}k_0n)dn = vol(N(\mathbb{A}_{fin}) \cap K_k)$. Hence, for $x \in L(\mathbb{A}_{fin})$ we get

**Lemma 2.8.** $\phi(x) := \int_{N(\mathbb{A}_{fin})} \overline{f}(xn)dn = \sum_{k \in \Omega/K'} vol(N(\mathbb{A}_{fin}) \cap K_k) \cdot 1_{(g_k^{-1}K_k \cap P(\mathbb{A}_{fin}))_L}(x)$.

**Step 3.** Next consider the orbital integral of the function $\phi$ defined on $L(\mathbb{A}_{fin})$

\[
O^L_\gamma(\phi) = \int_{L_\gamma(\mathbb{A}_{fin}) \cdot L(\mathbb{A}_{fin})} \phi(m^{-1}\gamma m)dm.
\]

By the definition of $\phi$ the value of $O^L_\gamma(\phi)$ is

\[
\sum_{k \in \Omega/K'} vol \left( N(\mathbb{A}_{fin}) \cap K_k \right) \cdot \int_{L_\gamma(\mathbb{A}_{fin}) \cdot L(\mathbb{A}_{fin})} \left| m^{-1} \gamma mL \in (g_k^{-1}K_k \cap P(\mathbb{A}_{fin}))_L \right| dm,
\]

or by Sect. 6.16 and the decomposition $S(\gamma, k) = \bigcup L_\gamma(\mathbb{A}_{fin}) \cdot \xi_{fin} \cdot K'(k)\mathbb{A}_{fin}$

\[
\sum_{k \in \Omega/K'} vol \left( N(\mathbb{A}_{fin}) \cap K_k \right) \cdot \sum_{\xi_{fin}} \frac{vol_{L,\gamma}(K'(k)\mathbb{A}_{fin})}{vol_{L,\gamma}(\xi_{fin}K'(k)\mathbb{A}_{fin} \cap L_\gamma(\mathbb{A}_{fin}))}.
\]

**Step 4.** To put things together. The function

\[
\overline{f}(x) = \int_{\Omega} f(k^{-1}xk)dk, \quad f(x) = vol_\Omega(K)^{-1}1_{Kg^{-1}K}(x)
\]
is $K$-bi-invariant on $G(\mathbb{A}_{fin})$. Define

$$\mathcal{T}_P^{(P)}(m) = |m|_{P_{fin}}^{p_P} \int_{N(\mathbb{A}_{fin})} \mathcal{T}(m)dn, \quad m \in L(\mathbb{A}_{fin})$$

$$O_L^L(\mathcal{T}^{(P)}) = \int_{L_\gamma(\mathbb{A}_{fin}) \backslash L(\mathbb{A}_{fin})} \mathcal{T}_P^{(P)}(m^{-1}\gamma m)dm,$$

assuming that the integrals are normalized by the conventions $\text{vol}_{P(\mathbb{A}_{fin})}(\Omega \cap P(\mathbb{A}_{fin})) = 1$, $\text{vol}(\Omega) = 1$, and $\text{vol}_{N(\mathbb{A}_{fin})}(\Omega \cap N(\mathbb{A}_{fin})) = 1$. See also [16], p. 144. The measure on $L_\gamma(\mathbb{A}_{fin})$ is $dg_{fin}$. Then the computation above proves that

$$O_L^L(\mathcal{T}^{(P)}) = O_\gamma.$$

### 2.6 Elliptic Traces

Recall $G$ is a connected reductive group over $\mathbb{Q}$ whose derived group is simply connected. Define elliptic “traces”

$$T_G^{\text{ell}}(f, \tau) = \sum_{\gamma \in G(\mathbb{Q})/\sim} \chi(\gamma) \lambda_G(f) \cdot \text{tr}(\gamma^{-1})$$

for a finite-dimensional complex representations $\tau$ of $G(\mathbb{Q})$ and $f \in C_\infty^c(G(\mathbb{A}_{fin}))$. If $\tau$ is an irreducible complex representation defined by a highest weight $\lambda$, we also write $T^{\text{ell}}_G(f, \lambda)$ instead of $T^{\text{ell}}_G(f, \tau)$. Hence, we do not distinguish between representations and their highest weights. The sum defining $T^{\text{ell}}_G(f, \tau)$ extends over the $G(\mathbb{Q})$-conjugacy classes of semisimple, strongly elliptic elements in $G(\mathbb{Q})$. The integrals $\lambda_G(f)$ in this sum are orbital integrals with respect to the group of finite adeles for functions $f \in C_\infty^c(G(\mathbb{A}_{fin}))$. The same definition defines elliptic traces $T^{\text{ell}}_L$ for the Levi subgroups $L$ of all standard $\mathbb{Q}$-parabolic subgroups $P = LN_P$ of $G$.

Let $\chi_G^\circ = \tau_G^\circ \circ H_P$ be defined by the characteristic function $\tau_G^\circ$ of the open positive Weyl chamber of $\mathcal{X}_L = X_s(\mathcal{X}_L)_{\mathbb{Q}} \otimes \mathbb{R}$, lifted to a function on $L(\mathbb{A}_{fin})$ via the Harish-Chandra homomorphism $H_L : L(\mathbb{A}_{fin}) \to \mathcal{X}_L$. Then Theorem 2.1 and the remarks following it imply

**Lemma 2.9.**

$$L(f, V_\lambda) = \sum_{P \leq L \leq G} T^{P}_{\text{ell}}(\mathcal{T}^{(P)}_P, \chi_G^\circ, \lambda),$$

where

$$T^{P}_{\text{ell}}(h, \lambda) = \sum_{w \in W_P} (-1)^{(w)} \cdot T^{L}_{\text{ell}}(h, w(\lambda + \rho_G) - \rho_L)$$

for $h \in C_\infty^c(L(\mathbb{A}_{fin}))$, and where $P = LN_P$ runs over the $\mathbb{Q}$-rational standard parabolic subgroups of $G$. 
Let \( \hat{\chi}^G_P = \hat{\tau}^G_P \circ H_P \) be the characteristic function \( \hat{\tau}^G_P \) of the open obtuse Weyl chamber in \( \mathcal{X}_L \), considered as a function on \( L(\mathbb{A}_{fin}) \). Notice \( \chi^G_P \leq \hat{\chi}^G_P \).

**Lemma 2.10.** Let the situation be as in Lemma 2.9. Then

\[
T^G_{el}(f, \lambda) = \sum_{P \subseteq Q \subseteq G} (-1)^{\text{rang}(Q) - \text{rang}(G)} L^Q(f^Q, \chi^G_Q, \lambda),
\]

where the sum is over the standard \( Q \)-parabolic subgroups of \( G \), where \( \text{rang}(Q) \) denotes the \( Q \)-split rank of the Levi subgroup of \( Q \), and the Lefschetz number \( L^Q \) for \( h \in C^\infty_c(L(\mathbb{A}_{fin})) \) and the parabolic group \( Q \) is defined by

\[
L^Q(h, V_\lambda) = \sum_{w \in W^Q} (-1)^{l(w)} \cdot L^L(h, V_{w(\lambda + \rho_G) - \rho_L}),
\]

and \( L^L(h, \cdot) \) is the Lefschetz number attached to coefficient systems for the symmetric space attached to the Levi subgroup \( L \) of the (standard) parabolic subgroup \( Q \).

Lemmas 2.9 and 2.10 were expected by Harder [39], pp. 144–145.

**Proof of Lemma 2.10.** Lemma 2.9 applied to the Levi subgroup \( L = LN \) gives

\[
L^Q(f^Q, \chi^G_Q, V_\lambda) = \sum_{w \in W^Q} (-1)^{l(w)} \cdot L^L(f^Q, \chi^G_Q, V_{w(\lambda + \rho_G) - \rho_L})
\]

\[
= \sum_{w \in W^Q} (-1)^{l(w)} \sum_{P_0 \cap L \subseteq P' \cap L' \subseteq L} T^P_{el}(f^Q, \chi^G_Q, V_{w(\lambda + \rho_G) - \rho_L})
\]

\[
= \sum_{P_0 \cap L \subseteq P' \cap L' \subseteq L} \sum_{w' \in W^{P'}} \sum_{w \in W^Q} (-1)^{l(w)} \cdot L^L(f^Q, \chi^G_Q, V_{w'(\lambda + \rho_G) - \rho_L'})
\]

\[
T^P_{el}(f^Q, \chi^G_Q, V_{w'(\lambda + \rho_G) - \rho_L'}).
\]

\( P \subseteq Q \) induces the parabolic group \( P' = P \cap L \) in the Levi component \( L \) of \( Q \), and all standard \( Q \)-parabolic groups \( P' \) are obtained in this way from the standard \( Q \)-parabolic subgroups \( P \subseteq Q \) such that the Levi components \( L' \) of \( P' \) and \( P \) coincide. Since \( sn(w) = (-1)^{l(w)} \) satisfies \( sn(w')sn(w) = sn(w'w) \), the inductivity \( W^{P'}W^Q = W^P \) and the formula \( f^P \circ G = (\hat{f}^Q, \hat{\chi}^G_Q) \) implies that the sum simplifies to

\[
L^Q(f^Q, \chi^G_Q, V_\lambda) = \sum_{P_0 \subseteq P' \cap L' \subseteq L} T^P_{el}(f^P, \chi^G_P, V_{\lambda}),
\]

The sum is over all \( Q \)-rational standard parabolic subgroups \( P \) of \( G \) contained in \( Q \). Notice in the formula above \( \chi^G_P \) is a function on \( \mathcal{X}_{L'} \), whereas \( \hat{\chi}^G_P \), which is defined
as a function on \(X_L\), is tacitly considered as a function on \(X_{L'}\) via the canonical projection map \(p_r : X_{L'} \rightarrow X_L\). Summing these formulas over the standard parabolic groups \(Q\), with the additional factors \((-1)^{\text{rang}(G) - \text{rang}(Q)}\),

\[
\sum_{P_0 \subseteq Q} (-1)^{\text{rang}(Q) - \text{rang}(G)} \cdot L^Q(\hat{f}^Q_{\tau_{\hat{G}}} \chi_{G}, V_{\lambda}),
\]

gives the desired result, by interchanging the order of summation. Fixing \(P\), the sum over all \(Q\) with \(P \subseteq Q \subseteq G\) gives zero except for \(P = G\). Indeed for fixed \(P \subseteq G\) the sum \(\sum_{P \subseteq Q \subseteq G} (-1)^{\text{rang}(G) - \text{rang}(Q)} \chi_{Q} \chi_{X_{L'}}\) is zero except for \(P = L'N_P = G\), where it is 1 instead. This is a well known result obtained by Arthur [1]. For the convenience of the reader we include the argument. 

**Proof.** Let \(P' \subseteq F = \Delta\) define \(P' \subseteq P \subseteq G\). Since the support \(\text{Supp}_{P'}\) of the characteristic function \(\hat{f}^Q_{\tau_{\hat{G}}}\) of the subset \(\sum_{\alpha \in F} R_{\geq 0} \alpha_1 + \sum_{j \in F' \setminus P} R_{\geq 0} \beta_j\) of \(X_{L'}\) is contained in \(\tau_{\hat{G}} X_{L'} = \sum_{\alpha \in F'} R_{\geq 0} \alpha_1\) (if \(F' \neq \Delta\)), \(\text{Supp}_{P'} = \tau_{\hat{G}} X_{L'}\) follows as an immediate consequence of the inequalities \((\alpha_i, \alpha_j) \leq 0\) for \(i \neq j\) and \((\beta_i, \beta_j) \geq 0\). For \(H \in X\) let \(\Delta_H\) denote the set of \(\alpha_i\) for which \((\alpha_i, H) > 0\). \(\Delta_H\) is nonempty for \(H \in \tau_{\hat{G}}\), since \(\tau_{\hat{G}} \cap \Delta = \{0\}\). Hence, \(\sum_{P \subseteq Q \subseteq G} (-1)^{\text{rang}(G) - \text{rang}(Q)} \chi_{Q} \chi_{X_{L'}}(H) = 0\) follows from \(\sum_{T \subseteq \Delta_H} (-1)^{|T|} = 0\).

**Corollary 2.2.** The elliptic trace \(T_{\text{ell}}^G(f, \lambda)\) is

\[
\sum_{P_0 \subseteq Q \subseteq G} \sum_{w \in W_Q} (-1)^{\text{rang}(Q) - \text{rang}(G) + 1(w)} \cdot \text{Tr} \left(\hat{f}^Q_{\tau_{\hat{G}}}, H^*(S_{LQ}, V_{w(\lambda + p_G) - p_L})\right).
\]

**Corollary 2.3.** The Lefschetz number \(L(f, V_{\lambda})\) is

\[
\sum_{P_0 \subseteq P \subseteq G} \sum_{w \in W_P} (-1)^{1(w)} \cdot T_{\text{ell}}^P(\hat{f}^P_{\tau_{\hat{G}}}, \chi_{P}, w(\lambda + p_G) - p_L).
\]

### 2.7 The Satake Transform

For a connected reductive group \(G\) over a non-Archimedean local field \(F_v\), let \(A\) be a maximal \(F_v\)-split torus in the center of \(G\). Let \(G^{ab}\) be the maximal Abelian quotient of \(G\). Write \(G_v = G(F_v)\), etc.

**ord\(_G\)**. There is a canonical homomorphism \(\text{ord}_G : G_v \rightarrow X_*(G) = \text{Hom}_{F_v \rightarrow \text{alg}} (G, G_m)\) (see [16], p. 134). We also write \(\text{ord}_G\) for the induced homomorphism \(G_v \rightarrow X_{G_v} = X_*(G) \otimes R\), and \(^0G\) for the kernel. The homomorphism \(\text{ord}_G\) is functorial in \(G\) and induces the field valuation in the case \(G = G_m\). It factorizes over the quotient \(G^{ab}\), and is trivial on compact subgroups. The kernel of the canonical map \(A_v \rightarrow G_v^{ab}\) is contained in the maximal compact subgroup \(^0A_v\). Hence, the
quotient group $A_v/\mathcal{O}_v$, which can be identified with the $F_v$-rational cocharacter lattice $X_*(A)$ of the torus $A$, is injected into $G_v^{ab}/\mathcal{O}_v^{ab}$ as a subgroup of finite index. Hence, the canonical maps $\mathcal{X}_{A_v} \rightarrow \mathcal{X}_{G_v} \rightarrow \mathcal{X}_{G_v}^{\mathcal{O}_v^{ab}}$ induce isomorphisms, which allows us to identify these vector spaces.

**The Map $S$.** Now assume $G_v = G(F_v)$ to be quasisplit, and split over a finite unramified extension field of $F_v$ such that the derived group is simply connected. Let $\Omega_v$ be a good maximal compact subgroup and $P = MN$ be a minimal $F_v$-rational parabolic subgroup of $G$ such that $G_v = P_v : \Omega_v$. To be precise, we demand $\Omega_v$ to be admissible relative to $M_v$ in the sense of [7], p. 9. The $\Omega_v$-bi-invariant functions on $G_v$ with compact support define the spherical Hecke algebra $\mathcal{H}(G_v, \Omega_v)$ of $G_v$.

Put $\Lambda = \mathcal{O}_v \setminus M_v$. For $f_\nu \in C_\infty(G_v)$ define $\mathcal{T}_v^{P_v}(m) = |m|_{\nu}^{\rho_m} \int_{\mathcal{X}} \mathcal{T}(mn)dn$ as on page 36 now locally for $F_v$. For elements $f_\nu$ in the spherical Hecke algebra of $G_v$ the Satake transform $S$ is defined by (see [16], p. 146, formula (19))

$$ f_\nu \mapsto S(f_\nu) = \mathcal{T}_v^{P_v}(\nu) $$

and defines a function $S(f_\nu)$ on $M_v(\mathbb{Q}_v)/M_v(\mathbb{Q}_v) \cap \Omega_v = \Lambda$. The group $\Lambda$ is a lattice, which contains and is commensurable with the cocharacter lattice $X_*(A)$ of the torus $A$ (see [16], p. 135) in $\mathcal{X}_{A_v} = X_*(A) \otimes \mathbb{R}$. The Satake transform defines an isomorphism between the spherical Hecke algebra of the group $G_v$ and the algebra $\mathbb{C}[\Lambda]^W$ ($W$-invariants in the group ring $\mathbb{C}[\Lambda]$) [16, Theorem 4.1]. Furthermore, for $\gamma$ regular in $M_v$ the Satake transform $S$ is given by the orbital integral up to a normalization factor

$$ S(f_\nu)(\gamma) = D_G(\gamma)^{1/2}O_{\gamma}^{G_v}(f_\nu). $$

For an arbitrary function $\chi : \mathcal{X}_{G_v} \rightarrow \mathbb{R}$ multiplication by $\chi$ determines a $\mathbb{C}$-linear endomorphism $f_\nu(x) \mapsto \chi(\text{ord}_L(x))f_\nu(x)$ of the Hecke algebra of $G_v$, which preserves the spherical Hecke algebra such that for the orbital integral

$$ O_{\gamma}^{G_v}(f_\nu \chi) = \chi(\text{ord}_G(\gamma)) \cdot O_{\gamma}^{G_v}(f_\nu) $$

holds, and also for the Satake transform $S(\chi f_\nu)(m) = \chi(\text{ord}_G(\gamma))(m)S(f_\nu)(m)$.

**Standard $F_v$-parabolic Groups.** Let $Q$ be a $F_v$-rational standard parabolic subgroup of $G$ with Levi component $L$. Let $A_Q$ be the maximal $F_v$-split torus in $Q$. The natural map $A_v \rightarrow L_v \rightarrow \mathcal{X}_L$ factorizes over the quotient $A_v/\mathcal{O}_v$, and hence induces a canonical $\mathbb{R}$-linear map

$$ \text{pr} : \mathcal{X}_{A_v} \rightarrow \mathcal{X}_{L_v}. $$

The following two properties characterize the projection $\text{pr}$. Firstly, the embedding $A_Q \hookrightarrow A_v$ induces a canonical embedding $i : \mathcal{X}_{L_v} = X_*(A_Q) \otimes \mathbb{R} \hookrightarrow \mathcal{X}_{A_v} = X_*(A) \otimes \mathbb{R}$ such that $\text{pr} : \mathcal{X}_{A_v} \rightarrow \mathcal{X}_{L_v}$ restricts us to the identity map on the
subspace $\mathcal{X}_{L_c} \subseteq \mathcal{X}_{M_c}$. Secondly $pr$ is zero on the subspace $X_s(A'_Q) \otimes \mathbb{R} \subseteq X_{M_c}$, where $A'_Q$ denotes the split torus $L_{der} \cap A$.

This gives the following formulation in terms of the Killing form. Let $\alpha_i \in \Delta(G_v, A_v)$ denote the simple $F_v$-roots attached to $P_v \subseteq G_v$, let $\langle , \rangle$ denote the Killing form, and let $\beta_j$ denote the dual basis $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Use the Killing form to identify $X_s(A) \otimes \mathbb{R}$ with its dual $X^*(A) \otimes \mathbb{R}$. The $F_v$-rational standard parabolic subgroups are in one-to-one correspondence with the subsets $F \subseteq \Delta(P_v, A_v)$. For $Q = Q_F$ the space $\mathcal{X}_{L_v} = X_s(A_Q) \otimes \mathbb{R}$ is given in $\mathcal{X}_{M_v} = X_s(A) \otimes \mathbb{R}$ by the equations $\langle \alpha_i, \alpha_i \rangle = 0$, $\alpha_i \in F$ (or $i \in F$ by abuse of notation) for a subset $F$ of the simple roots. $\mathcal{X}_{M_v}$ splits into the orthogonal direct sum of the two subspaces $\mathcal{X}_{L_v} = \sum_{j \notin F} \mathbb{R} \beta_j$ and the orthocomplement $\sum_{i \in F} \mathbb{R} \alpha_i$. $pr$ is the orthogonal projection defined by $pr(\sum_{j \notin F} x_j \beta_j + \sum_{i \in F} y_i \alpha_i) = \sum_{j \notin F} x_j \beta_j$.

Transitivity. Let $Q = LN$ be an $F_v$-rational parabolic subgroup of $G$. Let $\sigma_v$ be an irreducible admissible representation of $L_v$. The Hecke algebra $C_c^\infty(G_v)$ of locally constant functions with compact support on $G_v$ acts by convolution on the unitary normalized induced representation $\pi_v = \text{Ind}_{Q_0}^{G_v}(\sigma_v)$ such that (for measures suitably normalized) the adjunction formula (see, e.g., [44], Sect. 2, Lemma 1, the slightly different definition involving $f_v^*$ in the pairing in loc. cit. has no effect) holds

$$tr \text{Ind}_{Q_0}^{G_v}(\sigma_v)(f_v) = tr \sigma_v(\overline{f_v}(Q)),$$

where $f_v \in C_c^\infty(G_v)$ and by definition $\overline{f_v}(Q)(m) = |m|_e^{\rho_{Q_0}} \int_{\mathcal{X}_v} \overline{f_v}(mn)dn$.

The group $\Omega_v \cap L_v = (\Omega_v \cap Q_v)L_v$ is a good maximal compact subgroup of $L_v$, i.e., admissible with respect to $M_v$ (see [7], p. 9). $L_v$ is again quasisplit and splits over an unramified extension field. Hence, the spherical Hecke algebra $\mathcal{H}(L_v, \Omega_v \cap L_v)$ is defined. For $f_v \in \mathcal{H}(G_v, \Omega_v)$ the function $S_{L}^G(f_v) = \overline{f_v}(Q)$ is bi-invariant under $\Omega_v \cap L_v$, and hence the partial Satake transform $S = S_{M}^G : \mathcal{H}(G_v, \Omega_v) \to \mathcal{H}(M_v, 0 M_v)$ factorizes over the spherical Hecke algebra $\mathcal{H}(L_v, \Omega_v \cap L_v)$

$$S = S_{M}^G = S_{M}^L \circ S_{L}^G.$$ 

Absolute Support. In the following, a cone $C$ in Euclidean space is understood to be an open submonoid stable under multiplication by $\mathbb{R}_{>0}$ which does not contain a real line.

Lemma 2.11. Fix an arbitrary nonempty open cone $C \subseteq \mathcal{X}_{M}$, which is contained in the positive Weyl chamber attached to $P_v$. Let $\pi_v = \text{Ind}_{Q_0}^{G_v}(\sigma_v)$ be an unramified induced representation attached to an unramified character $\sigma_v$ of $M_v$ with spherical constituent $\pi_v^0$. Choose $x_0 \in C$. Then there exist spherical Hecke operators $f_v$ with the properties:

1. $tr \pi_v(f_v) = tr \pi_v^0(f_v) = 1.$
2. The support of the Satake transform $S(f_v)$ of $f_v$ is contained in the Weyl group orbit $\bigcup_{w \in W} w(x_0 + C)$ of the translated cone $x_0 + C$. 

Proof. It suffices to find \( f_v \in \mathcal{H}(G_v, \Omega_v) \) with \( tr \pi_v(f_v) \neq 0 \) such that (2) holds. \( tr \pi_v \), considered as a function on the spherical Hecke algebra \( C[\Lambda]^W \), is a finite sum of characters on the group \( \Lambda \). Up to a twist by \( \delta^{1/2}(x) \) these characters are in the \( W \)-orbit of the character \( \sigma_v \). This character sum is conjugation-invariant, and hence \( W \)-invariant. If the assertions of the lemma were false, there would exist finitely many different characters \( \chi_i, i = 1, \ldots, r \), of \( \Lambda \) and \( n_i \in \mathbb{C} \) such that
\[
\sum_{i=1}^r n_i \chi_i(x) = 0, \quad (n_1, \cdots, n_r) \neq 0
\]
holds for all \( x \in \Lambda \cap (x_0 + C) \). To see that this is impossible we can assume \( x_0 = 0 \), changing the coefficients \( n_i \) to \( n_i \chi_i(x_0) \), and then use induction on \( r \).

Since \( x, y \in C \) implies \( x + y \in C \) we can lower the length \( r \) of such a nontrivial character relation on \( C \) by considering \( \sum n_i (\chi_i(y) - \chi_i(x)) \chi_i(x) = 0 \), provided there exists \( y \in C \) with \( \chi_i(y) \neq \chi_1(y) \) if, say, \( n_i \neq 0 \). Because \( \chi = \chi_i/\chi_1 \) is a nontrivial character on \( \Lambda \), such a \( y \) exists, since otherwise \( \chi \) vanishes on \( C \cap \Lambda \), and hence on the generated group \( (C \cap \Lambda) = (C \cap \Lambda) = \Lambda \) holds for any nonempty open cone of \( \mathcal{X} \). This proves the lemma. \( \square \)

Relative Support. For \( f_v \in C^\infty_c(G_v) \) consider the support \( \Sigma \) of the orbital integral \( O^L_\gamma(\mathcal{T}^{(Q)}_v) \) as a function of \( \gamma \in L_v \). Notice the support of \( \mathcal{T}^{(Q)}_v \) itself is contained in \( \Sigma \). The image of \( \Sigma \) in \( \mathcal{X}_L \), the regular, semisimple subset of this support under \( \text{ord}_L : L_v \rightarrow \mathcal{X}_L \), will be called the relative support of \( f_v \) with respect to \( Q_v \). The relative support contains the image of the support of \( \mathcal{T}^{(Q)}_v \) in \( \mathcal{X}_L \), under the map \( \text{ord}_L \). Since the regular semisimple elements are dense in \( \Sigma \), and since the maximal compact subgroup of \( L_v \) is in the kernel of \( \text{ord}_L \), one could replace the support \( \Sigma \) by the regular, semisimple support of \( O^L_\gamma(\mathcal{T}^{(Q)}_v) \) for the definition of relative support above.

The relative support of \( f_v \) with respect to \( Q_v \) is a finite subset of the vector space \( \mathcal{X}_L \). Notice that \( \mathcal{T}^{(Q)}_v \) has compact support on \( L_v \), and \( \text{ord}_L \) is invariant under conjugation. Hence, the image \( \text{ord}_L(\Sigma) \) is relatively compact in \( \mathcal{X}_L \). On the other hand \( \text{ord}_L(L_v) \) is contained in a sublattice of \( \mathcal{X}_L \).

Lemma 2.12. Let \( f_v \in \mathcal{H}(G_v, \Omega_v) \) be a spherical function. Let \( Q = LN_0 \) be an \( F_v \)-rational standard parabolic subgroup of \( G \) containing the minimal \( F_v \)-parabolic subgroup \( P = MN \). Then \( x \in \mathcal{X}_L \) is in the relative support of \( O^L_\gamma(\mathcal{T}^{(Q)}_v) \) if and only if \( x \) is in the image of the support of the Satake transform \( S(f_v) \in \mathcal{H}(M_v, 0, M_v) \) under the map \( \text{pr} \circ \text{ord}_M \), where \( \text{pr} : \mathcal{X}_M \rightarrow \mathcal{X}_L \) is the canonical projection.

Proof. Let \( \chi_x(\lambda) \) be the function on \( \mathcal{X}_L \), which is not zero for \( \lambda = x \) and is zero otherwise. Then by definition the following statements are equivalent. By abuse of notation we consider \( \chi_x \) as a function on \( L_v \) using the map \( \text{ord}_L \). Then \( x \in \mathcal{X}_L \) is in the relative support of \( f_v \) if and only if
\[
\chi_x(\gamma)O^L_{\gamma, x}(\mathcal{T}^{(Q)}_v) = O^L_{\gamma, x}(\chi_x \cdot \mathcal{T}^{(Q)}_v)
\]
does not vanish identically for all semisimple, regular elements \( \gamma \in L_v \). Since \( f_v \) is a spherical function on \( G_v \), \( \mathcal{T}_v(Q) = S^G_L(f_v) \) is spherical on \( L_v \); hence, \( \chi_x \cdot \mathcal{T}_v(Q) = \chi_{x \cdot \mathcal{T}_v(Q)} \) is again spherical on \( L_v \). If \( O^{L_v}_v(\chi_x \mathcal{T}_v(Q)) \) does not vanish identically for all semisimple, regular elements \( \gamma \in L_v \), then \( \chi_x \mathcal{T}_v(Q) \) does not vanish identically on \( L_v \). Since \( \chi_x \mathcal{T}_v(Q) \) is spherical, this implies \( S^L_M(\chi_x \mathcal{T}_v(Q)) \neq 0 \); hence, \( O^{L_v}_v(\chi_x \mathcal{T}_v(Q)) \) does not vanish identically for all semisimple, regular elements \( \gamma \in M_v \subseteq L_v \). In other words, \( x \in X_L \) is in the relative support of \( f_v \) if and only if \( S^L_{M_v}(\chi_x \mathcal{T}_v(Q)) \neq 0 \). Obviously \( S^L_{M_v}(\chi_x \mathcal{T}_v(Q)) = \chi_x S^L_{M_v}(\mathcal{T}_v(Q)) = (\chi_x \circ \text{pr} \circ \text{ord}_M) \cdot S^L_M(S^G_L(f_v)) = (\chi_x \circ \text{pr} \circ \text{ord}_M) \cdot S(f_v) \). This does not vanish identically if and only if \( x \) is in the image of the support of \( S(f_v) \) in \( X_{M_v} \) under \( \text{pr} \). This proves the lemma. 

\[ \Box \]

2.7.1 Subdivision of the Weyl Chambers

Suppose \( Q = LN_Q \) is an \( F_t \)-rational standard parabolic subgroup \( Q = Q_F \) defined by \( F \subseteq \Delta(G_v, A_v) \), containing the minimal \( F_t \)-parabolic group \( P = MN \). Then an element \( x = \sum_{i \in \Delta(G_v, A_v)} x_i \alpha_i \) in \( X_{M_v} \) is contained in the support of the function

\[ \lambda^G_{Q_F} = \zeta^G_{Q_F} \circ \text{pr} \circ \text{ord}_M \]

if and only if its projection \( \text{pr}(x) = \sum_{i \notin F} x_i \alpha_i \in X_{L_v} \) is in the obtuse Weyl chamber \( \chi^{\Gamma}_{L_v} = \sum_{i \notin F} R_{x_0 + \alpha_i} \), which means \( x_i = \langle x_i, \beta_i \rangle > 0 \) for all \( i \notin F \).

The equations \( \alpha_i(x) = 0 \) and \( \beta_i(x) = 0 \) for \( \alpha_i \in \Delta(G_v, A_v) \) define hyperplanes in \( X_{M_v} \). The images of these hyperplanes under the action of the Weyl group on \( X_{M_v} \) define finitely many hyperplanes. The complement of these hyperplanes in \( X_{M_v} \) is a union of open connected cones. Each of these cones is the image under the Weyl group of a subcone of the open Weyl chamber \( X_{M_v}^{*} \). Pick one of these cones \( C \).

Example 2.3. For \( G_v = SL(3, F_v) \) the positive Weyl chamber contains two such cones.

Support Conditions. Suppose \( f_v \) is a spherical function on \( G_v \) such that its Satake transform is contained in the \( W \)-orbit of \( x_0 + C \subseteq X_{M_v} \) for some \( x_0 \in C \), as in Lemma 2.11. Then a regular semisimple element \( \gamma \) is in the support of \( O^L_v(\mathcal{T}_v(Q)) = \zeta^G_v \) if and only if \( x = \text{ord}_L(\gamma) \) is in \( \text{pr}(\bigcup_{w \in W} (x_0 + C)) \). If this is the case then \( x_i = \beta_i(x) > 0 \) for all \( i \notin F \). But then moreover, by our specific choice of the cone, we even get \( x_i > \text{const}(x_0) > 0 \) for all \( i \notin F \). Similarly, if \( \gamma \) is not in the support of \( O^L_v(\mathcal{T}_v(Q)) = \zeta^G_v \), then \( x_i < -\text{const}(x_0) \) holds for at least one \( i \notin F \). The constant \( \text{const}(x_0) \) which appears in these formulas of course depends on the choice of \( x_0 \in C \). By a suitable choice of \( x_0 \) it can be made arbitrarily large. A similar statement holds for the condition that \( x = \text{ord}(\gamma) \in X_{L_v} \) is in the support of \( O^{L'}_v(\mathcal{T}_v(P)) = \chi^G_{QP} \), \( L' \subseteq L \), \( P = L' N_F \), and \( Q = LN_Q \). In fact all values

\[ \text{const}(x_0) \]
2.7 The Satake Transform

\[ \alpha_i(w(x)), \beta_j(w(x)) \quad w \in W, \quad i, j \in \Delta(G_v, A_v) \]

are different from zero, and either \( > \text{const}(x_0) \) or \( < -\text{const}(x_0) \).

**Preferred Places** \( S' \). These facts can now be used in the global context to concentrate the effect of the adelic cutoff functions \( \hat{\chi}_{G_Q} \), as they appear in the formula of Corollary 2.2, to a finite set \( S' \) of “preferred” local non-Archimedean places in the sense that

\[ \text{tr}_s \left( \mathcal{F}^\mathcal{G}_{\chi_{Q}}; H^* (S_{L_Q}, V) \right) = \text{tr}_s \left( \mathcal{F}^\mathcal{G}_{\chi_{Q}}; H^* (S_{L_Q}, V) \right) \]

holds (in a suitable context). For this it would suffice to know that

\[ T_{\text{ell}} \left( \mathcal{F}^\mathcal{P}_{\chi_{Q}} L'_{\chi_{P'}} \right) = T_{\text{ell}} \left( \mathcal{F}^\mathcal{P}_{\chi_{Q}} L'_{\chi_{P'}} \right) \]

holds for all \( L' \subseteq L \), where \( L' \) is a Levi component of \( P = L' N_P \subseteq Q = L N_Q \) (Corollary 2.2). Alternatively (Corollary 2.3) it would be enough to know that

\[ O_{\gamma} \left( \mathcal{F}^\mathcal{P}_{\chi_{Q}} L'_{\chi_{P'}} \right) = O_{\gamma} \left( \mathcal{F}^\mathcal{P}_{\chi_{Q}} L'_{\chi_{P'}} \right) \]

Before we explain under which conditions this holds, we first recall certain definitions.

### 2.7.2 Global Situation

For \( P \) and \( Q \) of \( T \) the global cutoff function \( \hat{\chi}_{Q} \), which occurs in Corollary 2.3, was defined for \( P = L' N_P \) using the Harish-Chandra map \( H_P \) via

\[ L'(Q) \xrightarrow{\gamma} L' (\mathbb{A}) \xrightarrow{H_P} \mathcal{X}_{L'} . \]

In fact, by the product formula \( H_P(\gamma) = \text{log} |\gamma_\infty| - \sum_{v \neq \infty} q_v \cdot \text{ord}_L(\gamma_v) \), the global cutoff condition can be written as the condition on the point

\[ \sum_{v \neq \infty} q_v \cdot \text{ord}_L(\gamma_v) \in \mathcal{X}_{L'} \]

to lie in the support of \( \mathcal{F}^\mathcal{P}_{\chi_{Q}} \).

**Notation:** \( \gamma = (\gamma_v)_{v} \in L(\mathbb{A}_{\text{fin}}) \). \( q_v \) denotes the cardinality of the residue field, and \( \text{ord}_L(\gamma_v) \) the image of the local element \( \text{ord}_L(\gamma_v) \in \mathcal{X}_{L'_v} \) in \( \mathcal{X}_{L'} \) under the natural projection map \( \mathcal{X}_{L'_v} \to \mathcal{X}_{L'} \) (notice that locally the maximal \( F'_v \)-split torus may be larger than the maximal \( Q \)-split torus \( A_{L'_v} \)).
Assumptions. To be more specific about the concentration at specific places, let us assume \( f = \prod_{v \neq \infty} f_v \). Furthermore, suppose there are two finite disjoint sets \( S \) and \( S' \) of non-Archimedean places such that \( f_v \) is the unit element of the spherical Hecke algebra for all \( v \notin S \cup S' \). Suppose \( f_S = \prod_{v \in S} f_v \) has support in a fixed compact subset of \( G(\mathbb{A}_S) \). Finally, suppose that all \( f_v \) for \( v \in S' \) are spherical such that the Satake transform \( S_\lambda(f_v) \) has the following property.

Property (**). For all roots \( \alpha \) and all dual roots \( \beta \) in the set of \( \mathbb{Q} \)-rational simple roots of \( (G, P_0) \) and all elements \( w \in W \) the absolute value of the linear forms \( \alpha \circ w \) and \( \beta \circ w \) on \( \sum_{v \in S'} q_v \cdot \text{ord}_{L'}(\gamma_v) \in X_{L'} \) is larger than a fixed constant \( c > 0 \).

If \( c \) is sufficiently large compared with the support of \( f_S \), we obviously get

Lemma 2.13. Under the assumptions above, if the constant \( c \) is large enough depending only on the support of \( f_S \), the truncation condition concentrates on the places in \( S' \)

\[
O'(\mathcal{I}^{(P)}(\chi^G_{\lambda} \chi^G_{P'})) = O'(\mathcal{I}^{(P)}(\hat{\chi}^G_{\lambda} \chi^G_{P'}))_{S'}. 
\]

Notation. Let \( \mathcal{E}_v \) denote the set of irreducible constituents \( \rho = \rho_S \otimes \rho_{S'} \in \mathcal{E}_v \) of the admissible representation of \( G(\mathbb{A}_{fin}) \) on the cohomology group \( H^\bullet(S_L, V) \).

Corollary 2.4. Let the situation be as in Lemma 2.13. Then the truncated Lefschetz number \( \text{tr}_s(f^Q_{\lambda}(\chi^G_{\lambda}), \tau) \) is given by \( \text{tr}_s(f^Q_{S}(\hat{\chi}^G_{\lambda}), \tau) \), or alternatively by a sum

\[
\sum_v (-1)^v \cdot \sum_{\rho \in \mathcal{E}_v} \text{tr}(f^S \cdot \text{Ind}_{L(\mathbb{A}_S)}(\rho_S)) \cdot \text{tr}(\mathcal{I}^{(P)}(\hat{\chi}^G_{\lambda})), \rho_{S'}),
\]

where now \( f^{S'} = f_S \prod_{v \notin S', v \neq \infty} 1_w \).

Proof. The first statement follows from Corollary 2.3 together with Lemma 2.13, which implies \( T_{\text{ell}}(f^Q_{\lambda}, \tau) = T_{\text{ell}}(f_S \cdot (f^Q_{S}(\hat{\chi}^G_{\lambda})), \tau) \). The second formula then follows from the first assertion via the adjunction formula. \( \square \)

2.8 Automorphic Representations

Fix \( \lambda \) and a compact open subgroup \( K = \prod_{v \neq \infty} K_v \subseteq \Omega \) of \( G(\mathbb{A}_{fin}) \), which defines the “level,” the level group. The \( G(\mathbb{A}_{fin}) \)-module given by
the limit $H^\bullet(S(G), V_\lambda)$ is an admissible representation of $G(\mathbb{A}_{\text{fin}})$. Only finitely many irreducible constituents $\pi$ with the property $\pi^K \neq 0$ occur. The same holds for the finitely many Levi subgroups $L$, the induced level groups $K_L = (K \cap P(\mathbb{A}_{\text{fin}}))_L$, and the induced coefficient systems attached to the highest weights $\lambda' = w(\lambda + p\rho_G) - \rho_L$. Thus, the admissible representation

$$\Pi(\lambda) = \bigoplus_{P_0 \subseteq Q \subseteq G} \bigoplus_{w \in W^Q} \text{Ind}_Q^{G(\mathbb{A}_{\text{fin}})}(H^1(S(L), V_{w(\lambda + p\rho_G) - \rho_L}),$$

the “halo” of the $G(\mathbb{A}_{\text{fin}})$-module $H^\bullet(S(G), V_\lambda)$, again contains only finitely many irreducible $G(\mathbb{A}_{\text{fin}})$-constituents $\pi$ with the property $\pi^K \neq 0$. Let $\mathcal{P}$ be the set of equivalence classes of these representations of level $K$.

**Remark 2.10.** $\Pi(\lambda)$ should be considered as a superspace whose grading is induced by the sign defined by the parity of the sum of the number $\text{rank}(G) - \text{rank}(Q)$, the length $l(w)$ for $w \in W^P$, and the degree $i$.

Let $S_0$ be the set of places for which $K_v \neq \Omega_v$ (level primes). Outside $S_0$ representations in $\mathcal{P}$ are unramified. Fix a prime $p \notin S_0$, the “Frobenius” prime. For $\pi$ in $\mathcal{P}$ consider the representation $\pi^p$ of $G(\mathbb{A}_{\text{fin}})$ defined by $\pi = \pi^p \otimes \pi_p$. The set of places $S_0$ can be enlarged to a finite set $S$ of places not containing $p$ such that $\pi^p_1 \cong \pi^p_2 \iff (\pi_1)_S \cong (\pi_2)_S$. There exists $f_\mathbb{A} \in C^\infty_c(G(\mathbb{A}_{\text{fin}}))$, so $\text{tr} \pi_S(f_\mathbb{A}) = 0$ holds for all representations $\pi^p$ in $\mathcal{P}$ for which $(\pi^p)^p$ is not isomorphic to $\pi^p$, where $\pi$ is some fixed representation in $\mathcal{P}$. Furthermore, we can assume $\text{tr} \pi_S(f_\mathbb{A}) = 1$. For a suitable choice of $K$ (in a cofinal system, where $K^S$ is a product of special good maximal compact open subgroups), one can assume in addition that $f_\mathbb{A}$ is $K^S$-bi-invariant (see the Remark 4.3 on page 79). Now fix the $\pi^p$-projector $f_\mathbb{A}$. For a non-Archimedean place $v \notin S$ consider functions

$$f = f_\mathbb{A} \cdot h_p \cdot f_v \cdot \prod_{w \neq \infty \text{ else}} 1_w$$

in $C^\infty_c(G(\mathbb{A}_{\text{fin}}))$, where $h_p$ and $f_v$ are suitable functions in the spherical Hecke algebra $\mathcal{H}(G_p, \Omega_p)$, respectively, $\mathcal{H}(G_v, \Omega_v)$. $f_v$ is chosen subject to the conditions:

- Property $\ast$ (see the assumptions preceding Lemma 2.13) holds for $S' = \{v\}$ with respect to the fixed function $f_\mathbb{A}$ or more precisely its fixed support in $G(\mathbb{A}_S)$.
- $\text{tr} \pi_v(f_v) = 1$ holds for the unramified component $\pi_v$ of our fixed representation $\pi^p = \bigotimes_{w \neq p, \infty} \pi_w$.

Such functions $f_v$ exist, as explained on page 42, as a consequence of Lemma 2.11 choosing $x_0$ in the cone $C$ to be sufficiently large. The function $h_p$ is chosen to be either:

- $h_p = 1_p$ (unit element of $\mathcal{H}(G_p, \Omega_p)$) or
- $h_p^{(\alpha)} = b(\phi_h)$ (the local cyclic base change of the Kottwitz function $\phi_h$ on $G(E_p)$ of [51] under the unramified base change map homomorphism $b$ of spherical
Hecke algebras for some unramified local field extension $E_p/Q_p$ of degree $[E_p : Q_p] = n$ in the context where $G$ is attached to a Shimura variety as in [51] with reflex field $Q$ (for simplicity).

We claim that either for $h_p = 1$, $S' = \{v\}$, or for $h_p = h_p^{(n)}$, $S' = \{p, v\}$, and for sufficiently large $n \gg 0$, the assumptions preceding Lemma 2.13 are satisfied. For $h_p = 1$ this has already been explained. The case $h_p = h_p^{(n)}$ and $n \gg 0$ can be reduced to the ensuing Lemma 2.14. We leave this as an exercise. So taking this for granted, now assume $n \gg 0$ or $h_p = 1$.

Then we get from Corollary 2.4 an expression for the truncated Lefschetz numbers

$$\text{tr}_s(\mathcal{T}^{(Q)}_{\chi_Q} G, H^*(S_L, V))$$

in terms of

$$\sum_{\nu} \sum_{\rho \in \mathcal{E}_\nu} (-1)\nu \cdot \text{tr}(f^{S'}, \text{Ind}_{L(A)}^{G(A)}(\rho^{S'})) \cdot \text{tr}(\mathcal{T}^{(Q)}_{S'}(\chi_Q^G)_{S'}, \rho_{S'}).$$

This allows us to apply a theorem of Franke [27] which states that all irreducible representations $\rho$ of $L(A_{fin})$ which occur in $\mathcal{E}_\nu$ as constituents of the cohomology group $H^*(S_L, V)$ are automorphic representations of $L(A_{fin})$. Hence, all induced representations in

$$\text{Ind}_{L(A)}^{G(A)}(\rho^{S'})$$

are automorphic representations of $G(A^{S'})$, and are Eisenstein representations for $L \neq G$.

Therefore, if the fixed representation $\pi \in \mathcal{P}$ is cuspidal and not CAP, $\pi^{G}$ does not occur as a constituent in $\mathcal{P}$ from these induced representations in the case $L \neq G$. Since $f$ and $\mathcal{T}$ are $K_{S}$-bi-invariant, the trace of $f$ on $\Pi(\lambda)$ involves only constituents in $\mathcal{P}$, i.e., for the fixed level $K$. Since $f_S$ is a projector for $\pi^{G}$ among the representations in $\mathcal{P}$, this implies $\text{tr}(f_S, \text{Ind}_{L(A)}^{G(A)}(H^*(S_L, V))) = 0$. Hence, the truncated Lefschetz numbers

$$\text{tr}_s(\mathcal{T}^{(Q)}_{\chi_Q} G, H^*(S_L, V))$$

all vanish except for the case $G = Q$, where the truncated Lefschetz number is the trace $\text{tr}_s(\mathcal{T}, H^*(S_L, V))$ of $\mathcal{T}$ on the cohomology $H^*(S_L, V)$. Notice $\mathcal{T} = \mathcal{T}^{(G)} \neq f$ in general. However, $f$ and $\mathcal{T}$ have the same trace on every irreducible admissible representation. This follows from $O^G(\mathcal{T}) = O^G(f)$, since $\text{vol}(\Omega) = 1$. But then we can replace $\mathcal{T}$ by $f$. Then, since $f$ is $K_{S}$-bi-invariant, the remaining Lefschetz number is the trace of $f$ on the finite-dimensional space $H^*(S_K(G), V)$ for fixed level $K$, and it only involves the representations in $\mathcal{P}$. Since $f_S$ is a $\pi^{G}$-projector, the trace of $h_p f_v$ on this space is the trace of $h_p f_v$ on the generalized $\pi^{G}$-eigenspace of the cuspidal cohomology. Since $\text{tr} \pi_v(f_v) = 1$, this simplifies the formula for $T_{\lambda}^{G}(f_S f_v h_p, \lambda)$ of Corollary 2.2, and leaves only the term for $Q = G$ and $w = 1$. This proves
2.8 Automorphic Representations

Theorem 2.2. Suppose \( \pi \) is an irreducible cuspidal representation of \( G(\mathbb{A}_{\text{fin}}) \) and not CAP (see [69, 97]). Then for \( f = f_S f_v h_p \prod_{w \neq \infty} \chi_w \), where \( f_S, f_v, \) and \( h_p = 1 \) or \( h_p = h_p^{(n)} \) and \( n \gg 0 \) is chosen as above, we get for the trace on the \( \pi^p \)-constituents

\[
\text{tr}_a \left( h_p, H^\bullet(S_K(G), V_\lambda)(\pi^p) \right) = T_{\text{ell}}^G \left( f_S f_v, h_p, \lambda \right).
\]

This theorem will be used in Chap. 3 for the “Frobenius” prime \( p \).

Notice that \( f_S f_v \) again is a projector on \( \pi^p \) among the representations in \( P \). We write \( f_S f_v = f_{\pi^p} \), and call it a “good \( \pi^p \)-projector.”

Lemma 2.14. Let \( C \subseteq V \) be a cone in Euclidean space, \( W \) a finite group acting on \( V \), and \( L_1, \ldots, L_r \) a \( W \)-stable set of linear forms in \( V^* \) nonvanishing on \( C \). For \( x_0 \in C \) and \( x \in V \) and a bounded set \( M \subseteq V \), there exists an integer \( m \) depending on \( M \) and an integer \( N \) depending on \( m \) and \( M \) such that the following holds. Suppose \( v = v_1 + v_2 + v_3 \) for \( v_1 \in \bigcup_{w \in W} \{ n \cdot w(v) \mid n \geq N \} \), \( v_2 \in \bigcup_{w \in W} w(n \cdot x_0 + C) \), and \( v_3 \in M \). Then \( L_i(v) > 0 \) for some \( i = 1, \ldots, r \) holds if and only if \( L_i(v_1 + v_2) > 0 \) holds.

Proof. Obvious. \( \square \)

Remark 2.11. In the case \( h_p = 1 \), we may also omit the auxiliary prime \( p \) or choose \( p \) to be large so that the formula in Theorem 2.2 becomes

\[
\sum_v (-1)^{\text{dim}_C(H^\bullet(S_K(G), V_\lambda)(\pi))} = T_{\text{ell}}^G(f_{\pi^p}, \lambda)
\]

for a good \( \pi \)-projector \( f_{\pi^p} \in C^\infty_C(G(\mathbb{A}_{\text{fin}})) \).

Remark 2.12. In the Hermitian symmetric case there exists a formula analogous to Theorem 2.2 for the \( L^2 \)-cohomology instead of the Betti cohomology. In this case the \( L^2 \)-cohomology is finite-dimensional, so one can define the traces of Hecke operators on the \( L^2 \)-cohomology. Using the results in [33], one obtains a formula for the \( L^2 \)-Lefschetz numbers analogous to the one of Corollary 2.3. The relevant change in this case amounts to a subtler substitute of \( W_P \), which in the case of \( L^2 \)-cohomology also depends on the elements \( \gamma \). In fact one obtains the following formula for the \( L^2 \)-Lefschetz number:

\[
\sum_{P_0 \subseteq P = LN \subseteq G} \sum_{w \in W^P} (-1)^{\# I(w)} \cdot T_{\text{ell}}^{L^P} \left( \chi_{\lambda P}(w), w(\lambda + \rho_G) - \rho_L \right),
\]

where the cutoff functions \( \chi_{\lambda P}(w) \) now depend on \( w \in W^P \). They are defined as follows: \( \chi_{\lambda P}(w) \) is the characteristic function of the set of all \( \gamma \in L(Q) \), which satisfy \( I(\gamma) = I(w) \), for certain finite sets \( I(w) \) depending only on \( w, P, G, \) and \( \lambda \).
(see [33], p.474), and where $I(\gamma)$ is the set of simple roots $\alpha$ of $A_P$ in $N_P$ such that $|\alpha(\gamma)|_{f}^{-1} = |\alpha(\gamma)|_{\infty} \leq 1$ (see [33], p.471). Let $\chi_{P,i}$ denote the characteristic function of the set of all $\gamma \in L(\mathbb{R})$ such that $|\alpha_i(\gamma)|_{\infty} > 1$ for the simple root $\alpha_i$. Then the characteristic function $\chi_{P,i}(w)$ can be expressed in the form $\prod_{i \in I} \chi_{P,i}(1 - \chi_{P,i})$ for $I = I(w)$. It can therefore be expanded into a finite linear combination of the functions $\chi_K = \prod_{i \in K} \chi_{P,i}$ for $K \subseteq \Delta(G,A)$. For these finitely many global cutoff functions $\chi_K$, which now appear in the $L^2$-Lefschetz formula, the effect of the cutoff can now be concentrated at some preferred non-Archimedean places $v \in S'$ by the choice of a suitable good $\pi^S$-projector, modified at a single place $S' = \{v\}$ as in the discussion above using a variant of Lemma 2.13. This implies

**Corollary 2.5.** Suppose $G_{\infty}$ is of Hermitian symmetric type. Suppose $\pi$ is an irreducible cuspidal representation of $G(K_{\text{fin}})$ and not CAP. Then there exists a good $\pi$-projector $f_\pi$ such that the $L^2$-Lefschetz number of the $\pi$-constituents is

$$\text{tr}_s(H^2_\ell(G, V_\chi)(\pi)) = T_{\ell}^G(f_\pi, \lambda).$$

In particular, the alternating sums of the $\pi$-multiplicities on the cohomology and the $L^2$-cohomology coincide.

### 2.9 The Discrete Series Case

This is the case considered in [4]. Suppose $G$ is a connected reductive group over $\mathbb{Q}$, $G_{\text{der}}$ is simply connected, and $G$ contains a maximal $\mathbb{R}$-torus $B$, for which $B(\mathbb{R})/A_G(\mathbb{R})^0$ is compact (see [4], p. 262).

**Notation.** Let $2q(G)$ denote the real dimension of the symmetric domain attached to $G_{\infty}$ and $d(G)$ the cardinality of the packets of discrete series representations of $G_{\infty}$. Let $\tau$ be an irreducible complex representation of $G(\mathbb{Q})$ defined by the highest weight $\lambda \in X^+(B)_{\mathbb{C}}$. $\lambda$ defines a representation of $G(\mathbb{C})$, and hence of the compact inner form $\tilde{G}$ of $G$ over the field $\mathbb{R}$. Let $\tau^+$ denote the contragredient representation. Attached to the representation $\tau$ of $G$ is a packet $\Pi_{\text{disc}}(\tau)$ of discrete series representations $\pi_{\infty}$. Let $\pi^+_{\infty}$ denote the contragredient. Attached to $\tau$ and $\lambda$ is the function

$$f_\lambda = \frac{f_\lambda}{d(G)},$$

where $f_\lambda \in \mathcal{H}_{\text{disc}}(G_{\infty}, \xi^{-1}_{\lambda})$ (in the notation in [4], Lemma 3.1) is the stable cuspidal function (i.e., supported in discrete series, see [4], Sect. 4) defined by Clozel and Delorme. $f_\infty$ is compactly supported modulo $A_G(\mathbb{R})^0$ and is $K_{\infty}$-invariant. Then using the notation in [4], p. 271, formula (4.3), $tr \rho^*(f_\infty) = tr \rho^*(f_\infty) = (-1)^{q(G_{\infty})} tr \pi^+_{\infty}(\frac{1}{d(G)})$ for $\pi_{\infty} \in \Pi_{\text{disc}}(\rho)$ becomes $d(G)^{-1}$ if $\pi^+_{\infty} \in \Pi_{\text{disc}}(\tau)$ and is zero otherwise (see [4], Lemma 3.1). Notice $\pi^+_{\infty} \in \Pi_{\text{disc}}(\tau)$ if and only if $\rho^* \cong \tau$. Hence,
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\[ \text{tr } \rho^*(f_\infty) = d(G)^{-1} \]

if \( \rho \cong \tau^* \), and \( \text{tr } \rho^*(f_\infty) = 0 \) otherwise.

The orbital integral \( O^G(\gamma, \rho, \pi) = \int_{G(\mathbb{R})} f(x^{-1}gx)dx \), considered for fixed \( \gamma \in G(\mathbb{R}) \) as a distribution on \( \mathcal{H}_{ac}(G(\mathbb{R}), \xi_\gamma^{-1}) \), is denoted \( \Phi_G(\gamma, f) \) in [4], p. 269, and in [5], p. 325. Theorem 5.1 in [4] gives a formula valid for all \( \gamma \in G(\mathbb{R}) \) which expresses the orbital integral of stable cuspidal functions \( f_\infty \in \mathcal{H}_{ac}(G_\infty, \xi_\gamma^{-1}) \) in terms of the distributions \( \rho^*(f) \) discussed above,

\[ O^G(\gamma, f_\infty) = (-1)^{q(G)} d(G_\gamma) \cdot \text{vol}(\mathcal{O}_{G_\gamma, \infty})^{-1} \sum_{\rho} \Phi_G(\gamma, \rho) \cdot \text{tr } \rho^*(f_\infty), \]

for certain coefficients \( \Phi_G(\gamma, \rho) \). The sum runs over irreducible representations \( \rho \) of \( G(\mathbb{R}) \) in \( \Pi(\mathcal{O}_{G(\mathbb{R}), \xi_\gamma}) \). In particular, \( O^G(\gamma, f) \) is zero unless \( \gamma \) is semisimple and \( \gamma \in T(\mathbb{R}) \) for some maximal \( \mathbb{R} \)-torus of \( G \) such that \( T(\mathbb{R})/A_G(\mathbb{R})^0 \) is compact. Notice \( T(\mathbb{R}) \cong B(\mathbb{R}) \). On the regular part \( T_{reg}(\mathbb{R}) \) the function is \( \Phi_G(\gamma, \rho) = \text{tr } \rho(f) \) (see [4], p. 271). Since \( \Phi_G(\gamma, \rho) \) extends to a continuous function on \( T(\mathbb{R}) \) (see [4], Lemma 4.2), this holds for all \( \gamma \in T(\mathbb{R}) \). Hence, if \( O^G(\gamma, f) \) does not vanish a priori, one has \( \gamma \in T(\mathbb{R}) \), where \( T \) is a maximal \( \mathbb{R} \)-torus in \( G \) such that \( T(\mathbb{R})/A_G(\mathbb{R}) \) is compact. And for all \( \gamma \in T(\mathbb{R}) \) one has the formula

\[ O^G(\gamma, f_\infty) = (-1)^{q(G)} d(G_\gamma) \cdot \text{vol}(\mathcal{O}_{G_\gamma, \infty})^{-1} \text{tr } \tau^*(\gamma) d(G)^{-1}, \]

since only \( \rho \cong \tau^* \) contributes to the sum over all \( \rho \). Notice \( \text{tr } \tau^{-1} = \text{tr } \tau^*(\gamma) \) for the contragredient representation. Next, from the formula for the Euler numbers (see, e.g., [4], p. 281, formula (6.3), and also p. 282)

\[ \chi(G, df) = (-1)^{q(G)} d(G) \cdot \text{vol}(G(\mathbb{Q})A_G(\mathbb{R})^0 \setminus G(\mathbb{A})) \cdot \text{vol}(\mathcal{O}_{G(\mathbb{R}), \infty})^{-1}, \]

one obtains for \( f_{\text{fin}} \in C_c^\infty(G(\mathbb{A}_{\text{fin}})) \)

\[ \chi(G_\gamma) \cdot \text{tr } \tau^{-1} \cdot O^G(\gamma, f_{\text{fin}}) \]

\[ = (-1)^{q(G_\gamma)} d(G_\gamma) \cdot \text{vol}(\mathcal{O}_{G_\gamma, \infty})^{-1} \tau(G_\gamma) \cdot \text{tr } \tau^*(\gamma) \cdot O^G(\gamma, f_{\text{fin}}) \]

\[ = d(G) \tau(G_\gamma) O^G(\gamma) O^G_\gamma(f_{\text{fin}}) \]

\[ = d(G) \tau(G_\gamma) O^G(\gamma, f_{\text{fin}}), \]

provided the measure \( d\gamma_\infty \) is chosen such that \( d\gamma_\infty d\gamma_{\text{fin}} \) is the Tamagawa measure on \( G(\mathbb{A}) \). Hence, from the definition of \( T^G_\ell(\gamma, f_{\text{fin}}) \) we obtain

**Lemma 2.15.**

\[ T^G_\ell(f_{\text{fin}}, \tau) = d(G) \sum_{\gamma \in \tilde{\gamma}(G(\mathbb{Q}))} \tau(G_\gamma) O^G(\gamma, f_{\text{fin}}, f_\infty). \]
The summation is over all semisimple, strongly elliptic conjugacy classes of $G(\mathbb{Q})$. Here $\tau(G_{\gamma})$ is the Tamagawa number $\text{vol}(G, A_G(R)/G, A_G(\mathbb{A}))$, where the measure $dg_\infty$ is chosen such that $dg_\infty dg_{\text{fin}}$ is the Tamagawa measure on $G(\mathbb{A})$.

**Corollary 2.6.** With the assumptions and the notation used in Theorem 2.2 we get for $f_{\text{fin}} = h_p f_{\pi^p}$

$$tr(h_p, H^* (S_K(G), V_\lambda)(\pi^p)) = d(G) \sum_{\gamma \in G(\mathbb{A})/\sim} \tau(G_{\gamma}) O_G^G(A) (h_p f_{\pi^p} f_\infty).$$

The summation is over all semisimple, strongly elliptic conjugacy classes of $G(\mathbb{Q})$. The measures defining the orbital integrals are assumed to be Tamagawa measures on $G(\mathbb{A})$ and $G_{\gamma}(\mathbb{A})$.

**Remark 2.13.** The term $O_G^G(f_S f_{\pi^p} f_\infty)$ is independent of the chosen measures $dg_f$ and $dg_\infty$ provided $dg_\infty dg_f$ is the Tamagawa measure on $G(\mathbb{A})$. This follows from the definition of $f_{\text{fin}}$ and $f_\infty$. Hence, in applications we are now free to normalize the measures $dg_f$ and $dg_\infty$, e.g., such that $\text{vol}_{dg_f}(K) = 1$ following the convention of [51].

**Remark 2.14.** Assume that $Z_G/A_G$ is anisotropic over $\mathbb{R}$. If one considers a Shimura variety attached to $G$ (as in [51]) one replaces $S_K(G) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K, K$ by $G(\mathbb{Q}) \backslash G(\mathbb{A}) / \text{Zentr}(h)_\infty K$, where $h$ is the underlying structure homomorphism of the Shimura variety. For small $K$ this multiplies the trace by the index $[K : \text{Zentr}(h)^-_\infty]$. See also the remark on page 21! In fact $\gamma \epsilon_\infty \in K \text{Zentr}(h)^-_\infty$ for $\epsilon_\infty \in K^\infty$, and $\gamma \in G(\mathbb{Q})$ implies $\gamma \in Z_G(\mathbb{Q}) (K$ is small) and $\gamma \in K K^\infty$. Hence, $\gamma$ is finite, and hence is 1 ($K$ is small). Therefore, $\epsilon_\infty \in \text{Zentr}(h)^-_\infty$.

**Appendix 1**

Let $G$ be a reductive connected group over $\mathbb{Q}$. Let $K_\infty \subseteq G(A_{\text{fin}})$ be a compact open subgroup. For $g \in G(A_{\text{fin}})$ put $K_g = K_{g^{-1}} K g \cap K \subseteq K$. Consider $M = G(\mathbb{Q}) \backslash G(\mathbb{A})$, or some compactification, with continuous $G(A_{\text{fin}})$ left action $m \mapsto mg^{-1}$, $g \in G(A_{\text{fin}})$ together with the maps $p(m) = m$ and $p'(m) = mg^{-1}$

$$p : M/K_\infty \rightarrow M/K$$

$$p' : M/K' \rightarrow M/K.$$ The map $p$ (or the map $p'$) is equivariant with respect to the map $q$ (or the map $q'$) from $K' = K_\infty$ to $K$, defined by $k \mapsto k$ or $k \mapsto gkg^{-1}$. Two points $mK$ and $m'K$ in $M/K$ are related by the correspondence underlying $p, p'$ if there exists a point $m''K' \in M/K'$ such that...
\[ p(m'' K') = mK \text{ and } p'(m'' K') = m' K \text{ in } M/K. \]

This means that there exist \( k, k' \in K, k'' \in K' \) such that \( mkk'' = m'' \) and \( m''g^{-1} = m'(k')^{-1} \) holds. Hence, \( mkk''g^{-1}k' = m' \). Stated in other terms, \( m' = mx^{-1} \) for some \( x \in KgK \). There exists a finite decomposition \( KgK = \bigcup_j Kg_j \).

Hence,
\[ m' K = mg_j^{-1} K \]

for some \( j \). Conversely, suppose \( m' K = mkg^{-1}K \) for some \( k \in K \). Then for \( m'' := mKg \), we get \( p(m'' K_g) = mK = mK \) and \( p'(m'' K_g) = mkg^{-1}K = m' K \).

Put \( \Gamma = K \cap G(\mathbb{Q}) \). In general for \( \gamma \in G(\mathbb{Q}) \) the double coset \( \Gamma \gamma \Gamma = \bigcup_i \Gamma \gamma_i \Gamma \) decomposition gives \( K \gamma \Gamma = \bigcup_i K \gamma_i \Gamma \), again a disjoint union. Since \( k_{1 \gamma_1} = k_{2 \gamma_2} \) implies \( k_{2^{-1}k_1} = \gamma_2 \gamma_1^{-1} \in G(\mathbb{Q}) \cap K = \Gamma \), we get \( \Gamma \gamma_1 = \Gamma \gamma_2 \). Passing to the closure defines the subset \( K \gamma \Gamma \bigcap = \bigcup_i K \gamma_i \Gamma \) of \( K \gamma K = \bigcup_i Kg_j \), which might be smaller than \( KgK \) if \( \Gamma \neq K \). Therefore, to relate fixed points of the adelic correspondence to its classical analogue, one has to ensure that fixed points belong to cosets \( gK \) of the form \( \gamma K \) for some \( \gamma \in G(\mathbb{Q}) \) and in particular \( KgK = K \gamma K \).

However, this is the case (see page 24). Only rational cosets \( \gamma K \) contribute to the fixed points of the Goresky–MacPherson trace formula for the Lefschetz numbers.

**Appendix 2**

Let \( G_\infty \) be the group of real points of a reductive group over \( \mathbb{R} \). Let \( K_\infty \) be a maximal compact group, and let \( V_1 \subseteq Z_\infty \) be a vector group in the center \( Z_\infty \).

**Claim 2.1.** Then for every \( y \in K_\infty \cdot V_1 \), the set \( S \) of all \( x \in G_\infty \), such that \( x^{-1}yx \in K_\infty \cdot V_1 \), is either empty or
\[ S = G_{y, \infty} \cdot K_\infty. \]

Here \( G_{y, \infty} \) denotes the centralizer of \( y \) in \( G_\infty \).

**Proof.** The proof of this assertion is easily reduced to the case \( V_1 = 1 \). In fact, \( G_\infty = ^0G \cdot V \), where \( V \) is the maximal vector group in the center of \( G_\infty \) and \( ^0G \) is the normal subgroup of \( G_\infty \) with \( ^0G \cap V = \{e\} \) chosen as in [98], p. 19. Notice \( K_\infty \subseteq ^0G_{\infty} \).

This allows us to reduce the proof to the case where \( y \in K_\infty \) and \( x \) satisfies the equation \( x^{-1}yx \in K_\infty \). In fact, if \( x_0^{-1}y_0x_0 = k \cdot v_1 \) holds for some \( x = x_0 \) and \( k \in K_\infty, v \in V_1 \), we simply replace \( y \) by \( y_1 = x_0yv_1^{-1}x_0^{-1} \in K_\infty \) and \( x \) by \( x_1 = x_0^{-1}x \). Then \( x_1^{-1}y_1x_1 \in K_\infty V_1 \) is equivalent to \( x^{-1}yx \in K_\infty \cdot V_1 \). However \( x_1^{-1}y_1x_1 \in K_\infty V_1 \) if and only if \( x_1^{-1}y_1x_1 \in G_\infty \cap (K_\infty V_1) = K_\infty \). So we assume \( y \in K_\infty \) and \( x^{-1}yx \in K_\infty \).

Choose a Cartan involution \( \theta \) of \( G_\infty \) such that \( g \in K_\infty \) if and only if \( \theta(g) = g \) (see [98], Proposition 5). For \( x \) as above, the element \( z = \theta(x)x^{-1} \) is in \( G_{y, \infty} \), and
satisfies $\theta(z) = z^{-1}$. One can write $x = s \cdot \kappa$ for $\kappa \in K_\infty$ and $s = \exp(\sigma)$ and $\theta(\sigma) = -\sigma \in \text{Lie}(G_\infty)$ (follows from [98], Proposition 5). Then $z = \exp(-2\sigma) \in G_{y,\infty}$. Since $y \in K_\infty$, $y$ and hence also $G_{y,\infty}$ is $\theta$-stable. Therefore, there exists a symmetric one-parameter subgroup in $G_{y,\infty}$ passing through $z$. See, e.g., [98], p. 20. In other words we find a symmetric root $r = \exp(-\sigma) \in G_{y,\infty}$, $\theta(r) = r^{-1}$ of $z = r^2$ for $\theta(\sigma) = -\sigma \in \text{Lie}(G_{y,\infty})$. We conclude $1 = r^{-1}\theta(x)x^{-1}r^{-1} = \theta(rx)(rx)^{-1}$. Thus, $rx = k \in K_\infty$ and $x = r^{-1}k \in G_{y,\infty} \cdot K_\infty$, which proves the claim. □

**Corollary 2.7.** $S/(K_\infty \cdot V_1)$ is either empty or $G_{y,\infty}/(G_{y,\infty} \cap K_\infty)V_1$, where $G_{y,\infty} \cap K_\infty$.

**Proof.** Notice $V_1 \subseteq G_{y,\infty}$. □

**Remark 2.15.** Finally, there exists a diffeomorphism $G_\infty/(K_\infty \cdot V_1) \cong (0G_\infty/K_\infty) \times V/V_1$. In particular for $V_1 \subseteq A_G(R)^0$, we see that $G_\infty/(K_\infty \cdot V_1)$ is homotopic to $X_G = G_\infty/(K_\infty \cdot A_G(R)^0)$. 
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