Chapter 2
Plastic Behaviour of Sheet Metal

List of Special Symbols

\( a \) exponent in the Hershey and Hosford yield criteria
\( a, b \) coefficients in the Hill 1990 yield criterion
\( a, b, c, f, g, h \) material parameters in the Barlat 1991 yield criterion
\( a, b, c, h, p \) coefficients in the Barlat 1989 yield criterion
\( a, b, m, n, p, q \) parameters describing the planar anisotropy of the material in the Ferron yield criterion
\( a, b, c, d, e, f, g \) coefficients in the BBC 2000 yield criterion
\( a_1 \ldots a_4 \) coefficients in the Cazacu–Barlat yield criteria
\( a_1 \ldots a_{25} \) coefficients in the Soare yield criteria
\( A_0, \ldots, A_9 \) coefficients in the Gotoh yield criterion
\( b_1 \ldots b_{11} \) coefficients in the Cazacu–Barlat yield criteria
\( c \) weighting coefficient in the Karafillis–Boyce yield criterion
\( c, p, q \) coefficients in the Hill 1993 yield criterion
\( c_1, c_2, c_3 \) material coefficients describing the material anisotropy in the Barlat 1994 yield criterion
\( c_{12}, c_{13} \ldots c_{66} \) coefficients in the linear transformation in Barlat 2000 yield criterion
\( C \) elasticity tensor
\( C', C'' \) linear transformations in Barlat 2000 yield criterion
\( C_D \) material constant in the Drucker yield criterion
\( D \) strain-rate tensor
\( E \) elastic modulus
\( f, F, \varphi \) yield function
\( f, g, h, a, b, c \) coefficients in the Hill 1979 yield criterion
\( F, G, H, L, M, N \) coefficients in the Hill 1948 yield criterion
\( F_1, F_2 \) functions in the expression of the uniaxial yield stress (Barlat 1989 yield criterion)
\( F_b \) function used to define the biaxial yield stress and the biaxial anisotropy coefficient
$F_\theta$ function used to define the uniaxial yield stress and the anisotropy coefficient

g(\alpha) function used to define the Budiansky yield criterion

$g(\theta, \alpha)$ function used to define the Ferron yield criterion

$h$ scalar parameter which defines the plastic deformation accumulated in the material

$h_{ij}$ anisotropy coefficients in the von Mises 1928 yield criterion

$I_2, I_3$ second and third invariants of the stress tensor

$J_2, J_3$ second and third invariants of the stress tensor

$k$ exponent in the the Karafillis–Boyce and BBC yield criteria

$k_1, k_2$ invariants of the stress tensor

$l$ final gage length

$l_0$ initial gage length

$L$ linear transformation tensor in the Karafillis–Boyce yield criterion

$L, M, N$ function in the Comsa yield criterion

$M$ integer exponent used by the yield criteria

$M, N, P, Q, R, S, T$ coefficients in the BBC yield criteria

$m, n$ exponents used by the yield criteria

$p$ exponent in the generalized Drucker yield criterion

$p$ accumulated equivalent plastic strain

$p_1 \ldots p_8$ coefficients in the Comsa yield criterion

$R$ material parameter in the Lin–Ding yield criterion

$R$ isotropic hardening variable

$R, S, T$ shear yield stresses in the principal anisotropic directions (Hill 1948)

$r, R$ normal anisotropy coefficient

$r_b$ biaxial anisotropy coefficient

$r_\theta$ anisotropy coefficient associated to the direction $\theta$

$r_0, r_{45}, r_{90}$ anisotropy coefficients at 0, 45 and 90° from the rolling direction

$s$ exponent in the Lin–Ding yield criterion

$s$ deviatoric stress tensor in Barlat 2000 yield criterion

$S$ IPE stress tensor used by the Karafillis–Boyce yield criterion

$S$ stress deviatoric tensor

$S_1, S_2, S_3$ principal deviatoric stresses

$S_{11}, S_{22}, S_{33}$ components of the IPE stress tensor used by the Karafillis–Boyce yield criterion

$S_{12}, S_{23}, S_{31}$ Boyce yield criterion

$t_0, t$ initial and final thickness of the specimen

$t_1, t_2$ functions in the expression of the uniaxial anisotropy coefficient (Barlat 1989 yield criterion)

$T$ transformation matrix in Barlat 2000 yield criterion
Plastic Behaviour of Sheet Metal

\( w \) final width of the specimen
\( w_0 \) initial width of the specimen
\( W_f \) energy of distortion
\( W_p \) elastic potential energy
\( W_v \) volumetric change energy
\( X \) linear transformation stress tensor in Barlat 2000 yield criterion
\( X, Y, Z \) tensile yield stresses in principal anisotropic directions (Hill 1948)
\( Y \) yield stress
\( Y_\theta \) uniaxial yield stress in a sample inclined by \( \theta \) with respect to the rolling direction
\( Y_b \) theoretical biaxial yield stress
\( \alpha \) angle between principal stress \( \sigma_1 \) and rolling direction
\( \alpha, \beta, \gamma \) coefficients in the Wang yield criterion
\( \alpha = \sigma_2/\sigma_1 \) ratio of the principal stresses
\( \alpha \) back-stress tensor
\( \alpha_1, \alpha_2, \alpha_3 \) coefficients in the Barlat 1994 yield criterion
\( \alpha_1, \ldots, \alpha_8 \) coefficients in the Barlat 2000 yield criterion
\( \alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3, C \) parameters defining the anisotropy of the material in the Karafillis-Boyce yield criterion
\( \alpha_x, \alpha_y, \alpha_z \) coefficients in the Barlat 1994 yield criterion
\( \beta, \varphi, \delta, \gamma \) accuracy index of the yield criteria
\( \beta_1, \beta_2, \beta_3 \) auxiliary coefficients used to define the linear transformation tensor in the Karafillis-Boyce yield criterion
\( \Delta r \) variation of anisotropy coefficients
\( \varepsilon_e \) equivalent (effective) strain
\( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) principal (logarithmic) strains
\( \varepsilon, \varepsilon_e, \varepsilon_p \) tensors of total, elastic and plastic strain respectively
\( \Phi \) plastic potential
\( \phi \) invariant homogeneous function
\( \Gamma, \Psi, \Lambda \) function in the BBC yield criteria angle between the specimen longitudinal axis and the rolling direction
\( \lambda \) parameter of the Bézier function used in Vegter’s yield criterion
\( \lambda \) plastic multiplier in the flow rule
\( \mu \) Poisson’s ratio
\( \sigma \) stress tensor
\( \sigma_0, \sigma_{45}, \sigma_{90} \) uniaxial yield stress at 0, 45 and 90° from the rolling direction
\( \sigma_0 \) initial yield stress
\( \sigma_1, \sigma_2, \sigma_3 \) principal stresses
\( \sigma_b \) equibiaxial yield stress
\( \sigma_e \) equivalent (effective) stress
\( \sigma_k \) hardening stress
\[ \sigma_u \] uniaxial yield stress
\[ \sigma_{11}, \sigma_{22}, \sigma_{33}, \] components of the actual stress tensor
\[ \sigma_{12}, \sigma_{23}, \sigma_{31} \] shear yield stress

In the Sect. 2.8 the summation convention over repeated indices is used. Let \( \mathbf{A} \) denote second-order tensor and \( \mathbf{B} \) a forth-order tensor. One can define the double contracted tensor product as \( \mathbf{A} : \mathbf{A} = A_{ij}A_{ij} \) and \( (\mathbf{B} : \mathbf{A})_{ij} = B_{ijkl}A_{kl} \). The norm of \( \mathbf{A} \) is \( \| \mathbf{A} \| = \sqrt{\mathbf{A} : \mathbf{A}} \) and its direction is \( \mathbf{n} = \mathbf{A}/\| \mathbf{A} \| \). The time derivative is \( \dot{\mathbf{A}} = d\mathbf{A}/dt \).

### 2.1 Anisotropy of Sheet Metals

#### 2.1.1 Uniaxial Anisotropy Coefficients

Due to their crystallographic structure and the characteristics of the rolling process, sheet metals generally exhibit a significant anisotropy of mechanical properties. In fact, the rolling process induces a particular anisotropy characterised by the symmetry of the mechanical properties with respect to three orthogonal planes. Such a mechanical behaviour is called orthotropy. The intersection lines of the symmetry planes are the orthotropy axes. In the case of the rolled sheet metals, their orientation is as follows (see Fig. 2.1): rolling direction (RD); transverse direction (TD); normal direction (ND).

The variation of their plastic behavior with direction is assessed by a quantity called Lankford parameter or anisotropy coefficient [1]. This coefficient is determined by uniaxial tensile tests on sheet specimens in the form of a strip. The anisotropy coefficient \( r \) is defined by

\[
 r = \frac{\varepsilon_{22}}{\varepsilon_{33}} \tag{2.1}
\]

**Fig. 2.1** Orthotropy axes of the rolled sheet metals:
LD—longitudinal direction;
TD—transversal direction;
ND—normal direction
where, $\varepsilon_{22}, \varepsilon_{33}$ are the strains in the width and thickness directions, respectively. In the case of an isotropic material, the coefficient is one and the width and thickness strains have the same value. If the coefficient is greater than one, the width strains will be dominant (the ‘thinning resistance’ is more pronounced). On the other hand, for the materials having a coefficient less than one, the thickness strains will dominate.

Using the notations from Fig. 2.2, Eq. (2.1) can be written in the form

$$ r = \frac{\ln \frac{w}{w_0}}{\ln \frac{t}{t_0}} $$

(2.2)

where $w_0$ and $w$ are the initial and final width, while $t_0$ and $t$ are the initial and final thickness of the specimen, respectively.

As the thickness of the specimen is very small compared to its width (usually by at least one order), the relative errors of measurement of the two strains will be quite different. Therefore the above relationships are replaced by one implying quantities having the same order of magnitude: length and width of the specimen. Taking into account the condition of volume constancy

$$ \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 $$

(2.3)
the following form of Eq. (2.1) is obtained

\[ r = -\frac{\varepsilon_{22}}{\varepsilon_{11} + \varepsilon_{22}} \]  \hspace{1cm} (2.4)

and Eq. (2.2) becomes

\[ r = \frac{-\ln \frac{w}{w_0}}{\ln \frac{l}{l_0} + \ln \frac{w}{w_0}} \]  \hspace{1cm} (2.5)

where \( l_0 \) and \( l \) are the initial and final gage length. The length \( l_0 \) is specified by standards, see [2]. Equation (2.5) can be rearranged as follows:

\[ r = \frac{\ln \frac{w}{w_0}}{\ln \frac{l}{l_0} \cdot w_0 \cdot \frac{l_0 \cdot w_0}{l \cdot w}} \]  \hspace{1cm} (2.6)

This relationship is used in practice for evaluating the anisotropy coefficient.

Experiments show that \( r \) depends on the in-plane direction. If the tensile specimen is cut having its longitudinal axis inclined with the angle \( \theta \) to the rolling direction, the coefficient \( r_\theta \) is obtained (see Fig. 2.3). The subscript specifies the angle between the axis of the specimen and the rolling direction.

![Fig. 2.3 Tensile specimen prelevated at the angle \( \theta \) (measured from the rolling direction)](image-url)
Another important element characterising the performances of a yield criterion is the capability to predict the variations of the uniaxial yield stress and coefficient of plastic anisotropy in the plane of the sheet metal.

In order to assess this capability, we shall establish the relationships defining the dependence of the parameters mentioned above on the angle $\theta$ measured from the rolling direction (Fig. 2.4).

Let us denote by $Y_\theta$ the uniaxial yield stress corresponding to the direction inclined at the angle $\theta$ with respect to the rolling direction. In the case of a uniaxial load, the components of the stress tensor can be expressed as follows:

\begin{align*}
\sigma_{11} &= Y_\theta \cos^2 \theta; \\
\sigma_{22} &= Y_\theta \sin^2 \theta; \\
\sigma_{21} = \sigma_{12} &= Y_\theta \sin \theta \cos \theta
\end{align*} \hspace{1cm} (2.7)

By replacing Eq. (2.7) in the relationship defining the equivalent stress and taking into account its homogeneity, we obtain:

\[ \sigma = Y_\theta \cdot F_\theta, \] \hspace{1cm} (2.8)

where $F_\theta$ is a function depending on the angle $\theta$. Of course, $F_\theta$ is defined according to the specific formulation of the equivalent stress. If we combine Eq. (2.8) with the coherence condition,

\[ \Phi(\sigma, Y) := \sigma - Y(h) = 0, \] \hspace{1cm} (2.9)

where $\Phi(\sigma, Y)$ is the yield function associated to the yield criterion, $Y$—yield stress, $h$—scalar parameter defining the plastic strain accumulated by the material, we get:

\[ Y_\theta = \frac{Y(h)}{F_\theta}. \] \hspace{1cm} (2.10)
Equation (2.10) defines the uniaxial yield stress corresponding to the planar direction identified by the angle $\theta$. If the reference yield stress is selected to be the one corresponding to the rolling direction ($Y(h) = Y_0$), we obtain the following relationship:

$$Y_\theta = \frac{Y_0}{F_\theta}.$$  

(2.11)

In this case, the yield stress corresponding to some planar direction will depend only on the yield stress associated to the rolling direction and the function $F_\theta$ (which is related to the yield criterion adopted in the model). The determination of the function $F_\theta$ will be presented in the next subchapters, for each type of yield criterion.

In a similar way, we can establish the relationship defining the variation of the coefficient of plastic anisotropy in the plane of the sheet metal. Let us consider the specimen inclined at the angle $\theta$ with respect to the rolling direction (Fig. 2.4). According to Eq. (2.1), the instantaneous coefficient of plastic anisotropy $r_\theta$ is defined as the ratio of the plastic strain rates associated to the width (inclined at the angle $\theta+90^\circ$ with respect to the rolling direction), $\dot{\varepsilon}_{\theta+90}$, and thickness, $\dot{\varepsilon}_{33}$:

$$r_\theta = \frac{\dot{\varepsilon}_{\theta+90}}{\dot{\varepsilon}_{33}}.$$  

(2.12)

Taking into account the incompressibility restraint (see Eq. 2.3), as well as the expressions of the strain rate components along the principal directions,

$$\dot{\varepsilon}_2 = \dot{\varepsilon}_{11} \cos^2 \theta + \dot{\varepsilon}_{22} \sin^2 \theta + \dot{\varepsilon}_{12} \sin \theta \cos \theta,$$

$$\dot{\varepsilon}_{33} = -(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{22}),$$  

(2.13)

we obtain the relationship defining the coefficient of plastic anisotropy associated to the direction $\theta$:

$$r_\theta = \frac{\dot{\varepsilon}_{11} \cos^2 \theta + \dot{\varepsilon}_{22} \sin^2 \theta + \dot{\varepsilon}_{12} \sin \theta \cos \theta}{\dot{\varepsilon}_1 + \dot{\varepsilon}_2} - 1.$$  

(2.14)

Equation (2.14) can be rewritten in terms of the stress components if the associated flow rule is used:

$$r_\theta = \frac{\sigma_{11} \frac{\partial \sigma}{\partial \sigma_{11}} + \sigma_{22} \frac{\partial \sigma}{\partial \sigma_{22}} + \sigma_{12} \frac{\partial \sigma}{\partial \sigma_{12}}}{\sigma_{11} \frac{\partial \sigma}{\partial \sigma_{11}} + \sigma_{22} \frac{\partial \sigma}{\partial \sigma_{22}}} - 1.$$  

(2.15)

By coupling Eq. (2.15) with Euler’s identity, we obtain:

$$r_\theta = \frac{\sigma}{Y_\theta \left( \frac{\partial \sigma}{\partial \sigma_{11}} + \frac{\partial \sigma}{\partial \sigma_{22}} \right)} - 1.$$  

(2.16)
Finally, after replacing Eq. (2.8) in the last relationship, we get:

\[ r_\theta = \frac{F_\theta}{\frac{\partial \sigma}{\partial \sigma_{11}} + \frac{\partial \sigma}{\partial \sigma_{22}}} - 1. \] (2.17)

This formula defines the coefficient of plastic anisotropy as a dependence of the specimen inclination. In order to make use of it, we need the expression of the equivalent stress and the function \( F_\theta \), both of them being specific to the yield criterion adopted in the plasticity model. The planar distribution of the coefficient of plastic anisotropy will be determined in the next subchapters, for different yield criteria.

The average of the \( r \)-values obtained for different directions in the plane of the sheet metal represents the so-called coefficient of normal anisotropy \( r_n \). Having determined the values of \( r \) at specimens cut along three directions in the plane of the sheet metal (0°, 45°, 90°, respectively), the coefficient of normal anisotropy is determined by:

\[ r_n = \frac{r_0 + 2r_{45} + r_{90}}{4}. \] (2.18)

A measure of the variation of normal anisotropy with the angle to the rolling direction is given by the quantity:

\[ \Delta r = \frac{r_0 + r_{90} - 2r_{45}}{2}, \] (2.19)

known as planar anisotropy.

This quantity is related to the earring amplitude of the deep-drawn cups. More precisely, if the value of the anisotropy coefficient is the same along all the directions in the plane of the sheet metal, the earring phenomenon will not be observed. In order to get a more intuitive image of the planar anisotropy of sheet metals, a polar coordinate representation is frequently used. The diagram in Fig. 2.5 [3] gives such a representation for an aluminium alloy (AA5182-0). The polar diagrams are preferred due to their capability to offer direct information on the tendency of the sheet metal to form ears. One may notice the symmetry of the curve with respect to the coordinate axes. This fact is a proof of the orthotropic behaviour of the sheet metal.

By convention the \( r \)-values usually are determined at 20% elongation for the purpose of comparison. Modern tensile testing machines perform instantaneous measurement of the quantities required for evaluating the anisotropy coefficient and calculate it during the test. Values of \( r \) at 20% elongation as well as its variation with strain can be determined.

Figure 2.6 [4] shows the planar variation of the anisotropy coefficient for DDQ mild steel at different straining degrees. One may notice a significant variation of the \( r_{45} \) coefficient with respect to the straining degree.
2.1.2 Biaxial Anisotropy Coefficient

The experimental research has proved that the yield surfaces are not symmetric in the biaxial region [5, 6]. This fact is also a consequence of the plastic anisotropy. In order to give a quantitative description of such a behaviour, the so-called coefficient of biaxial anisotropy has been defined independently by Barlat et al. [5] and Poehlandt et al. [7]. Barlat and his co-workers [5] have proposed the use of
2.1 Anisotropy of Sheet Metals

Fig. 2.7 Strains measured on the specimen during the disk compression test [5]

A compression test for the experimental determination of this mechanical parameter. A set of circular specimens are subjected to a normal pressure. Due to the plastic anisotropy, the discs become elliptic during the compression. The phenomenon can be observed in Fig. 2.7 [5], showing the results of a compression test for an AA6111-T4 aluminium alloy. By measuring the major and minor axes of the elliptic specimen, the corresponding principal strains can be evaluated.

As in the case of the uniaxial tension, the ratio of the principal strains will define the coefficient of biaxial anisotropy:

$$r_b = \frac{\varepsilon_{22}}{\varepsilon_{11}}$$ (2.20)

If the material is isotropic, the coefficient will be one. The more pronounced is the anisotropy, the farther is the coefficient from unity. This parameter is a direct measure of the slope of the yield locus at the balanced biaxial stress state. Pöhlandt et al. [7] have proposed to use the biaxial tensile testing machine to determine the coefficient.

Figure 2.8 shows the method used for the determination of the principal strains on a biaxial testing machine. This experimental procedure is limited by the fact that the straining degree is rather small (less than 5%).
As for the uniaxial coefficient of plastic anisotropy, we shall define a general relationship defining the biaxial coefficient. The relationship will be usable for any yield criterion.

In the case of biaxial tension along the rolling and transverse directions, Eqs. (2.7) will get the following particular forms:

\[
\begin{align*}
\sigma_{11} &= \sigma_{22} = Y_b; \\
\sigma_{21} &= \sigma_{12} = 0.
\end{align*}
\]  

(2.21)

where, \(Y_b\) is the theoretical biaxial yield stress.

Consequently, Eq. (2.8) becomes:

\[
\sigma_b = Y_b \cdot F_b,
\]

(2.22)

Here, \(\sigma_b\) stands for the experimental biaxial yield stress, while \(F_b\) is a constant quantity depending on the yield criterion adopted in the plasticity model.

Equation (2.22) provides the theoretical biaxial yield stress as a dependence of the the experimental biaxial yield stress and the parameter \(F_b\):

\[
Y_b = \frac{\sigma_b}{F_b}.
\]

(2.23)

In a very close analogy with the case of the uniaxial coefficient of plastic anisotropy (see Eqs. 2.12, 2.13, 2.14, 2.15, 2.16, and 2.17), one may deduce the relationship defining the coefficient of biaxial plastic anisotropy:
2.2 Yield Criteria for Isotropic Materials

In this section the focus is on the conditions under which a material passes from the elastic to the plastic state and on the associated flow rules. In order to describe the plastic behavior of a material in a general stress state, three elements are needed:

a) a yield criterion expressing a relationship between the stress components at the moment when plastic ‘yielding’ occurs
b) an associated flow rule expressing the relationship between the components of the strain-rate and stress
c) a hardening rule describing the evolution of the initial yield stress during the forming process.

The transition from the elastic to the plastic state occurs when the stress reaches the yield point of the material. The yield point in uniaxial tension is established using the stress-strain curve of the material whereby a convention is necessary in order to define it, or by temperature measurement.

In case of a multiaxial stress state it is more difficult to define a criterion for the transition from the elastic to the plastic state. A relationship between the principal stresses is needed specifying the conditions under which plastic flow occurs. Such a relationship is usually defined in the form of an implicit function (known as the ‘yield function’):

\[ F(\sigma_1, \sigma_2, \sigma_3, Y) = 0 \]

(2.25)

where \( \sigma_1, \sigma_2, \sigma_3 \) are the principal stresses and \( Y \) is the yield stress obtained from a simple test (tension, compression or shearing).

Equation (2.25) can be interpreted as the mathematical description of a surface in the three dimensional space of the principal stresses usually called the ‘yield surface’. It must be closed, smooth and convex. For incompressible materials it is a cylinder the cross section of which depends on the material (only for the von Mises criterion—see below—it is a circular cylinder as shown in Fig. 2.9).

All the points located in the inside of the surface \((F < 0)\) are related to an elastic state of the material. The points belonging to the surface \((F = 0)\) are related to a plastic state. The points located outside the surface \((F > 0)\) have no physical meaning.
In the case of plane stress (e.g. $\sigma_3 = 0$) the yield surface reduces to a curve in the plane of the principal stresses $\sigma_1$ and $\sigma_2$.

The expression of the yield function is established on the basis of some phenomenological considerations concerning the transition from the elastic to the plastic state.

The most widely used yield criteria for isotropic materials have been proposed by Tresca (the ‘maximum shear stress criterion’) and Huber–von Mises (the ‘strain energy criterion’) [8].

Basically the yield function may be defined in two different ways [9]: either by assuming that plastic yield begins when some physical quantity (energy, stress, etc.) attains a critical value or by approximating experimental data by an analytical function.

The latter class of yield functions are not obtained from a calculus based on the crystallographic structure of the material; they are purely phenomenological functions. The advantages of using such phenomenological yield functions instead of those based on the crystallographic texture are [10]:

- they usually have a simpler mathematical form
- they are easy to understand and manipulate (‘user-friendly’)
- they may be easily implemented in finite element codes
- they may be generalized to describe the anisotropic behavior of the materials
- they may also be easily adapted to describe the behavior of fcc instead of bcc materials (for example, by simply changing the exponent of the Hershey family yield criteria).

The main disadvantage of the phenomenological approach is the poor accuracy under multi-axial and non-proportional loading conditions.
2.2 Yield Criteria for Isotropic Materials

2.2.1 Tresca Yield Criterion

The oldest yield criterion was proposed by Tresca in 1864 [11] on the basis of his observation that plastic strains appear by crystallographic gliding under acting shear stresses.

According to this criterion the material passes from an elastic to a plastic state when the maximum shear stress $\tau_{\text{max}}$ reaches a critical value.

In the general case, the criterion may be written as follows:

$$\max \{|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\} = \sigma_0$$  \hspace{1cm} (2.26)

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are principal stresses.

Under plane stress condition ($\sigma_3 = 0$), Eq. (2.26) becomes

$$\sigma_1 - \sigma_2 = \sigma_0 = 2K; \quad \sigma_1 > \sigma_2$$  \hspace{1cm} (2.27)

$$\sigma_2 - \sigma_1 = \sigma_0 = 2K; \quad \sigma_1 < \sigma_2$$  \hspace{1cm} (2.28)

Equations (2.27) and (2.28) can be unified:

$$\sigma_1 - \sigma_2 = \pm \sigma_0 = \pm 2K;$$  \hspace{1cm} (2.29)

Equation (2.29) represents a polygon in the plane of the principal stresses $\sigma_1$ and $\sigma_2$ and a hexagonal prism in the space, see Fig. 2.10.

By squaring Eq. (2.29) it is obtained

$$(\sigma_1 - \sigma_2)^2 = \sigma_0^2$$  \hspace{1cm} (2.30)

Fig. 2.10 Tresca yield surface in the stress space
In the case when the stress components \( \sigma_{11} \) and \( \sigma_{22} \) do not coincide with the principal stresses, the latter takes the following form:

\[
(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 = \sigma_0^2
\]  
(2.31)

### 2.2.2 Huber–Mises–Hencky Yield Criterion

This criterion was proposed independently by Huber [12] and von Mises [13] and further developed by Hencky [14]. It is based on the observation that a hydrostatic pressure cannot cause plastic yielding of the material. Thus, the conclusion that only the elastic energy of distortion influences the transition from an elastic to a plastic state comes naturally. This idea has been proposed first by Maxwell in a letter to Thomson [15] but it was not published at the time. For simplicity, in the following text the criterion will often be referred to as the Mises criterion.

This criterion can be formulated as follows: the material passes from an elastic to a plastic state when the elastic energy of distortion reaches a critical value that is independent of the type of the stress state.

The elastic potential energy, \( W_p \), has two components: the volumetric change energy \( W_v \) and the energy of distortion, \( W_f \).

\[
W_p = W_v + W_f
\]  
(2.32)

Thus \( W_f \) can be written in the form

\[
W_f = W_p - W_v
\]  
(2.33)

After replacing the expressions of the elastic potential energy and energy of distortion in Eq. (2.33), it is obtained

\[
W_f = \frac{1 + \mu}{6E} \cdot \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]
\]  
(2.34)

In the case of uniaxial tension (\( \sigma_2 = \sigma_3 = 0 \)) yielding of the material occurs if \( \sigma_1 = \sigma_0 \), where \( \sigma_0 \) is the uniaxial yield stress in direction 1. Thus, the critical value of the elastic energy of distortion at which the material passes from an elastic to the plastic state is:

\[
W_f = \frac{1 + \mu}{6E} 2\sigma_0^2
\]  
(2.35)

Then the Mises criterion may be written in the form:

\[
\frac{1 + \mu}{6E} 2\sigma_0^2 = \frac{1 + \mu}{6E} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right],
\]  
(2.36)
or

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2$$  \hspace{1cm} (2.37)

For plane stress ($\sigma_3 = 0$) Eq. (2.37) becomes

$$(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2 = 2\sigma_0^2$$  \hspace{1cm} (2.38)

rewritten in the form

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \cdot \sigma_2 = \sigma_0^2$$  \hspace{1cm} (2.39)

Another equivalent of the Mises criterion is

$$3 \left( \frac{\sigma_1 - \sigma_2}{\sigma_0} \right)^2 + \left( \frac{\sigma_1 + \sigma_2}{\sigma_0} \right)^2 = 4$$  \hspace{1cm} (2.40)

Equations (2.39) and (2.40) represent an ellipse in the plane of the principal stresses $\sigma_1 - \sigma_2$ which is circumscribed to the polygon given by Tresca criterion, see Fig. 2.11.

### 2.2.3 Drucker Yield Criterion

In order to represent the experimental data located between the Tresca and Mises yield surfaces, Drucker [16] proposed the following criterion:

$$J_2^3 - C_D J_3^2 = F$$  \hspace{1cm} (2.41)

where $J_2$ and $J_3$ are the second and third invariants of the stress tensor, respectively, and $C_D$ is a constant.
Equation (2.41) may be generalized in the form

\[ J_2^{3p} - C_D J_3^{2p} = F \]  

(2.42)

\( p \) being an integer.

### 2.2.4 Hershey Yield Criterion

Based on the Norton [17] and Bailey [18] laws used for non-linear creep, Hershey introduced a non-quadratic formulation of the yield criterion [19]

\[ (\sigma_1 - \sigma_2)^a + (\sigma_2 - \sigma_3)^a + (\sigma_3 - \sigma_1)^a = 2Y^a \]  

(2.43)

Here \( Y \) is the uniaxial yield stress and \( a \) is an exponent determined based on the crystallographic structure of the material. For \( a = 2 \), Eq. (2.43) reduces to the Mises yield condition, whereas for \( a = 1 \) and in the limit case \( a \to \infty \) it leads to the Tresca yield condition. For \( 2 < a < 4 \), the corresponding surface lies outside the Mises circular cylinder, whereas for \( 1 < a < 2 \) and for \( a > 4 \), it lies between Mises and Tresca. The Hershey formulation has been used later (1972) by Hosford [20].

The Hershey’s formulation has been generalized by Karafillis and Boyce [21] in the following form

\[ \Phi = (1 - c) \Phi_1 - c \Phi_2, \]  

(2.44)

where

\[ \Phi_1 = |S_1 - S_2|^{2k} + |S_2 - S_3|^{2k} + |S_3 - S_1|^{2k} = 2\sigma_e^{2k} \]  

(2.45)

and

\[ \Phi_2 = |S_1|^{2k} + |S_2|^{2k} + |S_3|^{2k} = \frac{2^{2k} + 2}{3^{2k}} \sigma_e^{2k} \]  

(2.46)

Here \( S_1, S_2, \) and \( S_3 \) are the principal deviatoric stresses, \( c \) is a weighting coefficient, and \( 2k \) is an exponent having the same significance as the exponent \( a \) in Hosford’s criterion.

For \( k = 1 \) the Eqs. (2.45) and (2.46) take the form given by von Mises, however, for \( k = \infty \) Eq. (2.45) becomes the Tresca function and (2.46) gives an upper limit of the yield surface.

The value of the coefficient \( c \) is in the range \([0, 1]\). It determines the weight of the functions \( \Phi_1 \) and \( \Phi_2 \) in the yield function \( \Phi \). As a consequence, there are two parameters \( k \) and \( c \) that may be used in order to ‘adjust’ the shape of the yield locus whereas the other criteria use only one parameter (exponent \( a \) or \( m \)) for this purpose. Therefore the new criterion is very flexible.

More examples of isotropic yield functions are reviewed by Žyczkowski [9] and Yu [22].
2.3 Classical Yield Criteria for Anisotropic Materials

The first yield criterion for anisotropic materials was proposed by von Mises in the form of a quadratic function [23]. Though it was initially used for describing the plastic behavior of an anisotropic single crystal, later it was also used for polycrystals.

This criterion is given by

\[ \Phi = h_{11}\sigma_{11}^2 + h_{22}\sigma_{22}^2 + h_{33}\sigma_{33}^2 + 2h_{44}\sigma_{12}^2 + 2h_{55}\sigma_{23}^2 + 2h_{66}\sigma_{31}^2 + 2h_{12}\sigma_{11}\sigma_{22} + 2h_{13}\sigma_{11}\sigma_{33} + 2h_{14}\sigma_{11}\sigma_{12} + 2h_{15}\sigma_{11}\sigma_{23} + 2h_{16}\sigma_{11}\sigma_{31} + 2h_{23}\sigma_{22}\sigma_{33} + 2h_{24}\sigma_{22}\sigma_{12} + 2h_{25}\sigma_{22}\sigma_{23} + 2h_{26}\sigma_{22}\sigma_{31} + 2h_{34}\sigma_{33}\sigma_{12} + 2h_{35}\sigma_{33}\sigma_{23} + 2h_{36}\sigma_{33}\sigma_{31} + 2h_{45}\sigma_{12}\sigma_{12} + 2h_{46}\sigma_{12}\sigma_{31} + 2h_{56}\sigma_{23}\sigma_{31} \]  

(2.47)

where \( h_{ij} (i, j = 1, 2, \ldots, 6) \) are coefficients of anisotropy which can be identified by mechanical tests. Equation (2.47) gives a quadratic function containing products implying both normal and shear stresses.

Olszak [24] gave a generalization of this function for non-homogeneous anisotropic materials. In the case of an orthotropic material, it can be reduced to a quadratic function having only six terms and coefficients of anisotropy. This is the same as the function proposed by Hill in 1948 [25].

2.3.1 Hill’s Family Yield Criteria

2.3.1.1 Hill 1948 Yield Criterion

In 1948 Hill [25] proposed an anisotropic yield criterion as a generalization of the Huber-Mises-Hencky criterion. The material is supposed to have an anisotropy with three orthogonal symmetry planes.

The yield criterion is expressed by a quadratic function of the following type:

\[ 2f (\sigma_{ij}) \equiv F (\sigma_{22} - \sigma_{33})^2 + G (\sigma_{33} - \sigma_{11})^2 + H (\sigma_{11} - \sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2 = 1, \]  

(2.48)

Here \( f \) is the yield function; \( F, G, H, L, M \) and \( N \) are constants specific to the anisotropy state of the material, and \( x, y, z \) are the principal anisotropic axes.

In the case of sheet metals, axis 1 is usually parallel to the rolling direction, 2 is parallel to the transverse direction and 3 is collinear with the normal direction.

If the tensile yield stresses in the principal anisotropy directions are denoted by \( X, Y \) and \( Z \) it can easily be shown that

\[ \frac{1}{X^2} = G + H; \quad \frac{1}{Y^2} = H + F; \quad \frac{1}{Z^2} = F + G \]  

(2.49)

From this equation, by some simple mathematical calculations the coefficients \( F, G \) and \( H \) are obtained as functions of the uniaxial yield stresses:
2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}; \quad 2G = \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}; \quad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}. \quad (2.50)

If R, S and T are the shear yield stresses associated to the same directions, then

\[ \begin{align*}
2L &= \frac{1}{R^2}, \quad 2M = \frac{1}{S^2}, \quad 2N = \frac{1}{T^2}. \quad (2.51)
\end{align*} \]

Only one of the parameters F, G, H can be negative. This situation rarely occurs in practice (it would cause great differences between the stresses); F > G if and only if X > Y, etc. L, M and N are always positive.

As a consequence, in order to give a complete description of the anisotropy of the material, six independent yield stresses (X, Y, Z, R, S and T) have to be known as well as the orientation of the principal anisotropy axes.

The yield criterion may be interpreted as a surface in a six-dimensional space of the stress components. The points located at the interior of the surface represent the elastic states of the material, while points belonging to the surface correspond to the plastic state.

For plane stress (\(\sigma_{33} = \sigma_{31} = \sigma_{23} = 0; \sigma_{11} \neq 0; \sigma_{22} \neq 0; \sigma_{12} \neq 0\)), the yield criterion becomes

\[ 2f(\sigma_{ij}) \equiv (G + H) \sigma_{11}^2 - 2H \sigma_{11} \sigma_{22} + (H + F) \sigma_{22}^2 + 2N \sigma_{12}^2 = 1. \quad (2.52) \]

After introducing the yield stress X, Y, Z and T, Eq. (2.52) may be rewritten as

\[ \frac{1}{X^2} \sigma_{11}^2 - \left( \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \right) \sigma_{11} \sigma_{22} + \frac{1}{Y^2} \sigma_{22}^2 + \frac{1}{T^2} \sigma_{12}^2 = 1, \quad (2.53) \]

When the principal directions of the stress tensor coincide with the principal anisotropic axes, the Hill 1948 yield criterion has the form

\[ \frac{1}{X^2} \sigma_{1}^2 - \left( \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \right) \sigma_{1} \sigma_{2} + \frac{1}{Y^2} \sigma_{2}^2 = 1, \quad (2.54) \]

where \(\sigma_1, \sigma_2\) are the non-zero principal stresses.

When simulating sheet metal forming processes, the anisotropy coefficients are denoted \(r_0, r_{45}, r_{90}\) and the yield stresses in the directions of the principal anisotropic axes are denoted as follows: \(X = \sigma_0, Y = \sigma_90\). The relations between the anisotropy coefficients and the coefficients \(F, G, H, \ldots\) may be easily obtained from the flow rule associated to the yield function:

\[ \begin{align*}
 r_0 &= \frac{H}{G}; \quad r_{90} = \frac{H}{F}; \quad r_{45} = \frac{N}{F + G} - \frac{1}{2}. \quad (2.55)
\end{align*} \]

It can be shown that the following relation between the yield stresses and the anisotropy coefficients applies:
\[ \sigma_0 = \sqrt{\frac{r_0 (1 + r_{90})}{r_{90} (1 + r_0)}} \] (2.56)

This equation implies that from \( r_0 > r_{90} \) it follows \( \sigma_0 > \sigma_{90} \) and the reciprocal, however, some materials do not satisfy this condition.

The last of the three Eqs. (2.55) leads to

\[ N = (F + G) \left( r_{45} + \frac{1}{2} \right) \] (2.57)

or, according to Eq. (2.57),

\[ 2N = \frac{1}{Z^2} \frac{2r_{45} + 1}{2} = \frac{1}{\sigma_o^2} \frac{r_0 + r_{90}}{r_{90} (1 + r_0)} (2r_{45} + 1). \] (2.58)

Finally it is obtained

\[ \frac{1}{\sigma_0^2} \sigma_{11}^2 - \left( \frac{1}{\sigma_{0}^2} + \frac{1}{\sigma_{90}^2} - \frac{1}{\sigma_0^2} r_{90} (1 + r_0) \right) \sigma_{12} \sigma_{22} + \frac{1}{\sigma_{90}^2} \frac{r_0 + r_{90}}{r_{90} (1 + r_0)} (2r_{45} + 1) \sigma_{12}^2 = 1 \] (2.59)

As \( \sigma_0 \) and \( \sigma_{90} \) are not independent, but related by (2.57), Eq. (2.59) may also be written as

\[ \sigma_{11}^2 - \frac{2r_0}{1 + r_0} \sigma_{12} \sigma_{22} + \frac{r_0 (1 + r_{90})}{r_{90} (1 + r_0)} \sigma_{22}^2 + \frac{r_0 + r_{90}}{r_{90} (1 + r_0)} (2r_{45} + 1) \sigma_{12}^2 = \sigma_0^2 \] (2.60)

In case that the principal directions of the stress tensor are coincident with the anisotropic axes \( (\sigma_{11} = \sigma_1, \sigma_{22} = \sigma_2, \sigma_{12} = 0) \), the Hill 1948 yield criterion can be written as a dependence of the principal stress in the form

\[ \sigma_1^2 - \frac{2r_0}{1 + r_0} \sigma_1 \sigma_2 + \frac{r_0 (1 + r_{90})}{r_{90} (1 + r_0)} \sigma_2^2 = \sigma_0^2 \] (2.61)

or, taking into account Eq. (2.56)

\[ \sigma_1^2 - \frac{2r_0}{1 + r_0} \sigma_1 \sigma_2 + \frac{r_0 (1 + r_{90})}{r_{90} (1 + r_0)} \sigma_2^2 = \frac{r_0 (1 + r_{90})}{r_{90} (1 + r_0)} \sigma_{90}^2. \] (2.62)

From Eqs. (2.61) and (2.62) it follows that in order to define the yield under plane stress condition, three mechanical parameters, namely the coefficients \( r_0 \) and \( r_{90} \) and one of the uniaxial yield stresses \( \sigma_0 \) and \( \sigma_{90} \) are needed.

Equations (2.61) or (2.62) geometrically represent families of ellipses depending on the parameters \( r_0 \) and \( r_{90} \). The influence of these parameters upon the yield loci is demonstrated in Figs. 2.12 and 2.13.
In case of a material exhibiting only normal anisotropy \((r_0 = r_{90} = r)\) Eq. (2.56) imposes that \(\sigma_0 = \sigma_{90}\) and Eqs. (2.61) and (2.62) take the same form:

\[
\sigma_1^2 - \frac{2r}{1+r} \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_u^2
\]  

(2.63)
where $\sigma_u$ is the uniaxial yield stress.

The influence of the parameters $r$ and $\sigma_u$ upon the shape of the yield locus is shown in Figs. 2.14 and 2.15.

It can be seen that if $r < 1$, the yield locus predicted by the Hill 1948 criterion is located inside the one given by von Mises; if $r > 1$ the Hill 1948 yield locus is outside the von Mises yield locus. Woodthrope and Pearce [26] and Pearce [27] noticed that some materials (in particular aluminum alloys) have the yield locus outside the von Mises surface though their $r$-coefficient was less than one.

This behavior cannot be properly described by the Hill 1948 yield criterion and materials exhibiting it are called ‘anomalous’.

Indeed, by rewriting Eq. (2.63) for the case of the equibiaxial tension ($\sigma_1 = \sigma_2 = \sigma_b$) one obtains

$$\sigma_b = \sigma_u \sqrt{\frac{1 + r}{2}}, \quad (2.64)$$

where $\sigma_b$ is the equibiaxial yield stress.

It follows that if $r > 1$, then $\sigma_b > \sigma_u$ and if $r < 1$ then $\sigma_b < \sigma_u$. This problem will be considered again below.

It can be seen from Fig. 2.15 that if the uniaxial yield stress increases, the yield surface expands uniformly. This is called isotropic hardening.

Equation (2.63) may be rewritten in the form

Fig 2.14 Influence of the normal anisotropic coefficient on the shape of the yield locus defined by the Hill 1948 criterion

![Graph showing the influence of the normal anisotropic coefficient on the shape of the yield locus defined by the Hill 1948 criterion.](image)
In Fig. 2.15 Influence of the uniaxial yield stress on the shape of the yield locus defined by the Hill 1948 criterion.

\[
(1 + 2r) \left( \frac{\sigma_1 - \sigma_2}{\sigma_u} \right)^2 + \left( \frac{\sigma_1 + \sigma_2}{\sigma_u} \right)^2 = 2(1 + r) \quad (2.65)
\]

In case of an isotropic material \( r = 1 \), Eqs. (2.63) and (2.65) reduce to the von Mises relationships (2.38) and (2.39), respectively.

In the above equations, the yielding condition is expressed by relations between components of the stress tensor. This defines the shape of the yield surface. Its extension in the space of the stress components is given by the equivalent or effective stress \( \sigma_e \). This is the stress associated to a simple mechanical test that causes the transition of the material from an elastic state to a plastic state. Yield criteria are frequently expressed using this parameter.

If we take into account Eqs. (2.52) and (2.7), the equivalent stress can be expressed as

\[
\sigma = Y_\theta [G \cos^2 \theta + F \sin^2 \theta + H(\cos^2 \theta - \sin^2 \theta) + 2N \sin^2 \theta \cos^2 \theta]^\frac{1}{2} \quad (2.66)
\]

Consequently, \( F_\theta \) will be defined by the relationship

\[
F_\theta = [G \cos^2 \theta + F \sin^2 \theta + H(\cos^2 \theta - \sin^2 \theta) + 2N \sin^2 \theta \cos^2 \theta]^\frac{1}{2} \quad (2.67)
\]

In the case of the Hill’48 yield criterion, the uniaxial yield stress corresponding to a direction inclined at the angle \( \theta \) with respect to the rolling direction is
2.3 Classical Yield Criteria for Anisotropic Materials

\[ Y_\theta = \frac{Y(h)}{[G \cos^2 \theta + F \sin^2 \theta + H(\cos^2 \theta - \sin^2 \theta) + 2N \sin^2 \theta \cos^2 \theta]^\frac{1}{2}} \]  

(2.68)

If the yield parameter \( Y(h) \) is set equal to the uniaxial yield stress \( \sigma_u \), the uniaxial yield stress predicted by this criterion is

\[ Y_0 = \frac{\sigma_u}{\sqrt{G + H}}. \]  

(2.69)

The expression of the uniaxial anisotropy predicted by the Hill’48 yield criterion is obtained by replacing Eq. (2.67) in Eq. (2.17):

\[ r_\theta = \frac{G \cos^4 \theta + F \sin^4 \theta + H \cos^2 2\theta + \frac{1}{2}N \sin^2 2\theta}{G \cos^2 \theta + F \sin^2 \theta} - 1. \]  

(2.70)

Equations (2.69) and (2.70) are used for predicting the uniaxial yield stress and coefficient of plastic anisotropy, in the case when the parameters \( F, G, H \) and \( N \) of the Hill 1948 yield criterion are related to the experimental yield stress \( \sigma_u \) and the experimental coefficients of plastic anisotropy \( r_0, r_{45}, r_{90} \). The identification of the yield criterion can be also performed by using three experimental values of the yield stress and one experimental value of the coefficient of plastic anisotropy.

2.3.1.2 Comments on the Hill’48 Yield Criterion

When describing the anisotropy of metals, the Hill 1948 yield criterion has the advantage that its basic assumptions are easy to understand. The parameters included in the yield functions have a direct physical meaning. This explains its wide use in practice. In addition, the model has a simple formulation for the 3D case. The criterion needs a small number of mechanical parameters for determining the yield function. Under plane stress conditions, four parameters are sufficient, namely, \( r_0, r_{45}, r_{90} \) and \( \sigma_0 \) or \( r_0, r_{45}, r_{90} \) and \( \sigma_90 \) because the parameters \( r_0, r_{45}, \sigma_0 \) and \( \sigma_90 \) are related by Eq. (2.56). In practice, the values of the anisotropy coefficients and an average of the uniaxial yield stress (\( \sigma_u \)) are used.

Besides its advantages, the criterion also has some drawbacks:

1. It cannot represent the ‘anomalous’ behavior observed by Woodthorpe and Pearce [26]: \( r < 1 \) and \( \sigma_b > \sigma_u \) (or the reciprocal) because the criterion predicts (see Eq. 2.64)

\[ \sigma_b = \sigma_u \sqrt{\frac{1 + r}{2}} \]  

(2.71)
2. It cannot represent the ‘second order anomalous’ behaviour (this new concept has been introduced in the literature by the author in the paper [28]): \( \frac{r_0}{r_{90}} > 1 \) and \( \frac{\sigma_0}{\sigma_{90}} < 1 \) (or vice-versa) because the criterion predicts:

\[
\frac{\sigma_0}{\sigma_{90}} = \sqrt{\frac{r_0 (1 + r_{90})}{r_{90} (1 + r_0)}}
\]

(2.72)

3. It can only be applied to materials forming four ‘ears’ in axisymmetric deep-drawing processes whereas in practice also different numbers of ears are observed.

4. The curves equivalent stress \( \sigma_e \) vs. equivalent strain \( \varepsilon_e \) for different materials depend on the loading path, although they should be unique and intrinsic for a given material.

5. In uniaxial tensile tests, the dependence of the yield stress on direction is poorly predicted by the theory, although the variation of the \( r \)-coefficient is properly determinate. There is also the possibility to perform the identification of the Hill 1948 model by using only yield stresses and not coefficients of plastic anisotropy. Of course, in such a case, the accuracy of the predictions related to the variability of the anisotropy coefficients is poorer. The model cannot use the same number of yield stresses and anisotropy coefficients in the identification procedure. This characteristic is an important drawback, especially in the case of materials exhibiting significant variations of the mechanical parameters mentioned above [28].

### 2.3.1.3 Hill 1979 Yield Criterion

As mentioned above the quadratic yield criteria cannot describe the plastic behavior of some materials such as aluminum alloys. According to Woodthrope and Pearce [26, 27], these metals, though having anisotropy coefficients less than unity, have an experimental yield surface located outside the surface predicted by the von Mises yield criterion (\( \sigma_b > \sigma_u \)). Since the Hill 1948 yield criterion cannot describe such a situation, the above authors improperly called this behavior ‘anomalous’ (see also the comments to Eq. 2.64).

Starting from this experimental observation, in the 1970s several researchers concluded independently from each other that only non-quadratic functions are suitable. In the next sections various criteria of this kind are presented in order to describe the so-called ‘anomalous’ behavior of some materials.

Whereas the quadratic criterion by Hill can be applied both to sheet metal and to round bars, the non-quadratic yield criteria described below generally can only be applied to sheet metal.

This function was expressed in its general form as well as in four special cases [29]. If the directions of the principal stresses coincide with the axes of anisotropy the criterion is written as
2.3 Classical Yield Criteria for Anisotropic Materials

\[ f \sigma_2 - \sigma_3|m + g \sigma_3 - \sigma_1|m + h \sigma_1 - \sigma_2|m + a |2\sigma_1 - \sigma_2 - \sigma_3|m + b |2\sigma_2 - \sigma_1 - \sigma_3|m + c |\sigma_3 - \sigma_1 - \sigma_2|m = \sigma_m^m \]  

(2.73)

Here \( f, g, h, a, b \) and \( c \) are anisotropy coefficients; the exponent \( m \) can be calculated from the non-linear relationship obtained from Eq. (2.73) written for equibiaxial tension (\( \sigma_1 = \sigma_2 = \sigma_b; \sigma_3 = 0 \)) [29]:

\[ \left( \frac{\sigma_b}{\sigma_m} \right)^m = \frac{1}{2} (1 + r) \cdot \left( 1 + \frac{(2^{m-1} - 2) \cdot (a - c)}{a + 2^{m-1} \cdot c + f} \right), \]  

(2.74)

This equation can be solved numerically.

If \( m \neq 2 \), there are seven parameters taking different values (according to the convexity conditions), ensuring an improved flexibility as compared to the Hill 1948 yield criterion.

For plane stress (\( \sigma_3 = 0 \)) Eq. (2.73) can be written in four particular forms given by Hill [29]. The most frequently used formulation is the so-called Case 4, according to Hill’s paper [29]. As a consequence, we shall describe only this model.

Case 4 is defined by the following constraint acting on the parameters: \( a = b = f = g = 0 \). Equation (2.73) reads now

\[ c |\sigma_1 + \sigma_2|m + h |\sigma_1 - \sigma_2|m = \sigma_m^m \]  

(2.75)

Lian, Zhou and Baudelet [30] proved that the four forms of the Hill 1979 yield criterion can be expressed as functions of only two coefficients depending on the parameters \( r \) and \( m \). The coefficients in Eq. (2.75) is given by

\[ c = \frac{r}{2 (1 + r)}, \quad h = \frac{1 + 2 \cdot r}{2 (1 + r)} \]  

(2.76)

These particular forms of the Hill’79 are based on the assumption of planar isotropy whereby the axes 1 and 2 can be arbitrarily oriented in the plane of the sheet metal and the terms associated to shear stress are not necessary.

However, it is possible to generalize the Hill 1979 criterion for taking into account planar anisotropy [31–33].

For planar isotropy, the most widely used expression of the Hill 1979 yield criterion is in the form (2.75). This expression shall be considered more thoroughly. It can be rewritten in the form

\[ |\sigma_1 + \sigma_2|m + (1 + 2 \cdot r) |\sigma_1 - \sigma_2|m = 2 (1 + r) \sigma_u^m \]  

(2.77)

The convexity condition requires that \( m \) be greater than unity. In this particular case Eq. (2.74) has the form

\[ \left( \frac{\sigma_b}{\sigma_u} \right)^m = \frac{1 + r}{2^{m-1}} \]  

(2.78)
By solving Eq. (2.78), usually non-integer values of \( m \) are obtained.

For the aluminum alloy DIN-AlMgSi1 having \( \sigma_u = 138 \text{ MPa} \), \( \sigma_b = 156 \text{ MPa} \), \( r = 0.50 \) it follows \( m = 1.358 \) [34]. For \( m = 2 \) the criterion is reduced to Eq. (2.63) associated to the Hill 1948 yield criterion.

In 1977 Bassani published a paper [35] where he proved independently of Hill that a family of yield functions depending on four parameters may be constructed in order to approximate a relatively broad category of transversally isotropic yield surfaces predicted by the Bishop–Hill theory [36].

He proposed the yield function:

\[
f = \left| \frac{\sigma_1 + \sigma_2}{2\sigma_b} \right|^n + \left| \frac{\sigma_1 - \sigma_2}{2\tau} \right|^m - 1
\]  

(2.79)

where \( \tau \) is the yield stress in pure shearing, while \( n \) and \( m \) are two constants greater than unity.

One may notice that this function is a generalised expression of Case 4 proposed by Hill in 1979 (for \( n = m \), Eq. (2.79) reduces to Eq. (2.75)). The difference consists in the way of defining the coefficients.

Bassani concluded that the proposed family of yield functions, for arbitrarily chosen values of \( m \) or \( n \) approximates the yield surfaces predicted by the Bishop-Hill theory.

Extensions of the Hill 1979 model for stress states with a planar shearing component have been proposed by Chu [32], Zhou [33], [37] and Monteillet [38].

**Advantages of the Hill 1979 yield criterion are** [30]:

- it can describe the Woodthrope-Pearce ‘anomalous’ behavior of materials
- it has a relatively simple form
- it leads to of an analytical expression of the associated flow rule and equivalent strain.

**Disadvantages are:**

- it cannot describe the behavior of the materials having \( \frac{\sigma_1}{\sigma_9} \neq 1 \) and \( \frac{\sigma_0}{\sigma_{90}} \neq 1 \)
- due to the non-integer value of the exponent \( m \), it requires numerical procedures even for the solution of quite simple cases
- although it allows the description of the ‘anomalous’ behavior, the predicted yield surfaces are sometimes far from the experimental surfaces predicted by the Bishop-Hill theory [36].

### 2.3.1.4 Hill 1990 Yield Criterion

As mentioned above, the yield criterion proposed by Hill in 1979 can only be used when the directions of the principal stresses are coincident with the orthotropic axes.
This causes severe limitations of application. The generalisation of the criterion by expressing it in a general coordinate system was realised by Hill in 1990 [39]:

\[
\varphi = |\sigma_{11} + \sigma_{22}|^m + \left(\sigma_b^m / \tau^m\right) \left| (\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 \right|^{m/2} + |\sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{12}^2 |^{(m/2)-1} \\
\times \left\{ -2a (\sigma_{11}^2 - \sigma_{22}^2) + b (\sigma_{11} - \sigma_{22})^2 \right\} = (2\sigma_b)^m
\]

(2.80)

Here \(\sigma_b\) is the yield stress in equibiaxial tension, \(\tau\) is the yield stress in pure shear deformation \((\sigma_1 = -\sigma_2)\), \(a\) and \(b\) are material constants.

The \(m\) exponent is obtained by solving the following equation

\[
\left( \frac{2 \cdot \sigma_b}{\sigma_{45}} \right)^m = 2 \cdot (1 + r_{45})
\]

(2.81)

The value of the \(m\) exponent is

\[
m = \frac{\ln [2 (r_{45} + 1)]}{\ln \frac{2\sigma_b}{\sigma_{45}}}
\]

(2.82)

The constants \(a\) and \(b\) are determined from the equations

\[
a = \frac{1}{4} \left| \left( \frac{2\sigma_b}{\sigma_{90}} \right)^m - \left( \frac{2\sigma_b}{\sigma_0} \right)^m \right| ;
\]

\[
b = \frac{1}{2} \left[ \left( \frac{2\sigma_b}{\sigma_0} \right)^m + \left( \frac{2\sigma_b}{\sigma_{90}} \right)^m \right] - \left( \frac{2\sigma_b}{\sigma_{45}} \right)^m.
\]

(2.83)

The ratio \(\sigma_b/\tau\) may be also expressed as a function of coefficient \(r_{45}\) (see Eq. 2.57):

\[
\left( \frac{\sigma_b}{\tau} \right)^m = 1 + 2 \cdot r_{45}
\]

(2.84)

It can be shown that the parameters \(a\) and \(b\) can also be determined as functions of the anisotropy coefficients \(r_0, r_{45}\) and \(r_{90}\):

\[
a = \frac{(r_0 - r_{90}) \left[ 1 - \left( (m - 2)/2 \right) \cdot r_{45} \right]}{(r_0 + r_{90}) - (m - 2) \cdot r_0 \cdot r_{90}} ;
\]

\[
b = \frac{m \cdot [2 \cdot r_0 \cdot r_{90} - r_{45} \cdot (r_0 + r_{90})]}{(r_0 + r_{90}) - (m - 2) \cdot r_0 \cdot r_{90}}.
\]

(2.85)
or as functions of the coefficients in Eqs. (2.50) and (2.51).

By using the methodology described in Sect. 2.1.1, the equivalent stress associated to the Hill 1990 yield criterion can be expressed as follows:

\[
\sigma = Y_\theta \frac{1}{2} \left[ 1 + \left( \frac{\sigma_b}{\tau} \right)^m - 2a \cos 2\theta + b \cos^2 2\theta \right]^{\frac{1}{m}}
\]  

(2.87)

The corresponding function \( F_\theta \) is

\[
F_\theta = \frac{1}{2} \left[ 1 + \left( \frac{\sigma_b}{\tau} \right)^m - 2a \cos 2\theta + b \cos^2 2\theta \right]^{\frac{1}{m}}
\]  

(2.88)

In the case of Hill 1990 model, the yield parameter \( Y(h) \) is set equal to the biaxial yield stress \( (Y(h) = Y_b) \). In order to preserve the original notations, we shall use the symbol \( \sigma_b \) for the biaxial yield stress. According to Eq. (2.10), the uniaxial yield stress is defined as follows:

\[
Y_\theta = \frac{4\sigma_b}{[1 + \left( \frac{\sigma_b}{\tau} \right)^m - 2a \cos 2\theta + b \cos^2 2\theta]^{\frac{1}{m}}}
\]  

(2.89)

The planar distribution of the uniaxial anisotropy coefficient is predicted by the formula

\[
\rho_\theta = \frac{\left( \frac{\sigma_b}{\tau} \right)^m - 1 + \frac{2}{m}b \cos^2 2\theta}{2 - 2a \cos 2\theta + \frac{m - 2}{m}b \cos^2 2\theta} - 1
\]  

(2.90)

Equations (2.89) and (2.90) allow the determination of the planar distribution of the uniaxial yield stress and coefficient of plastic anisotropy.

A detailed discussion of the computational methodology of the parameters in Hill’s 1990 yield function is presented in [40] and [41]. By comparing the yield loci computed with Eq. (2.80) using the coefficients \( a \) and \( b \) evaluated on the basis of the yield stresses (Eq. 2.83) as well as on the basis of the anisotropy coefficients (Eq. 2.85), with the yield loci given by the Taylor theory, Lin and Ding [40] concluded that the identification procedure using stresses ensured a better approximation.

Using Mohr’s circle, the yield function (2.80) can be expressed in principal stresses:

\[
\varphi = |\sigma_1 + \sigma_2|^m + \left( \frac{\sigma_b}{\tau^m} \right) |\sigma_1 - \sigma_2|^m + |\sigma_1^2 + \sigma_2^2|^{(m/2)-1} \cdot \left[ -2a \left( \sigma_1^2 - \sigma_2^2 \right) + b \left( \sigma_1 - \sigma_2 \right)^2 \cos 2\theta \right] \cos 2\theta = (2\sigma_b)^m
\]  

(2.91)
Here $\theta$ is the angle between the direction of $\sigma_1$ and the rolling direction. If $a = b$ or $\theta = \pi/4$, Eqs. (2.80) and (2.91) reduce to Case 4° of the Hill 1979 yield criterion (Eq. 2.75).

The flow rule associated to this yield criterion is presented in [41]. The Hill 1990 yield criterion preserves all the advantages of the Hill 1979 criterion. In addition, it includes all the planar components of the stress tensor ($\sigma_{11}$, $\sigma_{22}$ and $\sigma_{12}$). As a consequence it is possible to evaluate the distribution of the uniaxial yield stress and anisotropy coefficient in the plane of the sheet metal. From Eqs. (2.81) to (2.83) it can be seen that five parameters are needed for defining the yield function: $\sigma_0$, $\sigma_{45}$, $\sigma_{90}$, $\sigma_b$ and $r_{45}$.

If Eqs. (2.84) and (2.85) are used for calculating the coefficients $a$ and $b$, the mechanical parameters necessary for defining the yield function are $\sigma_{45}$, $\sigma_b$, $r_0$, $r_{45}$ and $r_{90}$ and again three uniaxial tensile tests and a biaxial tensile test are required.

In their analysis of Hill’s 1990 yield criterion, Lin and Ding [40] proposed a more general expression:

$$\varphi = |\sigma_1 + \sigma_2|^m + (1 + 2R) |\sigma_1 - \sigma_2|^m + |\sigma_1^2 + \sigma_2^2|^{m-\frac{1}{2}} \cdot (-2a (|\sigma_1|^s - |\sigma_2|^s) + b |\sigma_1 - \sigma_2|^s \cos 2\theta) \cos 2\theta = (2\sigma_b)^m$$

(2.92)

where $R$ and $s$ are material parameters. For $\theta = \pi/4$ and $s = 2$ Eq. (2.92) is simplified to Eq. (2.75) i.e. Hill’s 1979 criterion (Case 4°). The new characteristic of this criterion is that the additional terms are not quadratic. The exponent $s$ is calculated from the equation

$$s = -\frac{mr_0 \left\{2 (r_{90} - R) + r_{90} \left[ \left( \frac{2\sigma_b}{\sigma_{90}} \right)^m - 2 (1 + R) \right] \right\}}{(1 + R) (r_0 - r_{90}) - \frac{1}{2} (r_0 + r_{90} + 2r_0r_{90}) \left[ \left( \frac{2\sigma_b}{\sigma_{90}} \right)^m - 2 (1 + R) \right]}$$

(2.93)

where $R$ is determined from

$$R = \frac{1}{2} \left( \left( \frac{2\sigma_b}{\sigma_{45}} \right)^m - 1 \right).$$

(2.94)

If the identification of the parameters $a$ and $b$ is made using stresses then

$$a = \frac{1}{4} \left[ \left( \frac{2\sigma_b}{\sigma_{90}} \right)^m - \left( \frac{2\sigma_b}{\sigma_0} \right)^m \right];$$

$$b = \frac{1}{2} \left[ \left( \frac{2\sigma_b}{\sigma_0} \right)^m + \left( \frac{2\sigma_b}{\sigma_{90}} \right)^m \right] - 2 (1 + R).$$

(2.95)

where by $s$ is calculated from

$$s = \frac{m [(b - 2a) r_0 + 2 (r_0 - r_{45})]}{(1 + r_0) b}$$

(2.96)
The above equations show that the determination of the proposed yield function demands six parameters ($\sigma_0$, $\sigma_{90}$, $\sigma_b$, $r_0$, $r_{45}$ and $r_{90}$) in the first identification procedure, or $\sigma_{45}$, $\sigma_b$, $r_0$, $r_{45}$ and $r_{90}$ in the second one. In order to establish these parameters, three uniaxial tensile tests and a biaxial tensile test have to be carried out. The paper by Lin and Ding [40] also gives a comparison of the yield surfaces computed with coefficients $a$, $b$, $s$ and $R$ identified in the two manners described above with the yield surfaces predicted by the Taylor theory. The identification of the coefficients $a$ and $b$ could be made on the basis of the anisotropy coefficients [40]. In this case a better approximation of the yield surfaces predicted by the Taylor theory would be obtained (contrary to the situation occurring when the original Hill 1990 yield criterion is used). This is due to the non-quadratic terms added to the yield function.

Leacock [42] extended the formulation proposed by Hill in 1990 [39] by defining the following yield criterion:

$$
|\sigma_1 + \sigma_2|^m + A^m |\sigma_1 - \sigma_2|^m + |\sigma_1^2 + \sigma_2^2|^{(m/2)-2} \left( (\sigma_1^2 - \sigma_2^2) ^2 ight)
\{ H(\sigma_1^2 + \sigma_2^2) + I(\sigma_1^2 - \sigma_2^2) \cos 2\theta \} + (\sigma_1 - \sigma_2)^2 
\{ J(\sigma_1^2 + \sigma_2^2) + K(\sigma_1^2 - \sigma_2^2) \cos 2\theta \} \cos 2\theta \} = (2\sigma_b)^m. \tag{2.97}$$

The evaluation of the constants $A$, $H$, $I$, $J$, $K$ and $m$ is based on 6 mechanical parameters, namely $\sigma_0$, $\sigma_{45}$, $\sigma_{90}$, $\sigma_b$, $r_0$, $r_{45}$ and $r_{90}$.

Equation (2.97) can be reduced to the formulation proposed by Hill in 1990 (Eq. 2.80).

The uniaxial yield stress is defined by the relationship

$$
\sigma_\theta = 2\sigma_b \left( 1 + A^m + (H + I \cos 2\theta) + (J + K \cos 2\theta) \cos^2 2\theta \right)^{-1/m}. \tag{2.98}
$$

while the coefficient of uniaxial anisotropy is expressed as

$$
R_\theta = \frac{A^m + 0.5(H + I \cos 2\theta) \cos 2\theta + \left[ (m + 2)/(2m) \right] (J + K \cos 2\theta) \cos^2 2\theta}{2 \left[ 1 + 0.5(H + I \cos 2\theta) \cos 2\theta + \left[ (m - 2)/(2m) \right] (J + K \cos 2\theta) \cos^2 2\theta \right]^{-1/2}}. \tag{2.99}
$$

Due to the use of seven mechanical parameters in the identification procedure, Leacock’s model gives better predictions than Hill 1990 and Lin 1996 models both for the yield surfaces and the planar distributions of the uniaxial yield stresses and coefficient of plastic anisotropy.

Advantages of the Hill 1990 yield criterion are the following:

- it allows to describe both the ‘first order anomalous behavior’ ($r < 1, \sigma_b < \sigma_u$) and the ‘second order anomalous behavior’ ($r_0 < r_{90}$, $\sigma_0 > \sigma_{90}$ and vice-versa)
- it is able to describe very well the variation of the anisotropy coefficient and of the uniaxial yield stress in the plane of the sheet
it has a great flexibility due to the high number of the mechanical parameters incorporated

Leacock’s modification [42] is shown to explicitly deal with the second order anomalous behaviour.
However, there are also some disadvantages:

the formulation is not user-friendly

due to the trigonometric functions incorporated in their formulation, both Lin and Ding [40] and Leacock [42] extensions of the Hill 1990 model need larger CPU times when used in the numerical simulation of sheet metal forming processes.

2.3.1.5 Hill 1993 Yield Criterion

In 1993 Hill [43] improved the model of plastic behavior of textured sheet metals, namely those observed when complex loads are applied along the planar orthotropic axes.

There are some situations that cannot be dealt with completely by any of the criteria described above. A comprehensive analysis of these problems is given in [43] where Hill showed that for all the criteria described above the condition $\sigma_0 = \sigma_90$ enforces $r_0 = r_90$ and the reciprocal.

However, it has been observed experimentally that some materials, especially aluminum alloys and brass have almost equal yield stresses but different anisotropy coefficients in rolling and transverse direction. This applies to the results reported in [44] for a brass 70-30 sheet ($\sigma_0 = 126$ MPa, $\sigma_90 = 125$ MPa, $r_0 = 1.51, r_90 = 0.37$) and in [34] for AlMgSi1 sheet ($\sigma_0 = 143$ MPa, $\sigma_90 = 133$ MPa, $r_0 = 0.48, r_90 = 0.61$).

For this phenomenon the term ‘anomalous behavior of second order’ should be used [28], in order to distinguish it from the ‘anomalous behavior’ of the aluminum alloy described in [26, 27] whereby $r < 1$ though the equibiaxial yield stress is higher than the uniaxial one: $\sigma_b/\sigma_u > 1$.

For this reason Hill [43] proposed a new yield criterion that should preserve the generality of the one proposed in 1979, that is, the capability of modelling both the ‘anomalous behavior’ and the ‘anomalous behavior of second order’.

These constraints are satisfied by the following polynomial function, valid for stress states in the first quadrant (biaxial tension) which is most relevant for thin sheet metals:

$$\frac{\sigma_1^2}{\sigma_0^2} - \frac{c\sigma_1\sigma_2}{\sigma_0\sigma_90}\sigma_1\sigma_2 + \frac{\sigma_2^2}{\sigma_90^2} + \left\{ (p + q) - \frac{(p\sigma_1 + q\sigma_2)}{\sigma_b} \right\} \frac{\sigma_1\sigma_2}{\sigma_0\sigma_90} = \sigma_u^2$$  \hspace{1cm} (2.100)

where

$$\frac{c}{\sigma_0\sigma_90} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma_90^2} - \frac{1}{\sigma_b^2}$$ \hspace{1cm} (2.101)
while $p$ and $q$ are calculated with the normality condition of the strain rate tensor to the yield surface applied to function (2.100) at the intersection with the coordinate axes:

$$ p = \left[ \frac{2r_0 (\sigma_b - \sigma_{90})}{(1 + r_0) \sigma_0^2} - \frac{2r_0 \sigma_b}{(1 + r_{90}) \sigma_{90}^2} + \frac{c}{\sigma_0} \right] \frac{1}{\frac{1}{\sigma_0} + \frac{1}{\sigma_{90}} - \frac{1}{\sigma_b}}; \quad (2.102) $$

$$ q = \left[ \frac{2r_{90} (\sigma_b - \sigma_{90})}{(1 + r_{90}) \sigma_{90}^2} - \frac{2r_0 \sigma_0}{(1 + r_0) \sigma_0^2} + \frac{c}{\sigma_{90}} \right] \frac{1}{\frac{1}{\sigma_0} + \frac{1}{\sigma_{90}} - \frac{1}{\sigma_b}}; \quad (2.103) $$

From Eqs. (2.100) to (2.103) it follows that in order to define the yield function, five mechanical parameters are required ($r_0$, $r_{90}$, $\sigma_0$, $\sigma_{90}$ and $\sigma_b$). These parameters can be determined by two uniaxial tensile tests and an equibiaxial tensile test.

By expressing the modulus of the principal stress $\sigma_1$ and $\sigma_2$ from the third order term of Eq. (2.100) the criterion can be extrapolated to the other quadrants of the plane ($\sigma_1$, $\sigma_2$). This leads to some discontinuity of the yield locus which, however, can be tolerated if the discontinuity errors are within the limits of the experimental errors.

Equation (2.100) show that the proposed yield function is nonhomogeneous with respect to $\sigma_1$ and $\sigma_2$. Consequently it is not possible to get an explicit expression of the strain increment from the normality condition.

Advantages of the Hill 1993 yield criterion are the following:

- it allows to describe both the ‘first order anomalous behavior’ ($r < 1, \sigma_b < \sigma_u$) and the ‘second order anomalous behavior’ ($r_0 < r_{90}, \sigma_0 > \sigma_{90}$ and vice versa)
- it has a relatively simple and user-friendly expression; it has a great flexibility due to the five mechanical parameters incorporated.

However, there are also some disadvantages:

- the yield function is non-homogenous with respect to $\sigma_1$ and $\sigma_2$ and hence does not allow to obtain explicit expressions of the strain increments
- it can be used only if the directions of the principal stresses are coincident with the orthotropic axes
- it does not allow to describe the variation of the anisotropy coefficient and of the uniaxial yield stress in the plane of the sheet
- the yield surface predicted by this function is far from that obtained from polycrystal theories (Taylor or Bishop-Hill).

Due to the disadvantages mentioned above, the applicability of this yield criterion is limited.
2.3 Classical Yield Criteria for Anisotropic Materials

2.3.2 Yield Function Based on Crystal Plasticity
(Hershey’s Family)

Besides the family of Hill yield criteria presented in the previous subchapter, we have to mention another class of models based on the isotropic formulation proposed by Hershey [19]. Hosford ‘rediscovered’ Hershey’s model in 1972 [20] and used it for the development of an anisotropic yield criterion. Barlat et al. [5] and Banabic et al. [51] as well as some other researchers proposed further extensions of the model.

2.3.2.1 Hosford Yield Criterion

Independently of Hill, Hosford proposed a yield criterion in the form [31]

\[ F |\sigma_{22} - \sigma_{33}|^a + G |\sigma_{33} - \sigma_{11}|^a + H |\sigma_{11} - \sigma_{22}|^a = \sigma^a \]  

(2.104)

This is a generalization of his own isotropic criterion [20].

One may consider this criterion as a particular expression \((a = b = c = 0\) and \(f = g\)) of Case 4\(\alpha\) of Hill’s 1979 yield criterion (see Eq. 2.75).

The essential difference between the approaches by Hosford and Hill consists in the different way of determining the exponent \(a\). Hosford related \(a\) to the crystallographic structure of the material [45–48]. He concluded that the best approximation was given by \(a = 6\) for BCC materials and \(a = 8\) for FCC materials, see also [45].

For plane stress Eq. (2.104) reduces to

\[ r_90 |\sigma_{11}|^a + r_0 |\sigma_{22}|^a + r_0 r_90 |\sigma_{11} - \sigma_{22}|^a = r_90 (r_0 + 1) \sigma_0^a \]  

(2.105)

The main advantage of the Hosford 1979 yield criterion is that by fitting the value of the exponent \(a\) it ensures a good approximation of the yield locus computed through the Bishop-Hill theory [36] as well as from experimental data.

An important drawback of this criterion is caused by the lack of shear stress: it cannot predict the variation of the coefficient \(r\) with direction (planar anisotropy).

2.3.2.2 Barlat 1989 Yield Criterion

Barlat and Richmond [49] proposed a more general form of Hosford’s criterion for isotropic materials [20] by expressing it in an \(x, y, z\) coordinate system, not necessarily coincident with the principal directions (the so-called ‘tricomponent plane stress yield surface’):

\[ f = |k_1 + k_2|^M + |k_1 - k_2|^M + |2k_2|^M = 2\sigma_e^M \]  

(2.106)

Here \(k_1\) and \(k_2\) are invariants of the stress tensor while \(M\) is an integer exponent having the same significance as the exponent \(a\) used by Hosford; \(k_1\) and \(k_2\) are obtained from
\[ k_1 = \frac{\sigma_{11} + \sigma_{22}}{2} \]

\[ k_2 = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \]  \hspace{1cm} (2.107)

For \( \sigma_{12} = 0 \) it follows \( \sigma_{11} \rightarrow \sigma_1 \) and \( \sigma_{22} \rightarrow \sigma_2 \) and Eq. (2.106) is reduced to the isotropic Hosford criterion which now includes the shear stresses (incorporated in the invariant \( k_2 \)).

The yield function given by Eq. (2.106) was also extended to the case of *normal anisotropy* in the form:

\[ f = a |k_1 + k_2|^M + b |k_1 - k_2|^M + c |2k_2|^M = 2\sigma_e^M \]  \hspace{1cm} (2.108)

where \( a, b \) and \( c \) depend on the anisotropy coefficients while \( k_1 \) and \( k_2 \) are calculated from the equation

\[ a = b = 2 - c = \frac{2}{1 + r} \]  \hspace{1cm} (2.109)

In 1989, Barlat and Lian [50] published a generalisation of Eq. (2.106) for materials exhibiting *planar anisotropy* by introducing the following yield function:

\[ f = a |k_1 + k_2|^M + a |k_1 - k_2|^M + c |2k_2|^M = 2\sigma_e^M \]  \hspace{1cm} (2.110)

The coefficients \( k_1 \) and \( k_2 \) are given by

\[ k_1 = \frac{\sigma_{11} + h\sigma_{22}}{2}; \quad k_2 = \left[ \left( \frac{\sigma_{11} - h\sigma_{22}}{2} \right)^2 + p^2\sigma_{12}^2 \right]^{1/2} \]  \hspace{1cm} (2.111)

and \( a, c, h \) and \( p \) are material parameters identified by

\[ a = 2 - c = \frac{\left( \sigma_e \right)^M}{\left( \tau_{s2} \right)^M} - 2 \left( 1 + \frac{\sigma_e}{\sigma_{90}} \right)^M \]

\[ h = \frac{\sigma_e}{\sigma_{90}} \]

\[ p = \frac{\sigma_e}{\tau_{s1}} \left( \frac{2}{2a + 2Mc} \right)^{1/2} \]  \hspace{1cm} (2.112)

Here \( \tau_{s1} \) and \( \tau_{s2} \) are yield stresses for two different types of shear tests: \( \sigma_{12} = \tau_{s1} \) for \( \sigma_{11} = \sigma_{22} = 0 \) and \( \sigma_{12} = 0 \) for \( \sigma_{22} = -\sigma_{11} = \tau_{s2} \).

Using another identification procedure based on the coefficients \( r_0 \) and \( r_{90} \) it is obtained
\[ a = 2 - c = 2 - 2 \sqrt{r_0 \cdot \frac{1 + r_90}{1 + r_0}}; \]
\[ h = \sqrt{\frac{r_0}{1 + r_0} \cdot \frac{1 + r_90}{r_90}}. \]  

(2.113)

The coefficient \( p \) has to be calculated by a numerical procedure or by using Eq. (2.112) instead. For \( \sigma_{12} = 0 \), Eqs. (2.110) and (2.111) are practically reduced to the Hosford yield criterion in principal stresses (except of the constant \( h \), see Eq. 2.105).

In order to establish the expression of the uniaxial yield stress, Eq. (2.7) will be replaced in Eqs. (2.110) and (2.111). We get the relationship

\[ Y_\theta = \frac{Y_0}{[a(F_1 + F_2)^M + a(F_1 - F_2)^M + (1 - a)(2F_2)^M]^{\frac{1}{M}}}, \]  

(2.114)

where

\[ F_1 = \frac{h \sin^2 \theta + \cos^2 \theta}{2}; \quad F_2 = \left[ \left( \frac{h \sin^2 \theta - \cos^2 \theta}{2} \right)^2 + p^2 \sin^2 \theta \cos^2 \theta \right]. \]  

(2.115)

The function \( F_\theta \) is obtained from Eq. (2.114):

\[ F_\theta = \frac{a(F_1 + F_2)^M + a(F_1 - F_2)^M + (1 - a)(2F_2)^M}{[a(K_1 + K_2)^{M-1}(t_1 + t_2) + a(K_1 - K_2)^{M-1}(t_1 - t_2) + 2(a - 1)(2K_2)^{M-1}t_2]^{\frac{1}{M}}}, \]  

(2.116)

The yield parameter \( Y(h) \) in Eq. (2.114) has been set equal to the uniaxial yield stress corresponding to the rolling direction \( Y(h) = Y_0 \).

By replacing in Eq. (2.17) the \( F_\theta \) expression given by Eq. (2.116) and performing some computations, we get the relationship defining the coefficient of plastic anisotropy:

\[ r_\theta = \frac{[a(F_1 + F_2)^M + a(F_1 - F_2)^M + (1 - a)(2F_2)^M]^{\frac{1}{M}}}{a(K_1 + K_2)^{M-1}(t_1 + t_2) + a(K_1 - K_2)^{M-1}(t_1 - t_2) + 2(a - 1)(2K_2)^{M-1}t_2} - 1 \]  

(2.117)

where

\[ K_1 = \frac{h \sin^2 \theta + \cos^2 \theta}{2} Y_\theta; \quad K_2 = \left[ \left( \frac{h \sin^2 \theta - \cos^2 \theta}{2} \right)^2 + p^2 \sin^2 \theta \cos^2 \theta \right]. \]  

(2.118)

and \( t_1 \) and \( t_2 \) are
Equations (2.114) and (2.117) allow the calculation of the uniaxial yield stress and the coefficient of plastic anisotropy corresponding to different directions in the plane of the sheet metal.

Figures 2.16 and 2.17 shows that the anisotropy coefficient $r_0$ and $r_{90}$ act in a different manner on the yield locus. In case of the ‘tricomponent plane stress’ the influence of the exponent $M$ extends to the region of biaxial tension (Fig. 2.18). The figures demonstrates that the yield criterion by Barlat 1989 has a great flexibility. This is due to the large number of parameters (four material parameters and $M$ chosen in accordance with the crystallographic structure of the material).

The advantages of the Barlat 1989 yield criterion are:

- the reduced number of the mechanical parameters (four parameters) used in the identification
- relatively easy identification (except for the coefficient $p$)
- a relativ good prediction of the yield locus for aluminium alloys without high anisotropy;
- by correctly choosing the exponent $M$ a very good correlation with the yield locus predicted by the Bishop-Hill theory is obtained.
2.3 Classical Yield Criteria for Anisotropic Materials

The most important disadvantages are as follows:

- the coefficients of the yield function have not a direct and intuitive physical significance.
- the evaluation of the parameter $p$ can be performed only numerically, by solving a non-linear equation.
the model does not give accurate predictions of the biaxial yield stress, especially in the case of aluminium alloys exhibiting a pronounced anisotropy  
the model cannot capture simultaneously the planar variation of the uniaxial yield stress and uniaxial coefficient of plastic anisotropy  
the model does not give accurate predictions of the biaxial coefficient of plastic anisotropy in case of highly-anisotropic materials.

Despite its limitations, the Barlat 1989 yield criterion is still frequently used in the numerical simulation of sheet metal forming processes.

In 1991, Barlat proposed a 3D extension of his yield criterion [10] (see the next section). Banabic et al. [51–53] also proposed extensions of the Barlat 1989 yield criterion (with seven and eight coefficients), aiming to remove the disadvantages mentioned above (a detailed description of these formulations can be found in Sect. 2.4.2).

2.3.2.3 Barlat 1991 Yield Criterion

Barlat [10] proposed a general six-component yield criterion \((\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})\) that could be adopted with no restrictions to any stress state. With this aim in view, the isotropic Hershey criterion [19] is rewritten in a form containing the deviator principal stresses \(S_1, S_2, S_3\):

\[
\Phi = |S_1 - S_2|^m + |S_2 - S_3|^m + |S_3 - S_1|^m = 2 \cdot \sigma_e^m, \tag{2.120}
\]

After a complex number transformation and the Bishop-Hill notation (see [10]),

\[
A = \sigma_{22} - \sigma_{33}, \quad B = \sigma_{33} - \sigma_{11}, \quad C = \sigma_{11} - \sigma_{22}, \quad F = \sigma_{12}, \quad G = \sigma_{31}, \quad H = \sigma_{12}, \tag{2.121}
\]

Barlat obtained the following expression of the isotropic Hershey yield criterion:

\[
\Phi = \left(3I_2\right)^{\frac{2}{3}} \left\{ \left| 2 \cos \left(\frac{2\theta + \pi}{6}\right) \right|^m + \left| 2 \cos \left(\frac{2\theta + 3\pi}{6}\right) \right|^m + \left| -2 \cos \left(\frac{2\theta + 5\pi}{6}\right) \right|^m \right\} = 2\sigma_e^m, \tag{2.122}
\]

where

\[
\theta = \arccos \left( \frac{I_3}{I_2^{3/2}} \right), \tag{2.123}
\]

where \(I_2\) and \(I_3\) are the second and third invariant of the stress determinant, respectively:

\[
I_2 = \frac{F^2 + G^2 + H^2}{3} + \frac{(A - C)^2 + (C - B)^2 + (B - A)^2}{54}, \tag{2.124}
\]
The yield function defined above was generalised to the anisotropic case by using weighting coefficients \((a, b, c, f, g, h)\), multiplying the stress components given by Eq. (2.121). After this modification, the expressions of the yield function \(\phi\) (Eq. 2.122) and angle \(\theta\) (Eq. 2.123) remain the same. The expression of the invariants \(I_2\) and \(I_3\) become:

\[
I_2 = \frac{(C - B) \cdot (A - C) \cdot (B - A)}{54} + F \cdot G \cdot H - (C - B) \cdot F^2 + (A - C) \cdot G^2 + (B - A) \cdot H^2
\]

\[
I_3 = \frac{(C - B) \cdot F^2 + (A - C) \cdot G^2 + (B - A) \cdot H^2}{6}
\]

(2.125)

The weight factors \(a, b, c, f, g, h\) describe the anisotropy of the material. They are identified by three uniaxial tensile tests in the directions of the orthotropic axes \((a, b, c)\) and three pure shearing tests \((f, g, h)\).

For the case of plane stress, the number of coefficients diminishes to four \((a, b, c, h)\). They may be established by three uniaxial tensile tests (at \(0^\circ, 45^\circ\) and \(90^\circ\) with the rolling directions) and a uniaxial compressive test through the thickness.

Alternatively, the identification of the coefficients may be performed using the uniaxial yield stresses \((\sigma_0, \sigma_{45}, \sigma_{90})\) and the equibiaxial yield stress \((\sigma_b)\). By introducing these values in the expression of the yield function given by Eq. (2.120), one gets a set of four non-linear equations that can be solved numerically. Another identification method makes use of the anisotropy coefficients \(r_0, r_{45}\) and \(r_{90}\). In this case, the value of one coefficient is assumed as being known. The rest of three is obtained by numerically solving a set of non-linear equations.

The exponent \(m\) is above unity and is chosen in the same manner as in the case of Hershey’s yield criterion. For \(a = b = c = f = g = h = 1\) the criterion given by Eq. (2.122) reduces to the isotropic Hershey expression (Eq. 2.120).

The methodology used for obtaining the associative flow rules is rather difficult. It is fully detailed in [41] and [54].

More recently, Choi et al. [54] used the transformation proposed by Karafillis and Boyce [21] for his criterion. As will be shown below, the Barlat 1991 yield criterion may be considered as a special case of the Karafillis-Boyce yield criterion \((c = 0)\).

The Barlat 1991 criterion has the advantage of being general and flexible (like the Hill 1948 criterion it has six stress components); it predicts yield surfaces in good agreement with those calculated from polycristal theories (Taylor and Bishop-Hill).
it is easy to implement in finite-element codes; it gives a realistic estimation of the
distribution of the uniaxial yield stress and coefficient $r$ in the plane of the sheet.

The main disadvantage of the yield criterion is the complicated flow rule which
is not user-friendly.

A similar methodology was used by Lian and Chen [55] for generalising the Hill
1979 criterion in the case of the three-dimensional stress state. They established the
following six-component expression of the Hill 1979 criterion:

$$
2^{m} J = \frac{1}{2} \left\{ f \left| \sin \left( \frac{\varphi}{3} \right) \right|^m + g \left| \cos \left( \varphi / 3 + \pi / 6 \right) \right|^m + h \left| \cos \left( \varphi / 3 - \pi / 6 \right) \right|^m + 3^{m/2} \left[ a \left| \cos \left( \varphi / 3 \right) \right|^m + b \left| \cos \left( \varphi / 3 - \pi / 3 \right) \right|^m + c \left| \cos \left( \varphi / 3 + \pi / 3 \right) \right|^m \right\} = \sigma_e^m
$$

where $\varphi$ depends on the second and third invariant of the deviatoric stress tensor.
The $a, b, c, f, g$ and $h$ are the material coefficients, like in the Barlat model.

The methodology proposed by Lian and Chen is very general and can be applied
to any yield criterion written by using principal stresses in order to obtain a six-
component expression.

2.3.2.4 Yield Criteria by Barlat 1994 and 1996

Experimental studies showed that aluminum alloys have a plastic behavior that is
very difficult to model with the criteria above. At the beginning of 1990 several
researchers have focused their interest on this problem.

In [56] Barlat and co-workers proposed a more general expression of the yield
function introduced by himself in 1991. The generalization consisted in giving
weight factors to the terms of Eq. (2.120):

$$
\Phi = \alpha_x |S_y - S_z|^m + \alpha_y |S_z - S_x|^m + \alpha_z |S_x - S_y|^m = 2\sigma_e^m
$$

The weight factors $\alpha_x, \alpha_y$ and $\alpha_z$ are related to the anisotropy of the materials; $S_x,$ $S_y,$ and $S_z$ are normal component of the stress tensor modified with the linear trans-
formation like in the Karafillis-Boyce proposal [21]; $m$ has not the same significance
as in case of the Barlat 1991 criterion.

Assuming the shear stresses to be zero, the linear-transformation operator from
Karafillis-Boyce proposal (see [21]) becomes

$$
L = \begin{bmatrix}
\frac{c_2 + c_3}{3} & \frac{c_3}{3} & \frac{c_2}{3} \\
\frac{c_3}{3} & \frac{c_3 + c_1}{3} & \frac{c_1}{3} \\
\frac{c_2}{3} & \frac{c_2 + c_3}{3} & \frac{c_1 + c_2}{3}
\end{bmatrix}
$$

where $c_1, c_2$ and $c_3$ are material coefficients describing the anisotropy of the
material.

The generalization of this function in order to include the shear stresses
(six-component formulation) as obtained by rewriting Eq. (2.130) in the principal
stresses:
2.3 Classical Yield Criteria for Anisotropic Materials

\[
\Phi = \alpha_1 |S_2 - S_2|^m + \alpha_2 |S_3 - S_1|^m + \alpha_3 |S_1 - S_2|^m = 2\sigma_e^m
\]  

(2.131)

where the coefficients \(\alpha_1, \alpha_2\) and \(\alpha_3\) are computed from \(\alpha_x, \alpha_y\) and \(\alpha_z\) by the transformation:

\[
\alpha_k = \alpha_x p_{1k}^2 + \alpha_y p_{2k}^2 + \alpha_z p_{3k}^2,
\]  

(2.132)

The coefficients \(p_{1k}, p_{2k}, p_{3k}\) are components of a rotation matrix relating the anisotropy axes to the principal directions of the stress tensor \(S\).

In the case of plane stress six independent coefficients must be determined. As a consequence, four mechanical tests have to be made (three uniaxial tensile tests for establishing \(\sigma_0, \sigma_{45}, \sigma_{90}, r_0\) and \(r_{90}\) and one biaxial tensile test for establishing \(\sigma_b\)).

The yield surfaces predicted by this criterion are good approximations of those calculated by the Bishop-Hill theory and of experimental data. However, finite-element simulations based on this criterion [57] revealed some inaccuracies in predicting blank earing in deep-drawing.

In order to improve the performance of his criterion, Barlat and co-workers [58] proposed a generalisation of the 1994 criterion whereby \(\alpha_x, \alpha_y\) and \(\alpha_z\) are not constant anymore. They depend on the angles \(\beta_1, \beta_2, \beta_3\) between the principal directions of the stress tensor and the anisotropic axes:

\[
\begin{align*}
\alpha_x &= \alpha_{x0} \cos^2 \frac{2}{3} \beta_1 + \alpha_{x1} \cos \frac{2}{3} \beta_1; \\
\alpha_y &= \alpha_{y0} \cos^2 \frac{2}{3} \beta_2 + \alpha_{y1} \cos \frac{2}{3} \beta_2; \\
\alpha_z &= \alpha_{z0} \cos^2 \frac{2}{3} \beta_3 + \alpha_{z1} \cos \frac{2}{3} \beta_3. 
\end{align*}
\]  

(2.133)

where

\[
\begin{align*}
\alpha_{i0} &= \alpha_i \text{ for } \beta_i = 0; \\
\alpha_{i1} &= \alpha_i \text{ for } \beta_i = \pi/2. 
\end{align*}
\]  

(2.134)

and

\[
\begin{align*}
\cos^2 \frac{2}{3} \beta_1 &= \begin{cases} 
 y \cdot 1 \text{ if } |S_1| \geq |S_3| \\
 y \cdot 3 \text{ if } |S_1| < |S_3| 
\end{cases} \\
\cos^2 \frac{2}{3} \beta_2 &= \begin{cases} 
 z \cdot 1 \text{ if } |S_1| \geq |S_3| \\
 z \cdot 3 \text{ if } |S_1| < |S_3| 
\end{cases} \\
\cos^2 \frac{2}{3} \beta_3 &= \begin{cases} 
 x \cdot 1 \text{ if } |S_1| \geq |S_3| \\
 x \cdot 3 \text{ if } |S_1| < |S_3| 
\end{cases}
\]  

(2.135)

For plane stress, after applying the linear transformation introduced by Karafillis and Boyce (see [21]) the components of the IPE (isotropic plastic equivalent) stress tensor are written
\[ S_{11} = \frac{c_3 (\sigma_{11} - \sigma_{22}) + c_2 \sigma_{11}}{3}; \]
\[ S_{22} = \frac{c_1 \sigma_{22} + c_3 (\sigma_{11} - \sigma_{22})}{3}; \]
\[ S_{33} = \frac{-c_2 \sigma_{11} - c_1 \sigma_{22}}{3}; \]
\[ S_{12} = c_6 \sigma_{12}. \]

where \( c_1, c_2, c_3 \) and \( c_6 \) are material coefficients defining the anisotropy. The principal deviatoric stresses are computed as follows:

\[ S_{1,2} = \frac{S_{11} + S_{22}}{2} \pm \sqrt{\left( \frac{S_{11} + S_{22}}{2} \right)^2 + S_{12}^2}, \]  
\[ S_3 = -S_1 - S_2 = -S_{11} - S_{22}. \]  

In this case the coefficients \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are

\[ \alpha_1 = \alpha_x \cos^2 2\theta + \alpha_y \cos^2 2\theta; \]
\[ \alpha_2 = \alpha_y \cos^2 2\theta + \alpha_y \cos^2 2\theta; \]
\[ \alpha_3 = \alpha_z \cos^2 2\theta + \alpha_z \cos^2 2\theta. \]  

whereby \( \alpha_{z0} = 1 \) and

\[ \theta = \tan^{-1} \left( \frac{S_{11} - S_{22}}{S_{12}} \right). \]

The orthotropic axes 1, 2 and 3 are oriented along the rolling, transverse and normal direction, respectively. The methodology used for establishing the yield function is the same as that used by Karafillis and Boyce (see [21]).

Equations (2.136) and (2.138) show that in order to determine the yield function eight parameters are necessary: \( c_1, c_2, c_3, c_6, \alpha_x, \alpha_y \) and \( \alpha_{z0} \) and the exponent set in accordance with the crystallographic structure of the material. The great number of parameters ensures a good flexibility of the criterion but implies a large number of mechanical tests.

Simulations of deep-drawing of cylindrical cups using the new criterion [59] revealed a very good agreement of the predicted earing with experimental data. The computed yield surfaces are also in good agreement with those predicted by the Bishop-Hill theory and experiments. A very good agreement between theory and experiment has also been found for the distribution of the uniaxial yield stresses and anisotropy coefficients in the plane of sheet.

The most important disadvantages of these models are as follows:

- the convexity of the yield functions is not guaranteed
- the derivatives of the equivalent stress are difficult to obtain analytically and, consequently, the usability of the models in the numerical simulation codes is reduced.
• the application for full stress states leads to numerical problems due to the complexity of the yield functions
• the CPU time is considerably larger than in the case of simpler models. This is a major drawback in the numerical simulation of sheet metal forming processes.

2.3.2.5 Karafillis–Boyce Yield Criterion

As mentioned in Sect. 2.2.4 Karafillis and Boyce [21] proposed a very general yield criterion. Its originality is given by the expression of the yield function (a weighted combination between the von Mises and Tresca expression) as well as the use of a linear transformation in order to pass from the isotropic to the anisotropic case.

The proposed isotropic yield function is presented in the Sect. 2.2.4. In order to adopt the yield criterion for anisotropic materials, Karafillis and Boyce proposed an original method based on the linear transformation:

\[ \tilde{S} = \tilde{L} \cdot \tilde{\sigma}, \]  
(2.140)

where \( \tilde{S} \) is a deviatoric stress tensor associated to an ‘isotropic plastic equivalent’ (IPE) state, \( \tilde{\sigma} \) is the actual anisotropic stress tensor; \( \tilde{L} \) is a linear operator depending on the material which is defined as follows:

\[
\tilde{L} = C \begin{bmatrix}
1 & \beta_1 & \beta_2 & 0 & 0 & 0 \\
\beta_1 & \alpha_1 & \beta_3 & 0 & 0 & 0 \\
\beta_2 & \beta_3 & \alpha_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_3 \\
\end{bmatrix}
\]  
(2.141)

where

\[
\beta_1 = \frac{\alpha_2 - \alpha_1 - 1}{2},
\]
\[
\beta_2 = \frac{\alpha_1 - \alpha_2 - 1}{2},
\]
\[
\beta_3 = \frac{1 - \alpha_1 - \alpha_2}{2},
\]
(2.142)

and \( \alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3 \) and \( C \) are parameters defining the anisotropy of the metallic material.

Thus the Karafillis-Boyce yield function is defined by eight coefficients (\( \alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3, c, C \) and \( k \)).

In case of an isotropic material the parameters have the values

\[ c = \frac{2}{3}, \alpha_1 = \alpha_2 = 1, \gamma_1 = \gamma_2 = \gamma_3 = \frac{3}{2} \]  
(2.143)
For loading with negligible non-planar shear stresses ($\sigma_{32} = \sigma_{13} \approx 0$) it is obtained $\gamma_2 = \gamma_3 = 3/2$.

Equation (2.140) may be written in the form

$$
\begin{bmatrix}
S_{11}^i \\
S_{22}^i \\
S_{33}^i \\
S_{23}^i \\
S_{31}^i \\
S_{12}^i
\end{bmatrix} = C \cdot
\begin{bmatrix}
1 & \beta_1 & \beta_2 & 0 & 0 & 0 \\
\beta_1 & \alpha_1 & \beta_3 & 0 & 0 & 0 \\
\beta_2 & \beta_3 & \alpha_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_3
\end{bmatrix} \cdot
\begin{bmatrix}
\sigma_{11}^a \\
\sigma_{22}^a \\
\sigma_{33}^a \\
\sigma_{23}^a \\
\sigma_{31}^a \\
\sigma_{12}^a
\end{bmatrix}
$$

(2.144)

The ‘i’ and ‘a’ superscripts specify the ‘isotropic’ and ‘anisotropic’ state, respectively. In case of a plane-stress state, Eq. (2.144) becomes:

$$
\begin{bmatrix}
S_{11}^i \\
S_{22}^i \\
S_{12}^i
\end{bmatrix} = C \cdot
\begin{bmatrix}
1 & \frac{\alpha_2 - \alpha_1 - 1}{2} & 0 \\
\frac{\alpha_2 - \alpha_1 - 1}{2} & \alpha_1 & 0 \\
0 & 0 & \gamma_3
\end{bmatrix} \cdot
\begin{bmatrix}
\sigma_{11}^a \\
\sigma_{22}^a \\
\sigma_{12}^a
\end{bmatrix}
$$

(2.145)

and only six parameters needed for defining the yield surface ($\alpha_1$, $\alpha_2$, $\gamma_3$, $c$, $C$ and $k$), one more than for the Barlat 1991 criterion. Therefore the Karafillis-Boyce criterion is more flexible than Barlat 1991.

The linear transformation of an anisotropic stress state $\tilde{S}^a$ to an equivalent isotropic one, $\tilde{S}^i$, has been called by Karafillis and Boyce as ‘Isotropic Plasticity Equivalent’ (IPE). A similar transformation, although not in the same form, was used by Barlat in 1991 [10] in order to change the isotropic Hosford criterion into a six-component anisotropic one.

The methodology used to establish the Karafillis-Boyce yield function for plane stress is as follows:

1° Let $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{12}$ be the planar components of the anisotropic stress tensor

2° By using the linear transformation (2.145) the components of the IPE deviatoric stress tensor are obtained:

$$
\begin{align*}
S_{11} &= C \left( \sigma_{11} + \frac{\alpha_2 - \alpha_1 - 1}{2} \sigma_{22} \right); \\
S_{22} &= C \left( \frac{\alpha_2 - \alpha_1 - 1}{2} \sigma_{11} + \alpha_1 \sigma_{22} \right); \\
S_{12} &= \gamma_3 \cdot \sigma_{12}.
\end{align*}
$$

(2.146)

3° The principal deviatoric stresses are calculated:

$$
\begin{align*}
S_1 &= \frac{S_{11} + S_{12}}{2} + \sqrt{\left(\frac{S_{11} - S_{12}}{2}\right)^2 + S_{12}^2}; \\
S_2 &= \frac{S_{11} - S_{12}}{2} - \sqrt{\left(\frac{S_{11} - S_{12}}{2}\right)^2 + S_{12}^2}; \\
S_3 &= - (S_1 + S_2).
\end{align*}
$$

(2.147)
Using Eqs. (2.146) the principal deviatoric stress tensor \( \sigma \) is obtained. By inserting into Eq. (2.44), an expression of the yield function in terms of the components of the anisotropic stress tensor and the coefficients \( \alpha_1, \alpha_2, \gamma_3, c, C \) and \( k \) is obtained:

\[
\Phi = \Phi (\sigma_{11}, \sigma_{22}, \sigma_{12}, c, C, \alpha_1, \alpha_2, \gamma_3, k, \sigma_e) \quad (2.148)
\]

This methodology is used in order to establish the yield function for the Barlat 1994 and 1996 yield criteria (presented above).

Karafillis and Boyce applied the inverse transformation

\[
\tilde{D}^a = \tilde{L} \cdot \tilde{D}^i
\quad (2.149)
\]

in order to determinate the associated flow rules in the anisotropic state as functions to the isotropic ones. Here \( \tilde{D}^a \) is the anisotropic strain-rate tensor (for the beginning of plastic yielding) while \( \tilde{D}^i \) is the same tensor associated to the IPE material. \( \tilde{D}^i \) may be calculated from the associated flow rule assuming the yield function \( \Phi \) (Eq. 2.44):

\[
\tilde{D}^i = \lambda \frac{\partial \Phi}{\partial S} \quad (2.150)
\]

The transformation (2.145) is used for identifying the anisotropy coefficients \( r_0, r_{45} \) and \( r_{90} \). The numerical procedure used for the inverse determination of these coefficients in the operator \( L \) (Eq. 2.141) is presented in [21] as a flowchart.


The equivalent stress of the new yield function is defined in the following form:

\[
\sigma = \left( \sum_{k=1}^{K} \alpha^k (\sigma^k)^a \right)^{\psi^a} \quad (2.151)
\]

where the \( K \) functions \( \sigma^k \) are convex, positive and homogeneous of degree 1 and \( \alpha^k \) are positive coefficients (the sum of which is 1).

In their original paper [60], Bron and Besson use only two functions \( (K = 2) \). In this case, the general formulation of the criterion (2.151) reduces to

\[
\sigma^k = (\psi^k)^{1/b^k} \quad (2.152)
\]

\[
\psi^1 = \frac{1}{2} \left( |S_2^1 - S_3^1|^{b^1} + |S_3^1 - S_1^1|^{b^1} + |S_1^1 - S_2^1|^{b^1} \right) \quad (2.153)
\]

\[
\psi^2 = \frac{3b^2}{2b^2 + 2} \left( |S_1^2|^{b^2} + |S_2^2|^{b^2} + |S_3^2|^{b^2} \right) \quad (2.154)
\]
where \( S^k_{i=1-3} \) are the principal values of the stress deviator \( S^k \). The stress deviator is determined by a linear transformation \( L^k \) defined in the paper [60].

In the formulation proposed by Bron and Besson, a total number of 16 parameters are involved. Due to this fact, the model is very flexible. The convexity of the yield function has been proved.

A similar model with the Karafillis and Boyce one has been developed for anisotropic modelling of the polimeric foams by Wang and Pan [61]. An extra parameter is used to model the different yield behaviours under tension and compression.

The yield surfaces predicted by Karafillis-Boyce criterion are in very good agreement with experimental data as well as with the predictions of the Bishop-Hill theory [21]. The same agreement is also obtained when comparing the variation of the uniaxial yield stress and anisotropy coefficients in the plane of the sheet with experimental data [21]. Another advantage of the criterion is that it uses only uniaxial tensile tests for identifying the material parameters. From a mathematical point of view the method proposed by Karafillis and Boyce is both elegant and rigorous.

A disadvantage of the criterion is that the identification procedure of the tensor operator is complex and requires a numerical solution. But this is not a major difficulty when implementing the yield criterion into an FE code.

### 2.3.3 Yield Criteria Expressed in Polar Coordinates

#### 2.3.3.1 Budiansky Yield Criterion

Budiansky proposed a general yield criterion [62] in a form that seems to be attractive for applications, especially when using it together with appropriate planar constitutive equations. Planar isotropy allows that any two-dimensional yield criterion is expressed as a function of the stress points \((\sigma_2 + \sigma_1)\) and \((\sigma_2 - \sigma_1)\). Budiansky’s criterion can be written in the form

\[
\begin{align*}
x &= \frac{\sigma_1 + \sigma_2}{2\sigma_b} = g(\alpha) \cos \alpha; \\
y &= \frac{\sigma_2 - \sigma_1}{2\sigma_s} = g(\alpha) \sin \alpha.
\end{align*}
\]

(2.155)

where \( g(\alpha) > 0 \) is the radial coordinate of a point located on the yield surface, \( \alpha \) is the associated polar angle, \( \sigma_s \) is the yield stress in pure shear, and \( \sigma_b \) is the yield stress in equibiaxial tension.

The problem that arises is to establish the function \( g(\alpha) \). By using the ratios \( X = \sigma_b/\sigma_u \) and \( Y = \sigma_b/\sigma_s \) as non-dimensional parameters characteristic of the material Eq. (2.155) can be rewritten in the form:

\[
\begin{align*}
\frac{\sigma_2 + \sigma_1}{2\sigma_u} &= X g(\alpha) \cos \alpha \\
\frac{\sigma_2 - \sigma_1}{2\sigma_u} &= Y g(\alpha) \sin \alpha
\end{align*}
\]

(2.156)
Hence,

\[ \alpha = \tan^{-1}\left[ \frac{Y(\sigma_2 - \sigma_1)}{\sigma_2 + \sigma_1} \right] \]  

(2.157)

The associated flow rule is fully detailed in [62].

Ferron [63] proposed a yield criterion more general than by Budiansky [62] since it allows to include the case of planar anisotropy due to the dependence of \( g \) on two parameters \( \theta \) and \( \alpha \). The function \( g(\theta, \alpha) \) must satisfy the symmetry and convexity conditions imposed to the function \( g(\alpha) \) in Budiansky’s criterion and is determined starting from the isotropic criterion by Drucker [16].

Ferron [63] yield criterion is defined as follows:

\[ (1 - k)^{m/6} g(\theta, \alpha)^{-m} = \left[ F(\theta)^{m/6} + 2a \sin \theta \cos^{2n-1} \theta \cos 2\alpha + \sin^{2p} \theta \cos^{2q} \right] \]  

(2.158)

Here \( a, m, n, p \) and \( q \) are material parameters. The identification of these parameters is made by uniaxial tensile and shearing tests.

Among the parameters in Ferron’s yield function five are independent. This ensures a great flexibility of the criterion.

The obtained yield surfaces are in good agreement with experiments. The yield criterion can also model accurately the distribution of the uniaxial yield stress and coefficient \( r \) in the plane of the sheet [63].

### 2.3.4 Other Yield Criteria

In addition to the criteria described in the previous sections, several other non-quadratic yield criteria have been developed. With respect to their restrained use they are only described briefly.

#### 2.3.4.1 Gotoh Yield Criterion

In order to overcome the disadvantages of the Hill 1948 criterion, Gotoh proposed in 1977 [64] a polynomial yield function of fourth degree (instead of the quadratic one) for orthotropic rolled sheet metals by writing the yield function in the form given by Hill in 1950 [65].

\[ f = \sum_{i,j,k} A_{ijk} \sigma_{11}^i \sigma_{22}^j \sigma_{12}^{2k} \]  

(2.159)

where \( i, j, 2k \leq 4, x, y \) are the orthotropic axes, and \( A_{ijk} \) are constant coefficients.

The conditions of orthotropy and wrinkling of the blank in axisymmetric deep-drawing necessitate to write for the function \( f \)
\[ f = A_0 \left( \sigma_{11} + \sigma_{22} \right)^2 + \left[ A_1 \sigma_{11}^4 + A_2 \sigma_{11}^3 \sigma_{22} + A_3 \sigma_{11}^2 \sigma_{22}^2 + A_4 \sigma_{11} \sigma_{22}^3 + A_5 \sigma_{22}^4 + (A_6 \sigma_{11}^2 + A_7 \sigma_{11} \sigma_{22} + A_8 \sigma_{22}^2) \sigma_{12}^2 + A_9 \sigma_{12}^4 \right] \] (2.160)

The first term may be considered as a function of the mean normal pressure and thus, assuming an incompressible material, \( A_0 \) is obtained. The condition to avoid wrinkling leads to the equations [64]

\[
\begin{align*}
\cos \alpha \sin \alpha F(c) &= 0 \\
c &= \cos^2 \alpha \\
F(c) &= 4Ac^3 + 3Bc^2 + 2Cc + D
\end{align*}
\] (2.161)

\( A \ldots D \) are coefficients depending on \( A_1 \ldots A_9 \):

\[
\begin{align*}
A &= (A_1 + A_3 + A_5 + A_7 + A_9) - (A_2 + A_4 + A_6 + A_8) \\
B &= (A_2 + 3A_4 + A_6 + 3A_8) - 2(A_3 + 2A_5 + A_7 + A_9) \\
C &= (A_3 + 6A_5 + A_7 + A_9) - 3(A_4 + A_8) \\
D &= A_4 + A_8 - 4A_5
\end{align*}
\] (2.162)

Since the identification of the coefficients \( A_1 \ldots A_9 \) is made by means of an equibiaxial tensile test, this criterion can model the ‘anomalous behavior’ observed by Woodthrope and Pearce. This criterion is presented in detail in [64].

The large number of coefficients will generally allow a high flexibility of the yield criterion. The model captures the planar variation of both the uniaxial yield stress and uniaxial coefficient of plastic anisotropy. The advantages previously mentioned above explain why the model is frequently used, especially by the Japanese researchers.

A disadvantage of the criterion is its complicated form and the large number of mechanical tests needed for identifying the coefficients.

### 2.4 Advanced Anisotropic Yield Criteria

During the last years, the competition in the automotive and aeronautical industry has become more intense. This fact has lead to the development of new steel alloys (Bake Hardenable, Dual Phase, Complex Phase, Transformation Induced Plasticity-TRIP, Martensitic Steels, Hot-stamping boron-alloyed steels), aluminium alloys having better performances and increased interest on the use of magnesium and superplastic alloys. Since 2000, the modelling of the anisotropic behaviour of these materials has encouraged the research activities focused on the development of yield criteria. Several new models have been proposed during the last years. These models allow a very good description of the anisotropic behaviour both of steel alloys (BCC crystallographic structure), aluminium alloys (FCC structure) and magnesium alloys (HCP structure). The new yield criteria incorporate a large number of coefficients (usually, at least 8 coefficients). Due to this fact, they are able to
give an accurate description of the yield surface and follow closely the planar variations of the uniaxial yield stress and the coefficient of plastic anisotropy. Even more, some of the recently developed models can also capture the non-symmetric response in tension/compression specific to the HCP alloys. Due to the significant impact of these advanced yield criteria, they will be described in a separate subchapter entitled ‘Advanced Anisotropic Yield Criteria’.

### 2.4.1 Barlat Yield Criteria

In order to remove the disadvantages of the Barlat 1994 and Barlat 1996 yield criteria (see Sect. 2.3.2), but aiming to preserve their flexibility, Barlat proposed in 2000 [5, 66] a new model particularized for plane stress (2D).

Let us consider a linear transformation defined as follows:

\[ X = C \cdot s \]  

(2.163)

where \( s \) is the deviatoric stress tensor and \( X \) the linearly transformed stress tensor. This gives 9 independent coefficients for the general case and 7 for plane stress. However, applied to plane stress conditions, only one coefficient is available to account for \( \sigma_{45} \) and \( r_{45} \). As pointed out in Barlat et al. [5] additional coefficients in the context of linear transformations can be obtained by using two transformations associated to two different isotropic yield functions, respectively.

As a consequence, Barlat et al. [5] proposed a yield function expressed by the relationship

\[ \Phi = \Phi' + \Phi'' = 2\sigma^a, \]  

(2.164)

where

\[ \Phi' = |S_1 - S_2|^a \]  

(2.165)

\[ \Phi'' = |2S_2 + S_1|^a + |2S_2 + S_2|^a \]  

(2.166)

\( S_1 \) and \( S_2 \) are the principal deviatoric stresses and ‘\( a \)’ is an exponent determined based on the crystallographic structure of the material.

By applying a linear transformation to each of the isotropic functions defined by Eqs. (2.165) and (2.166), we obtain the yield function

\[ \Phi = \Phi' (X') + \Phi'' (X'') = 2\sigma^a \]  

(2.167)

where \( \sigma \) is the effective stress, ‘\( a \)’ is a material coefficient and

\[ \Phi' = |X'_1 + X'_2|^a \]  

(2.168)

\[ \Phi'' = |2X''_2 + X''_1|^a + |2X''_1 + X''_2|^a \]  

(2.169)
\[ X' = C'.s = C'.T.\sigma = L'.\sigma \]
\[ X'' = C''.s = C''.T.\sigma = L''.\sigma \]  \hspace{1cm} (2.170)

\( T \) is a matrix that transforms the Cauchy stress tensor \( \sigma \) to its deviator \( s \):

\[ T = \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (2.171)

\( C' \) and \( C' \) being the linear transformations.

In the reference frame associated with the material symmetry,

\[ \begin{bmatrix} X'_1 \\ X'_2 \\ X'_3 \end{bmatrix} = \begin{bmatrix} C'_1 \\ C'_2 \\ C'_3 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \\ s_{12} \end{bmatrix} \]  \hspace{1cm} (2.172)

and

\[ \begin{bmatrix} X''_1 \\ X''_2 \\ X''_3 \end{bmatrix} = \begin{bmatrix} C''_1 \\ C''_2 \\ C''_3 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \\ s_{12} \end{bmatrix} \]  \hspace{1cm} (2.173)

Because \( \Phi' \) depends on \( X'_1 - X'_2 \), only three coefficients remain independent in \( C' \) (see more details in [5]). In \( C'' \) are five independent coefficients. Totally, in both transformations, are 8 independent coefficients.

The principal values \( X_1 \) and \( X_2 \) of there \( X'' \) and \( X'' \) are the follows:

\[ X_1 = \frac{1}{2} \left( X_{11} + X_{22} + \sqrt{(X_{11} - X_{22})^2 + 4X_{12}^2} \right), \]  \hspace{1cm} (2.174)
\[ X_2 = \frac{1}{2} \left( X_{11} + X_{22} - \sqrt{(X_{11} - X_{22})^2 + 4X_{12}^2} \right). \]  \hspace{1cm} (2.175)

The coefficients of \( L' \) and \( L'' \) are expressed as follows

\[ \begin{bmatrix} L'_{11} \\ L'_{12} \\ L'_{21} \\ L'_{22} \\ L'_{66} \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & 0 \\ -1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_7 \end{bmatrix} \]  \hspace{1cm} (2.176)
Due to the fact that 8 coefficients are incorporated in the linear transformations, we need 8 material characteristics for evaluating them. The uniaxial tension test along the rolling, diagonal and transversal directions, together with the biaxial tension test can provide only 7 characteristics (3 uniaxial yield stresses, 3 coefficients of uniaxial anisotropy and the biaxial yield stress). Barlat adopted the coefficient of biaxial anisotropy \( r_b \) as the eighth characteristic in the identification procedure. The experimental procedure used for the determination of this mechanical parameter is described in Sect. 2.1.2.

By using the same methodology as the one described above, Aretz and Barlat [67] and Barlat et al. [68] proposed a 3D yield criterion called Barlat 2004-18p:

\[
\Phi = |s'_1 - s''_1|^a + |s'_2 - s''_2|^a + |s'_3 - s''_3|^a + |s'_1 - s'_3|^a + |s'_2 - s'_3|^a + |s'_3 - s'_1|^a + |s'_2 - s'_1|^a + |s'_3 - s'_2|^a + |s'_3 - s'_1|^a = 4\sigma^a,
\]

(2.178)

where, \( \sigma \) represent the uniaxial yield stress (any other yield stress may be use as reference yield stress) and \( a \) is an exponent determined based on the crystallographic structure of the material.

The associated linear transformation on the stress deviator is defined:

\[
C'' = \begin{bmatrix}
-2 & 2 & 8 & -2 & 0 & \alpha_3 \\
1 & -4 & -4 & 4 & 0 & \alpha_4 \\
4 & -4 & -4 & 4 & 0 & \alpha_5 \\
-2 & 8 & 2 & -2 & 0 & \alpha_6 \\
0 & 0 & 0 & 0 & 1 & \alpha_8 \\
\end{bmatrix}
\]

(2.177)

and \( C' \) and \( C'' \) are obtained by adding prime and double prime symbols.

Each transformation provides 9 coefficients and totally both transformations give 18 coefficients. In order to determine all this coefficients an the minimization of the error function method is used (see [51]). If only one linear transformation is assumed the Barlat 2004-18p formulation reduce to Barlat 1991 yield criterion.

The uniaxial yield stresses and anisotropy coefficients in seven directions in the plane of the sheets (0, 15, 30, 45, 60, 75 and 90 degree to the rolling direction), the biaxial yield stress, the biaxial anisotropy coefficient and four additional data characterizing out-of-plane properties (two tensile and two simple shear yield stresses) are used in the identification of all the coefficients. For determination of the out-of-plane parameters the crystal plasticity models are nedded (see [68]).
If we adopt a yield function defined by the relationship

\[
\Phi = |s'_1 - s''_2|^a + |s'_2 - s''_3|^a + |s'_3 - s''_1|^a - \{|s'_1|^a + |s'_2|^a + |s'_3|^a\} + |s''_1|^a + |s''_2|^a + |s''_3|^a = 4\sigma^a,
\]

(2.180)

the number of coefficients incorporated in the linear transformations will reduce to 13:

\[
C' = \begin{bmatrix}
0 & -1 & -c'_{13} & 0 & 0 & 0 \\
-c'_{21} & 0 & -c'_{23} & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c'_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c'_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & c'_{66}
\end{bmatrix}
\]

(2.181)

\[
C'' = \begin{bmatrix}
0 & -c''_{12} & -c''_{13} & 0 & 0 & 0 \\
-c''_{21} & 0 & -c''_{23} & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c''_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c''_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & c''_{66}
\end{bmatrix}
\]

(2.182)

For the plane stress case the number of the coefficients are reduced from 13 to 9.

The yield function has been tested for different aluminium alloys exhibiting a pronounced anisotropy. The model has proved its capability to provide an accurate prediction of the planar variations of the uniaxial yield stress and coefficient of plastic anisotropy.

The implementation of the Barlat 2004-18p model in finite-element codes [69] proved its capability to predict the occurrence of six and eight ears in the process of cup drawing. Barlat 2004-18p is one of the phenomenological model being able to capture more than 4 ears. This is the most important advantage of the yield criterion. Of course, it is possible to develop models incorporating more and more linear transformations and thus having a larger number of coefficients. The practical difficulty related to the use of such yield criteria consists in the experimental determination of the mechanical parameters needed for the evaluation of the coefficients.

The disadvantages of the models presented above are:

- due to the complexity of the formulation, they are not user-friendly
- they need crystal plasticity models for the evaluation of some parameters.

The Barlat 2004-18p is implemented in the LS Dyna commercial code.
2.4.2 Banabic–Balan–Comsa (BBC) Yield Criteria

In 2000 the members of the CERTETA\(^1\) team started a research programme having as principal objective the development of a model able to provide an accurate description of the yield surfaces predicted by texture computations. The new formulation was developed on the basis of the isotropic formulation proposed by Hershey. By adding weight coefficients to that model, the researchers succeeded in developing a flexible yield criterion. The last version incorporates a number of 8 coefficients and, consequently, its identification procedure uses 8 mechanical parameters (3 uniaxial yield stresses, 3 uniaxial coefficients of anisotropy, the biaxial yield stress and the biaxial coefficient of plastic anisotropy).

The first formulation of the yield criterion was proposed by Banabic et al. [51] in the form

\[
\sigma = \left[ a \left( b \Gamma + c \Psi \right)^{2k} + a \left( b \Gamma - c \Psi \right)^{2k} + (1 - a) \left( 2c \Psi \right)^{2k} \right]^{\frac{1}{2k}} \tag{2.183}
\]

where \(a, b, c,\) and \(k\) are material parameters, while \(\Gamma\) and \(\Psi\) are functions of the second and third invariants of a transformed stress tensor \(s' = L\sigma\), where \(L\) is a 4th order tensor. In this formulation anisotropy is described by means of the tensor \(L\), which satisfies: (i) the symmetry conditions \(L_{ijkl} = L_{jikl} = L_{jilk} = L_{klij} (i, j, k, l = 1 \ldots 3)\), (ii) the requirement of invariance with respect to the symmetry group of the material, and (iii) the three conditions \(L_{1k} + L_{2k} + L_{3k} = 0\) (for \(k = 1, 2,\) and \(3\)), which ensures that \(s'\) is traceless (see Karafillis–Boyce [21]). Hence, in the reference system associated with the directions of orthotropy, the tensor \(L\) has 6 non-zero components for 3D conditions and 4 components for plane stress state.

Let define \((1, 2, 3)\), the reference frame associated with orthotropy. For a rolled sheet, 1, 2, and 3 represent the rolling direction, the long transverse direction, and the short transverse direction, respectively. In the reference system \((1, 2, 3)\):

\[
\begin{align*}
    s'_{11} &= d\sigma_{11} + e\sigma_{22} \\
    s'_{22} &= e\sigma_{11} + f\sigma_{22} \\
    s'_{33} &= -(d + e)\sigma_{11} - (e + f)\sigma_{22} \\
    s'_{12} &= g\sigma_{12} \\
    s'_{13} &= s'_{23} = 0
\end{align*} \tag{2.184}
\]

where \(d,\) \(e,\) \(f,\) and \(g\) are the four independent components of the tensor \(L\).

The expressions of \(\Gamma\) and \(\Psi\) in terms of the stress components are

\[
\begin{align*}
    \Gamma &= M\sigma_{11} + N\sigma_{22} \\
    \Psi &= \sqrt{(P\sigma_{11} + Q\sigma_{22})^2 + R\sigma_{12}^2} \tag{2.185}
\end{align*}
\]

\(^1\) Research Centre in Sheet Metal Forming Technology belong the Technical University of Cluj Napoca, Romania (http://www.certeta.utcluj.ro).
where

\[
M = d + e \\
N = e + f \\
P = \frac{d - e}{2} \\
Q = \frac{e - f}{2} \\
R = g^2 \\
\text{(2.186)}
\]

(for more details see Banabic et al. [52]).

The convexity of the yield surface described by (2.183) is ensured if \( a \in [0, 1] \) and \( k \) is a strictly positive integer number.

The yield stress in uniaxial tension along an axis at orientation \( \theta \) to the rolling direction \( x \), the equibiaxial yield stress and the coefficient of plastic anisotropy associated to a direction inclined at an angle \( \theta \in [0, 90^\circ] \) with the rolling direction are presented in the paper [70]. Further details related to the evaluation of the coefficients as well as to the experimental validation of the BBC2000 yield criterion can be found in the paper [70].

The shape of the yield surface is defined by the material parameters \( a, b, c, d, e, f, g, \) and \( k \). Among these parameters, \( k \) has a distinct status. More precisely, its value is set in accordance with the crystallographic structure of the material [45]: \( k = 3 \) for BCC alloys, and \( k = 4 \) for FCC alloys. The other 7 parameters are determined such that the model reproduces as well as possible the following experimental characteristics of the orthotropic sheet metal: \( \sigma_{0\text{exp}}, \sigma_{90\text{exp}}, \sigma_{45\text{exp}}, \sigma_{0\text{exp}}, \sigma_{90\text{exp}}, \sigma_{45\text{exp}} \) and \( r_{45\text{exp}} \). It is possible to obtain their values by solving a set of seven non-linear equations.

An improvement of this criterion was proposed by Banabic et al. [53] in order to account for an additional mechanical parameter, namely, the biaxial anisotropy coefficient. The new formulation is given by:

\[
\sigma = \left[ a (\Gamma + \Psi) \right]^{2k} + a (\Gamma - \Psi) \right]^{2k} + (1 - a) (2\Lambda)^{2k} \right]^{\frac{1}{2k}} \text{(2.187)}
\]

where \( k \in \mathbb{N} \geq 1 \) and \( 0 \leq a \leq 1 \) are material parameters, while \( \Gamma, \Psi, \) and \( \Lambda \) are functions depending on the planar components of the stress tensor:

\[
\Gamma = \frac{\sigma_{11} + M \sigma_{22}}{2} \\
\Psi = \sqrt{\left( \frac{N \sigma_{11} - P \sigma_{22}}{2} \right)^2 + Q^2 \sigma_{12} \sigma_{21}} \\
\Lambda = \sqrt{\left( \frac{R \sigma_{11} - S \sigma_{22}}{2} \right)^2 + T^2 \sigma_{12} \sigma_{21}} \text{(2.188)}
\]

where the quantities \( M, N, P, Q, R, S, \) and \( T \) are also material parameters.

More details concerning the uniaxial yield stress and anisotropy coefficients variations are presented in the paper [53]. Further details related to the evaluation of the quantities mentioned above as well as to the convergence of
the identification procedure of the BBC2003 yield criterion can be found in the paper [71].

The other eight parameters are determined such that the model reproduces the experimental characteristics of the orthotropic sheet metal as well as possible, namely, $\sigma_0$, $\sigma_{45}$, $\sigma_{90}$, $\sigma_b$, $r_0$, $r_{45}$, $r_{90}$ and $r_b$. It is possible to obtain the value of these parameters by solving a set of eight non-linear equations. However, this set of equations has multiple solutions. A more effective strategy of identification is to impose the minimization of the following error function:

$$F(a, M, N, P, Q, R, S, T) = \left( \frac{\sigma_0^P}{\sigma_0} - 1 \right)^2 + \left( \frac{\sigma_{45}^P}{\sigma_{45}} - 1 \right)^2 + \left( \frac{\sigma_{90}^P}{\sigma_{90}} - 1 \right)^2 + \left( \frac{\sigma_b^P}{\sigma_b} - 1 \right)^2$$

where the superscript $(.)^P$ denotes the values predicted by the constitutive equation.

For the numerical minimization, the downhill simplex method proposed by Nelder and Mead [72] has been adopted because it does not need the evaluation of the gradients. The identification procedure can also use a reduced number of mechanical parameters (2, 4, 5, 6 or 7), as shown in Table 2.1. The particular set of mechanical parameters used by each identification strategy is specified in the table. The author have also developed identification procedures based on uniaxial and plane-strain experimental data [71].

A version of the BBC 2003 yield criterion has been improved by Aretz [73]. The BBC2003 yield criterion is reducible both to Hill 1948 and Barlat 1989 formulations (see more details in Sect. 2.3).

Barlat et al. [74] showed that the BBC 2003 and Barlat 2000 are the same. But one should notice that the development procedures adopted by the authors where different: the BBC models emerged in a classical manner by adding coefficients to Hershey’s formulation, while Barlat 2000 used two linear transformations.

The most important advantages of these models are:

- the yield functions have simple expressions
- they are able to describe accurately the yield surface and also give good predictions of the planar distribution of the uniaxial yield stress and uniaxial coefficient of plastic anisotropy

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$$F(a, M, N, P, Q, R, S, T) = \left( \frac{\sigma_0^P}{\sigma_0} - 1 \right)^2 + \left( \frac{\sigma_{45}^P}{\sigma_{45}} - 1 \right)^2 + \left( \frac{\sigma_{90}^P}{\sigma_{90}} - 1 \right)^2 + \left( \frac{\sigma_b^P}{\sigma_b} - 1 \right)^2$$

where the superscript $(.)^P$ denotes the values predicted by the constitutive equation.

For the numerical minimization, the downhill simplex method proposed by Nelder and Mead [72] has been adopted because it does not need the evaluation of the gradients. The identification procedure can also use a reduced number of mechanical parameters (2, 4, 5, 6 or 7), as shown in Table 2.1. The particular set of mechanical parameters used by each identification strategy is specified in the table. The author have also developed identification procedures based on uniaxial and plane-strain experimental data [71].

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Barlat et al. [74] showed that the BBC 2003 and Barlat 2000 are the same. But one should notice that the development procedures adopted by the authors where different: the BBC models emerged in a classical manner by adding coefficients to Hershey’s formulation, while Barlat 2000 used two linear transformations.

The most important advantages of these models are:

- the yield functions have simple expressions
- they are able to describe accurately the yield surface and also give good predictions of the planar distribution of the uniaxial yield stress and uniaxial coefficient of plastic anisotropy
• the predicted shape of the yield surface closely follows the results of the texture models
• the CPU time needed for the simulation of complex sheet metal forming processes is not considerably increased
• the models can be used also in the cases when less than 8 mechanical parameters are available (e.g., 2, 4, 5, 6 or 7 parameters)
• the models are reducible to classical formulations such as Hill 1948 or Barlat 1989.

The following disadvantages can be mentioned:

• the formulation of the yield criterion is not very user-friendly
• the coefficients of the yield function do not have a direct physical meaning
• the development of 3D extensions is difficult.

A modified version of this criterion (BBC 2005) has been implemented in the AutoForm 4.1 commercial Finite Element program (issued May 2007).

2.4.3 Cazacu–Barlat Yield Criteria

To introduce orthotropy in the expression of an isotropic criterion, Cazacu and Barlat [75] proposed an alternative method based on the theory of the representation of tensor functions. They developed a method of generalizations of the invariants of the stress deviator $J_2$ and $J_3$. Based on this method, an anisotropic yield criterion is obtained by substituting the expression of the stress deviator invariants in the isotropic criterion by their respective anisotropic forms.

The generalized forms of the invariants, $J_2^0$ and $J_3^0$, respectively, are:

$$J_3^0 = \frac{1}{27} \left[ (b_1 + b_2) \sigma_{11}^3 + \frac{1}{27} (b_3 + b_4) \sigma_{22}^3 + \frac{1}{27} \left[ 2 (b_1 + b_4) - b_2 - b_3 \right] \sigma_{33}^3 \right]$$

$$- \frac{1}{9} \left[ (b_1 \sigma_{22} + b_2 \sigma_{33}) \sigma_{11}^2 - \frac{1}{9} \left[ (b_3 \sigma_{33} + b_4 \sigma_{11}) \sigma_{22}^2 \right] \right]$$

$$- \frac{1}{9} \left[ (b_1 - b_2 + b_4) \sigma_{11} + (b_1 - b_3 + b_4) \sigma_{22} \right] \sigma_{33}$$

$$+ \frac{2}{9} \left[ (b_1 + b_4) \sigma_{11} \sigma_{22} \sigma_{33} - \frac{\sigma_{22}^2}{3} \left[ 2 b_9 \sigma_{22} - b_8 \sigma_{33} - (2 b_9 - b_8) \sigma_{11} \right] \right]$$

$$- \frac{\sigma_{12}^2}{3} \left[ 2 b_{10} \sigma_{33} + b_5 \sigma_{22} - (2 b_{10} + b_5) \sigma_{11} \right]$$

$$- \frac{\sigma_{23}^2}{3} \left[ (b_6 + b_7) \sigma_{11} - b_6 \sigma_{22} - b_7 \sigma_{33} \right] + 2 b_{11} \sigma_{12} \sigma_{13} \sigma_{23}.$$

(2.190)
where the coefficients $b_k \ (k = 1\ldots11)$ describe the anisotropy and they reduce to unity for isotropic conditions.

\begin{equation}
J_2^o = \frac{a_1}{6} (\sigma_{11} - \sigma_{22})^2 + \frac{a_2}{6} (\sigma_{22} - \sigma_{33})^2 + \frac{a_3}{6} (\sigma_{11} - \sigma_{33})^2 + a_4 \sigma_{12}^2 \\
+ a_5 \sigma_{13}^2 + a_6 \sigma_{23}^2
\end{equation}

where the coefficients $a_k \ (k = 1\ldots6)$ describe the anisotropy and they reduce to unity in the isotropic case. Note that $J_2^o$ is Hill’s [25] quadratic yield function.

In Cazacu and Barlat [75], this approach was used to extend Drucker’s [16] isotropic yield criterion to an orthotropic one. For this case the expression of the proposed orthotropic criterion is:

\begin{equation}
f^O = \left( J_2^o \right) ^3 - c \left( J_3^o \right) ^2 = k^2.
\end{equation}

where $c$ is a constant,

\begin{equation}
k^2 = 18 \left( \frac{Y}{3} \right) ^6
\end{equation}

and $Y$ is the uniaxial limit stress.

For the in-plane case the yield function may be written in the form:

\begin{equation}
f^O_2 = \left[ \frac{1}{6} (a_1 + a_3) \sigma_{11}^2 - \frac{a_1}{3} \sigma_{11} \sigma_{22} + \frac{1}{6} (a_1 + a_2) \sigma_{22}^2 + a_4 \sigma_{12}^2 \right] ^3 \\
- c \left\{ \frac{1}{27} (b_1 + b_2) \sigma_{11}^3 + \frac{1}{27} (b_3 + b_4) \sigma_{22}^3 + \frac{2}{3} \sigma_{12}^3 [(b_5 - 2 b_{10}) \sigma_{11} - b_5 \sigma_{22}] \right\} = k^2.
\end{equation}

where $a_1 - a_4$ and $b_1 - b_5$ and $b_{10}$ are coefficients describing the anisotropy, $c$ is a constant and $k$ is expressed by Eq. (2.193).

As one may see, the yield function incorporates 10 anisotropy coefficients and an extra constant $c$. The 10 anisotropy coefficients and the value of $c$ can be determined from the measured uniaxial yield stresses $\sigma_\theta$ and strain ratios $r_\theta$ in 5 different orientations and $\sigma_b$, the value of the equibiaxial tensile stress. In the 3D case, the model incorporates 18 coefficients.

The yield stress in uniaxial tension along an axis at orientation $\theta$ to the rolling direction is predicted by:

\begin{equation}
\sigma_\theta = k^{1/3} \left\{ \left[ \frac{1}{6} (a_1 + a_3) \cos^4 \theta + (a_4 - a_1/3) \cos^2 \theta \sin^2 \theta + \frac{1}{6} (a_1 + a_2) \sin^4 \theta \right] ^3 \right\}^{-1/6} \\
- c \left[ \frac{1}{27} (b_1 + b_2) \cos^6 \theta + \frac{1}{27} (b_3 + b_4) \sin^6 \theta + \frac{2}{3} \left[ (b_5 - 3 b_{10}) \cos^2 \theta + (b_4 - 3 b_5) \sin^2 \theta \right] \sin^2 \theta \cos^2 \theta \right] ^{2/3}
\end{equation}
and the biaxial yield stress by:

\[
\sigma_b = k\left[\left(\frac{a_2 + a_3}{6}\right)^3 - c \left(\frac{-2b_1 + b_2 + b_3 - 2b_4}{27}\right)^2\right]^{-\frac{1}{6}}
\] (2.196)

Yielding under pure shear parallel to the orthotropic axes occurs when \(\sigma_{xy}\) is equal to

\[
\tau = k\left(a_4\right)^{-\frac{1}{2}}
\] (2.197)

In order to predict the distribution of the anisotropy coefficient \(r_\theta\), the function \(f^0_2\) defined by Eq. (2.194) should be replaced in the relationship

\[
r_\theta = -\sin^2 \theta \frac{\partial f^0}{\partial \sigma_x} - \sin 2 \theta \frac{\partial f^0}{\partial \sigma_{xy}} + \cos^2 \theta \frac{\partial f^0}{\partial \sigma_y}
\] (2.198)

Cazacu and Barlat [76] also applied the representation theorems for transverse isotropy and cubic symmetries. The general expressions of the invariants of the stress deviators in these conditions are presented in detail in [75]. The method is applied for the extension of Drucker’s isotropic yield criterion to transverse isotropy and cubic symmetries.

Aiming to develop models of the asymmetrical tension/compression behaviour specific to the alloys having a Hexagonal Closed Packed-HCP structure, Cazacu and Barlat have successfully used the representation theory of tensor functions. They have proposed an isotropic yield function in the form [77]:

\[
f = (J_2)^{3/2} - cJ_3 = \tau_3
\] (2.199)

where \(\tau_Y\) is the yield stress in pure shear and \(c\) a constant. This constant can be expressed in the terms of the uniaxial yield stresses in tension \(\sigma_T\) and compression \(\sigma_C\), respectively, as follow:

\[
c = \frac{3\sqrt{3}(\sigma_T^3 - \sigma_C^3)}{2(\sigma_T^2 + \sigma_C^2)}.
\] (2.200)

Anisotropy was introduced in the formulation using the same method presented above.

For plane stress conditions, the yield locus is:

\[
\left[\frac{1}{3} \left(\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2\right)\right]^{3/2} - \frac{c}{27}[2\sigma_1^3 + \sigma_2^3 - 3(\sigma_1 + \sigma_2)\sigma_1 \sigma_2] = \tau_3,
\] (2.201)

where \(\sigma_1\) and \(\sigma_2\) are the principal stresses.
The expressions of the anisotropic yield function and of the uniaxial yield stresses in tension and compression along an axis at orientation $\theta$ to the rolling direction are presented in [77]. The predictions of the biaxial yield stresses corresponding to the tension and compression, as well as the planar distribution of the anisotropy coefficient are also presented.

The yield function defined by Eq. (2.190) is a third-order expression. The experimental researches [78] have shown that for some HCP alloys (e.g., titanium based alloys) the yield surface is better described by fourth order functions. As a consequence, in order to describe such a behaviour, Cazacu et al. [79] proposed an isotropic yield function for which the degree of homogeneity $a$ is not fixed:

$$\Phi = |S_1| - kS_1|^a + |S_2| - kS_2|^a + |S_3| - kS_3|^a,$$  \hspace{1cm} (2.202)

where $S_1, S_2, S_3$ are the principal values of the stress deviator, $a$—an positive integer and $k$—the strength differential parameter.

In order to extend the isotropic criterion defined by Eq. (2.202) to an anisotropic formulation, the principal values of the deviatoric stress ($S_1, S_2, S_3$) are replaced by the principal values of the transformed tensor ($\Sigma_1, \Sigma_2, \Sigma_3$), obtained after applying a linear transformation. In this way, the new anisotropic yield criterion (CPB05) can be written as

$$\Phi = |\Sigma_1| - k\Sigma_1 |^a + |\Sigma_2| - k\Sigma_2|^a + |\Sigma_3| - k\Sigma_3|^a.$$ \hspace{1cm} (2.203)

The paper [79] gives a detailed presentation of the relationships used to predict the uniaxial yield stresses and the coefficients of plastic anisotropy both for tension and compression states. Additional linear transformations can be incorporated into the CPB 2005 criterion for an improved representation of the anisotropy.

The most important advantage of this yield criterion consists in its capability to provide an accurate description of the tension/compression behaviour specific to the magnesium and titanium alloys.

### 2.4.4 Vegter Yield Criterion

Using points of the yield locus which had been determined directly by experiments, Vegter [80, 81] obtained the yield locus in the first quadrant by applying a Bezier interpolation. The Vegter criterion requires the determination of three parameters for each reference point (two principal stresses $\sigma_1$ and $\sigma_2$ and the strain vector $\rho = de_2/de_1$). In order to describe planar anisotropy Vegter’s criterion needs as many as 17 parameters.

The analytical expression of the criterion is

$$\left(\begin{array}{c}
\sigma_1 \\
\sigma_2
\end{array}\right) = (1 - \lambda)^2 \left(\begin{array}{c}
\sigma_1 \\
\sigma_2
\end{array}\right)_i + 2\lambda (1 - \lambda) \left(\begin{array}{c}
\sigma_1 \\
\sigma_2
\end{array}\right)_i^h + \lambda^2 \left(\begin{array}{c}
\sigma_1 \\
\sigma_2
\end{array}\right)_i^{r+1}$$ \hspace{1cm} (2.204)
for $\sigma_e$ and angle $\varphi$ where

$$
\left(\frac{\sigma_1}{\sigma_2}\right)^{r_{i+1}} = \sum_{j=0}^{m} \left(\frac{a_j^{r_i}}{a_j^{r_i}}\right) \cos (2j\varphi)
$$

(2.205)

is a trigonometric expansion associated to the reference point;

$$
R(\varphi) = \sum_{j=0}^{m} b^j \cos (2j\varphi)
$$

(2.206)

is cosine interpolation of the function $R(\varphi)$; $\varphi$ is the angle between the principal directions and the orthotropic axes; $\lambda$ is a parameter of the Bézier function; $r$ is a superscript denoting the reference point; $h$ is a superscript denoting the breaking point; $\left(\frac{a_j^{r_i}}{a_j^{r_i}}\right)$ are parameters of the trigonometric interpolation to be determined at the reference points; $b^j$ are parameters of the trigonometric interpolation of the $R$-function.

The most important advantage of the criterion is the flexibility ensured by the large number of parameters. Disadvantages are the unfriendly form of the yield function making it improper for analytical computation; the large number of experiments required (uniaxial tension, biaxial tension, plane strain and pure shearing) and the necessity of mathematical abilities of the user.

The Vegter’s model has been implemented in the PAMSTAMP FE commercial program.

Mollica and Srinivasa [82] proposed a method for generating the yield locus similar to the one presented above. A simple way to obtain a closed convex surface is to consider the intersection of a sufficient number of elementary convex surfaces. Each elementary surface is defined by an equation having the form $f_i = 0$. In order to avoid the sharp corners and edges a special regularization procedure is proposed. The method is illustrated in [82] for Hill 1948 [25] and Hosford [31] criteria.

### 2.4.5 Polynomial Yield Criteria

Hill [65] proposed in 1950 a general formulation of a plane-stress anisotropic yield criterion having the polynomial expression defined by Eq. (2.159). Gotoh [64] succeeded to apply that idea in the 1970s by developing a polynomial yield function of fourth degree (see Sect. 2.3.4).

During the last years, a new family of polynomial yield criteria has been created on the basis of Hill’s idea.
2.4.5.1 Hu Yield Criteria

Hu [83] proposed a yield criterion in the form

\[
f(\sigma_{ij}) = \frac{1}{\sigma_0^4} \sigma_{11}^4 - \frac{4R_0}{(1+R_0)\sigma_0^4} \sigma_{11}^3 \sigma_{22} + \left[ \frac{1}{\sigma_0^4} - \frac{1}{\sigma_{90}^4} \right] + \frac{4R_0}{(1+R_0)\sigma_{90}^4} + \right.

\[
+ \frac{4R_0}{(1+R_0)\sigma_0^4} \sigma_{11}^2 \sigma_{22}^2 - \frac{4R_0}{(1+R_0)\sigma_{90}^4} \sigma_{11} \sigma_{22}^3 + \frac{1}{\sigma_{90}^4} \sigma_{44}^2 + \left.

\right] 16 \left( \frac{1}{(1+R_{45})\sigma_{45}^4} - \frac{2}{\sigma_0^4} \right) \left( \sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11} \sigma_{22} \right) \sigma_{12}^2 + \left[ \frac{1}{\sigma_0^4} + \frac{16R_{45}}{(1+R_{45})\sigma_{45}^4} \right] \sigma_{12}^4 = 1.
\]

(2.207)

The planar distributions of the uniaxial yield stress and of the anisotropy coefficient are presented in [83]. Hu also succeeded to develop a 3D extension of his criterion [84]. A quadratic formulation of the yield function has been presented by Hu in [85].

2.4.5.2 Wang Yield Criterion

Wang (2005, Constitutive Modeling of Orthotropic Plasticity in Sheet Metals, private communication) proposed a 2D polynomial formulation of the yield function as:

\[
\sigma_{11}^{m+1} - F \sigma_{11}^m \sigma_{22} - \frac{G \beta}{\alpha} \sigma_{11} + \frac{\beta}{\alpha} \sigma_{22}^{m+1} + \frac{\gamma}{\alpha} \sigma_{12}^{m+1} = \sigma_Y^{m+1}
\]

(2.208)

where \( m \) is an odd integer 3, 5 or 7 (depending on the crystallographic structure), \( F, G, \beta/\alpha \) and \( \gamma/\alpha \) are anisotropic coefficients and \( \sigma_Y \) is the uniaxial yield stress in the rolling direction.

2.4.5.3 Comsa Yield Criterion

Comsa also developed in his PhD thesis [86] a polynomial criterion of sixth order:

\[
\Phi = \left[ p_1(L + M)(cL + M)(L + cM) + p_2(L + N)(cL + N)(L + cN) \right]^{1/6} - Y,
\]

(2.209)

where,

\[
L = (p_3\sigma_{11} - p_4\sigma_{22})^2 + \sigma_{12}\sigma_{21},
M = (p_5\sigma_{11} - p_6\sigma_{22})^2 + \sigma_{12}\sigma_{21},
N = (p_7\sigma_{11} - p_8\sigma_{22})^2.
\]

(2.210)

and \( c = (2 + \sqrt{3})^2 \), \( p_1 - p_8 (p_1 - p_8 > 0) \) are the material parameters and \( Y \) is the yield parameter. As one may see in the above relationship, the yield function incorporates
8 coefficients. These coefficients can be evaluated by using three uniaxial tensile tests and a biaxial tensile experiment.

The convexity of the yield surface is proved in [86]. The relationships used to evaluate the uniaxial/biaxial yield stress and the uniaxial/biaxial coefficients of plastic anisotropy are also presented.

The predictions of the model have been tested by comparison with experimental data for several types of materials (steel and aluminium alloys). The yield criterion has been also implemented by the author in the LS-DYNA programme and used for the numerical simulation of various sheet metal forming processes (deep-drawing and bending) [87].

### 2.4.5.4 Soare Yield Criteria

Soare [88] proposed three yield criteria expressed by polynomial functions of 4th, 6th and 8th order, respectively (Poly 4, 6 and 8).

The yield function is defined as

$$f(\sigma_{11}, \sigma_{22}, \sigma_{12}) := [P_n(\sigma_{11}, \sigma_{22}, \sigma_{12})]^{1/n},$$  

(2.211)

where, $P_n$ is the polynomial function and $n$—the order of the polynomial function.

The form of the orthotropic fourth order polynomial (Poly 4) is:

$$P_4 = a_1 \sigma_{11}^4 + a_2 \sigma_{11}^3 \sigma_{22} + a_3 \sigma_{11}^2 \sigma_{22}^2 + a_4 \sigma_{11} \sigma_{22}^3 + a_5 \sigma_{22}^4 + (a_6 \sigma_{11}^2 + a_7 \sigma_{11} \sigma_{22} + a_8 \sigma_{22}^2) \sigma_{12} + a_9 \sigma_{12}^4$$  

(2.212)

where $a_1 - a_9$ are coefficients describing the anisotropy.

The formulation proposed by Soare for the fourth order yield function differs from Gotoh’s model [64] only in one aspect: elimination of the first term in Eq. (2.160). Soare has paid a special attention to the convexity analysis. He has proposed an original identification strategy which is able to remove the large overall error affecting Gotoh’s procedure.

The 6th and 8th order polynomial functions developed by Soare are

$$P_6 = a_1 \sigma_{11}^6 + a_2 \sigma_{11}^5 \sigma_{22} + a_3 \sigma_{11}^4 \sigma_{22}^2 + a_4 \sigma_{11}^3 \sigma_{22}^3 + a_5 \sigma_{11}^2 \sigma_{22}^4 + a_6 \sigma_{11} \sigma_{22}^5 + a_7 \sigma_{22}^6 + (a_8 \sigma_{11}^4 + a_9 \sigma_{11}^3 \sigma_{22} + a_{10} \sigma_{11}^2 \sigma_{22}^2 + a_{11} \sigma_{11} \sigma_{22}^3 + a_{12} \sigma_{22}^4) \sigma_{12} + (a_{13} \sigma_{11}^2 + a_{14} \sigma_{11} \sigma_{22} + a_{15} \sigma_{22}^2) \sigma_{12}^2 + a_{16} \sigma_{12}^4$$  

(2.213)

respectively,

$$P_8 = a_1 \sigma_{11}^8 + a_2 \sigma_{11}^7 \sigma_{22} + a_3 \sigma_{11}^6 \sigma_{22}^2 + a_4 \sigma_{11}^5 \sigma_{22}^3 + a_5 \sigma_{11}^4 \sigma_{22}^4 + a_6 \sigma_{11}^3 \sigma_{22}^5 + a_7 \sigma_{11}^2 \sigma_{22}^6 + a_8 \sigma_{11} \sigma_{22}^7 + a_9 \sigma_{22}^8 + (a_{10} \sigma_{11}^6 + a_{11} \sigma_{11}^5 \sigma_{22} + a_{12} \sigma_{11}^4 \sigma_{22}^2 + a_{13} \sigma_{11}^3 \sigma_{22}^3 + a_{14} \sigma_{11}^2 \sigma_{22}^4 + a_{15} \sigma_{11} \sigma_{22}^5 + a_{16} \sigma_{22}^6) \sigma_{12} + (a_{17} \sigma_{11}^4 + a_{18} \sigma_{11}^3 \sigma_{22} + a_{19} \sigma_{11}^2 \sigma_{22}^2 + a_{20} \sigma_{11} \sigma_{22}^3 + a_{21} \sigma_{22}) \sigma_{12} + (a_{22} \sigma_{11} + a_{23} \sigma_{11} \sigma_{22} + a_{24} \sigma_{22}^2) \sigma_{12}^2 + a_{25} \sigma_{12}^4$$  

(2.214)
As one may notice, Poly 6 and 8 have 16 and 25 coefficients. The procedure used for evaluating them is based on the minimisation of an error-function. Due to the large number of coefficients, Poly 6 and 8 allow a better description of the plastic behaviour, even in the case of materials exhibiting a pronounced variation of the anisotropy characteristics. By implementing them in finite-element codes, the author has proved the ability of the new models to capture the occurrence of 6 or 8 ears in the deep-drawing process of cylindrical cups. The strength-differential effect into the yield surface has been also introduced in these formulations [89].

The most important advantages of these yield criteria are as follows:

- Simplicity of the formulation
- Direct formulation (use of the actual stress components)
- Flexibility ensured by the large number of coefficients (Poly 6 and 8)
- Easy extension to 3D stress states (see [88]).

The are also some disadvantages that should be mentioned:

- Not all the formulations are convex. Due to this fact, the variation range of some coefficients must be bounded
- The identification procedure is quite complex, especially for the Poly 6 and 8 models.

A quadratic yield model to describe the orthotropic behaviour of the sheet metals has been proposed by Oller et al. [90]. It deals with the case in which the yield stress in simple tension is different from the one in compression.

### 2.5 BBC 2005 Yield Criterion

#### 2.5.1 Equation of the Yield Surface

The sheet metal is assumed to behave as a plastically orthotropic membrane under plane stress conditions. By making this assumption, we can use the following description of the yield surface:

\[
\Phi \left( \sigma_{\alpha\beta}, Y \right) := \overline{\sigma} \left( \sigma_{\alpha\beta} \right) - Y = 0 \quad (2.215)
\]

where \( \overline{\sigma} \left( \sigma_{\alpha\beta} \right) > 0 \) is the BBC 2005 equivalent stress (see Sect. 2.5.3), \( Y > 0 \) is a yield parameter, and \( \sigma_{\alpha\beta} = \sigma_{\beta\alpha} \ (\alpha, \beta = 1, 2) \) are planar components of the stress tensor expressed in an orthonormal basis superimposed to the axes of plastic orthotropy: (1) rolling direction (RD), (2) transverse direction (TD), (3) normal direction (ND). The other components are subjected to the restriction

\[
\sigma_{3i} = \sigma_{i3} = 0, \ (i = 1, 2, 3) \quad (2.216)
\]
arising from the plane stress hypothesis. Whenever not clearly specified, we shall use the following convention: Greek indices take the values 1 and 2, while the Latin ones take the values 1, 2 and 3.

The BBC 2005 yield criterion does not enforce some special constraints on the choice of the yield parameter ($Y$). In fact, any quantity representing a yield stress can act as $Y$. For example, $Y$ may be the uniaxial yield stress $Y_\theta$ associated to a direction defined by the angle $\theta$ measured from RD, an average of several uniaxial yield stresses, or the biaxial yield stress $Y_b$ associated to RD and TD.

### 2.5.2 Flow Rule Associated to the Yield Surface

The flow rule associated to the yield surface described by Eq. (2.215) is

$$\sigma_{3i} = \sigma_{i3} = 0, \quad (i = 1, 2, 3) \quad (2.216)$$

$$\dot{\varepsilon}_{\alpha\beta}^{p} = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma_{\alpha\beta}}, \quad \alpha, \beta = 1, 2 \quad (2.217)$$

where $\dot{\varepsilon}_{\alpha\beta}^{p} = \dot{\varepsilon}_{\alpha\beta}^{p}$ ($\alpha, \beta = 1, 2$) are planar components of the plastic strain-rate tensor (expressed in the same basis as the corresponding components of the stress tensor), and $\dot{\lambda} \geq 0$ is a scalar multiplier (its significance is not essential for our discussion). The out of plane components of the plastic strain-rate tensor are subjected to the restrictions

$$\dot{\varepsilon}_{3\alpha}^{p} = \dot{\varepsilon}_{\alpha3}^{p} = 0, \quad \alpha = 1, 2 \quad (2.218)$$

$$\dot{\varepsilon}_{33}^{p} = - \dot{\varepsilon}_{11}^{p} - \dot{\varepsilon}_{22}^{p} \quad (2.219)$$

arising from the plane stress hypothesis and the isochoric character of the plastic deformation.

When using Eq. (2.217) we need the partial derivatives of the function $\Phi$ with respect to the planar components of the stress tensor. Equation (2.215) allows us to calculate them as partial derivatives of the equivalent stress:

$$\frac{\partial \Phi}{\partial \sigma_{\alpha\beta}} = \frac{\partial \sigma}{\partial \sigma_{\alpha\beta}}, \quad \alpha, \beta = 1, 2 \quad (2.220)$$

### 2.5.3 BBC 2005 Equivalent Stress

The equivalent stress used in Eq. (2.215) is defined by the following formula:

$$\bar{\sigma} = \left[ a (\Lambda + \Gamma)^{2k} + a (\Lambda - \Gamma)^{2k} + b (\Lambda + \Psi)^{2k} + b (\Lambda - \Psi)^{2k} \right]^\frac{1}{2k} \quad (2.221)$$
where \( k \in \mathbb{N} \geq 1 \) and \( a, b > 0 \) are material parameters, while \( \Gamma, \Lambda \) and \( \Psi \) are functions depending on the planar components of the stress tensor:

\[
\begin{align*}
\Gamma &= L \sigma_{11} + M \sigma_{22} \\
\Lambda &= \sqrt{(N \sigma_{11} - P \sigma_{22})^2 + \sigma_{12} \sigma_{21}} \\
\Psi &= \sqrt{(Q \sigma_{11} - R \sigma_{22})^2 + \sigma_{12} \sigma_{21}}
\end{align*}
\] (2.222)

The coefficients \( L, M, N, P, Q, \) and \( R \) involved in Eqs. (2.222) are also material parameters.

Despite the fact that Eqs. (2.221) and (2.222) do not enforce any constraint on the sign of the coefficients \( L, M, N, P, Q, \) and \( R \), the numerical tests performed by the authors have shown that positive values of these parameters lead to better predictions of the BBC 2005 yield criterion.

The conditions \( k \in \mathbb{N} \geq 1 \) and \( a, b > 0 \) ensure the convexity of the yield surface defined by Eqs. (2.215), (2.221) and (2.222). The parameters \( L, M, N, P, Q, \) and \( R \) are not subjected to any constraint from this point of view.

Nine material parameters are involved in the expression of the BBC 2005 equivalent stress: \( k, a, b, L, M, N, P, Q, \) and \( R \) (see Eqs. 2.221 and 2.222). The integer exponent \( k \) has a special status, due to the fact that its value is fixed from the very beginning in accordance with the crystallographic structure of the material: \( k = 3 \) for BCC materials, \( k = 4 \) for FCC materials.

The identification procedure calculates the other parameters \( (a, b, L, M, N, P, Q, \) and \( R) \) by forcing the constitutive equations associated to the BBC 2005 yield criterion to reproduce the following experimental data:

- The uniaxial yield stresses associated to the directions defined by \( 0^\circ, 45^\circ \) and \( 90^\circ \) angles measured from RD (denoted as \( Y_0, Y_{45} \) and \( Y_{90} \))
- The coefficients of uniaxial plastic anisotropy associated to the directions defined by \( 0^\circ, 45^\circ \) and \( 90^\circ \) angles measured from RD (denoted as \( r_0, r_{45} \) and \( r_{90} \))
- The biaxial yield stress associated to RD and TD (denoted as \( Y_b \))
- The coefficient of biaxial plastic anisotropy associated to RD and TD (denoted as \( r_b \)).

There are 8 constraints acting on 8 material parameters. The identification procedure has enough data to generate a set of equations having \( a, b, L, M, N, P, Q, \) and \( R \) as unknowns. The structure of this set of equations, as well as the solution strategy will be presented in Sect. 2.5.4.

When using the flow rule given by Eq. (2.217), we need the partial derivatives of the function \( \phi \) with respect to the planar components of the stress tensor. Equations (2.220), (2.221) and (2.222) lead to the following formula:

\[
\frac{\partial \Phi}{\partial \sigma_{\alpha\beta}} = \frac{\partial \sigma}{\partial \Gamma} \frac{\partial \Gamma}{\partial \sigma_{\alpha\beta}} + \frac{\partial \sigma}{\partial \Lambda} \frac{\partial \Lambda}{\partial \sigma_{\alpha\beta}} + \frac{\partial \sigma}{\partial \Psi} \frac{\partial \Psi}{\partial \sigma_{\alpha\beta}}, \quad \alpha, \beta = 1, 2
\] (2.223)
where
\[
\frac{\partial \sigma}{\partial \Gamma} = \frac{a}{\sigma^{2k-1}} \left[ (\Lambda + \Gamma)^{2k-1} - (\Lambda - \Gamma)^{2k-1} \right]
\]
\[
\frac{\partial \Lambda}{\partial \sigma} = \frac{1}{\sigma^{2k-1}} \left\{ a \left[ (\Lambda + \Gamma)^{2k-1} + (\Lambda - \Gamma)^{2k-1} \right] + b \left[ (\Lambda + \Psi)^{2k-1} + (\Lambda - \Psi)^{2k-1} \right] \right\}
\]
\[
\frac{\partial \Psi}{\partial \sigma} = \frac{b}{\sigma^{2k-1}} \left[ (\Lambda + \Psi)^{2k-1} - (\Lambda - \Psi)^{2k-1} \right]
\]

(2.224)

and
\[
\frac{\partial \Gamma}{\partial \sigma_{11}} = L, \quad \frac{\partial \Gamma}{\partial \sigma_{22}} = M, \quad \frac{\partial \Gamma}{\partial \sigma_{12}} = 0, \quad \frac{\partial \Gamma}{\partial \sigma_{21}} = 0,
\]
\[
\frac{\partial \Lambda}{\partial \sigma_{11}} = \frac{N (\sigma_{11} - P \sigma_{22})}{\Lambda}, \quad \frac{\partial \Lambda}{\partial \sigma_{22}} = - \frac{P (\sigma_{11} - P \sigma_{22})}{\Lambda},
\]
\[
\frac{\partial \Lambda}{\partial \sigma_{12}} = \frac{\sigma_{21}}{2\Lambda}, \quad \frac{\partial \Lambda}{\partial \sigma_{21}} = \frac{\sigma_{12}}{2\Lambda}, \quad (2.225)
\]
\[
\frac{\partial \Psi}{\partial \sigma_{11}} = \frac{Q (\sigma_{11} - R \sigma_{22})}{\Psi}, \quad \frac{\partial \Psi}{\partial \sigma_{22}} = - \frac{R (\sigma_{11} - R \sigma_{22})}{\Psi},
\]
\[
\frac{\partial \Psi}{\partial \sigma_{12}} = \frac{\sigma_{21}}{2\Psi}, \quad \frac{\partial \Psi}{\partial \sigma_{21}} = \frac{\sigma_{12}}{2\Psi}.
\]

Equations (2.221), (2.222), (2.223), (2.224) and (2.225) allow us to express the flow rule given by Eq. (2.217) as a dependency of the stress components \( \sigma_{\alpha\beta} \) \((\alpha, \beta = 1, 2)\).

### 2.5.4 Identification Procedure

As mentioned in Sect. 2.5.3, the parameters \( a, b, L, M, N, P, Q \), and \( R \) are obtained by constraining the constitutive equations associated to the BBC 2005 yield criterion to reproduce the following experimental data: \( Y_0 \), \( Y_{45} \), \( Y_{90} \), \( r_0 \), \( r_{45} \), \( r_{90} \), \( Y_b \), and \( r_b \). In fact, the identification procedure will solve the following set of 8 equations considering \( a, b, L, M, N, P, Q \), and \( R \) as unknowns:

\[
\begin{align*}
\tilde{Y}_0 &= Y_0, \quad \tilde{Y}_{45} = Y_{45}, \quad \tilde{Y}_{90} = Y_{90} \\
\tilde{r}_0 &= r_0, \quad \tilde{r}_{45} = r_{45}, \quad \tilde{r}_{90} = r_{90} \\
\tilde{Y}_b &= Y_b, \quad \tilde{r}_b = r_b
\end{align*}
\]

(2.226)

where:

- \( \tilde{Y}_0 \), \( \tilde{Y}_{45} \) and \( \tilde{Y}_{90} \) are the theoretical yield stresses corresponding to pure tension along the directions defined by 0°, 45° and 90° angles measured from RD.
- \( \tilde{r}_0 \), \( \tilde{r}_{45} \) and \( \tilde{r}_{90} \) are the theoretical coefficients of uniaxial plastic anisotropy associated to the directions mentioned above.
• \( \tilde{Y}_b \) is the theoretical yield stress corresponding to biaxial tension along RD and TD
• \( \tilde{r}_b \) is the theoretical coefficient of biaxial plastic anisotropy associated to RD and TD.

It is obvious that the identification procedure needs formulas for evaluating \( \tilde{Y}_0 \), \( \tilde{Y}_{45} \), \( \tilde{Y}_{90} \), \( \tilde{r}_0 \), \( \tilde{r}_{45} \), \( \tilde{r}_{90} \), \( \tilde{Y}_b \), and \( \tilde{r}_b \). These formulas will be presented below.

### 2.5.4.1 Theoretical Yield Stress in Pure Tension

Let \( \tilde{Y}_\theta \) be the theoretical yield stress corresponding to pure tension along a direction defined by the angle \( \theta \) measured from RD. The planar components of the stress tensor are

\[
\sigma_{11} = \tilde{Y}_\theta \cos^2 \theta, \quad \sigma_{22} = \tilde{Y}_\theta \sin^2 \theta, \quad \sigma_{12} = \sigma_{21} = \tilde{Y}_\theta \sin \theta \cos \theta
\]

The quantities \( \Gamma \), \( \Lambda \) and \( \Psi \) defined by Eqs. (2.222) become

\[
\Gamma = \tilde{Y}_\theta \Gamma_\theta, \quad \Lambda = \tilde{Y}_\theta \Lambda_\theta, \quad \Psi = \tilde{Y}_\theta \Psi_\theta
\]

where

\[
\Gamma_\theta = L \cos^2 \theta + M \sin^2 \theta
\]

\[
\Lambda_\theta = \sqrt{(N \cos^2 \theta - P \sin^2 \theta)^2 + \sin^2 \theta \cos^2 \theta}
\]

\[
\Psi_\theta = \sqrt{(Q \cos^2 \theta - R \sin^2 \theta)^2 + \sin^2 \theta \cos^2 \theta}
\]

Equations (2.221) and (2.228) lead to the following expression of the equivalent stress when pure tension is applied along the \( \theta \) direction:

\[
\tilde{\sigma} |_{\theta} = \tilde{Y}_\theta \cdot F(\theta)
\]

where

\[
F(\theta) = \left[ a (\Lambda_\theta + \Gamma_\theta)^{2k} + a (\Lambda_\theta - \Gamma_\theta)^{2k} + b (\Lambda_\theta + \Psi_\theta)^{2k} + b (\Lambda_\theta - \Psi_\theta)^{2k} \right]^{\frac{1}{2k}}
\]

\(\tilde{\sigma} |_{\theta} \) given by Eq. (2.230) should be replaced in Eq. (2.215). We thus obtain the desired formula of the theoretical yield stress \( \tilde{Y}_\theta \):

\[
\tilde{Y}_\theta = \frac{Y}{F(\theta)}
\]

\( \tilde{Y}_0 \), \( \tilde{Y}_{45} \) and \( \tilde{Y}_{90} \) can be calculated from Eqs. (2.232) and (2.231) using \( \theta = 0^\circ \), \( 45^\circ \) and \( 90^\circ \), respectively.
2.5.4.2 Theoretical Coefficient of Uniaxial Plastic Anisotropy

The theoretical coefficient of uniaxial plastic anisotropy associated to a direction inclined at the angle $\theta$ measured from RD is defined as follows:

$$\tilde{r}_\theta = \frac{\dot{\varepsilon}^p_{\theta + 90^\circ}}{\dot{\varepsilon}^p_{DN}}$$ (2.233)

where: $\dot{\varepsilon}^p_{\theta + 90^\circ}$ is the plastic strain-rate component associated to the direction defined by the angle $\theta + 90^\circ$, and $\dot{\varepsilon}^p_{DN}$ is the component of the same tensor associated to ND. After using the condition of plastic incompressibility

$$\dot{\varepsilon}^p_{\theta} + \dot{\varepsilon}^p_{\theta + 90^\circ} + \dot{\varepsilon}^p_{DN} = 0$$ (2.234)

Equation (2.233) becomes

$$\tilde{r}_\theta = -\frac{\dot{\varepsilon}^p_{\theta}}{\dot{\varepsilon}^p_{DN}} - 1$$ (2.235)

The symbol $\dot{\varepsilon}^p_{\theta}$ denotes the plastic strain-rate component associated to the $\theta$ direction. $\dot{\varepsilon}^p_{\theta}$ and $\dot{\varepsilon}^p_{DN}$ may be rewritten using the components of the plastic strain-rate tensor expressed in the orthotropy basis:

$$\dot{\varepsilon}^p_{\theta} = \dot{\varepsilon}^p_{11} \cos^2 \theta + \dot{\varepsilon}^p_{22} \sin^2 \theta + (\dot{\varepsilon}^p_{12} + \dot{\varepsilon}^p_{21}) \sin \theta \cos \theta$$
$$\dot{\varepsilon}^p_{DN} = \dot{\varepsilon}^p_{33} = -\dot{\varepsilon}^p_{11} - \dot{\varepsilon}^p_{22}$$ (2.236)

We can replace now $\dot{\varepsilon}^p_{\theta}$ and $\dot{\varepsilon}^p_{DN}$ given by Eqs. (2.236) into Eq. (2.235):

$$\tilde{r}_\theta = \frac{\dot{\varepsilon}^p_{11} \cos^2 \theta + \dot{\varepsilon}^p_{22} \sin^2 \theta + (\dot{\varepsilon}^p_{12} + \dot{\varepsilon}^p_{21}) \sin \theta \cos \theta}{\dot{\varepsilon}^p_{11} + \dot{\varepsilon}^p_{22}} - 1$$ (2.237)

The right-hand side of Eq. (2.237) should be expressed in terms of the planar stress components. This transformation is achieved using the flow rule (see Eqs. 2.217 and 2.220), as well as Eqs. (2.227) (they are valid because $\tilde{r}_\theta$ is defined for a uniaxial stress state):

$$\tilde{r}_\theta = \frac{1}{\tilde{Y}_{\theta}} \left( \frac{\sigma_{\alpha \beta} \frac{\partial \bar{\sigma}}{\partial \sigma_{\alpha \beta}}}{\frac{\partial \bar{\sigma}}{\partial \sigma_{11}} + \frac{\partial \bar{\sigma}}{\partial \sigma_{22}}} \right)_{\theta} - 1$$ (2.238)

The notation $(\cdot)|_{\theta}$ means that the expression enclosed by parentheses should be calculated for pure tension along the $\theta$ direction. The summation rule for tensor components has been used in Eq. (2.238).

The equivalent stress defined by Eqs. (2.221) and (2.222) is a homogeneous function of the stress components $\sigma_{\alpha \beta}$ ($\alpha, \beta = 1, 2$), its degree of homogeneity being one. Thus we can use Euler’s theorem:
\[ \bar{\sigma} = \sigma_{\alpha\beta} \frac{\partial \tilde{\sigma}}{\partial \sigma_{\alpha\beta}} \]  \hspace{1cm} (2.239)

Equations (2.238), (2.239) and (2.230) lead to the following formula for \( \tilde{r}_\theta \):

\[ \tilde{r}_\theta = \frac{F(\theta)}{\left( \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}} + \frac{\partial \tilde{\sigma}}{\partial \sigma_{22}} \right)_{\theta}} - 1 \]  \hspace{1cm} (2.240)

We shall express now \( \left( \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}} + \frac{\partial \tilde{\sigma}}{\partial \sigma_{22}} \right)_{\theta} \) as a dependency of the \( \theta \) angle. We start by rewriting Eq. (2.223) both for \( \alpha = \beta = 1 \) and \( \alpha = \beta = 2 \), assuming a uniaxial stress state along the \( \theta \) direction. We have two relationships that can be added, thus obtaining

\[
\left( \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}} + \frac{\partial \tilde{\sigma}}{\partial \sigma_{22}} \right)_{\theta} = \left( \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}} + \frac{\partial \tilde{\sigma}}{\partial \sigma_{22}} \right)_{\theta} + \left( \frac{\partial \tilde{\sigma}}{\partial \Lambda} \right)_{\theta} \left( \frac{\partial \Lambda}{\partial \sigma_{11}} + \frac{\partial \Lambda}{\partial \sigma_{22}} \right)_{\theta} + \left( \frac{\partial \tilde{\sigma}}{\partial \Psi} \right)_{\theta} \left( \frac{\partial \Psi}{\partial \sigma_{11}} + \frac{\partial \Psi}{\partial \sigma_{22}} \right)_{\theta} \]  \hspace{1cm} (2.241)

Equations (2.224), (2.228) and (2.230) allows us to express the derivatives \( \frac{\partial \tilde{\sigma}}{\partial \Lambda} |_{\theta} \) and \( \frac{\partial \tilde{\sigma}}{\partial \Psi} |_{\theta} \) as functions of the \( \theta \) angle:

\[
\frac{\partial \tilde{\sigma}}{\partial \Lambda} |_{\theta} = \frac{a}{[F(\theta)]^{2k-1}} \left[ (\Lambda_{\theta} + \Gamma_{\theta})^{2k-1} - (\Lambda_{\theta} - \Gamma_{\theta})^{2k-1} \right] + b \left[ (\Lambda_{\theta} + \Psi_{\theta})^{2k-1} - (\Lambda_{\theta} - \Psi_{\theta})^{2k-1} \right] \]  \hspace{1cm} (2.242)

where \( \Gamma_{\theta}, \Lambda_{\theta} \) and \( \Psi_{\theta} \) are defined by Eqs. (2.229). The other derivatives appearing in the right-hand side of Eq. (2.241) can be also expressed as functions of the \( \theta \) angle (see Eqs. 2.225, 2.227, and 2.228):

\[
\left( \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}} + \frac{\partial \tilde{\sigma}}{\partial \sigma_{22}} \right)_{\theta} = L + M, \]
\[
\left( \frac{\partial \Lambda}{\partial \sigma_{11}} + \frac{\partial \Lambda}{\partial \sigma_{22}} \right)_{\theta} = \frac{(N-P)(N \cos^2 \theta - P \sin^2 \theta)}{\Lambda_{\theta}}, \]  \hspace{1cm} (2.243)
\[
\left( \frac{\partial \Psi}{\partial \sigma_{11}} + \frac{\partial \Psi}{\partial \sigma_{22}} \right)_{\theta} = \frac{(Q-R)(Q \cos^2 \theta - R \sin^2 \theta)}{\Psi_{\theta}}. \]

After replacing the quantities given by Eqs. (2.242) and (2.243) into Eq. (2.241) and making some rearrangements, we get the following relationship for \( \left( \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}} + \frac{\partial \tilde{\sigma}}{\partial \sigma_{22}} \right)_{\theta} \):
\[
\left( \frac{\partial \bar{\sigma}}{\partial \sigma_{11}} + \frac{\partial \bar{\sigma}}{\partial \sigma_{22}} \right) \bigg|_\theta = \frac{G(\theta)}{[F(\theta)]^{2k-1}} \tag{2.244}
\]

where

\[
G(\theta) = a \left[ \frac{(N-P)(N \cos^2 \theta - P \sin^2 \theta)}{\Lambda_\theta} + L + M \right] (\Lambda_\theta + \Gamma_\theta)^{2k-1} +
\]

\[
a \left[ \frac{(N-P)(N \cos^2 \theta - P \sin^2 \theta)}{\Lambda_\theta} - L - M \right] (\Lambda_\theta - \Gamma_\theta)^{2k-1} +
\]

\[
b \left[ \frac{(N-P)(N \cos^2 \theta - P \sin^2 \theta)}{\Lambda_\theta} + \frac{(Q-R)(Q \cos^2 \theta - R \sin^2 \theta)}{\Psi_\theta} \right] (\Lambda_\theta + \Psi_\theta)^{2k-1} +
\]

\[
b \left[ \frac{(N-P)(N \cos^2 \theta - P \sin^2 \theta)}{\Lambda_\theta} - \frac{(Q-R)(Q \cos^2 \theta - R \sin^2 \theta)}{\Psi_\theta} \right] (\Lambda_\theta - \Psi_\theta)^{2k-1} \tag{2.245}
\]

We can now combine Eqs. (2.240) and (2.244) to obtain a formula for evaluating the coefficient of uniaxial plastic anisotropy:

\[
\tilde{r}_\theta = \frac{[F(\theta)]^{2k}}{G(\theta)} - 1 \tag{2.246}
\]

\(\tilde{r}_0, \tilde{r}_{45}, \text{ and } \tilde{r}_{90}\) can be calculated from Eqs. (2.246), (2.245) and (2.231) using \(\theta = 0^\circ, 45^\circ, \text{ and } 90^\circ\), respectively.

### 2.5.4.3 Theoretical Yield Stress in Biaxial Tension Along RD and TD

Let \(\tilde{Y}_b\) be the theoretical yield stress corresponding to biaxial tension along RD and TD. The planar components of the stress tensor are

\[
\sigma_{11} = \tilde{Y}_b, \quad \sigma_{22} = \tilde{Y}_b, \quad \sigma_{12} = \sigma_{21} = 0 \tag{2.247}
\]

The quantities \(\Gamma, \Psi, \text{ and } \Lambda\) defined by Eqs. (2.222) become

\[
\Gamma = \tilde{Y}_b \Gamma_b, \quad \Lambda = \tilde{Y}_b \Lambda_b, \quad \Psi = \tilde{Y}_b \Psi_b \tag{2.248}
\]

where

\[
\Gamma_b = L + M, \\
\Lambda_b = \sqrt{(N - P)^2} = |N - P|, \tag{2.249}
\]

\[
\Psi_b = \sqrt{(Q - R)^2} = |Q - R|
\]

Equations (2.221) and (2.248) lead to the following expression of the equivalent stress when biaxial tension is applied along RD and TD:
\( \bar{\sigma} | _b = \bar{Y}_b \cdot F_b \) \hspace{1cm} (2.250)

where

\[
F_b = \left[ a (\Lambda_b + \Gamma_b)^2 k + a (\Lambda_b - \Gamma_b)^2 k + b (\Lambda_b + \Psi_b)^2 k + b (\Lambda_b - \Psi_b)^2 k \right]^{\frac{1}{2}}
\]

(2.251)

\( \bar{\sigma} | _b \) given by Eq. (2.250) should be replaced in Eq. (2.215). We thus obtain the desired formula of the theoretical yield stress \( \bar{Y}_b \):

\[
\bar{Y}_b = \frac{Y}{F_b}
\]

(2.252)

\( \bar{Y}_b \) can be calculated from Eqs. (2.252) and (2.251).

### 2.5.4.4 Theoretical Coefficient of Biaxial Plastic Anisotropy

The theoretical coefficient of biaxial plastic anisotropy associated to RD and TD is defined as follows:

\[
\tilde{\tau}_b = \frac{\dot{\varepsilon}_p^{TD}}{\dot{\varepsilon}_p^{RD}}
\]

(2.253)

where \( \dot{\varepsilon}_p^{RD} \) and \( \dot{\varepsilon}_p^{TD} \) are the components of the plastic strain-rate tensor corresponding to RD and TD, respectively. The choice of the orthonormal basis allows us to write the equalities

\[
\dot{\varepsilon}_p^{RD} = \dot{\varepsilon}_p^{11}, \quad \dot{\varepsilon}_p^{TD} = \dot{\varepsilon}_p^{22}
\]

(2.254)

We can replace now \( \dot{\varepsilon}_p^{RD} \) and \( \dot{\varepsilon}_p^{TD} \) given by Eqs. (2.254) into Eq. (2.253):

\[
\tilde{\tau}_b = \frac{\dot{\varepsilon}_p^{22}}{\dot{\varepsilon}_p^{11}} = \frac{\dot{\varepsilon}_p^{11} + \dot{\varepsilon}_p^{22}}{\dot{\varepsilon}_p^{11}} - 1
\]

(2.255)

The right-hand side of Eq. (2.255) should be expressed in terms of the planar stress components. This transformation is achieved using the flow rule (see Eqs. 2.217 and 2.220), as well as Eqs. (2.247) (they are valid because \( \tilde{\tau}_b \) is defined for a biaxial stress state):

\[
\tilde{\tau}_b = \frac{1}{\bar{Y}_b} \left( \frac{\partial \bar{\sigma}}{\partial \sigma_{ab}} \right)_{b} - 1
\]

(2.256)
The notation \((\cdot)|_b\) means that the expression enclosed by parentheses should be calculated for biaxial tension along RD and TD. Eqs. (2.256), (2.239) and (2.250) lead to the following formula for \(\tilde{r}_b\):

\[
\tilde{r}_b = \frac{F_b}{\left(\frac{\partial \bar{\sigma}}{\partial \sigma_{11}}\right)|_b} - 1
\]  

(2.257)

We shall find now the expression of the denominator \(\left(\frac{\partial \bar{\sigma}}{\partial \sigma_{11}}\right)|_b\). We start by rewriting Eq. (2.223) for \(\alpha = \beta = 1\), assuming a biaxial stress state along RD and TD:

\[
\left(\frac{\partial \bar{\sigma}}{\partial \sigma_{11}}\right)|_b = \frac{\partial \tilde{\sigma}}{\partial \Gamma}|_b \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}}|_b + \frac{\partial \tilde{\sigma}}{\partial \Lambda}|_b \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}}|_b + \frac{\partial \tilde{\sigma}}{\partial \Psi}|_b \frac{\partial \tilde{\sigma}}{\partial \sigma_{11}}|_b
\]  

(2.258)

Equations (2.224), (2.248) and (2.250) allows us to express the derivatives \(\frac{\partial \tilde{\sigma}}{\partial \Gamma}|_b\), \(\frac{\partial \tilde{\sigma}}{\partial \Lambda}|_b\) and \(\frac{\partial \tilde{\sigma}}{\partial \Psi}|_b\):

\[
\frac{\partial \tilde{\sigma}}{\partial \Gamma}|_b = \frac{a}{F_b^{2k-1}} \left[ (\Lambda_b + \Gamma_b)^{2k-1} - (\Lambda_b - \Gamma_b)^{2k-1} \right]
\]

\[
\frac{\partial \tilde{\sigma}}{\partial \Lambda}|_b = \frac{1}{F_b^{2k-1}} \left\{ a \left[ (\Lambda_b + \Gamma_b)^{2k-1} + (\Lambda_b - \Gamma_b)^{2k-1} \right] + b \left[ (\Lambda_b + \Psi_b)^{2k-1} + (\Lambda_b - \Psi_b)^{2k-1} \right] \right\}
\]

\[
\frac{\partial \tilde{\sigma}}{\partial \Psi}|_b = \frac{b}{F_b^{2k-1}} \left[ (\Lambda_b + \Psi_b)^{2k-1} - (\Lambda_b - \Psi_b)^{2k-1} \right]
\]  

(2.259)

where \(\Gamma_b\), \(\Lambda_b\) and \(\Psi_b\) are defined by Eqs. (2.249). The other derivatives appearing in the right-hand side of Eq. (2.258) can be also expressed from Eqs. (2.225), (2.247) and (2.248):

\[
\frac{\partial \Gamma}{\partial \sigma_{11}}|_b = L, \quad \frac{\partial \Lambda}{\partial \sigma_{11}}|_b = \frac{N (N - P)}{\Lambda_b}, \quad \frac{\partial \Psi}{\partial \sigma_{11}}|_b = \frac{Q (Q - R)}{\Psi_b}
\]  

(2.260)

After replacing the quantities given by Eqs. (2.259) and (2.260) into Eq. (2.258), we get the following relationship for \(\left(\frac{\partial \tilde{\sigma}}{\partial \sigma_{11}}\right)|_b\) (see also Eqs. 2.249):

\[
\left(\frac{\partial \tilde{\sigma}}{\partial \sigma_{11}}\right)|_b = \frac{G_b}{F_b^{2k-1}}
\]  

(2.261)
where

\[ G_b = a \left[ \frac{N(N-P)}{\Lambda_b} + L \right] (\Lambda_b + \Gamma_b)^{2k-1} + a \left[ \frac{N(N-P)}{\Lambda_b} - L \right] (\Lambda_b - \Gamma_b)^{2k-1} + b \left[ \frac{N(N-P)}{\Lambda_b} + \frac{Q(Q-R)}{\Psi_b} \right] (\Lambda_b + \Psi_b)^{2k-1} + b \left[ \frac{N(N-P)}{\Lambda_b} - \frac{Q(Q-R)}{\Psi_b} \right] (\Lambda_b - \Psi_b)^{2k-1} \]

(2.262)

We can combine Eqs. (2.257) and (2.261) to obtain a formula for evaluating the coefficient of biaxial plastic anisotropy:

\[ \tilde{r}_b = \frac{F_b^{2k}}{G_b} - 1 \]  

(2.263)

Now we have all the quantities needed to construct the identification conditions (see Eqs. 2.226). At first, we shall refer to Eqs. (2.226.1), (2.226.2), (2.226.3) and (2.226.7) (the constraints associated to the yield stresses). Equations (2.232) and (2.252) allow us to rewrite Eqs. (2.226.1), (2.226.2), (2.226.3) and (2.226.7) in a more convenient form:

\[ [F(0^\circ)]^{2k} = y_0^{2k}, \quad [F(45^\circ)]^{2k} = y_{45}^{2k}, \quad [F(90^\circ)]^{2k} = y_{90}^{2k}, \quad F_b^{2k} = y_b^{2k}. \]  

(2.264)

where

\[ y_0 = \frac{Y}{Y_0}, \quad y_{45} = \frac{Y}{Y_{45}}, \quad y_{90} = \frac{Y}{Y_{90}}, \quad y_b = \frac{Y}{Y_b} \]  

(2.265)

are normalized values of the experimental yield stresses.

In a similar way, Eqs. (2.244), (2.263) and (2.264) lead to the following expressions of Eqs. (2.226.4), (2.226.5), (2.226.6) and (2.226.8) (the constraints associated to the coefficients of plastic anisotropy):

\[ G(0^\circ) = \frac{1}{r_0 + 1} y_0^{2k}, \quad G(45^\circ) = \frac{1}{r_{45} + 1} y_{45}^{2k}, \quad G(90^\circ) = \frac{1}{r_{90} + 1} y_{90}^{2k}, \quad G_b = \frac{1}{r_b + 1} y_b^{2k}. \]  

(2.266)

Finally, we use Eqs. (2.231), (2.229), (2.251), (2.249), (2.245) and (2.262) to put into evidence the unknown material parameters \( a, b, L, M, N, P, Q \), and \( R \) in the left-hand side of Eqs. (2.264) and (2.266):
\[
a(N + L)^{2k} + a(N - L)^{2k} + b(N + Q)^{2k} + b(N - Q)^{2k} = y_0^{2k}
\]
\[
a \left[ \sqrt{(N - P)^2 + 1} + L + M \right]^{2k} + a \left[ \sqrt{(N - P)^2 + 1} - L - M \right]^{2k} +
\]
\[
b \left[ \sqrt{(N - P)^2 + 1} + \sqrt{(Q - R)^2 + 1} \right]^{2k} +
\]
\[
b \left[ \sqrt{(N - P)^2 + 1} - \sqrt{(Q - R)^2 + 1} \right]^{2k} = (2y_{45})^{2k}
\]
\[
a(P + M)^{2k} + a(P - M)^{2k} + b(P + R)^{2k} + b(P - R)^{2k} = y_{90}^{2k}
\]
\[
a(N - P + L + M)^{2k} + a(N - P - L - M)^{2k} +
\]
\[
b(N - P + Q - R)^{2k} + b(N - P - Q + R)^{2k} = y_b^{2k}
\]
\[
a(P - M)(N + L)^{2k-1} + a(P + M)(N - L)^{2k-1} +
\]
\[
b(P + R)(N + Q)^{2k-1} + b(P - R)(N - Q)^{2k-1} = \frac{r_0}{r_{0+1}}y_{90}^{2k}
\]
\[
a\sqrt{(Q - R)^2 + 1} \left\{ \left[ \sqrt{(N - P)^2 + 1} + L + M \right]^{2k-1} +
\right.
\]
\[
\left. \left[ \sqrt{(N - P)^2 + 1} - L - M \right]^{2k-1} \right\} +
\]
\[
b \left\{ \left[ \sqrt{(N - P)^2 + 1} + \sqrt{(Q - R)^2 + 1} \right]^{2k} -
\right.
\]
\[
\left. \left[ \sqrt{(N - P)^2 + 1} - \sqrt{(Q - R)^2 + 1} \right]^{2k} \right\} =
\]
\[
\sqrt{(N - P)^2 + 1} \sqrt{(Q - R)^2 + 1} \frac{r_{45} + 1/2}{r_{45} + 1} (2y_{45})^{2k}
\]
\[
a(N - L)(P + M)^{2k-1} + a(N + L)(P - M)^{2k-1} +
\]
\[
b(N + Q)(P + R)^{2k-1} + b(N - Q)(P - R)^{2k-1} = \frac{r_0}{r_{0+1}^3}y_{90}^{2k}
\]
\[
a(N + L)(N - P + L + M)^{2k-1} + a(N - L)(N - P - L - M)^{2k-1} +
\]
\[
b(N + Q)(N - P + Q - R)^{2k-1} +
\]
\[
b(N - Q)(N - P - Q + R)^{2k-1} = \frac{1}{r_b^{3}}y_{b}^{2k}
\]
\[
(2.267)
\]

Equations (2.267) form together a set of 8 non-linear equations. The identification procedure uses Newton’s method to obtain its numerical solution.

From now on we shall manipulate Eqs. (2.267) in a generic form:

\[
f_i(a, b, L, M, N, P, Q, R) = 0, \quad i = 1, 2, \ldots, 8 \quad (2.268)
\]
where

1. \( f_1 = a (N + L)^{2k} + a (N - L)^{2k} + b (N + Q)^{2k} + b (N - Q)^{2k} - y_0^{2k} \)
2. \( f_2 = a \left[ \sqrt{(N - P)^2 + 1 + L + M} \right]^{2k} + a \left[ \sqrt{(N - P)^2 + 1 - L - M} \right]^{2k} + b \left[ \sqrt{(N - P)^2 + 1 + (Q - R)^2 + 1} \right]^{2k} + b \left[ \sqrt{(N - P)^2 + 1 - (Q - R)^2 + 1} \right]^{2k} - (2 y_{45})^{2k} \)
3. \( f_3 = a (P + M)^{2k} + a (P - M)^{2k} + b (P + R)^{2k} + b (P - R)^{2k} - y_0^{2k} \)
4. \( f_4 = a (N - P + L + M)^{2k} + a (N - P - L - M)^{2k} + b (N - P + Q - R)^{2k} + b (N - P - Q + R)^{2k} - y_b^{2k} \)
5. \( f_5 = a (P - M) (N + L)^{2k-1} + a (P + M) (N - L)^{2k-1} + b (P + R) (N + Q)^{2k-1} + b (P - R) (N - Q)^{2k-1} - \frac{r_o}{r_{90} + \tau} y_0^{2k} \)
6. \( f_6 = a \sqrt{(Q - R)^2 + 1} \left\{ \left[ \sqrt{(N - P)^2 + 1 + L + M} \right]^{2k-1} + \left[ \sqrt{(N - P)^2 + 1 - L - M} \right]^{2k-1} \right\} + b \left\{ \left[ \sqrt{(N - P)^2 + 1 + (Q - R)^2 + 1} \right]^{2k} - \left[ \sqrt{(N - P)^2 + 1 - (Q - R)^2 + 1} \right]^{2k} \right\} \right\} - \sqrt{(N - P)^2 + 1} \sqrt{(Q - R)^2 + 1} \frac{r_{45} + 1/2}{r_{45} + 1} (2 y_{45})^{2k} \)
7. \( f_7 = a (N - L) (P + M)^{2k-1} + a (N + L) (P - M)^{2k-1} + b (N + Q) (P + R)^{2k-1} + b (N - Q) (P - R)^{2k-1} - \frac{r_{90}}{r_{90} + 1} y_{90}^{2k} \)
8. \( f_8 = a (N + L) (N - P + L + M)^{2k-1} + a (N - L) (N - P - L - M)^{2k-1} + b (N + Q) (N - P + Q - R)^{2k-1} + b (N - Q) (N - P - Q + R)^{2k-1} - \frac{1}{r_b + \tau} y_b^{2k} \)

As mentioned in Sect. 2.5.3, the parameters involved in the expression of the equivalent stress are subjected to constraints: \( a > 0, b > 0, L > 0, M > 0, N > 0, P > 0, Q > 0, \) and \( R > 0. \) Aiming to ensure a natural treatment of these restrictions, the identification procedure replaces the parameters \( a, b, L, M, N, P, Q, \) and \( R \) with the following substitutes:
\[ a = (a')^2, \quad b = (b')^2, \quad L = (L')^2, \quad M = (M')^2, \]
\[ N = (N')^2, \quad P = (P')^2, \quad Q = (Q')^2, \quad R = (R')^2, \quad \text{(2.270)} \]
\[ a', b', L', M', N', P', Q', R' \in \mathbb{R} \]

In this way, \( f_1, f_2, \ldots, f_8 \) will become functions of the variables \( a', b', L', M', N', P', Q', \) and \( R' \). As a consequence, Eqs. (2.268) should be rewritten in the form
\[
f_i \left[ a (a'), b (b'), L (L'), M (M'), N (N'), P (P'), Q (Q'), R (R') \right] = 0, \quad i = 1, 2, \ldots, 8 \quad \text{(2.271)}
\]

The identification procedure will solve Eqs. (2.271) considering \( a', b', L', M', N', P', Q', \) and \( R' \) as unknowns. After finding the numerical solution, the values of the actual parameters \( a, b, L, M, N, P, Q, \) and \( R \) can be obtained from Eqs. (2.270).

Let \( a'_k, b'_k, L'_k, M'_k, N'_k, P'_k, Q'_k, \) and \( R'_k \) be the approximations of the numerical solution corresponding to the \( k \)-th Newton iteration. Eqs. (2.271) are linearised in the vicinity of this approximation using a truncated Taylor expansion:

\[
\begin{align*}
\frac{\partial f_i}{\partial a_k} \mid_k \frac{\partial a}{\partial a'} \mid_k (\Delta a'_k) + \frac{\partial f_i}{\partial b_k} \mid_k \frac{\partial b}{\partial b'} \mid_k (\Delta b'_k) + \\
\frac{\partial f_i}{\partial L_k} \mid_k \frac{\partial L}{\partial L'} \mid_k (\Delta L'_k) + \frac{\partial f_i}{\partial M_k} \mid_k \frac{\partial M}{\partial M'} \mid_k (\Delta M'_k) + \\
\frac{\partial f_i}{\partial N_k} \mid_k \frac{\partial N}{\partial N'} \mid_k (\Delta N'_k) + \frac{\partial f_i}{\partial P_k} \mid_k \frac{\partial P}{\partial P'} \mid_k (\Delta P'_k) + \\
\frac{\partial f_i}{\partial Q_k} \mid_k \frac{\partial Q}{\partial Q'} \mid_k (\Delta Q'_k) + \frac{\partial f_i}{\partial R_k} \mid_k \frac{\partial R}{\partial R'} \mid_k (\Delta R'_k) = - f_i \mid_k, \\
i = 1, 2, \ldots, 8
\end{align*}
\]

The symbol \( \mid_k \) means that the associated expression should be evaluated considering
\[ a' = a'_k, b' = b'_k, L' = L'_k, M' = M'_k, N' = N'_k, P' = P'_k, Q' = Q'_k, \] and \( R' = R'_k \).

The unknowns of the linearised set (see Eqs. 2.271 and 2.270) are the corrections \( \Delta a'_k, \Delta b'_k, \Delta L'_k, \Delta M'_k, \Delta N'_k, \Delta P'_k, \Delta Q'_k, \) and \( \Delta R'_k \). After adding them to \( a'_k, b'_k, L'_k, M'_k, N'_k, P'_k, Q'_k, \) and \( R'_k \), respectively, we obtain a new approximation of the numerical solution that should be used in the next iteration:
2.5 BBC 2005 Yield Criterion

\[
a'_{k+1} = a'_k + \Delta a'_k \\
b'_{k+1} = b'_k + \Delta b'_k \\
L'_{k+1} = L'_k + \Delta L'_k \\
M'_{k+1} = M'_k + \Delta M'_k \\
N'_{k+1} = N'_k + \Delta N'_k \\
P'_{k+1} = P'_k + \Delta P'_k \\
Q'_{k+1} = Q'_k + \Delta Q'_k \\
R'_{k+1} = R'_k + \Delta R'_k
\]

Two convergence criteria are used to stop the iterations:

\[
\sqrt{\frac{(\Delta a'_k)^2 + (\Delta b'_k)^2 + (\Delta L'_k)^2 + (\Delta M'_k)^2 + (\Delta N'_k)^2 + (\Delta P'_k)^2 + (\Delta Q'_k)^2 + (\Delta R'_k)^2}{(a'_{k+1})^2 + (b'_{k+1})^2 + (L'_{k+1})^2 + (M'_{k+1})^2 + (N'_{k+1})^2 + (P'_{k+1})^2 + (Q'_{k+1})^2 + (R'_{k+1})^2}} < 10^{-7}
\]

\[
\sqrt{\frac{8}{\sum_{i=1}^{8} (f_{i|k})^2}} < 10^{-5}
\]

The convergence of the Newton iterations is strongly influenced by the initial guess.

2.5.5 Particular Formulations of the BBC 2005 Yield Criterion

We can reduce BBC 2005 to Hill 1948 yield criterion if we choose the material parameters as follows:

\[
Y = Y_0 \\
k = 1 \\
Y = \begin{cases} 
1 + r_0 & \text{if } r_0 \leq 1 \\
1 + r_90 & \text{if } r_90 \leq 1 \\
\frac{1 + r_0}{r_90} & \text{if } r_0 > 1 \\
\frac{1 + r_90}{r_0} & \text{if } r_90 > 1
\end{cases} \\
a = \sqrt{\frac{1 + r_0}{r_0} \frac{1 + r_90}{r_90} - 1} \left( 1 + \frac{r_0}{r_90} \right) \left( r_{45} + \frac{1}{2} \right) \\
b = \sqrt{\frac{1 + r_0}{r_0} \frac{1 + r_90}{r_90} - 1} \left( \frac{a}{r_90} \right) \\
L = N = Q = \frac{1}{2\sqrt{a + b}} \\
M = P = R = \frac{1}{2} \sqrt{\frac{r_0}{1 + r_0} \frac{1 + r_90}{r_90} \frac{a}{a + b}}
\]

\[\text{(2.275)}\]
In this case, the identification procedure needs only \( r_0, r_{45} \) and \( r_{90} \) as input data. The yield criterion proposed by Barlat and Lian in 1989 can be also obtained by enforcing the following constraints on the material parameters:

\[
Y = Y_0, \ k = 3 \text{ or } 4, \ L = N = Q, \ M = P = R \tag{2.276}
\]

As above, the identification procedure needs only \( r_0, r_{45} \) and \( r_{90} \) as input data. Another situation of practical interest is the so-called normal anisotropy \( (r_0=r_{45}=r_{90}=r, Y_0=Y_{45}=Y_{90}=Y) \). In this case, BBC 2005 also reduces to the Hill 1948 or Barlat 1989 yield criteria (depending on the value of the exponent \( k \)):

\[
k = 1 \ (\text{Hill 1948}), \ k = 3 \text{ or } 4 \ (\text{Barlat 1989}), \\
a = \frac{1}{1+r}, \ b = \frac{r}{1+r}, \ L = N = Q = M = P = R = \frac{1}{2} \tag{2.277}
\]

There are many situations when the coefficient of biaxial plastic anisotropy \( (r_b) \) is not available as input data. The most convenient strategy for handling such cases consists in replacing Eq. (2.267.8) with the following constraint:

\[
N = P \tag{2.278}
\]

In this way, the number of identification equations is preserved. As a consequence, the solution procedure discussed in Sect. 2.5.4 will remain usable. The numerical tests performed for several sorts of sheet metals have shown that the above constraint leads to well-shaped yield loci (i.e. the predicted value of \( r_b \) is not far from 1).

The situations when the biaxial yield stress \( (Y_b) \) and the coefficient of biaxial plastic anisotropy \( (r_b) \) are not available can be handled by replacing Eqs. (2.267.4) and (2.267.8) with the following constraints:

\[
L + M = 2N, \ N = P \tag{2.279}
\]

This strategy has the same advantages as those mentioned above.

The situations when the uniaxial yield stresses \( Y_{45} \) and \( Y_{90} \), as well as the coefficient of biaxial plastic anisotropy \( r_b \) are not available can be handled by replacing Eqs. (2.267.2), (2.267.3) and (2.267.8) with the following constraints:

\[
M = R, \ L + M = 2N, \ N = P \tag{2.280}
\]

### 2.6 BBC 2008 Yield Criterion

In order to enhance the flexibility of the BBC 2005 yield criterion, a new version of this model has been developed [91]. The model is expressed as a finite series that can be expanded to retain more or less terms, depending on the volume of experimental
2.6 BBC 2008 Yield Criterion

Different identification strategies (using 8, 16, 24, etc. input values) could be used in order to determine the coefficients of the yield function.

2.6.1 Equation of the Yield Surface

The sheet metal is assumed to behave as a plastically orthotropic membrane under plane-stress conditions. We use the following description of the yield surface:

$$\bar{\sigma}(\sigma_{\alpha\beta}) - Y = 0 \quad (2.281)$$

where $\bar{\sigma}(\sigma_{\alpha\beta}) \geq 0$ is the equivalent stress defined in Sect. 2.6.2, $Y > 0$ is the yield parameter, and $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ ($\alpha, \beta = 1, 2$) are planar components of the stress tensor expressed in an orthonormal basis superimposed to the axes of plastic orthotropy: (1) rolling direction (RD), (2) transverse direction (TD), (3) normal direction (ND). The other components are subjected to the constraint

$$\sigma_{3i} = \sigma_{i3} = 0, \quad i = 1, 2, 3 \quad (2.282)$$

arising from the plane-stress hypothesis. Whenever not specified, the following convention will be adopted: Latin subscripts take the values 1, 2 and 3, while the Greek ones take only the values 1 and 2.

The equivalent stress defined in Sect. 2.6.2 does not enforce constraints on the choice of the parameter $Y$. In fact, any quantity representing a yield stress can act as $Y$. For example, $Y$ may be the uniaxial yield stress $Y_{\theta}$ associated to a planar direction defined by the angle $\theta$ measured from RD, an average of several uniaxial yield stresses, or the biaxial yield stress corresponding to the tension along RD and TD.

The flow rule associated to the yield surface described by Eq. (2.281) is

$$\dot{\varepsilon}^{(p)}_{\alpha\beta} = \dot{\lambda} \frac{\partial \bar{\sigma}}{\partial \sigma_{\alpha\beta}} \quad (2.283)$$

where $\varepsilon^{(p)}_{\alpha\beta} = \varepsilon^{(p)}_{\beta\alpha}$ are planar components of the plastic strain-rate tensor (expressed in the same basis as the corresponding components of the stress tensor), and $\dot{\lambda} \geq 0$ is a scalar multiplier (its significance is not essential for our discussion). The out of plane components of the plastic strain-rate are subjected to the constraints

$$\dot{\varepsilon}^{(p)}_{3i} = \dot{\varepsilon}^{(p)}_{i3} = 0, \quad \dot{\varepsilon}^{(p)}_{33} = -\dot{\varepsilon}^{(p)}_{11} - \dot{\varepsilon}^{(p)}_{22} \quad (2.284)$$

arising from the plane-stress hypothesis and the isochoric character of the plastic deformation [29].
2.6.2 BBC 2008 Equivalent Stress

The equivalent stress used in Eq. (2.281) is defined as follows:

\[
\sigma_{eq}^2 = \sum_{i=1}^{s} \left[ \left( L^{(i)} + M^{(i)} \right)^{2k} + \left( L^{(i)} - M^{(i)} \right)^{2k} \right] + \sum_{i=1}^{s} \left[ \left( M^{(i)} + N^{(i)} \right)^{2k} + \left( M^{(i)} - N^{(i)} \right)^{2k} \right]
\]

where

\[
k, s \in \mathbb{N}^* \quad w = \left( \frac{3}{2} \right)^{1/s} > 1
\]

\[
L^{(i)} = \ell_1^{(i)} \sigma_{11} + \ell_2^{(i)} \sigma_{22}
\]

\[
M^{(i)} = \sqrt{m_1^{(i)} (\sigma_{11} - \sigma_{22})^2 + m_3^{(i)} (\sigma_{12} + \sigma_{21})^2}
\]

\[
N^{(i)} = \sqrt{n_1^{(i)} (\sigma_{11} - \sigma_{22})^2 + n_3^{(i)} (\sigma_{12} + \sigma_{21})^2}
\]

\[
\ell_1^{(i)}, \ell_2^{(i)}, m_1^{(i)}, m_2^{(i)}, m_3^{(i)}, n_1^{(i)}, n_2^{(i)}, n_3^{(i)} \in \mathbb{R}
\]

The quantities denoted \(k, \ell_1^{(i)}, \ell_2^{(i)}, m_1^{(i)}, m_2^{(i)}, m_3^{(i)}, n_1^{(i)}, n_2^{(i)}, n_3^{(i)}\) \((i = 1, \ldots, s)\) are material parameters. One may prove that \(k \in \mathbb{N}^*\) is a sufficient condition for the convexity of the yield surface defined by Eqs. (2.281) and (2.285). From this point of view, there is no constraint acting on the admissible values of the other material parameters.

It is easily noticeable that Eq. (2.285) reduces to the isotropic formulation proposed by Barlat and Richmond [49] if

\[
\ell_1^{(i)} = \ell_2^{(i)} = m_1^{(i)} = m_2^{(i)} = m_3^{(i)} = n_1^{(i)} = n_2^{(i)} = n_3^{(i)} = 1/2, \quad i = 1, \ldots, s
\]

Under these circumstances, the exponent \(k\) may be chosen as in Barlat and Richmond’s model, i.e. according to the crystallographic structure of the sheet metal: \(k = 3\) for BCC materials (\(2k = 6\)), and \(k = 4\) for FCC materials (\(2k = 8\)).

The other parameters involved in Eq. (2.285) result from an identification procedure (see 2.6.3). Their number \(n_p\) is defined by the summation limit \(s\):

\[
n_p = 8s
\]

Let \(n_e\) be the number of experimental values describing the plastic anisotropy. The summation limit should be chosen according to the following constraint:

\[
n_p = 8s \leq n_e
\]
2.6 BBC 2008 Yield Criterion

i.e.

\[ s \leq n_e/8, \quad s \in N^* \]  \hspace{1cm} (2.289)

Apparently, Eq. (2.285) is usable only when \( n_e \geq 8 \). In fact, it also works with less experimental values. When such a situation occurs, the summation limit should be \( s = 1 \), and the \( n_e < 8 \) identification constraints arisen from experiments should be accompanied by at least \( 8 - n_e \) artificial conditions involving the material parameters. For example, if \( n_e = 6 \), we may enforce the equalities \( m_1^{(1)} = n_1^{(1)} \) and \( m_2^{(1)} = n_2^{(1)} \).

2.6.3 Identification Procedure

Due to the expandable structure of the yield criterion, many identification strategies can be devised. We shall restrict our discussion to a procedure that uses only normalized yield stresses and \( r \)-coefficients obtained from uniaxial and biaxial tensile tests.

Let \( Y_\theta \) be the yield stress predicted by the yield criterion in the case of a uniaxial traction along the direction defined by the angle \( \theta \) measured from RD. The planar components of the stress tensor are in this case

\[ \sigma_{11}|_\theta = Y_\theta \cos^2 \theta, \quad \sigma_{22}|_\theta = Y_\theta \sin^2 \theta, \]
\[ \sigma_{12}|_\theta = \sigma_{21}|_\theta = Y_\theta \sin \theta \cos \theta \] \hspace{1cm} (2.290)

After replacing them in Eq. (2.285), we get the associated equivalent stress

\[ \bar{\sigma}|_\theta = Y_\theta F_\theta \] \hspace{1cm} (2.291)

where \( F_\theta \) is defined by the relationships

\[ \frac{F^{2k}}{w^{-1}} = \sum_{i=1}^{s} \left\{ w^{s-i} \left\{ \left[ L^{(i)}_\theta + M^{(i)}_\theta \right]^{2k} + \left[ L^{(i)}_\theta - M^{(i)}_\theta \right]^{2k} \right\} \right\} + \]
\[ w^{s-i} \left\{ \left[ M^{(i)}_\theta + N^{(i)}_\theta \right]^{2k} + \left[ M^{(i)}_\theta - N^{(i)}_\theta \right]^{2k} \right\} \}

\[ L^{(i)}_\theta = \ell_1^{(i)} \cos^2 \theta + \ell_2^{(i)} \sin^2 \theta \] \hspace{1cm} (2.292)
\[ M^{(i)}_\theta = \sqrt{\left[ m_1^{(i)} \cos^2 \theta - m_2^{(i)} \sin^2 \theta \right]^2 + \left[ m_3^{(i)} \sin 2\theta \right]^2} \]
\[ N^{(i)}_\theta = \sqrt{\left[ n_1^{(i)} \cos^2 \theta - n_2^{(i)} \sin^2 \theta \right]^2 + \left[ n_3^{(i)} \sin 2\theta \right]^2} \]
Equations (2.281) and (2.291) lead to the following expression of the normalized uniaxial yield stress:

\[ y_\theta = \frac{Y_\theta}{Y} = \frac{1}{F_\theta} \]  

(2.293)

The \( r \)-coefficient corresponding to the uniaxial traction along a direction inclined at the angle \( \theta \) measured from RD is defined by the formula

\[ r_\theta = \frac{\dot{\varepsilon}_{\theta+90^\circ}}{\dot{\varepsilon}_{ND}} \]  

(2.294)

where \( \dot{\varepsilon}_{\theta+90^\circ} \) is the plastic strain-rate component associated to the \( \theta + 90^\circ \) planar direction, and \( \dot{\varepsilon}_{ND} \) is the through-thickness component of the same tensor. After some simple mathematical manipulations, Eq. (2.294) becomes

\[ r_\theta = F_\theta \frac{G_\theta}{G_\theta - 1} \]  

(2.295)

where \( G_\theta \) is defined by the relationships

\[
\begin{align*}
F_\theta^2 k^{\theta - 1} G_\theta &= \sum_{i=1}^{s} \left\{ w^{i-1} \left[ \hat{L}^{(i)} - \hat{M}^{(i)} \right] \left[ L^{(i)} - M^{(i)} \right]^{2k-1} + w^{s-i} \left[ \hat{M}^{(i)} + \hat{N}^{(i)} \right] \left[ M^{(i)} + N^{(i)} \right]^{2k-1} \right. \\
&\left. + w^{s-i} \left[ \hat{N}^{(i)} - \hat{M}^{(i)} \right] \left[ N^{(i)} - M^{(i)} \right]^{2k-1} \right\} \\
\hat{L}^{(i)} &= \ell^{(i)} + 1 \\
\hat{M}^{(i)} &= m_1^{(i)} - m_2^{(i)} \left[ m_1^{(i)} \cos^2 \theta - m_2^{(i)} \sin^2 \theta \right] / M^{(i)} \\
\hat{N}^{(i)} &= n_1^{(i)} - n_2^{(i)} \left[ n_1^{(i)} \cos^2 \theta - n_2^{(i)} \sin^2 \theta \right] / N^{(i)}
\end{align*}
\]

(2.296)

together with Eq. (2.292).

Let us denote by \( Y_b \) the yield stress predicted in the case of a biaxial traction along RD and TD. The corresponding planar components of the stress tensor are

\[ \sigma_{11}|_b = Y_b, \quad \sigma_{22}|_b = Y_b, \quad \sigma_{12}|_b = \sigma_{21}|_b = 0 \]  

(2.297)

After replacing them in Eq. (2.285), we get the associated equivalent stress
\[ \bar{\sigma} |_b = Y_b F_b \]  

(2.298)

where \( F_b \) is defined by the relationships

\[
\frac{F^2_b}{w-1} = \sum_{i=1}^{s} \left\{ w^{i-1} \left\{ \left[ L^{(i)}_b + M^{(i)}_b \right]^{2k} + \left[ L^{(i)}_b - M^{(i)}_b \right]^{2k} \right\} + \right. \\
\left. \left. w^s-i \left\{ \left[ M^{(i)}_b + N^{(i)}_b \right]^{2k} + \left[ M^{(i)}_b - N^{(i)}_b \right]^{2k} \right\} \right\} \right\} 
\]

(2.299)

\[
L^{(i)}_b = \hat{\ell}^{(i)}_1 + \hat{\ell}^{(i)}_2, \quad M^{(i)}_b = m^{(i)}_1 - m^{(i)}_2, \quad N^{(i)}_b = n^{(i)}_1 - n^{(i)}_2
\]

Equations (2.281) and (2.298) lead to the following expression of the normalized biaxial yield stress:

\[
y_b = \frac{Y_b}{Y} = \frac{1}{F_b} 
\]

(2.300)

The \( r \)-coefficient corresponding to the biaxial traction along RD and TD is defined by the formula

\[
r_b = \frac{\dot{\varepsilon}^{(p)}_{TD}}{\dot{\varepsilon}^{(p)}_{RD}} 
\]

(2.301)

where \( \dot{\varepsilon}^{(p)}_{RD} \) and \( \dot{\varepsilon}^{(p)}_{TD} \) are the plastic strain-rate components associated to the rolling and transverse directions, respectively. After some simple mathematical manipulations, Eq. (2.301) becomes

\[
r_b = \frac{\bar{F}_b}{G_b} - 1 
\]

(2.302)

where \( G_b \) is defined by the relationships

\[
\frac{\bar{F}^{2k-1}_b}{w-1} = \sum_{i=1}^{s} \left\{ w^{i-1} \left[ \hat{\ell}^{(i)}_b - \hat{M}^{(i)}_b \right] \left[ L^{(i)}_b + M^{(i)}_b \right]^{2k-1} + \right. \\
\left. \left. w^s-i \left[ \hat{M}^{(i)}_b - \hat{N}^{(i)}_b \right] \left[ M^{(i)}_b + N^{(i)}_b \right]^{2k-1} \right\} \right\} \\
\hat{L}^{(i)}_b = \ell^{(i)}_1, \quad \hat{M}^{(i)}_b = m^{(i)}_1, \quad \hat{N}^{(i)}_b = n^{(i)}_1
\]

(2.303)
An identification procedure that strictly enforces a large number of experimental constraints on the yield criterion would be inefficient in practical applications. The failure probability of such a strategy increases when the external restrictions become stronger. Taking into account this aspect, the authors have developed an identification procedure based on the minimization of the following error-function:

\[
E \left[ \ell_1^{(i)}, \ell_2^{(i)}, m_1^{(i)}, m_2^{(i)}, m_3^{(i)}, n_1^{(i)}, n_2^{(i)}, n_3^{(i)} \right]_{i = 1, \ldots, s} =
\sum_{\theta_j} \left( \frac{y_0^{(exp)}}{y_{\theta_j}} - 1 \right)^2 + \sum_{\theta_j} \left( \frac{r_0^{(exp)}}{r_{\theta_j}} - r_{\theta_j} \right)^2 + \left[ \frac{y_b^{(exp)}}{y_b} - 1 \right]^2 + \left[ \frac{r_b^{(exp)}}{r_b} - r_b \right]^2
\]  

(2.304)

where \( \theta_j \) represents an individual element from a finite set of angles defining the orientation of the specimens used in the uniaxial tensile tests. One may notice that Eq. (2.304) describes a square-distance between the experimental and predicted values of the anisotropy characteristics.

Two versions of the BBC 2008 yield criterion have been evaluated from the point of view of their performances (see [91]). They include 8 and 16 material coefficients, respectively, and correspond to the smallest values of the summation limit (\( s = 1 \) and \( s = 2 \)). The identification of the BBC 2008 (16 parameters) model has been performed using the following mechanical parameters: \( y_0^{(exp)}, y_{15^o}^{(exp)}, y_{30^o}^{(exp)}, y_{45^o}^{(exp)}, y_{60^o}^{(exp)}, y_{75^o}^{(exp)}, y_{90^o}^{(exp)}, y_b^{(exp)}, r_0^{(exp)}, r_{15^o}^{(exp)}, r_{30^o}^{(exp)}, r_{45^o}^{(exp)}, r_{60^o}^{(exp)}, r_{75^o}^{(exp)}, r_{90^o}^{(exp)} \) and \( r_b^{(exp)} \). In the case of BBC 2008 (8 parameters), the input data has been restricted to the values \( y_0^{(exp)}, y_{45^o}^{(exp)}, y_{90^o}^{(exp)}, y_b^{(exp)}, r_0^{(exp)}, r_{45^o}^{(exp)}, r_{90^o}^{(exp)} \) and \( r_b^{(exp)} \).

The predictions of the BBC 2008 model with 16 parameters are superior to those given by the 8-parameters version. The improvement is noticeable especially in the case of the \( r \)-coefficients. This capability of the 16-parameter version is relevant for the accurate prediction of the thickness when simulating sheet metal forming processes. For the materials exhibiting a distribution of the anisotropy characteristics that would lead to the occurrence of 8 ears in a cylindrical deep-drawing process [69] the planar distribution of the \( r \)-coefficient predicted by the BBC 2008 yield criterion with 8 parameters is very inaccurate (see [91]). This model would not be able to predict the occurrence of more than 4 ears at the top edge of a cup deep-drawn from a circular blank. In contrast, the variation of the \( r \)-coefficient described by BBC 2008 with 16 parameters closely follows the reference data. In conclusion, this model would predict the occurrence of 8 ears as reported by Yoon et al. [69].

As compared with other formulations described in the literature, the new model does not use linear transformations of the stress tensor. Due to this fact, its computational efficiency should be superior in the simulation of sheet metal forming processes.
2.7 Recommendations on the Choice of the Yield Criterion

As concerns the modelling of the anisotropic plasticity, the present state-of-the-art is somewhat confusing since most of the above-described yield criteria are still being used. The most important factors that must be taken into account when choosing the yield criterion are as follows:

- Accuracy of the prediction both of the yield locus and the uniaxial yield stress and uniaxial coefficient of plastic anisotropy
- Computational efficiency and ease of implementation in numerical simulation codes
- Flexibility of the yield criterion
- Degree of generality
- Number of mechanical parameters needed by the identification procedure
- Robustness of the identification procedure
- Experimental difficulties caused by the determination of the mechanical parameters involved in the identification procedure
- User-friendliness of the yield criterion
- Acceptance of the yield criterion in the scientific/industrial community.

We shall now analyse some of these factors.

2.7.1 Comparison of the Yield Criteria

At present, the most frequently used yield criteria are Hill 1948, Hill 1990 and Barlat 1989. Due to this fact, they have been chosen as reference models for comparison with one of the advanced yield criteria, namely BBC 2000. The comparison is focused on the following performance aspects: prediction of the yield locus geometry and description of the planar distribution both for uniaxial yield stress and uniaxial coefficient of plastic anisotropy. The aluminium alloy AA3103-0 has been chosen as a test material.

Figure 2.19 shows the normalised yield loci predicted by Hill 1948, Hill 1990, Barlat 1989 and BBC 2000 yield criteria together with experimental data [6, 70]. One may notice that the best predictions are provided by the BBC 2000 and Hill 1990 models, while the performances of the Hill 1948 and Barlat 1989 models are unsatisfactory, especially in the biaxial tension region.

The planar distribution of the uniaxial yield stresses predicted by the models is shown in Fig. 2.20. As one may notice, BBC 2000 and Hill 1990 have better performances. The predictions of the Hill 1948 and Barlat 1989 are in poorer agreement with the experimental data, due to the fact that only the uniaxial yield stress corresponding to the rolling direction is used by their identification procedure.
As concerns the prediction of the uniaxial coefficient of plastic anisotropy (Fig. 2.21), the BBC 2000, Hill 1948 and Barlat 1989 have the best performances. Due to the fact that only the coefficient corresponding to the diagonal direction is used in its identification procedure, Hill 1990 cannot provide good predictions.

We can conclude that the best quality of the predictions will be ensured by the yield criteria having an identification procedure based both on uniaxial and biaxial tension experimental data. As concerns the experimental data obtained by uniaxial tension tests, the identification should use at least the yield stresses and the coefficients of plastic anisotropy corresponding to three planar directions (0, 45 and 90°). Under such circumstances, the yield criterion will have to fit a minimum number of seven mechanical parameters.
2.7 Recommendations on the Choice of the Yield Criterion

The yield criteria using at least seven mechanical parameters in the identification procedure provide almost the same predictions of the anisotropic behaviour for usual materials. This fact can be noticed by comparing three of the most recently developed models, namely Barlat 2000 (Yld2000-2d) [5], Cazacu–Barlat 2001 (CB 2001) [75] and BBC 2000 [51]. The AA3103-0 aluminium alloy has been used in order to evaluate the accuracy of the predictions (see [92, 93]). Figure 2.22 shows the yield locus for this material measured in the first quadrant using biaxial tensile testing of cruciform specimens [92] and predicted with the Yld2000-2d [5], CB 2001 [75] and BBC 2000 [51] yield functions.

Figure 2.23 shows the experimental normalized yield stress and \( r \) value as a function of the tensile direction for a 3103-0 aluminium alloy sheet sample. The three
models are able to capture tensile anisotropy and the yield locus shape of this alloy very well.

The yield criteria that use a larger number of mechanical parameters in the identification (13 or even more—Barlat 2004, Soare 2007, BBC 2008 etc.) are able to provide highly accurate descriptions of the anisotropic behaviour. It is especially notable their capability to capture the occurrence of six or eight ears in the case of deep-drawing of cylindrical cups (see [69, 89]).

### 2.7.2 Evaluating the Performances of the Yield Criteria

As emphasized in the previous section, the performances of the yield criteria must be evaluated by thorough comparisons with experimental data. These comparisons should not be limited to the analysis of the yield locus shape. They should also envisage the planar distributions of the uniaxial yield stress and uniaxial coefficient of plastic anisotropy. In order to have a comprehensive evaluation tool, the members of the CERTETA research team have developed a global accuracy index defined as follows [70, 71, 94]:

\[
\beta = \phi + \delta + \gamma \%
\]  

(2.305)

where: $\phi$ is an accuracy index associated to the prediction of the yield locus shape in the plane of the principal stresses; $\delta$ is the accuracy index associated to the prediction of the planar distribution of the uniaxial yield stress; $\gamma$ is the accuracy index associated to the prediction of the planar distribution of the uniaxial coefficient of plastic anisotropy.
2.7 Recommendations on the Choice of the Yield Criterion

Fig. 2.24 Projection $Q_i$ of an experimental point $P_i$ onto the yield locus predicted by a yield criterion

$\varphi$ is computed by using the formula

$$\varphi = \sqrt{\frac{\sum_{i=1}^{n} d^2 (P_i, Q_i)}{Y}} \times 100 \text{ [\%]} \quad (2.306)$$

where $Y$ is the reference yield stress, $d^2(P_i, Q_i)$ is the squared distance from an experimental point $P_i$ and its projection $Q_i$ onto the yield locus predicted by the yield criterion under testing (see Fig. 2.24), and $n$ is the total number of the available experimental points.

$\delta$ is computed by using the formula

$$\delta = \sqrt{\frac{\sum_{i=1}^{n} \left( \frac{\sigma_{\theta_i}^{\text{exp}} - \sigma_{\theta_i}^{t}}{\sigma_{\theta_i}^{\text{exp}}} \right)^2}{n}} \times 100 \text{ [\%]} \quad (2.307)$$

where; $\sigma_{\theta_i}^{\text{exp}}$ is the experimental uniaxial yield stress corresponding to the direction defined by the angle $\theta_i$ (measured from the rolling direction); $\sigma_{\theta_i}^{t}$ is the predicted uniaxial yield stress associated to the same direction; $n$ is the total number of experimental points.

$\gamma$ is computed by using the formula

$$\gamma = \sqrt{\frac{\sum_{i=1}^{n} \left( \frac{r_{\theta_i}^{\text{exp}} - r_{\theta_i}^{t}}{r_{\theta_i}^{\text{exp}}} \right)^2}{n}} \times 100 \text{ [\%]} \quad (2.308)$$

where; $r_{\theta_i}^{\text{exp}}$ is the experimental anisotropy coefficient corresponding to the direction defined by the angle $\theta_i$ (measured from the rolling direction); $r_{\theta_i}^{t}$ is the predicted anisotropy coefficient corresponding to the same direction; $n$ is the total number of experimental points.

The practical use of the global accuracy index is exemplified by comparing four yield criteria: Hill 1948, Hill 1990, Barlat 1989 and BBC 2000. The identification procedure of the BBC 2000 model is based on the minimisation of an
error-function. The comparison has been performed in the case of three aluminium alloys: AA3103-0, AA5182-0 and AA6111-T4. The values of the individual accuracy indices are listed in Table 2.2. The best overall performance corresponds to the lowest value of the global index $\beta$. Table 2.2 shows that BBC 2000 has better performances as compared to the other yield criteria (the corresponding overall index is three times smaller than in the case of the Hill 1948 yield criterion).

### 2.7.3 Mechanical Parameters Used by the Identification Procedure of the Yield Criteria

Table 2.3 shows the mechanical parameters needed for the identification of several yield criteria. On the basis of this list, we can estimate the amount of experimental tests and the costs required for identification of various yield criteria. The main question of interest is whether a biaxial yield stress and biaxial anisotropy coefficient have to be determined since this requires a special apparatus: either for cross tensile tests, hydraulic bulge tests or for disk compression, respectively.

Table 2.3 makes reference only to plane-stress models (2D). The following notations have been used in the table: 3D signifies the fact that the model is extendable to spatial stress states; A1 shows that the yield criterion is able to describe ‘the first order anomalous behaviour’ (see [26]); A2 shows that the yield criterion is able to describe ‘the second order anomalous behaviour’ (see [28]). The yield criteria belonging to Hershey family use an exponent chosen in accordance with the crystallographic structure of the material.

### 2.7.4 Implementation of the Yield Criteria in Numerical Simulation Programmes

The main criteria for selecting the yield function for implementation in Finite Element code are the prediction precision of the anisotropic behaviour and the CPU
2.7 Recommendations on the Choice of the Yield Criterion

Table 2.3  The mechanical parameters needed for the identification of several yield criteria

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The table above shows the mechanical parameters needed for the identification of several yield criteria. The time efficiency of the program. Table 2.4 presents the main commercial FE software and the anisotropic yield criteria implemented in them. The Barlat 2000, Vegter and BBC 2005 models have been implemented by various users in the material subroutines of ABAQUS and LS-DYNA.

Table 2.4  Various yield criteria implemented in some commercial codes

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2.7.5 Overview of the Anisotropic Yield Criteria Developing

The yield criterion proposed by Hill in 1948 [25] is still very used in industrial applications. Its general acceptance is due to the mathematical simplicity and the direct significance of the coefficients. The identification procedure of this yield criterion uses only four mechanical parameters. As a consequence, it cannot accurately describe the planar distribution of both the uniaxial yield stress and the uniaxial coefficient of plastic anisotropy. In addition, it cannot capture the biaxial yield stress.

The discussion in the previous sections of the chapter has shown the stages followed by the development of the anisotropic yield criteria:

Methods used for transforming isotropic formulations into anisotropic ones:

- inclusion of new coefficients into isotropic models: Hill 1948 [25], Hill 1979 [29], Hosford 1979 [31]
- use of linear transformations: Barlat 1991 [10], Karafillis-Boyce [21]
- representation functions: Cazacu–Barlat 2001 [75]
- geometrical methods: Vegter [80], Mollica [82]
- extension of the yield criteria outside the orthotropy axes by using coordinate transformations: Barlat 1989 [50], Hill 1990 [39].

Identification strategies:

- numerical procedures:
  - minimization of an error-function: BBC 2000 [51], BBC 2008 [91]
  - genetic algorithms: Chaparro et al. [95].

Further information referring to the most recent anisotropic yield criteria can be found in the synthesis papers/monographs [96–101].

2.7.6 Perspectives

As it can be seen from the previous sub-chapters, advanced yield criteria allow accurate prediction of the anisotropic behaviour of materials. On the one hand, it is possible to simultaneously describe both the uniaxial yield stress variation and the anisotropic coefficient in the sheet. On the other hand, it is also possible to model both ‘first and second order anisotropic behaviour anomalies’. As well as this, the yield criteria have been extended to 3D. The asymmetry of the yield loci can be accurately predicted, thus allowing modeling the strength differential effect specific to materials with hexagonal close packed structure.
In the future the research in this field of study will be oriented towards developing new models which include special properties (superplastic materials, shape memory materials etc.). By including the evolution of the coefficients in yield functions it will be possible to predict the yield loci for nonlinear loading. Stochastic modeling will be used for a more robust prediction of the yield loci (taking into account the variability of the mechanical parameters). Coupling of the phenomenological models with the ones based on crystal plasticity will allow better simulation of the parameters evolution in technological processes (these include temperature, strain rate, strain path, structural evolution).

Therefore, the virtual process chain will be described more accurately, allowing it to be used in real fabrication processes.

2.8 Modeling of the Bauschinger Effect

2.8.1 Reversal Loading in Sheet Metal Forming Processes

In a typical sheet metal forming process a considerable amount of material undergoes non-proportional loading. Such as during passing over a tool radius or through a draw bead material is subjected to the multiple bending-unbending, which means reverse tension-compression loading over the sheet thickness. Figure 2.25 shows the stress distribution for these two cases, calculated with the commercial FE-code ABAQUS [102].

It is well known that the mechanical response of metals depends not only on the current state but also on the previous deformation history. One of the phenomena related to the change of mechanical properties during a non-proportional loading is the Bauschinger effect. It is usually associated with conditions where the yield strength of a metal decreases when the direction of strain is changed. Figure 2.26a illustrates the Bauschinger effect for the uniaxial tension-compression

![Fig. 2.25 Reversal tension-compression loading (red for the tension stress) during passing over a tool radius (left) or through a draw bead. Arrows show the direction of drawing](image)
Fig. 2.26 Schematic of the Bauschinger effect (a) and its influence on the stress evolution during passing over a die radius (b)

First material hardens in tension to the stress $\sigma_h$ and then loaded in compression. Plastic deformation occurs before negative yield strength (dashed line presents material behavior without the Bauschinger effect). Similar reversal stress-strain history occurs when material passes over a die radius (Fig. 2.26b). During the first bending there is a tension on the outer side of a sheet and a compression inside. During the second unbending the stress state over the sheets thickness reverses and is influenced directly by the Bauschinger effect.

Especially for a springback prediction an accurate description of the material behavior during reversal loading is essential: on the one hand it is important to know the exact stress distribution at the end of the forming process before unloading starts, on the other hand it is necessary to model unloading with the proper stress-strain response [103]. Since the magnitude of a springback depends on the yield strength and the Young’s modulus, more proper material modeling including the Bauschinger effect has become necessary particularly for the new materials like high strength steels and aluminium alloys.

2.8.2 Experimental Observations

The experimental procedure in order to get cyclic tension-compression curves for sheet metals is complex as sheet metal tends to buckle during compression loading. This makes the measurement delicate, especially for large strains. Several techniques to perform such tests have been developed recently [103–105]. Based on these investigations one can separate the Bauschinger effect into three partial effects: early re-plastification, transient softening and workhardening stagnation leading to
2.8 Modeling of the Bauschinger Effect

Fig. 2.27 Uniaxial tension and tension-compression curves with different pre-strains for DP600 steel

reduced yielding. Figure 2.27 shows tension and two tension-compression curves at different pre-strains of 8 and 14% for the high strength steel DP600. A special representation of the compression curves as the positive effective stress over the accumulated true strain permits the observation of the workhardening stagnation, which is typical for many materials and looks like a delay in the hardening for a certain amount of strain.

Noticeable there is a non-linear character on the stress-strain curve directly after the load reversal, so that the tangential modulus is lower than the elastic modulus. One call this phenomena can early re-yielding or early re-plastification [104–106].

Fig. 2.28 The Bauschinger effect during reversal shear loading and the cross effect during orthogonal loading for mild steel DC06 [107]
Another phenomenon related to the Bauschinger effect is the so-called reduction of the Young’s modulus depending on the pre-strain. In [104, 105] it has been shown that the initial modulus typically reduces exponentially with the accumulated pre-strain. In [107] not only the sign of the loading but also its direction has been verified. During the orthogonal loading (uniaxial tension followed by simple shear) the increase of the yield stress has been observed (see Fig. 2.28). This phenomena is the so-called cross effect.

### 2.8.3 Physical Nature of the Bauschinger Effect

Before dealing with phenomenological modeling of the Bauschinger effect, it is advisable to get a good understanding of the physical origin of it, what may lead to more refined plasticity models and may ultimately improve the simulation results. As for the metal plasticity at room temperature in general, the main source for the Bauschinger effect is a dislocation structure.

As deformation occurs, dislocations move, activating slip on the energetically favorable slip systems, and the dislocation density increases gradually. Dislocations overlap, accumulate at obstacles producing dislocation tangles and pile-ups (Fig. 2.29). This increases the resistance to further dislocation motions and causes a hardening of the metal.

The Bauschinger effect can be generally ascribed to long-range effects, such as internal stresses due to dislocation interactions, dislocation pile-ups at grain boundaries or Orowan loops around strong precipitates, and to short-range effects, such as directionality of mobile dislocations in their resistance to motion or annihilation of the dislocations during the reverse loading.

![Fig. 2.29](image-url) Edge dislocation (a) and dislocation pile-ups on the grain boundaries (b)
The primary driving force of the Bauschinger effect can be explained by the motion of the less stable dislocation structures such as pile-ups. Pile-up occurs as a cluster of dislocations is unable to move past the barrier. As accumulated dislocations generate microscopic back-stresses, they will assist the movement of dislocations in the reverse direction and the yield strength becomes lower. This occurs directly after the change of load direction or during unloading and takes place simultaneously with elastic deformation. With this microscopic mechanism one can explain such macroscopic phenomena as the transient softening, the early re-plastification and the reduction of the Young’s modulus.

Another mechanism is, when the strain direction is reversed, dislocations of the opposite sign can be produced from the same source that produced the slip-causing dislocations in the initial direction. Dislocations with opposite signs can attract and annihilate each other. Since strain hardening is related to an increased dislocation density, reducing the number of dislocations reduces strength.

The workhardening stagnation can be explained by the partial disintegration of the performed dislocation cell structures and the subsequent resumption of work-hardening to the formation of new dislocation structures [105, 107].

The so-called cross effect during orthogonal loading is referred to the fact that the dislocation structures which developed during pre-loading in a given direction act as obstacles to slip on systems activated in the orthogonal direction after the change of loading direction [107].

Other mechanisms beside the crystallographic slip can also macroscopically contribute to the Bauschinger effect. Twinning is crucial particularly for the metals with hexagonal close-packed lattice such as magnesium or zircon. During the cold forming of the magnesium alloys the twinning under compression can occur, which leads to the essential reduction of the yield strength. Other factors which contribute to such material behavior on the macroscopic level could be a change of the crystallographic texture during plastic deformation, stress induced phase transformation or porosity evolution.

### 2.8.4 Phenomenological Modelling

The Bauschinger effect is usually being predicted by using the anisotropic (also known as distortional) hardening, which describes in general a transformation of the yield surface during the plastic deformation. However, as the experimental observations show, the distortion of the yield surface is very complex and includes its translation and change of its size and shape [108]. To describe such a behavior phenomenologically a large amount of experiments is needed, however with reasonable effort one can describe the translation of the yield surface (kinematic hardening) and the change of its size (isotropic hardening, Fig. 2.30).

It is necessary to distinguish between the aspects: the word isotropic can be related to both yield criteria and to the evolution of yield surface, while kinematic is applied only to the evolution. A position of the yield surface will be described with the back-stress tensor \( \alpha \). It is obvious, that as soon as plastic pre-strain occurs
and the back-stress tensor has a value, the kinematic hardening leads to the plastic anisotropy even if isotropic yield criterion is used.

Before starting with particular models describing kinematic hardening, let us first refer to the classical framework usually being used for elastoplastic modeling with isotropic hardening.

Assuming small elastic and large plastic deformations for metals, the rate of deformation can be decomposed into elastic and plastic parts as

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$$ (2.309)

The stress-strain response in elastic range is assumed to be linear

$$\dot{\sigma} = C : \dot{\varepsilon}_e$$ (2.310)

where $C$ is the elasticity tensor.

The plastic flow occurs when the yield criterion is valid:

$$f = \phi(\sigma) - R - \sigma_0 = 0$$ (2.311)

where $\sigma_0$ is the initial yield stress, $R$ is the scalar function, which introduces hardening and $\phi$ is the homogeneous function, which indicates the form of the yield surface. For simplicity, let us assume the von Mises isotropic yield function derived from the second invariant of the stress tensor

$$\phi = J_2(\sigma) = \sqrt{\frac{3}{2}} ||S||$$ (2.312)

where $S$ is stress deviator tensor.
The consistency condition during the plastic flow is

\[ \dot{f} = \frac{\partial f}{\partial \sigma} \dot{\sigma} + \frac{\partial f}{\partial R} \dot{R} = 0 \]  

(2.313)

The relation between plastic strains and stresses is defined by the flow rule:

\[ \dot{\varepsilon}_p = \lambda \frac{\partial \Phi}{\partial \sigma} \]  

(2.314)

where \( \lambda \) is the Lagrange multiplier and \( \Phi \) is the plastic potential, which defines the direction of plastic flow. From (2.314) follows the normality of \( \dot{\varepsilon}_p \) to the yield surface for the associated flow rule (\( \Phi = f \)).

The equivalent plastic strain rate is defined as

\[ \dot{\rho} = -\dot{\lambda} \frac{\partial \Phi}{\partial R} \]  

(2.315)

For the associated flow rule and von Mises plasticity one can now derive the plastic multiplier

\[ \dot{\lambda} = \dot{\rho} = \sqrt{\frac{2}{3} \| \dot{\varepsilon}_p \|} \]  

(2.316)

Finally substituting (2.315) into (2.310) one can write

\[ \dot{\sigma} = C : (\dot{\varepsilon} - \dot{\lambda} \frac{\partial \Phi}{\partial \sigma}) \]  

(2.317)

Additionally, the hypothesis of plastic incompressibility (independence on the hydrostatic stress) will be assumed:

\[ \frac{\partial \Phi}{\partial (tr[\sigma])} = 0 \]  

(2.318)

### 2.8.4.1 Prager’s Model

The hardening in this formulation corresponds to the pure translation of the yield surface in stress space and will be described by the back-stress tensor \( \alpha \). The yield condition in this case is

\[ f = \sqrt{\frac{3}{2} \| S - \alpha \| - \sigma_0} = 0 \]  

(2.319)

In the theory of Prager, it is assumed that during loading the back-stress develops in the same direction as the plastic strain increment [108]:
\[ \dot{\alpha} = c \dot{\varepsilon}_p \] (2.320)

where \( c \) is a material constant. This model is the simplest case of the pure kinematic hardening. However, it can not accurately describe material behavior using the linear hardening.

### 2.8.4.2 Model of Armstrong and Frederick

To get a better description of a hardening and to make a smoother transition from elastic to inelastic behaviour, Armstrong and Frederick [109] suggested to enhance the evolution equation (2.320) to one of the Bailey-Orowan type:

\[ \dot{\alpha} = c \left( \frac{2}{3} a \dot{\varepsilon}_p - \alpha \dot{\rho} \right) \] (2.321)

where \( c \) and \( a \) are the material parameters. The additional recovery term with the accumulated plastic strain \( p \) describes a kind of memory effect and leads to an exponential evolution character of the back-stress, which corresponds much better to experimental observations.

### 2.8.4.3 Chaboche’s Model

The previous models described the hardening only by the translation of the yield surface. With further improvement, Chaboche presented a mixed hardening rule as a superposition of nonlinear isotropic and kinematic hardening [108, 110].

Then the yield criterion is

\[ f = \sqrt{\frac{3}{2}} ||S - \alpha|| - R - \sigma_0 = 0 \] (2.322)

with the back-stress

\[ \dot{\alpha} = c \left( \frac{2}{3} a \dot{\varepsilon}_p - \alpha \dot{\rho} \right) \] (2.323)

and the isotropic part of the hardening

\[ \dot{R} = b(Q - R) \dot{\rho} \] (2.324)

where \( c, a, b \) and \( Q \) are the material constants.

As it has been mentioned in [110] one non-linear term of the back-stress is not sufficient for the description of large-strain hardening. To improve it one can use several back-stresses simultaneously

\[ \alpha = \sum_{m=1}^{M} \alpha^{(m)} \] (2.325)
\[ \dot{\alpha}^{(m)} = c^{(m)} \left( \frac{2}{3} \alpha^{(m)} \dot{\varepsilon}_p - \alpha^{(m)} \dot{p} \right) \] (2.326)

including one linear

\[ \dot{\alpha}^{(M)} = \frac{2}{3} H \dot{\varepsilon}_p \] (2.327)

The model of Chaboche with many back-stress components is able to describe the transient softening quite accurately. However, the phenomena of the early re-plastification and the workhardening stagnation are still uncovered.

### 2.8.4.4 Yoshida–Uemori Model

The model of Yoshida and Uemori [105, 111] is a further improvement of the Chaboche’s model which additionally describes the early re-plastification and the workhardening stagnation.

This model assumes two surfaces: the yield surface \( f \) develops within the bounding surface \( F \). Based on the wide experimental investigations Yoshida and co-workers apply the pure kinematic hardening for the yield surface since the early re-plastification occurs at a very early stage of load reversal. At the same time the bounding surface has mixed isotropic-kinematic hardening:

\[ f = \sqrt{\frac{3}{2}} \| S - \alpha \| - \sigma_0 = 0 \] (2.328)
\[ F = \sqrt{\frac{3}{2}} \| S - \beta \| - R - B = 0 \] (2.329)

where \( \beta \) denotes the center of the bounding surface, and \( B \) and \( R \) are its initial size and isotropic hardening.

The flow rule is defined as

\[ \dot{\varepsilon}_p = \dot{\lambda} \frac{\partial f}{\partial \sigma} \] (2.330)

One can write the back-stress as the sum

\[ \alpha = \alpha_* + \beta \] (2.331)

where \( \alpha_* \) is the relative motion of the yield surface with respect to the bounding surface

\[ \dot{\alpha}_* = \sqrt{\frac{2}{3}} c a (n_p - \sqrt{\frac{\alpha_*}{a}} n_a) \dot{p} \] (2.332)

in terms of the non-dimensional measures.
\[ n_p = \frac{S - \alpha}{\| S - \alpha \|}, \quad n_s = \frac{\alpha}{\| \alpha \|} \] (2.333)

with

\[ \dot{p} = \sqrt{\frac{2}{3}} \| \dot{\varepsilon}_p \|, \quad \bar{\alpha}_s = \sqrt{\frac{3}{2}} \| \alpha_s \|, \quad a = B + R - \sigma_0 \] (2.334)

Kinematic hardening for the bounding surface is

\[ \dot{\beta} = m \left( \sqrt{\frac{2}{3}} b n_p - \beta \right) \dot{p} \] (2.335)

and its isotropic hardening

\[ \dot{R} = m(R_{sat} - R) \dot{p} \] (2.336)

To describe the workhardening stagnation Yoshida and Uemori proposed to modify the evolution of the bounding surface by providing a new surface \( g \) in the stress space

\[ g = \sqrt{\frac{3}{2}} \| S - q \| - r = 0 \] (2.337)

where \( q \) and \( r \) denote the center and the size of the surface.

It will be assumed that the center of the bounding surface \( \beta \) exists either on or inside of the surface \( g \) and the isotropic hardening of the bounding surface \( R \) takes place when \( \beta \) stays on the surface \( g \), namely \( \dot{R} > 0 \) when

\[ g = \sqrt{\frac{3}{2}} \| \beta - q \| - r = 0 \text{ and } \frac{\partial g}{\partial \beta} : \dot{\beta} > 0 \] (2.338)

and \( \dot{R} = 0 \) otherwise (see Fig. 2.31).

The kinematic motion of the surface \( g \) is defined as

\[ \text{Fig. 2.31} \quad \text{Schematic illustration of the surface } g \text{ in stress space} \]
\[ \dot{q} = \mu (\beta - q) \]  

(2.339)

with

\[ \mu = \frac{\Gamma - \dot{r}}{r} \text{ and } \Gamma = \frac{3(\beta - q) : \dot{\beta}}{2r} \]  

(2.340)

the evolution of \( r \) is

\[ \dot{r} = h\Gamma \text{ when } \dot{R} > 0 \]
\[ \dot{r} = 0 \text{ when } \dot{R} = 0 \]  

(2.341)

where \( h(0 \leq h \leq 1) \) is a material parameter.

After analysing the ability of the above model to describe experimental results it was found that the early re-plastification phenomenon can not be covered well enough. The proposed solution is to use the variable Young’s modulus as a function of the accumulated plastic strain \( p \):

\[ E = E_0 - (E_0 - E_a) \left( 1 - e^{-\xi p} \right) \]  

(2.342)

where \( E_0 \) and \( E_a \) are the values of Young’s modulus for virgin and infinitely large pre-strained material, respectively, and \( \xi \) is a material parameter. For the definition of the Young’s modulus Yoshida suggests an average value in the stress range \( 0 \leq \sigma \leq 0.95 \sigma_r \), where \( \sigma_r \) indicates the stress reversal point.

### 2.8.4.5 AutoForm-Model

To improve the springback prediction a novel approach to model the Bauschinger effect has been developed and implemented in the commercial code AutoForm [112]. The main idea of the model is to use the same evolution equation for the entire unloading and reverse loading path, including the area, which is treated as an elastic in conventional models. As the model is part of an undisclosed research, the principal of it will be presented here for the uniaxial tension-compression case using fictive values of the reversal stress \( \sigma_r \) and the reversal strain \( \epsilon_r \) with the coordinate center in the point of load reversal

\[ \epsilon_r = \frac{\sigma_r}{E_l(p)} + K \arctan h^2 \left( \frac{\sigma_r}{2\sigma_h(p)} \right)^2 \]  

(2.343)

with \( \sigma_h(p) \) as hardening stress used to describe tension curve (see Fig. 2.32) and \( E_l \) as the initial Young’s modulus at the moment of load reversal. Analog to the Yoshida-Uemori model \( E_l \) is a function of the accumulated plastic strain \( p \):

\[ E_l = E \left( 1 - \gamma \left( 1 - e^{-\xi p} \right) \right) \]  

(2.344)
where $\gamma$ and $\chi$ are two material parameters and $E$ is the elastic modulus.

One can then write the definition for tension and compression parts of the curve

$$\sigma = \begin{cases} 
\sigma_h(p) & \text{for tension} \\
\sigma_h(p) - \sigma_r & \text{for compression}
\end{cases}$$  \hspace{1cm} (2.345)

On the contrary to the previous models, here the evolution equations for the hardening stress and for the back-stress are much more decoupled. The definition of the hardening during proportional loading is independent from the definition of the back-stress and hardening curve can be defined even in tabular form.

As in the model of Yoshida-Uemori, the pure kinematic hardening is assumed here. However, the model in AutoForm uses the non-linear stress-strain response within the yield surface and can describe the early re-plastification very accurately.

In order to model the workhardening stagnation, the accumulated equivalent plastic strain $p$ is replaced by a new hardening parameter $p_d$, which behaves as follows: $p_d$ is identical with $p$ during proportional deformation and develops slower than $p$ during reverse or non-proportional deformation. To determine if a deformation is reverse or non-proportional, a storage surface $g$ is introduced (Fig. 2.33). During the initial proportional loading (time $t$), the progression of plastic strain extends the storage surface and the strain-like variable $q$ lies on the surface $g$. During the further reverse or non-proportional loading (time $t + \Delta t$), the tensor $q$ traverses the storage surface; during the time it is inside, the variable $p_d$ develops slower than $p$. This behavior is controlled by a material parameter $\xi (0 \leq \xi \leq 1)$, which characterizes the fraction of forward strain that can be reversed with the delay of work hardening.

The set of equations is:

$$\dot{p}_d = f \cdot \dot{p}$$  \hspace{1cm} (2.346)
2.8 Modeling of the Bauschinger Effect

Fig. 2.33 Schematic illustration of the storage surface $g$

\[ f = \text{sign}(\tilde{f})\tilde{f}^2 \quad (2.347) \]
\[ \tilde{f} = 1 - \frac{3}{4} \frac{r}{g} \quad (2.348) \]

In the beginning is $g = 0$ and grows proportionally to $\dot{p}_d$:
\[ \dot{g} = \xi \dot{p}_d \quad (2.349) \]

$r$ in Eq. (2.348) is a scalar value
\[ r = \phi (g - q) \quad (2.350) \]

where $q$ is a tensor representing the position within the surface $g$
\[ \dot{q} = \dot{g} + \dot{\rho} \left[ \frac{\partial r}{\partial g} \right]^{-1} \quad (2.351) \]

and
\[ \dot{g} = \frac{\dot{g}}{\varphi(\sigma)} \quad (2.352) \]

Considering Eqs. (2.350) and (2.351) and since $q$ cannot exceed the surface $g$, one derives the rate of $r$ as
\[ \dot{r} = -\dot{\rho} \text{ when } r > 0 \]
\[ \dot{r} = 0 \text{ when } r = 0 \quad (2.353) \]

Finally, assuming all the discussed above phenomena, the model is able to describe accurately experimental tension-compression curves. Figure 2.34a shows the uniaxial tension and tension-compression data for the high-strength steel DP600
Fig. 2.34 Uniaxial tension and tension-compression curves (a) and evolution of the tangential modulus (b) during compression for DP600 steel calculated with AutoForm-model together with experimental results

Table 2.5 Kinematic hardening parameters for DP600 steel

<table>
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<tr>
<th></th>
<th>$K$</th>
<th>$\xi$</th>
<th>$\gamma$</th>
<th>$\chi$</th>
<th>$E_l$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.011</td>
<td>1</td>
<td>0.13</td>
<td>30</td>
<td>200</td>
</tr>
</tbody>
</table>
for two tests with different pre-strains of 8 and 14%. The second picture on the figure presents the evolution of tangent modulus starting from the reversal point, which allows to see the non-linear character of the stress-strain response directly after the load reversal.

Table 2.5 presents the model parameters for the DP600 steel. For the description of tensile hardening curve tabulated data from the tension test has been used.

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